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Maximal smoothings of real plane curve singular points

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To V.A.Rokhlin who guided certain of us in the marvelous world behind reals.

Abstract

The local Harnack inequality bounds from above the number of ovals which can appear in a small perturbation of a singular point. As is known, there are real singular points for which this bound is not sharp. We show that Harnack inequality is sharp in any complex topologically equisingular class: every real singular point is complex deformation equivalent to a real singularity for which Harnack inequality is sharp. For semi-quasi-homogeneous and some other singularities we exhibit a real deformation with the same property. A refined Harnack inequality and its sharpness are discussed as well.

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1 Introduction

The Harnack theorem [6] states that a real plane projective curve of degree d has at most $\frac{1}{2}(d-1)(d-2)+1$ connected components, and for any degree d curves with this number of components (M-curves) do exist. Every M-curve of degree d is nonsingular and can be considered as a nonsingular perturbation of an ordinary homogeneous singularity of multiplicity d: pick a generic system of affine coordinates and consider the family $f_\varepsilon(x,y) = \varepsilon^d f(\varepsilon^{-1}x, \varepsilon^{-1}y)$, where $f=0$ is an equation of the curve.

Small nonsingular real perturbations (further on referred to as *smoothings*) of an isolated real plane curve singular point is subject to the local analogue of the Harnack bound: the number v of closed components (further called *ovals*) of a smoothing satisfies the inequality

$$2v \le \begin{cases} \mu - r + 1, & \text{if the singularity has a real branch,} \\ \mu - r + 3, & \text{otherwise} \end{cases} \tag{1}$$

where μ is the Milnor number and r the number of complex branches of the singular point (see [1, 16]). The question on the sharpness of this estimate (i.e., on the existence of M-smoothings) turned to be more subtle than for projective curves: as it is proven in [16], (1) is sharp for unibranch singular points (see [15] for a different proof); on the other hand, there are singular points which have no M-smoothing (see [9]). It brings us before the task to describe the class of singular points for which (1) is sharp. The present paper is devoted to this problem (additional information is found in [10] and [4]; [4] contains some mistaken statements, easily recognizable, cf., remarks below).

We prove that for any singular point there exists a singular point which is topologically equivalent to it over $\mathbb{C}$ and has M-smoothing (more detailed statement is given in Theorem 1, section 3.2). Another result of the present paper (Theorem 2(2) in section 4) states that varying a real nondegenerate semi-quasi-homogeneous point through the real nondegenerate semi-quasi-homogeneous points one can obtain a point which has M-smoothing. The proofs are constructive. The proof of the first statement is based on a local version of the original Harnack construction for curves (cf., [16, 10]). The proof of the second one is based on the Viro patchworking method, which is a far-reaching generalization of the Harnack one (see [22]). Under some additional hypothesis on the singular point the above methods provide M-smoothings without any preliminary equisingular deformation of the singularity (Proposition 1 in section 3.1 and Theorem 2(1, 3) in section 4).

In addition, in section 5 we consider an improved local Harnack bound, which takes into account the arrangement of nonclosed components of smoothings, and we show that this improved bound is sharp in the real equisingular class of any Newton-nondegenerate singular point (Theorem 5 in section 5).

In the last section we discuss some open problems.

Acknowledgment. We are grateful to S. Orevkov for stimulating discussions: Theorem 1 answers one of his questions.

2 Preliminaries

2.1 Smoothings of singular points

A holomorphic curve C in $\mathbb{C}^2$ or in another complex surface with complex conjugation is called real, if C is invariant under complex conjugation. A *real isolated singular point* is a germ $(C, z) \subset (\mathbb{C}^2, z)$, $z \in \mathbb{R}^2$, of a real plane reduced holomorphic curve C. With usual for isolated singularities ambiguity, we denote by $B(C, z)$ a Milnor ball (see [14] or [2]; recall that each branch of C meets $\partial B(C, z)$ transversally and along a smooth circle) whose radius we diminish, if necessary, when making perturbations.

A real holomorphic curve C' in the Milnor ball $B(C, z)$ is called a *real smoothing of* C if there exists a real-analytic 1-parameter family C_t, $t \in [0, 1]$, of real holomorphic curves in $B(C, z)$ such that $C_0 = C$, $C' = C_1$ and each C_t with $t > 0$ is nonsingular and transversal to the boundary of $B(C, z)$. We call such a family a *smoothing out deformation of* (C, z).

The real part $C'_{\mathbb{R}}$ of C' consists of finitely many ovals and nonclosed components. The number of nonclosed components is equal to the number $r_{\mathbb{R}}$ of real branches of (C, z). The number v of the ovals satisfies (1), and C' is called an *M-smoothing* if (1) turns into equality, i.e., $v = \frac{1}{2}(\mu - r + 1)$ if $r_{\mathbb{R}} > 0$, and $v = \frac{1}{2}(\mu - r + 3)$ if $r_{\mathbb{R}} = 0$.

Recall that $\frac{1}{2}(\mu - r + 1)$ is equal to the genus of smoothings (as well as to the number δ of virtual double points, which is the maximal number of double points appearing in small perturbations of the singularity, diminished by $r - 1$). If an isolated plane curve singular point is given by a polynomial or power series equation $f = 0$ and the truncations of f to the edges of the Newton diagram $\Gamma(f)$ of f have no critical points in $(\mathbb{C}^*)^2$, then

$$\frac{1}{2}(\mu - r + 1) = \#(\mathrm{Int}(\mathcal{D}) \cap \mathbb{Z}^2) \ , \tag{2}$$

where $\mathcal{D}$ is the domain bounded in $\mathbb{R}^2$ by $\Gamma(f)$ and the coordinate axes and Int states for the interior (this formula can be found already in [3], for far-reaching generalizations and a modern exposition see [11, 12]). Singular points satisfying the hypothesis of the above statement are called *Newton nondegenerate* (shortly ND).

An isolated singular point $(C, 0)$ is called *semi-quasi-homogeneous* if in some local coordinates it is given by an equation $f = 0$ such that the truncation f^γ of f to an edge γ of $\Gamma(f)$ has no critical points in $\mathbb{C}^2 \setminus \{(0, 0)\}$. It is equivalent to

$$f^\gamma(x, y) = \alpha x^a y^b \prod_{i=1}^{s} (y^c - \alpha_i x^d), \tag{3}$$

where: c and d are coprime; $\alpha_1, \ldots, \alpha_s$ are distinct and nonzero; $\alpha \neq 0$ and $a, b \in \{0, 1\}$. The numbers $\alpha_1, ..., \alpha_s$ are called *peripheral roots* of f. Note, that adding x^i or y^j with big i, j we don't change the singularity up to isomorphism (see [21]) and, in particular, can make $\Gamma(f)$ compact, i.e., containing vertices on the both axes.

2.2 Topological equivalence of singular points

We distinguish equivalences over $\mathbb{C}$ and over $\mathbb{R}$. A *topological equivalence over* $\mathbb{C}$ (resp., *over* $\mathbb{R}$) of two (real, in the case of the real equivalence) singular points (C, z) and (D, w) is a homeomorphism $\varphi : B(C, z) \to B(D, w)$ which takes $C \cap B(C, z)$ to $D \cap B(D, w)$ (and commutes with the complex conjugation in the real case). Singularities (C, z) and (D, z) are called *topologically equisingular* (or deformation equivalent) *over* $\mathbb{C}$ (*over* $\mathbb{R}$) if there exists a real-analytic family of topologically equivalent over $\mathbb{C}$ (resp., over $\mathbb{R}$) complex (resp., real) singularities $(C_t, z) \subset B(C, z)$, $t \in [0, 1]$, connecting (C, z) and (D, z). As is known, a real-analytic family satisfies this property (both in the complex and real cases) if, and only if, the Milnor number is constant (see, for example, [13]).

Recall also the following known facts: (1) topologically equivalent isolated plane curve singularities are topologically equisingular over $\mathbb{C}$ (see [19]); (2) if two real unibranch singular points are topologically equivalent over $\mathbb{C}$, they are topologically equisingular over $\mathbb{R}$; (3) any real plane curve singular point is topologically equivalent over $\mathbb{C}$ to a singular point with all local branches real; (4) two real ND singular points are topologically equivalent over $\mathbb{R}$ if their equations have the same Newton diagram and for any edge of the diagram the truncations of the both have the same numbers of positive and negative peripheral roots; (5) replacing a real isolated singular point $f = 0$ by $Tf = 0$ where Tf is the Taylor polynomial of f of degree $\geq \mu + 1$ we do not change the singularity up to real analytic coordinate transformation (see [21]).

3 M-smoothings

3.1 Prélude: blowing-up construction

The blowing-up method for construction of smoothings of an isolated real plane curve singularity is described in details in [16]. Here, we give an application of this method, which allow us to deliver some information on the class of singularities which have M-smoothing and to recall the principal ingredients of the method.

Proposition 1 *Any singular point whose branches all are real and nonsingular, has M-smoothing.*

Proof. We make induction by blow-ups introducing a stronger statement which takes into account the position of the smoothing with respect to a straight line.

Let (C, z) be a singular point with r branches which all are real and nonsingular, and L be a real straight line through z transversal to the branches of C. Let us fix an orientation of $L_{\mathbb{R}}$ and denote the components of $B(C, z)_{\mathbb{R}} \backslash L$ by B_+ and B_-. We say that a real smoothing C' of (C, z) is of type (ε, δ) with respect to L, where $\varepsilon, \delta = \pm 1$, if C' has a nonclosed real component which starts in B_ε, then successively intersects L at r points ordered in accordance with the fixed orientation of $L_{\mathbb{R}}$, and ends in B_δ (see Figure 1; it specifies also the convention distinguishing B_+ and B_-).

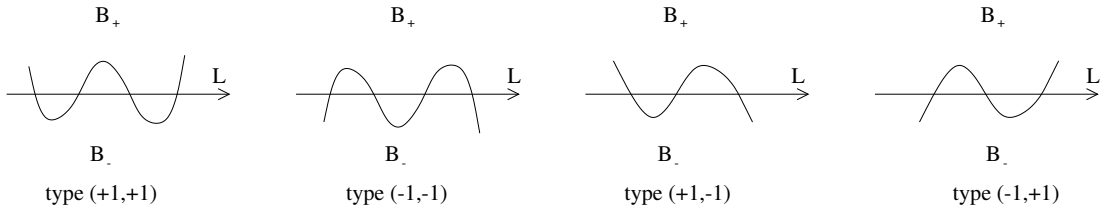


Figure 1: M-smoothings of a singular point with nonsingular branches

Prove that under the above hypothesis on (C, z) and L, given $\varepsilon, \delta = \pm 1$ such that $\varepsilon\delta = (-1)^r$, there exists an M-smoothing of (C, z) of type (ε, δ) with respect to L.

Proceed by induction on $\mu(C, z)$. If $\mu(C, z) \leq 1$ the statement is trivial, so assume that $\mu(C, z) > 1$. Blow up the point z. The strict transform C^* of C intersects the exceptional divisor E at real points $z_1, ..., z_m$, and all local branches of C^* are real, nonsingular and transversal to E. Without loss of generality suppose that $\delta = +1$, and fix an orientation of $E_{\mathbb{R}}$ as in Figure 2a. By the induction assumption the singular points (C^*, z_1), (C^*, z_2), ..., (C^*, z_m) have M-smoothings $C'_1, ..., C'_m$ of types $(+1, \varepsilon_1)$, $(\varepsilon_1, \varepsilon_2)$, ..., $(\varepsilon_{m-1}, \varepsilon_m)$, respectively, with respect to E (see Figure 2b).

Now, to finish the proof use the recursive Harnack procedure. Pick up $r-1$ real holomorphic nonsingular curves $L_1^{(0)}, ..., L_{r-1}^{(0)}$ crossing E transversally at $r-1$ distinct points positioned as in Figure 2b (these curves are shown dashed), and deform the union of the smoothings $C'_1, ..., C'_m$ in the family $\widetilde{C}_1 = C'_1 ... C'_m + tL_1^{(0)} ... L_{r-1}^{(0)}$. If t is small and of proper sign, the curve $\widetilde{C}_1$ has only one real component intersecting $E \cup L^*$; this component is nonclosed and crosses $E_{\mathbb{R}}$ at $r-1$ points and $L^*_{\mathbb{R}}$ at one point as shown in Figure 2c. In the next step pick up $r-2$ real holomorphic nonsingular curves $L_1^{(1)}, ..., L_{r-2}^{(1)}$ crossing E transversally at $r-2$ distinct points positioned as in Figure 2c (these curves are shown dashed), and deform $\widetilde{C}_1$ in the family $\widetilde{C}_1 + tL_1^{(1)} ... L_{r-2}^{(1)}$. Thus, one obtains a curve $\widetilde{C}_2$, whose only real component intersecting $E \cup L^*$ is non-closed, meets $E_{\mathbb{R}}$ at $r-2$ points and $L^*_{\mathbb{R}}$ at two points and is located as shown in Figure 2d. Repeat this procedure alternating the position of auxiliary curves with respect to L^* until obtaining a curve $\widetilde{C}_m$ whose only real component intersecting $E \cup L^*$ does not meet $E_{\mathbb{R}}$. Then, this real component crosses $L^*_{\mathbb{R}}$ at r points as shown in Figure 2e. Blowing E down transforms $\widetilde{C}_m$ into an M-smoothing of (C, z) of type $(\varepsilon, +1)$ (computations similar to those done in the proof of Theorem 1 show that the total number of ovals obtained is $\frac{1}{2}(\mu - r + 1)$). $\square$

Remark 1 A similar result, but for a more restricted class of singularities is contained in [4].

It may be interesting to study the class of M-smoothings obtained by the blowing-up construction. More precisely, let us call *a BM-smoothing* an M-smoothing obtained by the algorithm proposed in [16] or by another inductive algorithm which depends only on the topology of the intermediate germs of curves with respect to the resolution trees. Then, as it follows from the simultaneous resolution theorem

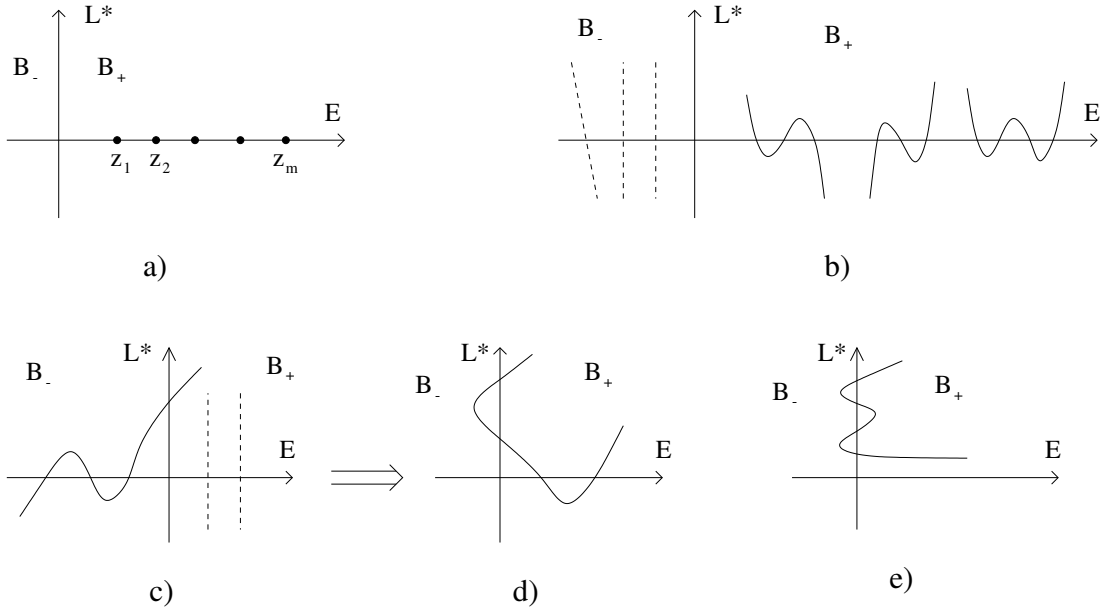


Figure 2: Construction of M-smoothing for a singular point with nonsingular branches

(see [20]), *the family of isolated real plane singularities which have BM-smoothing is closed under equisingular deformations over* $\mathbb{R}$.

3.2 M-smoothing versus complex equisingular deformation

Theorem 1 *Any real isolated plane curve singular point can be connected by a topologically equisingular over* $\mathbb{C}$ *family of complex singular points to a real singular point which has M-smoothing.*

According to remark (3) in section 2.2, we may assume that the initial real singular point have only real branches. Thus, throughout this section we consider *only the real singular points which have no imaginary branches.*

Note also that due to remark (1) in 2.2, instead of connecting singularities by topologically equisingular families it is sufficient to check their topological equivalence. This is the point of view which we adopt in the main steps of the proof of Theorem 1 below.

Definition 1. Let L_1, L_2 be two real straight lines through a real singular point z of real plane curves C and D which do not contain L_1, L_2 as components. The germs (C, z) and (D, z) are called topologically $(L_1 L_2)$-equivalent over $\mathbb{C}$ (or, shortly, $(L_1 L_2)$-equivalent) if there is a homeomorphism $\varphi : B \to B$, $B = B(C, z) = B(D, z)$, such that $\varphi(C) = D$, $\varphi(L_1) = L_1$, $\varphi(L_2) = L_2$. Given only one line L_1 through z, we similarly define (topological) L_1-equivalence over $\mathbb{C}$.

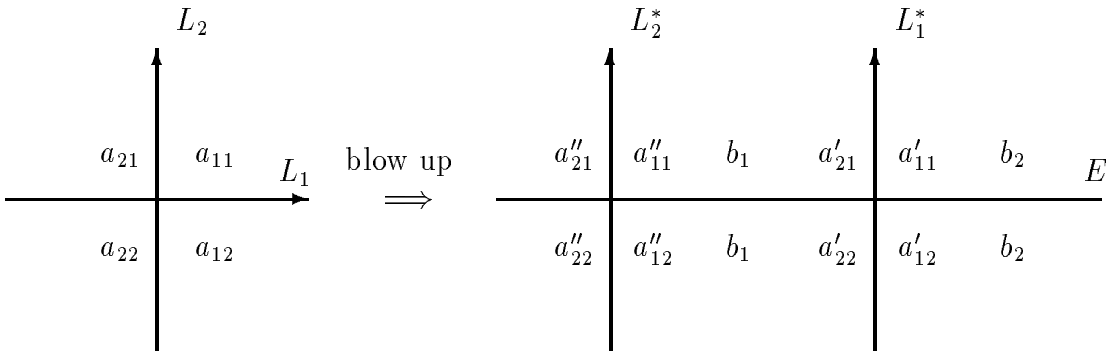


Figure 3: Blowing up a singular point

In the above notation, assume that L_1, L_2 are the coordinate axes in some local coordinate system (x, y). For the singular point (C, z) we define the 2×2 matrix $A(C, z) = (a_{ij})_{i,j=1,2}$ over $\mathbb{Z}/2\mathbb{Z}$ so that a_{ij} is the mod 2 residue of the number of the real demi-branches (components of $C \cap (B(C, z)_{\mathbb{R}} \backslash \{z\})$) of C in $\mathbb{R}^2_{i,j} = \{x(-1)^i < 0, \quad y(-1)^j < 0\}$.

Proposition 2 *If $A(C, z) \neq 0$ then there is a germ (D, z) such that (C, z) and (D, z) are $(L_1 L_2)$-equivalent over $\mathbb{C}$ and*

$$A(D, z) = A(C, z) + J, \quad J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

Proof. A nonzero matrix $A(C, z)$ contains two or four units. If $A(C, z)$ has two units, one obtains the required germ (D, z) by means of transformations $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, -y)$.

If all the entries of $A(C, z)$ are units we apply induction on the Milnor number $\mu(C, z)$. In the case $\mu(C, z) = 1$, which corresponds to an ordinary node, it is sufficient to rotate one line (or to make a linear transformation).

Assume that $\mu(C, z) > 1$ and blow up the point z. Denote by E the exceptional divisor, by U a neighborhood of $E_{\mathbb{R}}$ in the blown up $\mathbb{R}^2$, by C^*, L_1^*, L_2^* the strict transforms of C, L_1, L_2, respectively, and by $z_1^* = E \cap L_1^*$, $z_2^* = E \cap L_2^*$ the intersection points. Denote by $a'_{ij} \in \mathbb{Z}/2\mathbb{Z}$ (resp. by $a''_{ij} \in \mathbb{Z}/2\mathbb{Z}$), $i, j = 1, 2$, the mod 2 residue of the number of real demi-branches of (C^*, z_1^*) (resp. (C^*, z_2^*)) in the corresponding component of $U \backslash E_{\mathbb{R}}$ (see Figure 3). Finally, denote by $b_i \in \mathbb{Z}/2\mathbb{Z}$, $i = 1, 2$, the mod 2 residue of the number of real demi-branches of C^* which are centered on $E_{\mathbb{R}} \backslash \{z_1^*, z_2^*\}$ and contained in the corresponding component of $U \backslash E_{\mathbb{R}}$ (see Figure 3). In the above notation

$$A(C, z) = A' + A'' + \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix} , \quad A' = \begin{pmatrix} a'_{21} & a'_{11} \\ a'_{12} & a'_{22} \end{pmatrix} , \ A'' = \begin{pmatrix} a''_{11} & a''_{22} \\ a''_{21} & a''_{12} \end{pmatrix} .$$

Since the Milnor numbers of the points of C^* are strictly less than $\mu(C, z)$, the induction assumption applies to them. If $A' \neq 0$ (or $A'' \neq 0$), by induction one

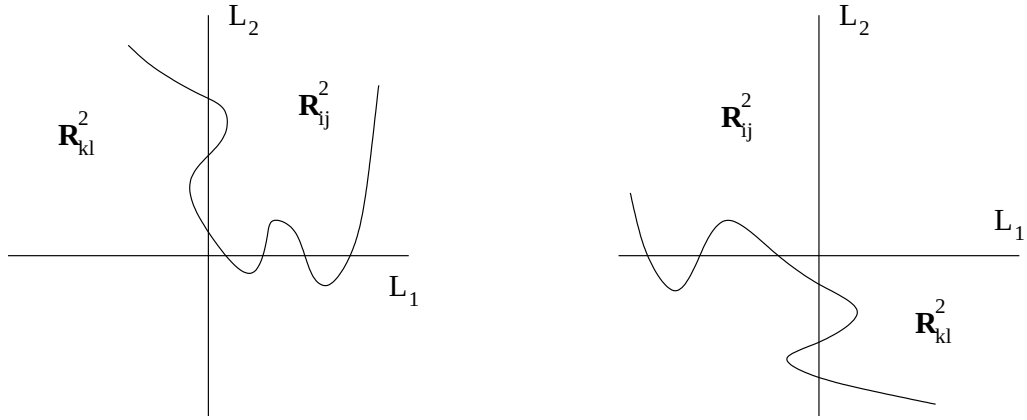


Figure 4: (ij, kl)-regular M-smoothing

can convert A' into $A' + J$ (resp. A'' into $A'' + J$), then blow down and obtain the required germ with $A(D, z) = 0$. If $A' = A'' = 0$ then at least one of b_1 and b_2 is equal to 1, and in this case we move all the points $C^* \cap (E_{\mathbb{R}} \backslash \{z_1^*, z_2^*\})$ along $E_{\mathbb{C}}$ into one interval of $E_{\mathbb{R}} \backslash \{z_1^*, z_2^*\})$, obtaining after blowing down the required germ. $\square$

Definition 2. Let (C, z) be a singular point, L_1, L_2 be two distinct real straight lines through z which are not components of C, and $n_i = (C \circ L_i)_z$, $i = 1, 2$, be the intersection numbers. Choose real local coordinates x, y so that $L_1 = \{y = 0\}$, $L_2 = \{x = 0\}$ and assume that $A(C, z) \neq J$. An M-smoothing C' of (C, z) is called *(ij, kl)-regular (with respect to L_1, L_2)*, where $i, j, k, l, \in \{1, 2\}$, if C' contains a non-closed branch λ which goes from $\mathbb{R}^2_{ij}$ to $\mathbb{R}^2_{kl}$ and, first, successively crosses L_1 at n_1 points x_s, $s = 1, ..., n_1$, such that

$$(-1)^i x_1 > ... > (-1)^i x_{n_1} > 0$$

and then successively crosses L_2 at n_2 points y_s, $s = 1, ..., n_2$, such that

$$0 < (-1)^l y_1 < ... < (-1)^l y_{n_2}$$

(see Figure 4). Clearly, $k - i \equiv n_2 \mod 2$ and $l - j \equiv n_1 \mod 2$.

Note that if there exists an M-smoothing (ij, kl)-regular with respect to two lines, then $A(C, z) \neq J$.

Proposition 2 and Proposition 3 below complete the proof of Theorem 1.

Proposition 3 *Let (C, z), L_1, L_2 be as in Definition 2.*

- *If $A(C, z) = 0$, there exists a singular point (D, z) such that: (C, z) and (D, z) are $(L_1 L_2)$-equivalent , $A(D, z) = 0$ and (D, z) has a $(11, 11)$-regular M-smoothing.*

- *If $A(C, z) \neq J$ and $a_{ij} = a_{kl} = 1$ (in particular, $(A(C, z) \neq 0)$, there exists a singular point (D, z) such that: (C, z) and (D, z) are $(L_1 L_2)$-equivalent, $A(D, z) = A(C, z)$ and (D, z) has an (ij, kl)-regular M-smoothing.*

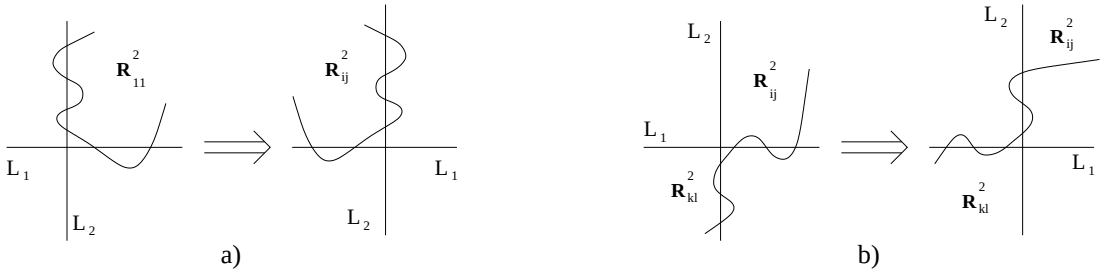


Figure 5: Reflections with respect to L_1, L_2

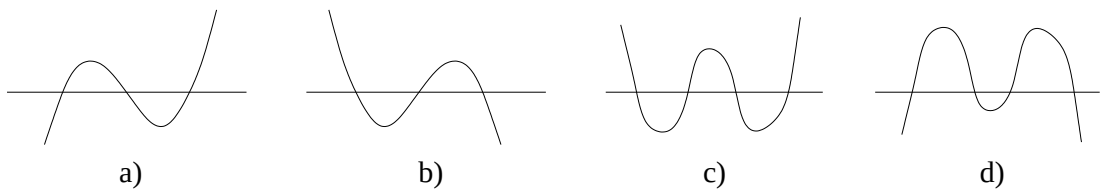


Figure 6: Smoothings regular with respect to a line

Proof. We start with the following observations:

(1) If a singular point (D, z) with $A(D, z) = 0$ has a $(11, 11)$-regular M-smoothing, then one of the reflections $(x, y) \mapsto (\pm x, \pm y)$ transform (D, z) in a $(L_1 L_2)$-equivalent singularity which has an (ij, ij)-regular M-smoothing for prescribed $i, j = 1, 2$ (see Figure 5a).

(2) If a singular point (D, z) with $A(D, z)$ different from 0 and J has an (ij, kl)-regular M-smoothing D', then the non-trivial reflection $T(x, y) = (\pm x, \pm y)$ such that $A(T(D), z) = A(D, z)$ takes D' into a (kl, ij)-regular M-smoothing of $(T(D), z)$ (see Figure 5b). Similarly, if the statement of Proposition 3 is proven for germs with $A(C, z) = A$ different from 0 and J, then it holds for germs with $A(C, z) = A + J$.

(3) Let (C, z) be a singular point and L be a real straight line through z which is not a component of C. If Proposition 3 is proven for (C, z) equipped with any L_1, L_2 as in Definition 2, then, appending any generic real straight line L' through z, one derive from Propositions 2, 3 that if the intersection number $(C \cdot L)_z = n$ is odd (even) then there exists a singular point (D, z) such that (C, z) and (D, z) are L-equivalent, all the branches of (D, z) are real and (D, z) has an M-smoothing with a non-closed branch intersecting L at n real points in a prescribed way as shown in Figure 6a, b (resp. Figure 6c, d).

Now, proceed by induction on the Milnor number $\mu(C, z)$ (using, as in the proof of Proposition 2, that μ decreases after a blow-up). The case $\mu(C, x) = 1$ is trivial. Assume that $\mu(C, z) > 1$.

Blow up the point z. In the notation of the proof of Proposition 2, the strict transform C^* of (C, z) decomposes into germs (C^*, z_1^*), (C^*, z_2^*) and germs centered

at points on $E\backslash\{z_1^*, z_2^*\}$ (some of them may be empty).

In what follows we replace the germs of C^* by (E, L_1^*)-, or (E, L_2^*)-, or E-equivalent germs, and move the germs (C^*, w), $w \in E\backslash\{z_1^*, z_2^*\}$, along $E\backslash\{z_1^*, z_2^*\}$. These operations give after blowing down a singular point (D, z) which is (L_1, L_2)-equivalent to (C, z) and has $A(D, z) = A(C, z)$ or $A(C, z) + J$. Then, according to the second observation above, it remains to construct a regular M-smoothing of (D, z). (the case $(A(C, z) = 0, A(D, z) = J)$ is excluded, since, as it is shown below, (D, z) has a regular M-smoothing).

Denote by $[z_1^*, z_2^*]$ the segment in $E_{\mathbb{R}}$ which is the common boundary of the images of $\mathbb{R}^2_{11}$ and $\mathbb{R}^2_{22}$ in the blown up plane (see Figure 3), and place all the germs of C^* centered on $E_{\mathbb{R}}\backslash\{z_1^*, z_2^*\}$ to the interval $E_{\mathbb{R}}\backslash[z_1^*, z_2^*]$ so that the germs (C^*, w) with odd intersection numbers $(C^* \circ E)_w$ are in a neighborhood of z_1^* and the germs with even $(C^* \circ E)_w$ are in a neighborhood of z_2^*. For the sake of simplicity we use the same symbol C^* in the notation of new germs.

Construction of a regular M-smoothing depends on the matrices $A(C^*, z_1^*)$, $A(C^*, z_2^*)$ with entries a'_{ij}, a''_{ij} distributed as shown in Figure 3 and the multiplicity m of (C, z), which is equal to the total intersection number of C^* with E. By the induction assumption and according to Proposition 2 and the remarks made in the beginning of the proof, for any possible combination of $m, A(C^*, z_1^*), A(C^*, z_2^*)$, we can replace each germ (C^*, w), $w \in E$, with a suitable regular M-smoothing as shown in Figures 7, 8 (changing (C^*, w) if necessary in its topological (E, L_1^*)-, (E, L_2^*)-, or E-equivalence class), where the triples $m, A(C^*, z_1^*), A(C^*, z_2^*)$ are encoded by symbols (odd, A_i, A_j) or $(even, A_i, A_j)$ with

$$A_0 = 0, \quad A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Denote the union of the smoothed germs by $\widetilde{C}_0$.

Since $\delta = \frac{1}{2}(\mu + r - 1)$ drops by $\frac{1}{2}m(m-1)$ when blowing up (see [7]), the total number of ovals in $\widetilde{C}_0$ is

$$v_0 = \sum_{w \in E} \frac{\mu(C^*, w) - r(C^*, w) + 1}{2}$$

$$= \frac{\mu(C, z) - r(C, z) + 1}{2} + \#(C^* \cap E) - 1 - \frac{m(m-1)}{2}.$$

Next we deform $\widetilde{C}_0$ in the following m-step recursive procedure. At the first step, we vary $\widetilde{C}_0 \cup E$ in a linear pencil generated by $\widetilde{C}_0 \cup E$ and a real divisor intersecting E at $m-1$ points. Namely, in terms of the equations in an affine chart of U, we define $\widetilde{C}_1 = E\widetilde{C}_0 + tL_1^{(0)} ... L_{m-1}^{(0)}$, where t is a small real parameter and $L_1^{(0)}, ..., L_{m-1}^{(0)}$ are real holomorphic curves crossing E transversally at $m-1$ distinct points in the interior of $[z_1^*, z_2^*]$ (one can take the strict transforms of appropriate straight lines

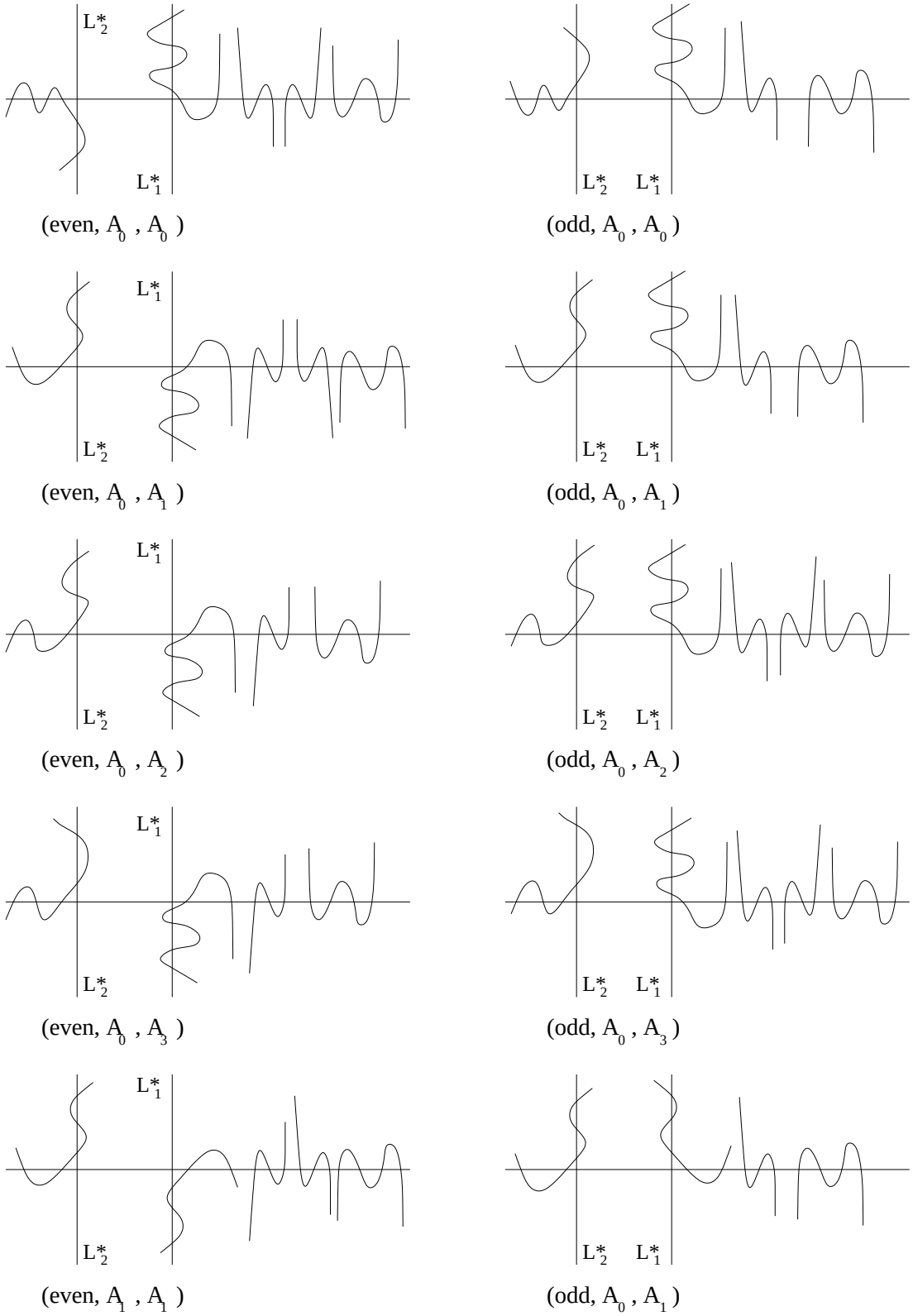


Figure 7: Regular M-smoothings in the blown up plane I

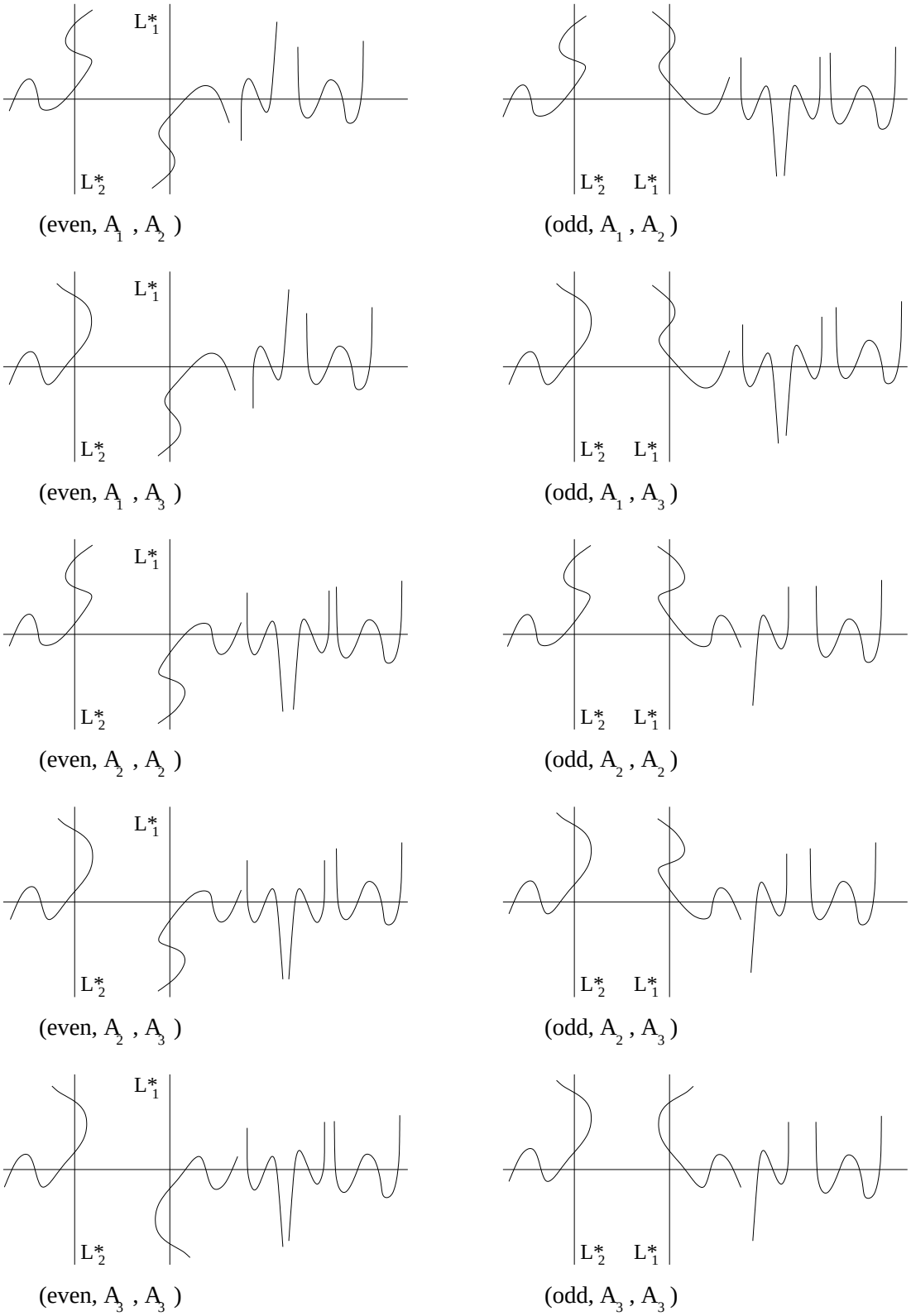


Figure 8: Regular M-smoothings in the blown up plane II

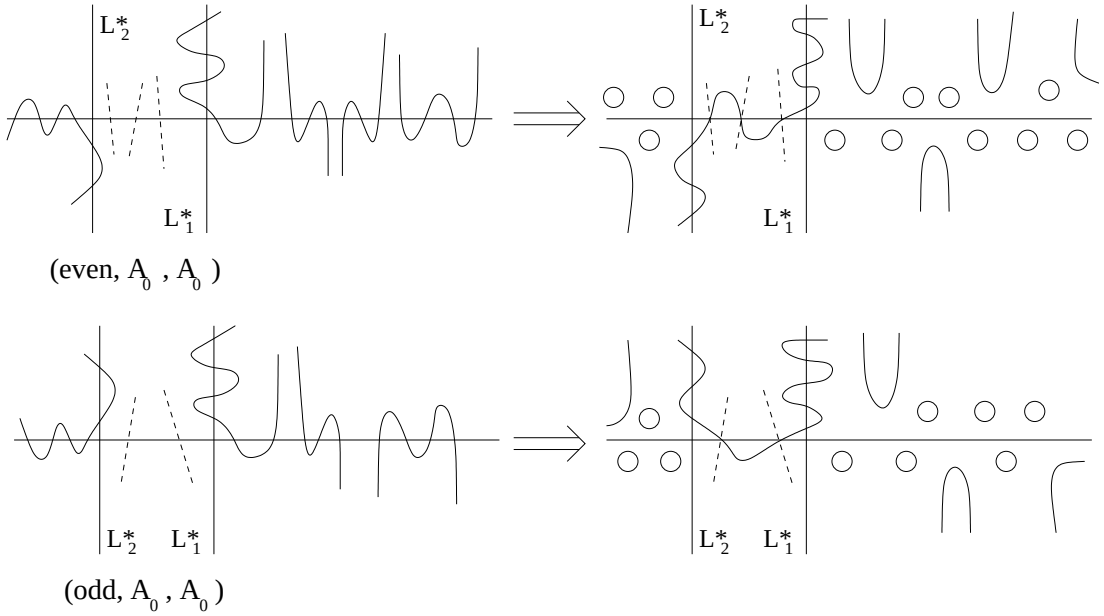


Figure 9: Smoothing $\widetilde{C}_1$

through z). In any of the situations shown in Figures 7, 8 and under proper choice of the sign of t, $\widetilde{C}_1$ has

$$v_1 = v_0 + m - \#(C^* \cap E) = \frac{\mu(C,z) - r(C,z) + 1}{2} - \frac{(m-1)(m-2)}{2}$$

ovals, $\#(C^* \cap E) - 1$ non-closed real branches which do not intersect $E \cup L_1^* \cup L_2^*$, and one non-closed real branch which intersects: L_1^* at $(C^* \circ L_1^*)_{z_1^*} + 1$ real points, L_2^* at $(C^* \circ L_2^*)_{z_2^*} + 1$ real points, and E at $m-1$ real points (see Figure 9 for the cases (even,A_0,A_0), (odd,A_0,A_0); lines $L_1^{(0)}, ..., L_{m-1}^{(0)}$ are shown dashed).

Let us assume that $\widetilde{C}_k$, $1 \le k < m$, is a real holomorphic curve in a neighborhood of E such that: it has

$$v_k = \frac{\mu(C,z) - r(C,z) + 1}{2} - \frac{(m-k)(m-k-1)}{2}$$

ovals, $\#(C^* \cap E) - 1$ non-closed real branches which do not intersect $E \cup L_1^* \cup L_2^*$, and a non-closed real branch which is shaped as shown in Figure 9 and intersects L_1^* at $(C^* \circ L_1^*)_{z_1^*} + k$ real points, L_2^* at $(C^* \circ L_2^*)_{z_2^*} + k$ real points, and E at $m-k$ real points. Then, we define

$$\widetilde{C}_{k+1} = \widetilde{C}_k E + t L_1^{(k)} ... L_{m-k-1}^{(k)},$$

where t is a small real parameter, $L_1^{(k)}, ..., L_{m-k-1}^{(k)}$ are real holomorphic curves meeting E transversally at $m-k-1$ distinct real points in the interior of that interval between z_1^*, z_2^* which does not contain the points of $\widetilde{C}_k \cap E$. It is easily seen that $\widetilde{C}_{k+1}$ possesses the same properties as $\widetilde{C}_k$ with substitution of $k+1$ for k.

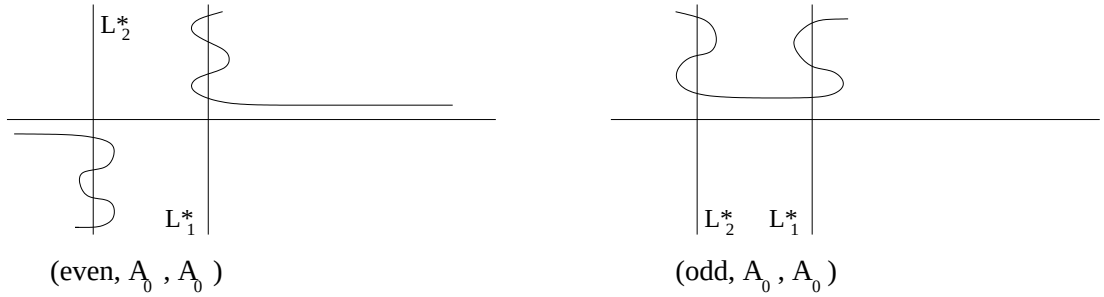


Figure 10: Smoothing $\widetilde{C}_m$

The curve $\widetilde{C}_m$ has $v_m = \frac{1}{2}(\mu(C,z) - r(C,z) + 1)$ ovals, $\#(C^* \cap E) - 1$ non-closed real branches which do not intersect $E \cup L_1^* \cup L_2^*$, and a non-closed real branch which intersects L_1^* at $(C^* \cdot L_1^*)_{z_1^*} + m = (C \circ L_1)_z$ real points, L_2^* at $(C^* \circ L_2^*)_{z_2^*} + m = (C \circ L_2)_z$ real points, and does not intersect E. This branch is located as shown in Figure 10 for the cases (even,A_0,A_0), (odd,A_0,A_0), and similarly in the other cases. Blowing down E, one converts $\widetilde{C}_m$ into a required regular M-smoothing. $\square$

4 M-smoothing of semi-quasi-homogeneous and Newton nondegenerate singular points

Theorem 2 *(1) Any real semi-quasi-homogeneous singular point which has no peripheral real roots of different signs admits an M-smoothing.*

(2) Any real semi-quasi-homogeneous singular point is deformation equivalent over $\mathbb{R}$ to a singular point admitting an M-smoothing.

(3) Any real Newton nondegenerate singular point without real branches has an M-smoothing.

Remark 2 Similar to Theorem 1, the first and third parts of Theorem 2 can be proven by methods of [16] (cf., lemmas IV.3.3 and IV.3.4 in [4], where the statements are not proven as stated and, in part, are wrong without additional hypotheses). Here we prove all the parts in one manner using Viro method (cf., [17]).

4.1 Patch-working of polynomials

Here and further, we denote by $\Delta(F)$, where F is a polynomial or a power series in two variables, the Newton polygon of F.

Recall the notion of charts of polynomials, which is crucial for Viro patch-working method. Given a real polynomial F with $\operatorname{Int}\Delta(F) \neq \emptyset$, consider the union Δ_* of the (mirror) images $\Delta_{\varepsilon,\delta}$, $\varepsilon, \delta = \pm 1$, of $\Delta = \Delta(F)$ with respect to reflections $(x,y) \mapsto (\varepsilon x, \delta y)$ and introduce the map $\mu_\Delta : (\mathbb{R}^*)^2 \to \Delta_*$ defined by

$$(x,y) \in (\mathbb{R}^*)^2 \mapsto \mu_\Delta(x,y) = \frac{\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2}(i\cdot \mathrm{sign} x, j\cdot \mathrm{sign} y)|x|^i|y|^j}{\sum_{(i,j)\in\Delta\cap\mathbb{Z}^2}|x|^i|y|^j};$$

it takes $(\mathbb{R}^*)^2$ diffeomorphically onto $\bigcup \mathrm{Int}\,\Delta_{\varepsilon,\delta}$. The chart $\mathrm{Ch}F$ of F is the closure of $\mu_\Delta(\{F=0\}\cap(\mathbb{R}^*)^2)$ (more precisely, the pair $(\Delta, \mathrm{Ch}F)$) considered up to homeomorphisms of Δ_* preserving each $\Delta_{\varepsilon,\delta}$ and its edges and vertices.

If F and its truncations to the edges of $\Delta(F)$ have no singular points in $(\mathbb{R}^*)^2$, then $\mathrm{Ch}F$ is a topological curve with boundary properly embedded in Δ_*. Polynomials satisfying the hypothesis of the above statement are called completely Newton nondegenerate (shortly CND).

Theorem 3 (see [22]) *Let $F_1,\dots,F_N$ be CND polynomials whose Newton polygons $\Delta_1,\dots,\Delta_N$ have non empty interior and form a subdivision of a convex polygon Δ. If all F_i are the truncations of the same polynomial $\Phi = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} A_{ij}x^iy^j$ and there is a convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$ whose linearity domains are $\Delta_1,...,\Delta_N$, then for any sufficiently small t the polynomial $F(x,y) = \sum_{(i,j)\in\Delta\cap\mathbb{Z}^2} A_{ij}t^{\nu(i,j)}x^iy^j$ is a CND polynomial and $\mathrm{Ch}V = \bigcup \mathrm{Ch}F_i$.*

Remark 3 The function ν in theorem 3 can be corrected by a linear function in a way that on two prescribed neighboring facets σ_1,σ_2 of Δ, or in some Δ_k, the coefficients of V will coincide with A_{ij}.

The patch-working construction from Theorem 3 feats into framework of toric varieties, see [22]. In particular, with any convex lattice polygon Δ is associated its toric surface $K\Delta$, which is a real algebraic surface obtained by some natural identification of edges of Δ_*. This surface has at most isolated singular points, they correspond to a subset of vertices of Δ. To each edge γ of Δ there corresponds a real divisor $K\gamma$, which is topologically a circle. It intersects only the circles of the neighboring edges, and each one only at one point. These 2 points divide $K\gamma$ in 2 intervals $K\gamma_\pm$. For any CND polynomial F with Newton polygon Δ the image $K\mathrm{Ch}F$ of $\mathrm{Ch}F$ in $K\Delta$ is a closed topological 1-submanifold of the smooth part of $T\Delta$. The intersection points of $K\mathrm{Ch}F$ with $K\gamma_+$ are the positive and with $K\gamma_-$ the negative peripheral roots of F.

Under integral translations of Δ the pair $(K\Delta, K\mathrm{Ch}F)$ does not change, and under a $SL_2(\mathbb{Z})$ transformation of Δ it is replaced by a canonically homeomorphic pair and the canonical homeomorphism respects the stratification of $K\Delta$.

If, in notation of Theorem 3, $K\mathrm{Ch}F$ is an M-curve (Harnack maximal), then $K\mathrm{Ch}F_i$ is an M-curve for any $i, 1\le i\le N$. The inverse is not true. To remedy such a difficulty we introduce some natural regularity conditions.

A real quasihomogeneous polynomial is called *steady*, if it has no critical points in $(\mathbb{C}^*)^2$ and has no real peripheral roots of different signs (in particular, it can have only imaginary roots). A polynomial with Newton polygon Δ not reduced to an edge is called *steady* if it is a real CND polynomial with *steady* truncations to the edges of Δ.

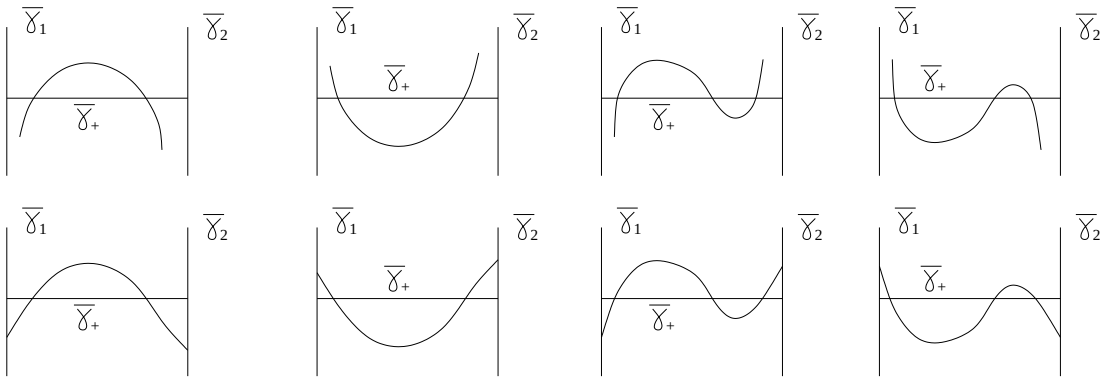


Figure 11: Weakly regular and regular intersections

Let F be a real CND polynomial with $\mathrm{Int}\Delta(F) \neq \emptyset$. Suppose that F^γ for some edge γ of Δ is steady. Denote by γ_1, γ_2 the neighboring to γ edges of Δ. We speak of the *weakly regular intersection* of F and γ, if there is a component of $K\mathrm{Ch}F \setminus \bigcup_{\gamma' \neq \gamma} K\gamma'$ which passes through all the points of $K\mathrm{Ch}F \cap K\gamma$ and these points are placed in $K\gamma$ in the same order as in this component. We call a weakly regular intersection of F and γ *regular*, if the above component continued joins the point of $K\mathrm{Ch}F \cap K\gamma$ most close to γ_1 with a point on γ_1 and the point of $K\mathrm{Ch}F \cap K\gamma$ most close to γ_2 with a point on γ_2 (see the lower part of Figure 11). If F^γ has at least one (resp. two) real roots, then there are exactly two isotopy types of regular (resp, weakly regular) intersection of F with γ (see Figure 11).

Let $\gamma = \Delta_1 \cap \Delta_2$ be a common edge of Newton polygons Δ_1, Δ_2 of polynomials F_1, F_2 with the same truncation on γ. Assume that both polygons have a nonempty interior. Regular, or weakly regular intersections of F_1, F_2 with γ are called *compatible*, if $\mathrm{Ch}F_1 \cup \mathrm{Ch}F_2$ contains $\#(\gamma \cap \mathbb{Z}^2) - 2$ ovals intersecting γ and its symmetric copies, and not intersecting the other edges (see Figure 12).

Theorem 3 is applied to patch-work smoothings of singularities. In particular, it has the following straightforward consequence.

Theorem 4 (see [22]) *If in notation of Theorem 3, Δ_N intersects both the coordinate axes and $\Delta_1, \ldots, \Delta_{N-1}$ fill the domain $\mathcal{D}$ bounded by Δ_N and the coordinate axes, then the singular point $(C, 0)$, where C is defined by $F_N = 0$, has a smoothing C' with $(B_{\mathbb{R}}(C, 0), C'_{\mathbb{R}})$ homeomorphic to $(\mathcal{D}_*, \bigcup_{i=1}^{N-1} \mathrm{Ch}F_i)$.*

($\mathcal{D}_*$ states for the union of the four mirror copies of $\mathcal{D}$.)

4.2 Auxiliary M-polynomials with Newton triangles

A CND polynomial F is called *M-polynomial* if $\mathrm{Ch}F \cap (\mathbb{R}^*)^2$ has $\#(\mathbb{Z}^2 \cap \mathrm{Int}\Delta(F)) + 1$ ovals for the case when F has no real peripheral roots, or has $\#(\mathbb{Z}^2 \cap \mathrm{Int}\Delta(F))$ ovals otherwise. Note that in the second case all noncompact components of $\mathrm{Ch}F \cap (\mathbb{R}^*)^2$ belong to one component of $K\mathrm{Ch}F$.

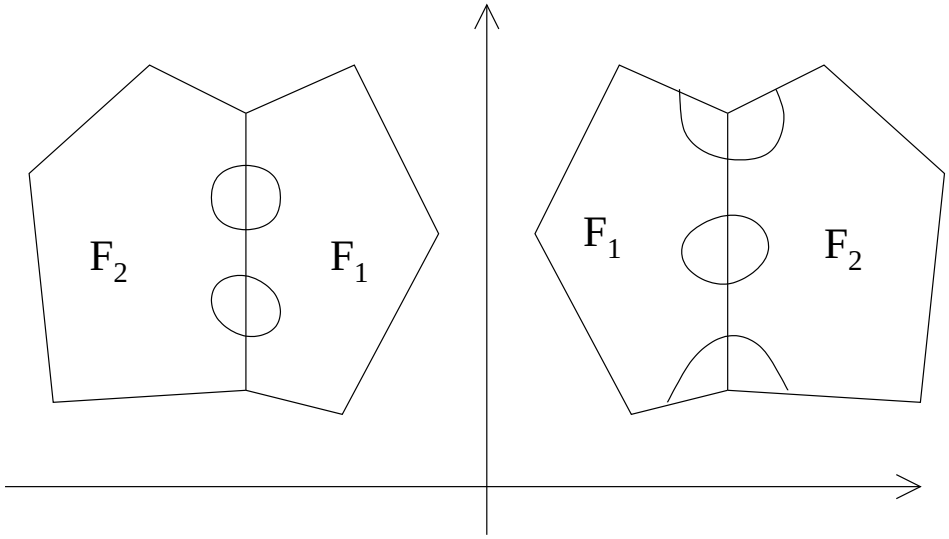


Figure 12: Compatible regular intersection

Proposition 4 *Let V_1, V_2, and V_3 be integral points of $\mathbb{R}^2_+$ which form a nondegenerate triangle T with edges $S_1 = [V_2, V_3]$, $S_2 = [V_1, V_3]$, and $S_3 = [V_1, V_2]$, and let h be a steady quasihomogeneous polynomial with $\Delta(h) = S_1$ and with at least one real peripheral root. Then there exists an M–polynomial F with $\Delta(F) = T$ such that: $h = F^{S_1}$; F^{S_2} and F^{S_3} are steady and have only real peripheral roots; the intersections of F with S_1, S_2, and S_3 are regular, and the isotopy type of the regular intersection with S_1 can be prescribed.*

Proof. We proceed by induction on the number s of integral points inside T. If $s = 0$, a suitable $SL_2(\mathbb{Z})$ transformation combined with a translation takes T into a triangle with vertices $(0,0), (0,1), (k,0)$, or $(0,0), (2,0), (0,2)$. For such a triangle the statement is obvious.

Assume that $s > 0$. Take the closest to S_1 integral point $V \in \operatorname{Int} T$ and divide T into the triangles

$$\tau_1 = \operatorname{conv}\{V, V_2, V_3\}, \quad \tau_2 = \operatorname{conv}\{V, V_1, V_3\}, \quad \tau_3 = \operatorname{conv}\{V, V_1, V_2\}$$

(conv states for the convex hull). Inside each of τ_1, τ_2, τ_3 there are less than s integral points, and we can apply the induction assumption. First, take a steady M-polynomial F_1 with: $\Delta(F_1) = \tau_1$, $F_1^{S_1} = h$ and the isotopy type of regular intersection with S_1 which is prescribed for F. Thus, in a prescribed quadrant $\mathbb{R}^2_\varepsilon$ the components of $\operatorname{Ch} F$ intersecting S_1 are placed as shown in Figure 13a. In addition, $\operatorname{Ch} F_1$ has in some other quadrant, say, in the positive one, an arc connecting the edges $[V, V_2]$, $[V, V_3]$ (see Figure 13a). Then, pick a steady quasihomogeneous polynomial h' with: $\Delta(h') = [V, V_1]$, all peripheral roots real and the same coefficient at V as in F_1. By induction assumption, there exist steady M-polynomials F_2, F_3 with $\Delta(F_2) = \tau_2, \Delta(F_3) = \tau_3$ and $F_2^{[V,V_1]} = F_3^{[V,V_1]} = h'$, and which have a compatible

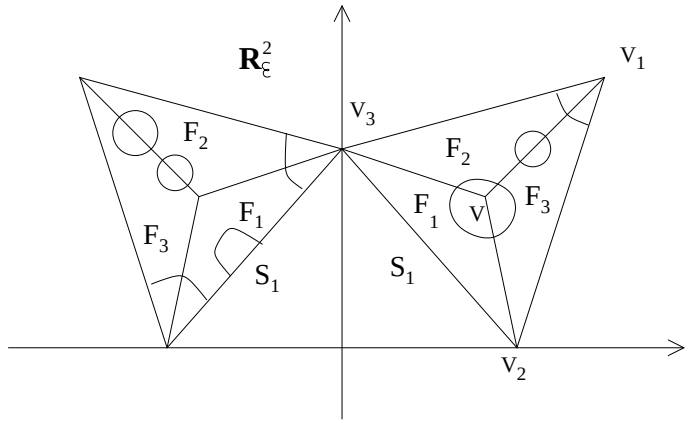


Figure 13: M-polynomials with Newton triangles

regular intersection with $[V, V_1]$ such that $\mathrm{Ch}F_2$ (resp. $\mathrm{Ch}F_3$) has in τ_2 (resp. τ_3) an arc connecting the edges $[V, V_1]$, $[V, V_3]$ (resp. $[V, V_1]$, $[V, V_2]$). By construction, V, V_2, V_3 are the only integral points on the edges $[V, V_2]$, $[V, V_3]$, and the sign of the coefficient of F_1, F_2 (resp. F_1, F_3) at V_3 (resp. V_2) is the same. Hence we can equalize the coefficients of F_1, F_2 (resp. F_1, F_3) at V_3 (resp. V_2) by a suitable transformation

$$F_2(x, y) \mapsto \lambda_0 F_2(\lambda_1 x, \lambda_2 y), \quad F_3(x, y) \mapsto \lambda'_0 F_2(\lambda'_1 x, \lambda'_2 y), \quad \lambda_i, \lambda'_i > 0, \ i = 0, 1, 2, \tag{4}$$

keeping the truncation h' on $[V, V_1]$. The isotopy type of regular intersection of F_2 and F_3 with this edge is preserved by such transformations.

Now we apply Theorem 3 and Remark 3 to F_1, F_2, F_3 and get a polynomial V with $V_1^S = h$: the convex function ν can be constructed by a perturbation of a linear one. The polynomial V thus obtained is an M-polynomial, since $\mathrm{Ch}V$ has: $\sum \#(\mathrm{Int}(\tau_i) \cap \mathbb{Z}^2)$ ovals coming from $\bigcup(\mathrm{Ch}F_i \cap (\mathbb{R}^*)^2)$; $\#([V, V_1] \cap \mathbb{Z}^2) - 2$ ovals coming from gluing $\mathrm{Ch}F_2$ and $\mathrm{Ch}F_3$ along $[V, V_1]$ and its symmetric copies; and one more oval appearing around the point V (see Figure 13a), which gives a total of

$$\sum \#(\mathrm{Int}(\tau_i) \cap \mathbb{Z}^2) + \#([V, V_1] \cap \mathbb{Z}^2) - 2 + 1 = \#(\mathrm{Int}(T) \cap \mathbb{Z}^2)$$

ovals.

Further, F has regular intersections with S_1, S_2, S_3 and the type of the regular intersection with S_1 is as prescribed. Indeed, $\mathbb{R}^2_\varepsilon$ is not the positive quadrant (see above), and, thus, the only arc of $\mathrm{Ch}F_2 \cap \mathbb{R}^2_\varepsilon$ with the endpoint on $[V, V_3]$ does not go to the edge $[V, V_1]$ (the intersection of F_2 with this edge is regular); hence, it goes to the edge S_2, completing the prescribed regular intersection of F with S_1. $\square$

Proposition 5 *Let $T \subset \mathbb{R}^2$ be a triangle whose vertices V_1, V_2, V_3 have nonnegative integral coordinates and the edges $S_2 = [V_1, V_3]$, $S_3 = [V_1, V_2]$ contain no integral points except for V_1, V_2, V_3. If h is a steady quasihomogeneous polynomial with $\Delta(h) = [V_2, V_3]$ and without real peripheral roots, then there exists an M-polynomial F with $\Delta(F) = T$ and $h = F^{S_1}$, where $S_1 = [V_2, V_3]$.*

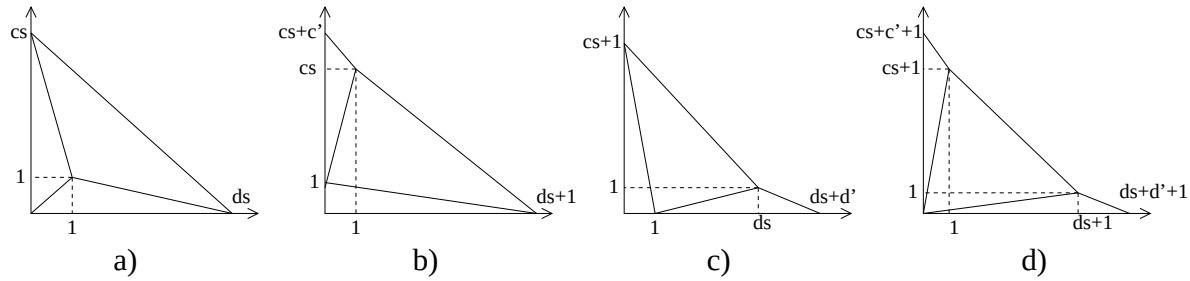


Figure 14: M-smoothing of regular semiquasihomogeneous singular point

The proof is a word-by-word copy of the proof of Proposition 4. Note, that *for any M-polynomial F as in Proposition 5,* $\mathrm{Ch}F$ *has in T and in one of its copies, $T_{\varepsilon,\delta}$, an arc joining S_2 with S_3.*

Proposition 6 *Let T be a triangle with integral vertices $V_1 = (0,0)$, $V_2 = (0, 2p+1)$, $V_3 = (2q, 1)$ and h a steady quasihomogeneous polynomial with $\Delta(h) = [V_2, V_3]$ which has no any real peripheral root. Then there exists an M-polynomial F with $\Delta(F) = T$ and $F^{[V_2,V_3]} = h$ whose chart crosses the vertical coordinate axis.*

Proof. Apply Theorem 3 to suitable polynomials F_1, F_2, F_3 with $\Delta_1 = \mathrm{conv}\{V, V_2, V_3\}$, $\Delta_2 = \mathrm{conv}\{V_1, V, V_3\}$, $\Delta_3 = \mathrm{conv}\{V_1, V, V_2\}$, where $V = (1,2)$. Namely, choose as F_1 an M-polynomial from Proposition 5 having in Δ_1 an arc connecting $[V, V_3]$ with $[V, V_2]$, and as F_2, F_3 polynomials from Proposition 4 having regular intersections with $[V_1, V]$, containing an arc connecting $[V_1, V]$ with $[V_1, V_3]$, $[V_1, V_2]$, respectively, and such that $F_3^{[V_1,V_2]}$ has real roots. (The coefficients of F_2 and F_3 are equalized by transformation (4), as in the proof of Proposition 4.) The charts $\mathrm{Ch}(F_1)$, $\mathrm{Ch}(F_2)$, $\mathrm{Ch}(F_3)$ have arcs which glue into an oval of $\mathrm{Ch}F$ embracing the point V. This oval together with other $\#((\mathrm{Int}(\Delta_1) \cup \mathrm{Int}(\Delta_2) \cup \mathrm{Int}(\Delta_3)) \cap \mathbb{Z}^2)$ ovals of F_1, F_2, F_3 in $(\mathbb{R}^*)^2$ gives a total of $\#(\mathrm{Int}(T) \cap \mathbb{Z}^2)$ ovals. Hence, F is an M-polynomial. $\square$

4.3 Proof of Theorem 2(1)

Adding x^i and y^j with big i and j, if necessary, and making a coordinate change, we may suppose that the Newton diagram of the given semi-quasi-homogeneous singular point has vertices at the points

$$\begin{gathered}(0,\ cs),\quad (ds,\ 0),\quad \text{or}\\ (0,\ cs+c'),\quad (1,\ cs),\quad (ds+1,\ 0),\quad \text{or}\\ (0,\ cs+1),\quad (ds,\ 1),\quad (ds+d',\ 0),\quad \text{or}\\ (0,\ cs+1+c`),\quad (1,\ cs+1),\quad (ds+1,\ 1),\quad (ds+d'+1,\ 0),\end{gathered}$$

where $c' > c/d$, $d' > d/c$. In all these cases we construct the required M-smoothing applying Theorem 4: divide the domain bounded by the Newton diagram and the

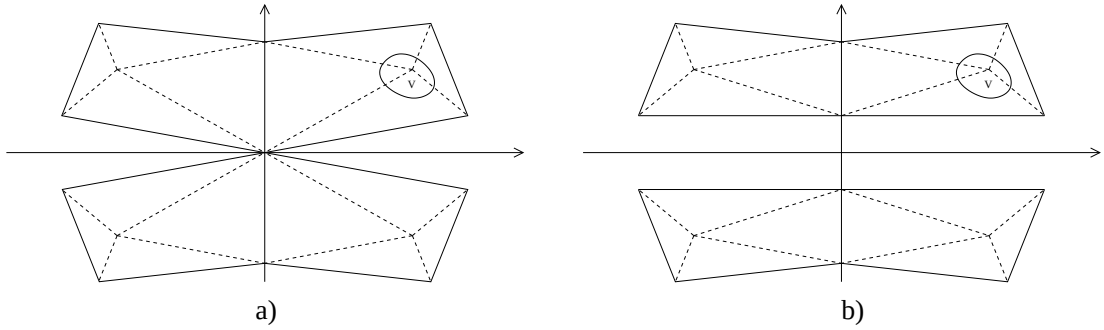


Figure 15: M-polynomials with Newton quadrangles I

coordinate axes into triangles as shown in Figure 14 (the existence of an appropriate convex function ν follows from the convexity of the Newton diagram) and then patch-work suitable M-polynomials from Propositions 4, 5 with these Newton triangles and given truncations on the edges on the Newton diagram (note that the edges inside the diagram are without integral points in their interior and recall that $\frac{1}{2}(\mu(C,z) - r(C,z)+1)$ is equal to the number of integral points in the interior of the domain bounded by the coordinate axes and the Newton diagram). For instance, in the case when the singularity is given by equation (3) with $a = b = 0$ and cs, ds both even, if the peripheral roots are all imaginary we use the subdivision into three triangles shown in Figure 14a (in each triangle two edges have no integral points in their interior) and patch-work three M-polynomials given by Proposition 5. The result provides an M-smoothing with $\frac{1}{2}(\mu(C,z) - r(C,z)+3)$ ovals: namely, $\frac{1}{2}(\mu(C,z) - r(C,z)+1) - 1$ ovals come from the ovals in $(\mathbb{R}^*)^2$ of the polynomial with Newton triangle $\mathrm{conv}\{(1,1),(0,cs),(ds,0)\}$, one more oval appears around the point $(1,1)$, and one more around $(-1,-1)$. $\square$

4.4 Auxiliary M-polynomials with Newton quadrangles

Let p, q, s be positive integers. Consider the polygons $Q'_{pqs} = \mathrm{conv}\{A,B,C,D\}$ and $Q''_{pqs} = \mathrm{conv}\{A',B,C,D\}$ where $A = (0,0)$, $B = (0,p)$, $C = (q,p+1)$, $D = (q+s,1)$, and $A' = (0,1)$. Note that the edges $[AD]$ and $[BC]$ contain no integral point in their interior, and, thus, real polynomials corresponding to them are all steady, each of them has only one peripheral root and this root is real.

Proposition 7 *Let h_1 and h_2 be steady quasi-homogeneous polynomials with $\Delta(h_1) = [BC]$ and $\Delta(h_2) = [CD]$. Assume that p and s are even, h_1 and h_2 have the same coefficient of $x^q y^{p+1}$, and h_2 has no real peripheral roots. Then there exist M-polynomials F' and F'' with $\Delta(F') = Q'_{pqs}$, $\Delta(F'') = Q''_{pqs}$, $(F')^{[BC]} = h_1$, and $(F'')^{[CD]} = h_2$.*

Proof. Take the closest to $[CD]$ integral point V of $\mathrm{Int}(Q'_{pqs})$ and divide Q'_{pqs} into four triangles joining V with the vertices of Q'_{pqs} (see Figure 15a): $\tau_1 = \mathrm{conv}\{B,C,V\}, \tau_2 = \mathrm{conv}\{A,B,V\}, \tau_3 = \mathrm{conv}\{C,D,V\}, \tau_4 = \mathrm{conv}\{A,D,V\}$.

Let F_1 be an M-polynomial with $\Delta(F_1) = \tau_1$ and $(F_1)^{[BC]} = h_1$ given by Proposition 4. Its chart has an arc connecting $[BC]$ and $[CD]$ in τ_1 or in one of the symmetric copies of τ_1; without lost of generality, we may suppose that it happens in τ_1. Proposition 4 provides an M-polynomial F_2 with $\Delta(F_2) = \tau_2$, $(F_2)^{[BV]} = (F_1)^{[BV]}$, and such that F_1 and F_2 have a compatible regular intersection with $[BV]$. In particular, $\mathrm{Ch}(F_2)$ has in τ_2 an arc connecting $[BV]$ and $[AV]$. By Proposition 5 there exists an M-polynomial F_3 with $\Delta(F_3) = \tau_3$, $(F_3)^{[CD]} = h_2$, and $\mathrm{Ch}(F_3)$ joining $[CV]$ and $[DV]$ in τ_3 by an arc.

By construction, $[CV]$ has the endpoints as its only integral points, the coefficients at the monomial $x^q y^{p+1}$ corresponding to C in F_1 and F_3 coincide, and the coefficients at the monomial $x^u y^v$ corresponding to V in F_1 and F_3 have the same sign. One can equalize the latter coefficients by a suitable transformation (4) applied to F_3, preserving all the properties of F_3 cited above. Finally, Proposition 4 provides an M-polynomial F_4 with $\Delta(F_4) = \tau_4$, $(F_4)^{[AV]} = (F_2)^{[AV]}$, and such that F_4 and F_2 have a compatible intersection with $[av]$. In particular, $\mathrm{Ch}(F_3)$ has in τ_4 an arc joining $[AV]$ and $[DV]$. It implies that the coefficients at the monomial $x^{q+s}y$ corresponding to D in F_3 and F_4 have the same sign, and to equalize the truncations of F_3 and F_4 onto $[DV]$, one has only to correct the absolute value of this coefficient in F_4, which can be made as in (4).

Now, it remains to apply Theorem 3 and patch-work F_1, F_2, F_3, and F_4 (a suitable convex function ν is obtained by a small variation of a linear function). The result is an M-polynomial, since its chart has: $\bigcup_{i=1}^4 \#(\mathrm{Int}(\tau_i) \cap \mathbb{Z}^2)$ ovals which come from the ovals of the glued charts, $\#((\mathrm{Int}[AV] \cup \mathrm{Int}[BV]) \cap \mathbb{Z}^2)$ ovals which come from gluing compatible intersections along $[AV]$ and $[BV]$ and their symmetric images, and one more oval around V.

The polynomial F'' is constructed in the same way (see Figure 15b). $\square$

Proposition 8 *Let h_1 and h_2 be steady quasi-homogeneous polynomials with $\Delta(h_1) = [BC]$ and $\Delta(h_2) = [CD]$. Assume that p, s are coprime and h_1, h_2 have the same coefficient of $x^q y^{p+1}$. If p is odd and q, s are even, assume additionally that the peripheral roots of h_1, h_2 have the same sign. Then there exist M-polynomials F' and F'' with $\Delta(F') = Q'_{pqs}$, $\Delta(F'') = Q''_{pqs}$, $(F')^{[BC]} = h_1$, and $(F'')^{[CD]} = h_2$.*

Proof. Assume that B, C, and D are not collinear $\mod 2$, i.e.,

$$pq - s \equiv 1 \mod 2 \ . \tag{5}$$

Divide Q'_{pqs} into two triangles: $\tau_1 = \mathrm{conv}\{B, C, D\}$ and $\tau_2 = \mathrm{conv}\{A, B, D\}$ (see Figure 16a). By Proposition 4 there exists an M-polynomial F_1 with $\Delta(F_1) = \tau_1$, $F_1^{[BC]} = h_1$ and regular intersection with $[BD]$. Since the vertices of τ_1 are not collinear $\mod 2$, suitable transformations (4) followed by $F_1(x, y) \mapsto \pm F_1(\pm x, \pm y)$ makes the coefficient at $x^{q+s}y$ corresponding to D in F_1 equal to that in h_2, keeping the truncation to $[BC]$. Complete the construction of F', patch-working F_1 with an M-polynomial F_2 which has $\Delta(F_2) = \tau_2$ and $F_2^{[BD]} = F_1^{[BD]}$, and whose chart intersects $[BD]$ compatibly with $\mathrm{Ch}(F_1)$.

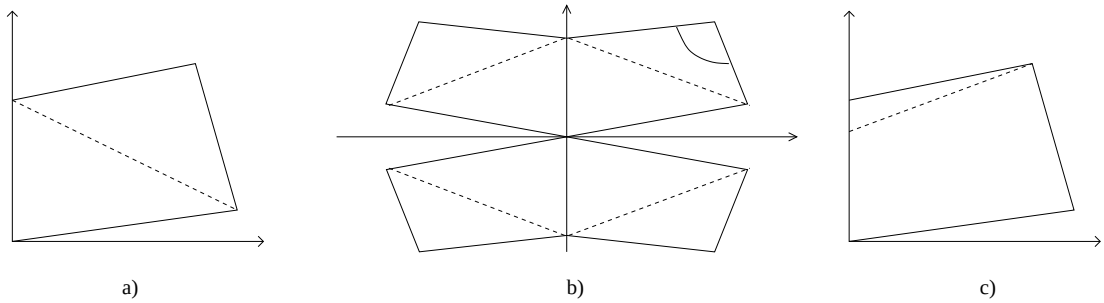


Figure 16: M-polynomials with Newton quadrangles II

Assume (5) does not hold and h_1, h_2 have peripheral roots of the same sign, say, all the roots are positive. Then, by Proposition 4 there exists an M-polynomial F_1 with $\Delta(F_1) = \tau_1$, $F_1^{[BD]} = h_1$, and whose chart has in τ_1 an arc joining $[BC]$ and $[CD]$ (see Figure 16b). This means, in particular, that the sign of the coefficient at $x^{q+s}y$ in h_2 and F_1 is the same, hence one can equalize these coefficients by a transformation (4) applied to F_1, and, thus, complete the construction of F' as in the previous case.

In the only remaining situation p, q, s are odd, and the peripheral roots of h_1, h_2 have different signs. Divide Q'_{pqs} into $\tau_1 = \mathrm{conv}\{B', B, C\}$ and $\widetilde{Q}_{pqr} = \mathrm{conv}\{A, B', C, D\}$, where $B' = (0, p-1)$ (see Figure 16c). Take any M-polynomial G with $\Delta(G) = \tau$ and $G^{[BC]} = h_1$. Since $[B'C]$ does not contain integral points in its interior, and the points B', C, D are not collinear $\mod 2$, the construction as in the case (5) provides an M-polynomial $\widetilde{F}$ with $\Delta(\widetilde{F}) = \widetilde{Q}_{pqs}$, $\widetilde{F}^{[CD]} = h_2$, and $\widetilde{F}^{[B'C]} = G^{[B'C]}$, and it remains to patch-work G and $\widetilde{F}$.

The polynomial F'' with Newton polygon Q''_{pqs} is constructed in the same way, except for the case $p = 1$, q, s are odd, but in this case the statement is trivial, since Q''_{1qs} is a triangle without integral points in the interior. $\square$

4.5 Proof of Theorem 2(3)

Given a Newton nondegenerate singular point with Newton diagram Γ, we divide the domain $\mathcal{D}$ bounded by Γ and the coordinate axes by the segments $[(i,j),(0,j-1)]$ for any nonsmooth point $(i,j) \in \Gamma$ (see Figure 17). Note that, up to vertical shifts, the upper triangle in this subdivision is as described in Proposition 6, and the other patterns are quadrangles as in Proposition 7. To obtain the required M-smoothing we apply Theorem 4 and patch-work the charts of M-polynomials from Propositions 6, 7 (the existence of a suitable convex function ν follows from the convexity of Newton diagram). Since the segments dividing $\mathcal{D}$ have no internal integral points, the smoothing obtained has $\#(\mathrm{Int}(\mathcal{D}) \cap \mathbb{Z}^2)$ ovals which come from the ovals of the glued charts in $(\mathbb{R}^*)^2$ and at least one more oval which intersects the coordinate axes. Hence, the total number of ovals is $\geq \#(\mathrm{Int}(\mathcal{D}) \cap \mathbb{Z}^2) + 1$. Thus, according to (2) and (1), it is equal to $\frac{1}{2}(\mu - r + 3)$ $\square$

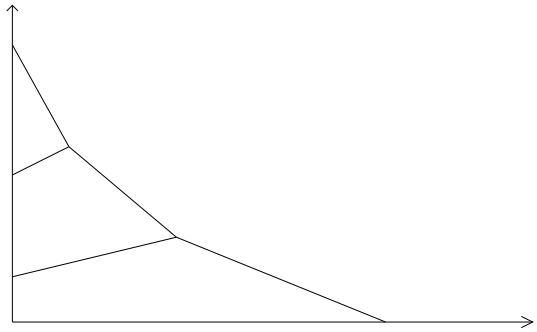


Figure 17: M-smoothing of isolated Newton nondegenerate singular point

4.6 Proof of Theorem 2(2)

We assume that c, d in (3) describing the leading part f^γ of the given real semi-quasi-homogeneous singularity satisfy:

- $c, d > 1$, since otherwise, say, for $c = 1$ a transformation $y \mapsto y - Ax^d$ makes all the real peripheral roots of f^γ positive and, hence, reduces the problem to Theorem 2(1);
- d is odd, since c, d are coprime and we can interchange x, y if necessary.

Assume also that f^γ has $2k_0$ imaginary, $k_+ > 0$ positive and $k_- > 0$ negative peripheral roots (otherwise, Theorem 2(2) follows from Theorem 2(1)). Trace a segment $[(i,j),(0,j-1)]$ for any integral point $(i,j) \in \gamma$, $i,j > 0$. We obtain a polygon Δ which is either a triangle (case $a = 0$) or a quadrangle (case $a = 1$), has γ as an edge, and is subdivided in polygons (also triangles and quadrangles) Δ_i, $i = 1, ..., k = 2k_0 + k_+ + k_-$, which we number from up downwards. Put

$$Q_0 = \Delta_1 \cup ... \cup \Delta_{2k_0}, \quad Q_i = \Delta_{2k_0+i},\ i = 1, ..., k_+ + k_-.$$

Pick: a polynomial h_0 with $\Delta(h_0) = \gamma \cap Q_0$ which has $2k_0$ distinct imaginary peripheral roots; polynomials h_i, $i = 1, \ldots, k_+$, with $\Delta(h_i) = \gamma \cap Q_i$ which have positive peripheral roots; and polynomials h_i, $i = k_+ + 1, \ldots, k_+ + k_-$, with $\Delta(h_i) = \gamma \cap Q_i$ which have negative peripheral roots. Note that up to vertical shifts: Q_i, $i = 1, ..., k_+ + k_- - 1$, are quadrangles of type Q'_{pqs}; Q_0 is a triangle as in Proposition 6 or a quadrangle of type Q'_{pqs}, according as $a = 0$ or 1 in (3); and $Q_{k_+ + k_-}$ is a quadrangle of type Q'_{pqs} or Q''_{pqs}, according as $b = 1$ or 0 in (3). Note also that all these polygons satisfy the corresponding conditions of Propositions 6, 7, 8, and the case which is not covered by Proposition 8 (i.e. p is odd, q, s are even, the roots of h_1, h_2 have different signs) never occurs because of initial adjustment of c and d.

Take an M-polynomial with $\Delta(F_0) = Q_0$ and $F_0^{\gamma \cap Q_0} = h_0$ which satisfies the conditions of Proposition 6 or 7, according as Q_0 is triangle or quadrangle. By induction, construct M-polynomials F_i, $i = 1, ..., k_+ + k_-$, with $\Delta(F_i) = Q_i$ and $F_i^{\gamma \cap Q_i} = h_i$ satisfying the conditions of Proposition 8 so that F_i truncated on the upper edge of Q_i coincides with F_{i-1} truncated on the lower edge of Q_{i-1}.

Theorem 3 applied to F_i as above gives us an M-polynomial F with $\Delta(F) = \Delta$: a suitable function ν is constructed by induction, and F is an M-polynomial since

$$\sum_{i=0}^{k_+ + k_-} \#(\mathrm{Int} Q_i \cap \mathbb{Z}^2) = \#(\mathrm{Int}\Delta \cap \mathbb{Z}^2). \tag{6}$$

The truncation $\tilde{f} = F^{\gamma \cap Q}$ is a quasi-homogeneous polynomial obtained by patchworking polynomials h_i. Thus, $\widetilde{f}$ is Newton nondegenerate and it has the same numbers of positive, negative and imaginary peripheral roots as f^γ. Therefore, $\widetilde{f} = 0$ defines at the origin a singular point deformation equivalent over $\mathbb{R}$ to $f = 0$. It remains to observe that $F = 0$ can be considered as an M-smoothing of $\widetilde{f} = 0$: a smoothing family is given by $F_t(x, y) = t^u F(xt^{-c}, yt^{-d})$, $u = (2k_0 + k_+ + k_-)cd + ac + bd$, and it is an M-smoothing due to (2) and (6). $\square$

Remark 4 The end of the proof can be replaced by reference to Theorem 4. If $a = 1$ or $b = 1$, then, first, Δ should be extended by one or two triangles, as well as by the convex envelope of the Newton diagram, to fill completely the domain between the diagram and the coordinate axes. The deformation type over $\mathbb{R}$ of the singularity remains unchanged.

5 Weak M-smoothing of Newton nondegenerate singular points

The local Harnack inequality (1) can be refined in the following way (see, for instance, [8]). Given a real isolated singular point (C, z), the intersection of C with the circle $\partial B(C, z)_{\mathbb{R}}$ is decomposed in pairs of points belonging to one branch of C. Given a real smoothing C' of (C, z), this decomposition is transported to the intersection of C' with $\partial B(C, z)_{\mathbb{R}}$: two boundary points of $C'_{\mathbb{R}}$ belong to the same pair, and called *boundary equivalent*, if they belong to the same component of $C' \cap \partial B(C, z)$. The refined inequality states that

$$v \leq \frac{\mu - r + 3}{2} - l \ , \tag{7}$$

where l is the number of topological circles obtained from the union of nonclosed components of $C'_{\mathbb{R}}$ by identification of boundary equivalent points. A smoothing C' is called a *weak M-smoothing* if it provides an equality in (7).

The Harnack inequality for CND polynomials is refined in a similar way. Given a real CND polynomial F in 2 variables, call two ends of $\{F = 0\} \cap (\mathbb{R}^*)^2 = \mathrm{Ch}(F) \cap (\mathrm{Int}\Delta(F))_*$ (as usually, $()_*$ stands for the union of four mirror images) equivalent if they merge into the same end of $\{F = 0\} \cap (\mathbb{C}^*)^2$, i.e., define the same point on $K\Delta$. The refined inequality states that

$$v \leq \#(\mathrm{Int}\Delta(F) \cap \mathbb{Z}^2) - l + 1 \ , \tag{8}$$

where v is the number of ovals of $\{F=0\}\cap(\mathbb{R}^*)^2$ and l is the number of topological circles obtained from the union of nonclosed components of $\{F=0\}\cap(\mathbb{R}^*)^2$ by identification of equivalent ends. A real CND polynomial is called a *weak M-polynomial* if it provides an equality in (8). (For related information see section 6.)

The following simple lemma and its corollary are well known (see, [10]).

Lemma 9 *A smoothing C' (respectively, a real CND polynomial F) is a weak M-smoothing (respectively, a weak M-polynomial) if, and only if, $C'_{\mathbb{C}}\setminus C'_{\mathbb{R}}$ (respectively, $\{F=0\}\cap(\mathbb{C}^*)^2\setminus\{F=0\}\cap(\mathbb{R}^*)^2$) is the disjoint union of two spheres with holes.*

Corollary 9 *If in notation of Theorem 3 (respectively, Theorem 4) $F_1,\dots,F_N$ (respectively, $F_1,\dots,F_{N-1}$) are weak M-polynomials, the adjacency graph of the decomposition $\Delta=\Delta_1\cup\dots\cup\Delta_N$ (respectively, $\Delta_1\cup\dots\cup\Delta_{N-1}$) is a tree, and the adjacency edges $\Delta_i\cap\Delta_j$ are without internal integral points, then F is a weak M-polynomial (respectively, weak M-smoothing).*

Theorem 5 *Any Newton nondegenerate singular point is deformation equivalent over $\mathbb{R}$ to a Newton nondegenerate singular point which has a weak M-smoothing.*

Remark 5 The number l in (7) is 0 if $r_{\mathbb{R}}=0$ and ≥ 1 if $r_{\mathbb{R}}\geq 1$. For the weak M-smoothings which we construct in the proof of Theorem 5 it can be estimated as follows. Call a real branch of a singular point a (0,1)-branch (or (1,0)-branch) if its intersection number with a real line through the singular point is even (resp. odd) if the line is tangent and odd (resp. even) otherwise. Then, $l\leq 1+\min\{\#((0,1)\text{-branches}),\ \#((1,0)\text{-branches})\}$. Note also that a weak M-smoothing with $l=1$ is an M-smoothing.

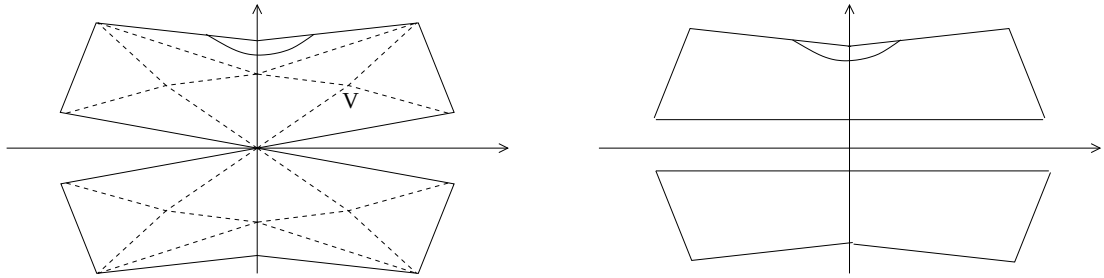
In the proof of Theorem 5 we use the following auxiliary polynomials.

Proposition 10 *Let p,g,s be positive integers and h_1,h_2 polynomials with $\Delta(h_1)=[BC]$ and $\Delta(h_2)=[CD]$, where $B=(0,p)$, $C=(q,p+1)$, and $D=(q+s,1)$. Assume that p and s are coprime, q and s are even, p is odd, the (only) peripheral root of h_1 is positive, and the (only) peripheral root of h_2 is negative.*

(1) There exists a weak M-polynomial F with $\Delta(F)=Q'_{pqs}$, $Q'_{pqs}=\mathrm{conv}(A,B,C,D)$, $A=(0,0)$, and such that: $F^{[BC]}=h_1$, $F^{[CD]}=h_2$, $\mathrm{Ch}F$ has $\#(\mathrm{Int}(Q'_{pqs})\cap\mathbb{Z}^2)-1$ ovals and a connected component of $\mathrm{Ch}(F)$ joins $[BC]$ and $B\bar{C}$, $\bar{C}=(-q,p+1)$, as shown in Figure 18a.

(2) If $p>1$, then there exists a weak M-polynomial F with $\Delta(F)=Q''_{pqs}$, $Q''_{pqs}=\mathrm{conv}(A',B,C,D)$, $A'=(0,1)$, and such that: $F^{[BC]}=h_1$, $F^{[CD]}=h_2$, $\mathrm{Ch}F$ has $\#(\mathrm{Int}(Q''_{pqs})\cap\mathbb{Z}^2)-1$ ovals and a connected component of $\mathrm{Ch}(F)$ joins $[BC]$ and $B\bar{C}$, $\bar{C}=(-q,p+1)$, as shown in Figure 18b.

(3) The same statements (1) and (2) hold true if the root of h_1 is negative, the root of h_2 is positive, and the above charts are replaced by their symmetric images with respect to the horizontal axis.



a) b)

Figure 18: M-polynomials with Newton quadrangles III

Proof. Divide Q'_{pqs} into a triangle $\tau = \mathrm{conv}\{B', B, C\}$ and a quadrangle $\widetilde{Q}_{pqs} = \mathrm{conv}\{AB'CD\}$, where $B' = (0, p-1)$. Choose the closest to $[B'C]$ integral point $V \in \mathrm{Int}(\widetilde{Q}_{pqs})$ and divide $\widetilde{Q}_{pqs}$ into four triangles by segments joining V with the vertices of $\widetilde{Q}_{pqs}$ (see Figure 18a).

Pick a polynomial h_3 with $\Delta(h) = [B'C]$ which has its both peripheral roots imaginary and distinct and whose coefficient at $x^q y^{p+1}$ is the same as in h_1. By Proposition 5 there exists an M-polynomial F_1 with $\Delta(F_1) = \tau$ and $F_1^{[B'C]} = h_3$ whose chart $\mathrm{Ch}(F_1)$ has an arc joining $[BC]$ and $[B'B]$ in τ. Then, $\mathrm{Ch}(F_1)$ has an arc connecting $[B'B]$ and $B\bar{C}$. The union of these arcs is the arc required in Proposition 10. In addition, this implies that the (only) peripheral root of $F_1^{[BC]}$ is positive; hence the sign of the coefficient at y^p in F_1 is as in h_1, and one can equalize these coefficients by a transformation (4) applied to F_1.

Now, construct an M-polynomial F_2 with $\Delta(F_2) = \widetilde{Q}_{pqs}$ and truncations h_2, h_3 on the corresponding edges. This can be done in the same manner as in the proof of Proposition 7 with the same role of the chosen point V. Patch-working F_1 and F_2 provides the required polynomial F.

The same construction applies to prove statements (2), (3). □

Proof of Theorem 5. Let $(C, 0)$ be a Newton nondegenerate singular point given by $f = 0$. Without loss of generality, suppose that the Newton diagram $\Gamma(f)$ of f has vertices on the both coordinate axes. Denote by $\gamma_1, ..., \gamma_n$ the edges of $\Gamma(f)$ numbered from up downwards, and by $2k_u, k_u^+, k_u^-$, $1 \le u \le n$, the number of imaginary, positive, and negative peripheral foots of f^{γ_u}. Divide the domain $\mathcal{D}$, bounded by $\Gamma(f)$ and the coordinate axes, by the segments $[(i,j),(0,j-1)]$ taken for any integral point (i,j), $i,j > 0$, in $\Gamma(f)$. Then, as in the proof of Theorem 2(2), for any γ_u, $1 \le u \le n$, define the polygons $Q_u^{(i)}$, $i = 0, ..., k_u^+ + k_u^-$, and the quasihomogeneous polynomials $h_u^{(i)}$ with $\Delta(h_u^{(i)}) = Q_u^{(i)} \cap \gamma_u$, $i = 0, ..., k_u^+ + k_u^-$, such that each $h_u^{(0)}$ has $2k_u^0$ imaginary peripheral roots, the peripheral root of each $h_u^{(i)}$, $i = 1, ..., k_u^+$, is positive, and the peripheral root of each $h_u^{(i)}$, $i = k_u^+ + 1, \ldots, k_u^+ + k_u^-$, is negative. We suppose also, that the coefficients of a common monomial in any two such quasihomogeneous polynomials coincide.

Examining the sequence of polygons $Q_u^{(i)}$ from up downwards, construct a sequence of polynomials $F_u^{(i)}$ with $\Delta(F_u^{(i)}) = Q_u^{(i)}$ such that: their truncations to the edges $Q_u^{(i)} \cap \gamma_u$ are $h_u^{(i)}$, their truncations to adjacent edges of neighboring polygons coincide, and each $F_u^{(i)}$ satisfies the conditions of Propositions 4, 6, 7, 8, 10, respectively.

For each $u = 1, \ldots, n$ apply Theorem 3 to patch-work $F_u^{(i)}$, $i = 0, ..., k_u^+ + k_u^-$, into a polynomial F_u with $\Delta(F_u) = Q_u = \bigcup_i Q_u^{(i)}$ (a suitable convex function ν is constructed by induction). The truncations of F_u and F_{u+1} on $\sigma = [(i,j),(0,j-1)] = Q_u \cap Q_{u+1}$ are

$$F_u^\sigma = t_1 A y^{j-1} + t_2 B x^i y^j, \quad F_{u+1}^\sigma = t_3 A y^{j-1} + t_4 B x^i y^j, \quad t_1, t_2, t_3, t_4 > 0.$$

Equalize them successively applying the transformation (4) to $F_2, ..., F_n$, and keeping the signs of the peripheral roots of $F_u^{\gamma_u}$, $u = 2, ..., n$.

Let $\widetilde{f}$ be a polynomial such that $\gamma_1 \cup ... \cup \gamma_n$ is a part of the boundary of its Newton polygon, and $\widetilde{f}^{\gamma_u} = F_u^{\gamma_u}$, $u = 1, ..., n$. By construction, $\widetilde{f}$ and f has the same numbers of imaginary, positive and negative peripheral roots on each edge γ_u. Therefore, $\widetilde{f}$ defines a singular point $(\widetilde{C}, 0)$ deformation equivalent over $\mathbb{R}$ to $(C, 0)$. To obtain a smoothing $\widetilde{C}'$ of $(\widetilde{C}, 0)$ it remains to apply Theorem 4 to F_u, $1 \le u \le n$ (a suitable convex function ν is constructed as in the proof of Theorem 2(3)). As it follows from Corollary 9, $\widetilde{C}'$ is a weak M-smoothing. □

Remark 6 To check that $\widetilde{C}'$ in the above proof is a weak M-smoothing one may count explicitly the corresponding ν and l and verify simultaneously that they turn (7) into an equality. Note, first, that the number of those ovals in $\widetilde{C}'_{\mathbb{R}}$ which come from the ovals of $F_s^{(i)}$ in $(\mathbb{R}^*)^2$ is

$$\sum_{u,i} \#(\mathrm{Int}(Q_u^{(i)}) \cap \mathbb{Z}^2) - l' = \#(\mathrm{Int}(\mathcal{D}) \cap \mathbb{Z}^2) - l' = \frac{1}{2}(\mu - r + 1) - l', \qquad (10)$$

where l' counts how many polynomials as in Proposition 10 occur in the set $\{F_u^{(i)}\}_{s,i}$. Renumber the polynomials $F_u^{(i)}$ as $F^{(1)}, ..., F^{(N)}$ following the order of the polygons $Q_u^{(i)}$ from up downwards. Assume that $F^{(k_i)}$, $i = 1, ..., l'$, are as in Proposition 10. If two real branches $\mathcal{P}_1, \mathcal{P}_2$ of $\widetilde{C}_{\mathbb{R}}$ correspond to peripheral roots of $F^{(j)}, F^{(k)}$ such that $j < k_i \le k$ for some $i = 1, ..., l'$, then neither of the endpoints of $\mathcal{P}_1$ on $B(C, z)_{\mathbb{R}}$ belongs to the same connected component of $\widetilde{C}'_{\mathbb{R}}$ as an endpoint of $\mathcal{P}_2$. Indeed, if there were such a component, it would have passed in $Q_k^{(i)}$ from the edge $[(0,p),(q,p+1)]$ (or its symmetric image) to the edge $[(q,p+1),(q+s,1)]$ or $[(0,0),(q+s,1)]$ (or their symmetric images), but it is not the case as shown in Figure 18. Therefore, if one of $F^{(j)}$ with $j < k_1$ has a real peripheral root, then $l \ge 1 + l'$, which implies that (7) turns into an equality. If the polynomials $F^{(j)}$, $j = 1, ..., k_1 - 1$, has only imaginary peripheral roots, then $l \ge l'$. In this case the arc in $\mathrm{Ch}(F^{(k_1)})$ shown in Figure 18 continued in the union of the charts of $F^{(j)}$, $j = 1, ..., k_1 - 1$, gives one more oval in $\widetilde{C}'_{\mathbb{R}}$ which together with previously counted ovals (10) and $l \ge l'$ provides the equality in (7).

6 Some open problems

1. Consider a bouquet of three real ordinary cusps with common vertex at 0 (each one topologically equivalent over $\mathbb{R}$ to $x^2 - y^3 = 0$). In the case of distinct tangents, such singularities form two real topologically equisingular over $\mathbb{R}$ families: bouquets which are contained in a half-plane with the boundary line through 0 and those which are not. The singularities of the first class have M-smoothing, while, as shown in [9], the singularities of the second class (called in [9] Sirler singularities) have no M-smoothing. The proof given in [9] exploits various particular geometrical properties of Sirler singularities. So, *is there any general geometric phenomenon behind the nonexistence of M-smoothing?*

Look at intermediate positions of three cusps. The construction in the proof of Theorem 5 applied to a bouquet of 3 cusps with 2 different tangents, where the cusps with the same tangent are directed into opposite sides, shows that this singular point is deformation equivalent over $\mathbb{R}$ to a singularity which has M-smoothing. *What are topological and geometrical properties of the stratum of singularities which have M-smoothing? Say, in the versal deformation of a singular point?*

As it follows from above examples, this stratum is not necessarily closed or open. However, in a μ-constant families, i.e., inside families of real topologically equisingular over $\mathbb{R}$ singularities, the set of singularities which have M-smoothing is closed. *Is it open?* i.e., *Is it true that the real singularities which are topologically equisingular over $\mathbb{R}$ either all have or all have no M-smoothing?*

2. In view of Theorem 2 and the above question, it is natural to ask: *Does any semi-quasi-homogeneous singular point admit an M-smoothing?* Though the construction set up in the proof of Theorem 2 provides M-smoothings only for special values of the peripheral roots, the following example shows that the Viro theorem with a different subdivision of the domain under the Newton diagram can give M-smoothings for the whole range of the roots. The same example is the simplest singularity where the algorithm of [16] fails, i.e. the singularity which has M-smoothing but not BM-smoothing.

Example. For a singular point $(y^2 + ax^3)(y^2 + bx^3) = 0$ with arbitrary $a < 0 < b$ there is a smoothing with $\frac{1}{2}(\mu - r + 1) = 7$ ovals.

Proof. First, construct a polynomial F_1 with $\Delta(F_1) = \mathrm{conv}\{(0,0),(0,2),(4,0)\}$ and the chart shown in Figure 19a. We get it by patch-working two polynomials: F_1', which has $\Delta(F_1') = \mathrm{conv}\{(0,2),(4,0),(3,0)\}$, truncation $(y + ax^2)(y + bx^2)$ on $[(0,2),(4,0)]$, and vertical tangent at the intersection point with the horizontal axis; and F_1'', which has $\Delta(F_1'') = \mathrm{conv}\{(0,0),(0,2),(3,0)\}$ and the chart as in Figure 19a. The shift $(x,y) \mapsto (x+\alpha, y)$ with a suitable α transforms F_1 in F_2, which has $\Delta(F_2) = \mathrm{conv}\{(0,2),(1,0),(4,0)\}$, the same truncation on the edge $[(0,2),(4,0)]$ as F_1, and the chart as in Figure 19b. Consider the polynomial $F_3(x,y) = x^{-2}y^4F_2(x^2y^{-1}, x)$. It has $\Delta(F_3) = \mathrm{conv}\{(0,3),(0,4),(6,0)\}$ and its truncation on $[(0,4),(6,0)]$ is $(y^2 + ax^3)(y^2+bx^3)$. Its chart is shown in Figure 19c. Patch-working F_3 with a polynomial, which has $\Delta(F_3) = \mathrm{conv}\{(0,0),(0,3),(6,0)\}$, the chart shown in Figure 19d and prescribed truncation with negative roots on $[(0,3),(6,0)]$ (such a polynomial is

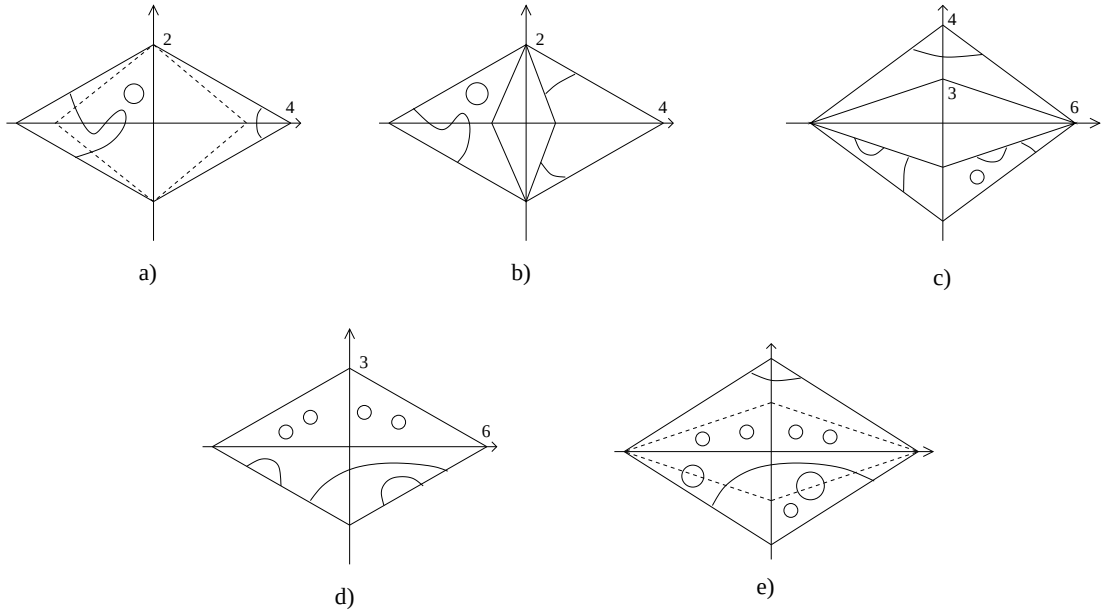


Figure 19: M-smoothing of $(y^2 + ax^3)(y^2 + bx^3) = 0$

found in [22] or [23]), we obtain the desired M-smoothing (Figure 19e).

3. Every Sirler singularity (see **1** above) admits a weak M-smoothing. On the other hand, Theorem 5 says that up to equisingular deformation over $\mathbb{R}$, a wide class of singular points have weak M-smoothing. So, *does any real plane curve singular point have a weak M-smoothing?* Or, at least, *is any real plane curve singular point topologically equivalent over $\mathbb{R}$ to a singular point which has a weak M-smoothing?*

4. The improved Harnack bounds (7) and (8) are contained in the following general bounds for pairs (X, A) with $\mathbb{Z}/2$-action $c : (X, A) \to (X, A)$, which state that

$$\dim H_*(X_{\mathbb{R}}; \mathbb{Z}/2) \leq \dim H^1(\mathbb{Z}/2; H_*(X; \mathbb{Z}/2)) \leq \dim H_*(X; \mathbb{Z}/2) - 2a,$$

where $X_{\mathbb{R}}$ is the fixed point set of c, $2a = \dim H_*(A; \mathbb{Z}/2) - \dim K - (\dim H^1(\mathbb{Z}/2; H_*(A; \mathbb{Z}/2)/K)$, K is the kernel of the map $H_*(A; \mathbb{Z}/2) \to H_*(X; \mathbb{Z}/2)$ induced by inclusion $A \subset X$ and H^1 stands for the Galois cohomology of a $\mathbb{Z}/2$-vector space with $Z/2$-action. The first bound is Swan's inequality, see [18], the second one is an easy consequence of the usual Ker $-$ Coker sequence, cf. [5]. The pairs which turn both the bounds in equalities can be characterized by the following properties: each c-invariant element of $H_*(X; \mathbb{Z}/2)$ is realized by a c-invariant cycle and $\dim H_*(X_{\mathbb{R}}; \mathbb{Z}/2) = \dim H_*(X; \mathbb{Z}/2) - 2a$, where a is the dimension of the image of $(1 + c_*)H_*(A; \mathbb{Z}/2)$ in $H_*(X; \mathbb{Z}/2)$.

To get (7), (8), and Lemma 9 take $X = C'_{\mathbb{C}}$ and $A = C'_{\mathbb{R}} \cup \partial C'_{\mathbb{C}}$.

Since the above bound by $\dim H_*(X; \mathbb{Z}/2) - 2a$ is, in addition, has the same extremal properties as (7) does in the case of plane curve singularities (cf., [8]), it is

natural to ask *is this version (or its stronger form, with $a = 0$) of the Harnack-Smith-Thom inequality sharp?* Recent results on the sharpness of the Harnack-Smith-Thom upper bound

$$\dim H_*(X_{\mathbb{R}}, \mathbb{Z}/2\mathbb{Z}) \leq \dim H_*(X_{\mathbb{C}}, \mathbb{Z}/2\mathbb{Z})$$

for projective nonsingular hypersurfaces and complete intersections (Itenberg-Viro, unpublished) seem to show the sharpness of the above local version for homogeneous hypersurface and complete intersections singularities, up to equisingular deformation over $\mathbb{C}$.

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