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An introduction to Frobenius manifolds, moduli spaces of stable maps and quantum cohomology

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Introduction

The aim of these survey notes is to try to motivate and to introduce two beautiful geometric concepts that have already proved, despite their relative youth, to be rich and useful, namely (Dubrovin's) Frobenius manifolds and (Kontsevich's) moduli spaces of stable maps.

According to a philosophy which I have learned in Givental's papers [21, 23], there are two big families of Frobenius manifolds

- those defined by unfoldings (Landau-Ginzburg, or B -models, according to the “physical” terminology),
- those defined by quantum cohomology (field theory or A -models)

and the famous mirror conjecture¹ postulates the existence of a correspondence between these two families.

The two parts of these notes can be considered as a description of these two families (respectively), through concrete examples.

Although the definition of a Frobenius manifold somehow looks like some French cooking recipes (many ingredients and a rather intricate mixture of them, see *e.g.* [17, 12]), the concept itself is quite natural, and, in some sense classical. Singularists, for instance, are dealing with Frobenius manifolds for ages (or, at least since K. Saito's paper [41]). The Frobenius structure is what you see when you look at the product on the Jacobian algebra and at the residue form as being a ring structure on the tangent sheaf to the space of parameters of the deformation and a 1-form on this space, the main point being Saito's theorem, that is, that the residue form and the product define a *flat* metric.

A Frobenius structure on a manifold M is, indeed, a ring structure on the tangent sheaf together with a flat metric subject to a couple of compatibility conditions. In very down-to-earth terms, the Landau-Ginzburg models are those in which the multiplication is rather simple, but for which it is hard to find flat coordinates, while in quantum cohomology, the flat coordinates are the natural coordinates, while the product is very hard to define. In the language which is used in the Frobenius world, in Landau-Ginzburg models, you are given the “canonical coordinates” while in quantum cohomology, you are given the “flat coordinates”.

Unfoldings belong to the world of Lagrangian singularities (caustics) but this is also the case of all Frobenius manifolds: a product on vector fields defines a *spectral cover* of the manifold, a subvariety of T^*M , which is very often Lagrangian, and roughly speaking describes the way the rings $T_\xi M$ split as sums of simple summands according to the values of ξ . Thus, Lagrangian subvarieties definitely belong to the Frobenius landscape.

The first chapter of the present survey is devoted to the description of the Lagrangian subvarieties defined by unfoldings, of spectral covers and to the definition of a Frobenius manifold. I spend many pages on examples, *e.g.* on 2-dimensional examples and on the versal unfolding of z^{n+1} .

The second chapter is a description of the Frobenius structure on the cohomology of some projective varieties given by the quantum product. It is well-known that the idea of quantum cohomology comes from the work of physicists trying to deform the cup-product using rational

¹For a more classical formulation, see *e.g.* [48, 7].

curves present in the variety (see [47]). The fact that this is related to the “associativity equations” (*i.e.* to a Frobenius structure) is beautifully explained in [14]. In this text, I describe Kontsevich’s moduli spaces of stable maps and how these moduli spaces can be used to construct the Gromov-Witten potential, which is the main ingredient in cooking up the Frobenius structure. I also give some examples of computations of Gromov-Witten invariants and quantum rings using stable maps.

Some remarks and further examples are the subject of the Appendix.

I have been very impressed by the remarkable idea (of Kontsevich [31]) of considering spaces of stable maps, so that I have tried to convince the readers that this was actually a very natural idea and I have spent some place (maybe too much for professional algebro-geometers) on motivating the definition by the investigation of plane cubics (see Appendix A). I also explain some relations of the spaces of stable maps with configuration spaces and algebraic loop spaces (Appendix B).

I also spend a few pages to discuss whether the relations defining the quantum rings should or should not be an integrable system (Appendix C). Then I turn to the semi-simplicity problem, in Appendix D, where I give some partial results in direction of the semi-simplicity of some quantum cohomology rings.

In addition to the papers I have already mentioned, there are many good papers on the topics discussed here, *e.g.* [17, 27, 40] on Frobenius structures, [20] on stable maps and quantum cohomology, and [35] on both Frobenius manifolds and quantum cohomology. Therefore, there can be nothing original in the present survey, except for the presentation of rather sophisticated concepts through natural examples.

I am pleased to acknowledge the influence of Sasha Givental’s ideas. I also thank the people with whom I have discussed the matters presented here, especially Claude Sabbah and the participants of the “isomonodromy conference” at CIRM in July 1996, of the seminars on “mirror symmetry” and on “rational curves” in Strasbourg in 1995, 1996, 1997 and of the winter school in Srní in January 1997.

Special thanks are due to Holger Spielberg for his careful reading of various preliminary versions of this paper. His comments have been very useful to improve the presentation given here and to correct some of the mistakes it contained.

Many thanks are due to Alain Sartout for his help with pictures in postscript.

Chapter I

From Lagrange projections to Frobenius manifolds

Frobenius manifolds have been the object of investigations of Dubrovin and others in the last few years. It seems clear that this is a really important notion, as it is related to a lot of different aspects of geometry (including some aspects of field theory and isomonodromy problems) as can be seen in *e.g.* [17, 27, 40, 35].

Roughly speaking, the structure of a Frobenius manifold is the structure of a ring sheaf on the tangent sheaf together with a flat connection allowing to relate the structures of the various

fibers. The aim of this first part is to introduce its definition by a rich and already well-known family, that of Lagrange projections — alias unfoldings.

Following an idea of Givental, I will then explain why Lagrange subvarieties definitely belong to the Frobenius landscape.

I will then give the definition and a few examples of Frobenius manifolds.

1. Stable germs of Lagrange projections

1.1. Lagrangian submanifolds in a cotangent bundle

The total space of the cotangent bundle of a manifold M carries a canonical 1-form, the Liouville form λ : recall that, if

$$\pi : T^*M \longrightarrow M$$

denotes the projection and (ξ, φ) denotes a point in T^*M (consisting of a point ξ of M and a linear form $\varphi : T_\xi M \rightarrow \mathbf{C}$),

$$\lambda_{(\xi, \varphi)}(X) = \varphi(T_\xi \pi(X)).$$

It is a “universal” 1-form in the sense that, if α is a 1-form on M that we consider as a section of the projection π :

$$\alpha : M \longrightarrow T^*M,$$

then $\alpha^* \lambda = \alpha$.

The derivative $\omega = d\lambda$ is a symplectic form (*i.e.* a non degenerate closed 2-form on T^*M). A submanifold L of T^*M is called *Lagrangian* if it is totally isotropic (that is to say that ω vanishes on L) of maximal dimension (half the dimension of T^*M , that is, the dimension of M).

Generating functions. — The Liouville form λ defines a closed 1-form on Lagrangian submanifolds. For instance, the graph of a 1-form α is a Lagrangian submanifold of T^*M if and only if α is a closed form. The simplest examples of Lagrangian submanifolds are thus the graphs of exact 1-forms, that is, the graphs of the differentials of functions

$$F : M \longrightarrow \mathbf{C}.$$

Such a function F is called a *generating function* for the Lagrangian submanifold it defines.

These examples (graphs of differentials of functions) are very special, since the graph L of dF obviously has the property that the composition

$$L \subset T^*M \xrightarrow{\pi} M$$

is a diffeomorphism. To construct more general examples, one slightly modifies the construction, adding parameters to the function F .

Generating families. — Consider a function

$$\begin{aligned} F : \mathbf{C}^k \times M &\longrightarrow \mathbf{C} \\ (z_1, \dots, z_k, \xi) &\longmapsto F_\xi(z). \end{aligned}$$

The graph of its differential is a Lagrangian subvariety of $T^*(\mathbf{C}^k \times M)$.

Assume now that this Lagrangian is transversal to the conormal bundle of the projection $\mathbf{C}^k \times M \longrightarrow M$: in local coordinates (ξ_1, \dots, ξ_N) on M , this is to say that the $k \times (k + N)$ -matrix

$$\left(\left(\frac{\partial^2 F}{\partial z_i \partial z_j} \right) \left(\frac{\partial^2 F}{\partial z_i \partial \xi_j} \right) \right)$$

has rank k . The intersection of the graph of dF with the conormal bundle of the projection is the image of the submanifold

$$\Lambda = \{(z, \xi) \in \mathbf{C}^k \times M \mid d_z F = 0\} \subset \mathbf{C}^k \times M.$$

The mapping

$$\begin{aligned} \Lambda &\longrightarrow T^*M \\ (z, \xi) &\longmapsto (\xi, d_\xi F) \end{aligned}$$

is a Lagrange immersion. This is indeed more general than the case of generating functions as any germ of Lagrange immersion in T^*M can be described this way (see [49]).

1.2. Singularities and unfoldings

Let us change our point of view and consider now that the generating family F is a deformation of $F_0 = f$ (a function of the k variables (z_1, \dots, z_k)). We will consider only germs, so that we can assume that $M = \mathbf{C}^N$. Assume also that $0 \in \Lambda$. This is to say that f has a singularity at 0.

Consider the projection

$$\varphi : \Lambda \longrightarrow T^*\mathbf{C}^N \longrightarrow \mathbf{C}^N.$$

It is singular at 0 if and only if the Hessian matrix of f at 0 is degenerate. Moreover, the dimension of the kernel $\text{Ker } T_0\varphi$ is k if and only if the Hessian matrix is zero.

Before to look at the converse construction, we need some notation.

Notation. — The Jacobian ideal of a germ f is denoted by $J(f)$ (*i.e.* $J(f)$ is the ideal generated by the first order partial derivatives of f at 0). The Jacobian algebra is the quotient of the ring of convergent series by the Jacobian ideal and is denoted by Q_f . So if $f \in \mathbf{C}\{z_1, \dots, z_k\}$,

$$Q_f = \mathbf{C}\{z_1, \dots, z_k\} / \left\langle \frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_k} \right\rangle.$$

Suppose now that $f : \mathbf{C}^k \rightarrow \mathbf{C}$ is a polynomial with an isolated singularity at 0 and assume that the Hessian of f at 0 is 0. Then there exists a basis of the Jacobian algebra Q_f that contains the classes of $\{1, z_1, \dots, z_k\}$ modulo $J(f)$.

Choose polynomials $g_{k+1}, \dots, g_N : \mathbf{C}^k \rightarrow \mathbf{C}$ such that the classes of

$$\{1, z_1, \dots, z_k, g_{k+1}, \dots, g_N\}$$

generate the vector space² Q_f . Define

$$F(z, \xi) = F_\xi(z) = f(z) + \xi_1 z_1 + \dots + \xi_k z_k + \xi_{k+1} g_{k+1}(z) + \dots + \xi_N g_N(z).$$

This is a function on $\mathbf{C}^k \times \mathbf{C}^N$. The germ

$$\begin{array}{ccc} F : \mathbf{C}^k \times \mathbf{C}^N & \longrightarrow & \mathbf{C} \\ (z_1, \dots, z_k, \xi_1, \dots, \xi_N) & \longmapsto & F_\xi(z) \end{array}$$

is a *versal unfolding* of f (this is the definition) and is also a generating family for a Lagrangian Λ as above. It is said that the germ of the Lagrangian projection

$$\varphi : \Lambda \longrightarrow T^*\mathbf{C}^N \longrightarrow \mathbf{C}^N$$

is described by the unfolding F_ξ .

Notice that in this case, with the choices taken, the transversality condition is automatically fulfilled, since

$$\left(\left(\frac{\partial^2 F}{\partial z_i \partial z_j} \right) \left(\frac{\partial^2 F}{\partial z_i \partial \xi_j} \right) \right) = \left(\left(\frac{\partial^2 F}{\partial z_i \partial z_j} \right) (\text{Id}_k) (\star) \right),$$

obviously has rank k .

Degree. — Being defined by polynomials, the Lagrangian projection

$$\varphi : \Lambda \longrightarrow \mathbf{C}^N$$

is a finite map. Its degree is the multiplicity μ of f at 0, the “number” of critical points of $z \mapsto F_\xi(z)$. For a generic ξ , F_ξ actually has μ distinct critical points, each one having multiplicity 1: this is true already for $f(z) + \xi_1 z_1 + \dots + \xi_k z_k$ (see Appendix B in [37]).

Stable germs. — A germ of a *Lagrange mapping*

$$\varphi : L \subset T^*\mathbf{C}^N \longrightarrow \mathbf{C}^N$$

is *stable* if all nearby germs are equivalent to it. It is well known (see *e.g.* [1] p. 278) that the stable germs are the germs that can be described by generating families that are *versal unfoldings*. In other words, the germ of φ at 0 is stable if and only if $F_\xi(z) + \xi_0$ is a versal unfolding of $F_0(z)$ (that we denoted $f(z)$), in the sense that the classes of the functions

$$1 \text{ and } \frac{\partial F_\xi}{\partial \xi_i} \Big|_{\xi=0} \text{ (for } 1 \leq i \leq N)$$

generate the vector space underlying the Jacobian algebra Q_f .

²Note that the multiplicity of f at 0 is then at most $N + 1$.

Remark. — The number k we have used so far is determined by the germ, being the dimension of $\text{Ker } T_0\varphi$. The number N can be as big as we want. This does not change anything to Q_f but changes the dimension of the Lagrangian: this amounts to add excedentary variables ξ . For instance, unfolding z^3 gives Lagrangian folds in any dimensions (see § 1.3 below).

Notation. — Denote $F_\xi + \xi_0 = F_{\tilde{\xi}}$.

1.3. Examples of Lagrange projections

Non singular projections. — The zero section of $T^*\mathbf{C}^N$ is a Lagrange submanifold. The projection has no singularity ($k = 0$). Any constant function $\mathbf{C}^N \rightarrow \mathbf{C}$ is a generating function.

Folds and the A_n -family. — Here, $k = 1$, $f(z) = z^3$, the Jacobian algebra Q_f is $\mathbf{C}[z]/z^2$ and $(1, z)$ is a basis. The versal unfolding is

$$F_{\tilde{\xi}}(z) = z^3 + \xi_1 z + \xi_0$$

and the generating function is

$$F_\xi(z) = z^3 + \xi_1 z$$

considered as a function of $(z, \xi_1, \dots, \xi_N) \in \mathbf{C} \times \mathbf{C}^N$ (even if it depends only on ξ_1). The Lagrangian L is described by

$$\begin{array}{ccc} \mathbf{C} \times \mathbf{C}^{N-1} & \longrightarrow & T^*\mathbf{C}^N \\ (z, \xi_2, \dots, \xi_N) & \longmapsto & (-3z^2, \xi_2, \dots, \xi_N; z, 0, \dots, 0) \end{array}$$

It has a fold singularity (it is shown of Figure 2 below).

One can consider more generally $f(z) = z^{n+1}$ in place of $f(z) = z^3$. Then

$$F_\xi(z) = z^{n+1} + \xi_{n-1} z^{n-1} + \dots + \xi_1 z.$$

We will come back to these examples in §§ 1.4 and 3.3.

The D_4 -singularity. — Here is an example with two variables. The function f is

$$f(z_1, z_2) = z_1^3 + z_1 z_2^2.$$

The Jacobian algebra has dimension 4 (as the subscript in D_4 indicates it) and is generated by $\{1, z_1, z_2, z_1^2\}$. As a versal unfolding, we thus chose

$$F_\xi(z) = f(z) + \xi_1 z_1 + \xi_2 z_2 + \xi_3 z_1^2.$$

It defines a Lagrangian in $T^*\mathbf{C}^N$ for $N \geq 3$, which is the product of the zero section in $T^*\mathbf{C}^{N-3}$ with a Lagrangian in $T^*\mathbf{C}^3$ that we describe now. This is the image of

$$\Lambda = \left\{ (z, \xi) \mid 3z_1^2 + z_2^2 + \xi_1 + 2\xi_3 z_1 = 0 \text{ and } 2z_1 z_2 + \xi_2 = 0 \right\}$$

under the mapping

$$\begin{array}{ccc} \Lambda & \longrightarrow & T^*\mathbf{C}^3 \\ (z, \xi) & \longmapsto & (\xi_1, \xi_2, \xi_3, z_1, z_2, z_1^2) \end{array}$$

that is,

$$L = \left\{ (-2z_1^2 - z_2^2 - 2\xi_3 z_1, -2z_1 z_2, \xi_3, z_1, z_1^2) \right\}.$$

The caustic. — The caustic is the set of singular points of φ in the target, or the set of all ξ 's such that two critical points of F_ξ coincide: the map

$$\begin{aligned} \varphi : \Lambda &\longrightarrow \mathbf{C}^N \\ (z, \xi) &\longmapsto \xi \end{aligned}$$

is a submersion at $(z, \xi) \in \Lambda$ if and only if

$$\begin{aligned} T_{(z, \xi)}\varphi : T_{(z, \xi)}\Lambda &\longrightarrow \mathbf{C}^N \\ (Z, \Xi) &\longmapsto \Xi \end{aligned}$$

is injective, that is, if and only if the system

$$\sum_{j=1}^k \frac{\partial^2 F_\xi}{\partial z_i \partial z_j} Z_j = 0 \quad (1 \leq i \leq k)$$

has no non trivial solution (Z_1, \dots, Z_k) . In this way, we find the values of ξ for which all the critical points of F_ξ are non degenerate. The caustic is a discriminant.

For instance, Figure 1 shows the discriminant in \mathbf{C}^3 of the unfolding of D_4 described above.

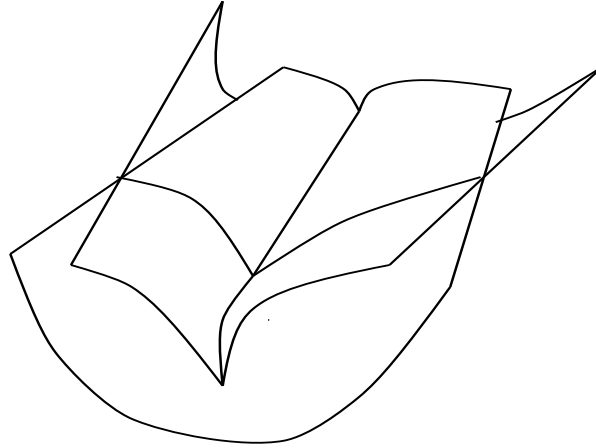


Figure 1: a discriminant (an umbilic)

1.4. Structures on \mathbf{C}^μ

Consider now a germ $f : \mathbf{C}^k \rightarrow \mathbf{C}$ with an isolated singularity at 0 of multiplicity μ . Put $m = \mu - 1$.

To write down the versal unfolding of f , we chose a *basis* $1, g_1, \dots, g_m$ of the vector space Q_f . Let us use as above the notations $\xi = (\xi_0, \xi) \in \mathbf{C} \times \mathbf{C}^m$ and $F_\xi = F_\xi + \xi_0$ for this unfolding.

The aim of this § is to recall that the unfolding F_ξ gives the parameter space \mathbf{C}^μ a very rich structure.

The product \star_ξ . — To choose a basis of Q_{F_ξ} amounts to choosing an isomorphism between this vector space and $T_\xi \mathbf{C}^\mu$. This gives $T_\xi \mathbf{C}^\mu$ the structure of a commutative algebra with unit. Thus we get a multiplication on the vector fields on \mathbf{C}^μ . Denote \star_ξ the product at the point $\tilde{\xi} \in \mathbf{C}^\mu$.

Remark. — We need the ξ_0 -axis here, as this is what gives the identity: the vector field $\partial/\partial\xi_0$ is the unit in all the $T_{\tilde{\xi}}\mathbf{C}^\mu$.

A flat metric. — Recall the definition of the linear form³ $\theta_{\tilde{\xi}}$ on $\mathbf{C}\{z_1, \dots, z_k\}$

$$\begin{aligned} \theta_{\tilde{\xi}}: \mathbf{C}\{z_1, \dots, z_k\} &\longrightarrow \mathbf{C} \\ \alpha &\longmapsto \left(\frac{1}{2i\pi}\right)^k \int_{\Gamma} \frac{\alpha dz_1 \wedge \dots \wedge dz_k}{\frac{\partial F_{\tilde{\xi}}}{\partial z_1} \dots \frac{\partial F_{\tilde{\xi}}}{\partial z_k}} \end{aligned}$$

where Γ is the *distinguished boundary* of a small enough neighbourhood of the critical points of $F_{\tilde{\xi}}$:

$$\Gamma = \left\{ z \mid \left| \frac{\partial F_{\tilde{\xi}}}{\partial z_i} \right| = \varepsilon_i \right\} \quad \varepsilon_i > 0.$$

The value of $\theta_{\tilde{\xi}}(\alpha)$ depends only of the class of α in $Q_{F_{\tilde{\xi}}}$, so that it is possible to consider the family of $\theta_{\tilde{\xi}}$, letting $\tilde{\xi}$ vary, as a 1-form θ on \mathbf{C}^μ . At a point $\tilde{\xi}$ where the μ critical points a_1, \dots, a_μ of $F_{\tilde{\xi}}$ are distinct (and thus non degenerate), it is possible to write (see § 5 of [1]):

$$\theta_{\tilde{\xi}}(\alpha) = \sum_{i=1}^{m+1} \frac{\alpha(a_i)}{\mathcal{H}_{\tilde{\xi}}(a_i)}$$

where $\mathcal{H}_{\tilde{\xi}}(a_i)$ is the value at a_i of the Hessian of $F_{\tilde{\xi}}$.

At these points, it is clear that the bilinear form

$$(\alpha, \beta) \longmapsto \theta_{\tilde{\xi}}(\alpha\beta)$$

is non degenerate. It can be shown that this is true everywhere.

Moreover, a theorem of K. Saito [41] asserts that there is an isomorphism of vector bundles over \mathbf{C}^μ , which, at any $\tilde{\xi}$, is an isomorphism $Q_{F_{\tilde{\xi}}} \rightarrow T_{\tilde{\xi}}\mathbf{C}^\mu$, and which transports the bilinear form defined by θ to a *flat metric* on \mathbf{C}^μ . The proof is easy in the case of the A_n family, let us thus concentrate on this case.

The A_n -example: the flat metric. — Consider, as in § 1.3, the polynomial $f(z) = z^{n+1}$ and its unfolding

$$F_{\xi}(z) = z^{n+1} + \xi_{n-1}z^{n-1} + \dots + \xi_0$$

(forgetting \sim 's and noting $n = \mu$ to save on typing). The space of parameters is the affine space of all the polynomials F_{ξ} ($\xi \in \mathbf{C}^n$). Its tangent space at any point is the vector space of polynomials of degree $\leq n-1$. It is endowed with the multiplication of $Q_{F_{\xi}} = \mathbf{C}[z]/\langle F'_{\xi} \rangle$: $\alpha \star_{\xi} \beta$ is the remainder of $\alpha\beta$ in the Euclidean division by F'_{ξ} . The 1-form θ_{ξ} is

$$\alpha \longmapsto \frac{1}{2i\pi} \int_{\Gamma} \frac{\alpha dz}{F'_{\xi}} = -\text{Res}_{\infty} \frac{\alpha dz}{F'_{\xi}}.$$

³It was used mainly by Arnold and Khovanski *e.g.* to solve problems in real algebraic geometry [29].

Notice that, in this case, the equality

$$\theta_\xi(\alpha) = \sum \frac{\alpha(a_i)}{F_\xi''(a_i)}$$

(when the critical points are distinct) is obvious. Let us now show that θ defines a *flat* metric, simply by the exhibition of coordinates in which the metric is constant (these are called *flat* coordinates). The idea is to solve the equations

$$\begin{cases} w^{n+1} &= F_\xi(z) \\ w &= z + O(z^{-1}) \end{cases}$$

near $z = \infty$ and to expand the solution

$$z = w + \frac{t_{n-1}}{w} + \cdots + \frac{t_0}{w^n} + O\left(\frac{1}{w^{n+1}}\right),$$

defining t_0, \dots, t_{n-1} (a basis of the vector space of symmetric polynomials). The claim is that t_0, \dots, t_{n-1} are the flat coordinates we are looking for. That they are (global) coordinates is a simple consequence of the definition, which implies that

$$\xi_i = -t_i + A_i(t_{i+1}, \dots, t_{n-1}) \quad 0 \leq i \leq n-1$$

or equivalently, that

$$t_i = -\xi_i + B_i(\xi_{i+1}, \dots, \xi_{n-1}) \quad 0 \leq i \leq n-1.$$

That they are flat coordinates is a simple computation:

$$\frac{\partial}{\partial t_i}(F_\xi) = F'_\xi(z(w, t)) \frac{\partial z}{\partial t_i} = F'_\xi(z(w, t)) w^{-n+i}$$

so that

$$g\left(\frac{\partial}{\partial t_i}(F_\xi), \frac{\partial}{\partial t_j}(F_\xi)\right) = -\text{Res}_{z=\infty} F'_\xi(z(w, t)) w^{-2n+i+j} dz.$$

But $F'_\xi(z) dz = (n+1)w^n dw$, so that

$$g\left(\frac{\partial}{\partial t_i}(F_\xi), \frac{\partial}{\partial t_j}(F_\xi)\right) = -(n+1) \text{Res}_{w=\infty} w^{-n+i+j} dw = (n+1)\delta_{i+j, n-1}.$$

This shows that g is non degenerate everywhere and that it is constant in the t_i 's, the latter being flat coordinates. \square

2. Multiplication of vector fields and spectral covers

2.1. Spectral covers

We consider now an N dimensional manifold M and assume that all its tangent spaces are endowed with the structure of a commutative ring with unit. The product on $T_\xi M$ is denoted

\star_ξ and the identity element 1_ξ . We assume M to be complex analytic (usually, $M = \mathbf{C}^N$), the identity element

$$\xi \longmapsto 1_\xi$$

to be a vector field, and

$$\begin{aligned} \Omega : TM &\longrightarrow \text{End}(TM) \\ (\xi, \alpha) &\longmapsto \alpha \star_\xi \cdot \end{aligned}$$

to define an $\text{End}(TM)$ -valued 1-form. Commutativity and associativity of \star_ξ imply:

2.1 LEMMA. $\Omega \wedge \Omega = 0$.

Proof. — By definition, $\Omega_\xi(\alpha) \cdot (\Omega_\xi(\beta) \cdot \gamma) = \alpha \star_\xi (\beta \star_\xi \gamma)$, so that

$$\begin{aligned} (\Omega \wedge \Omega)_\xi(\alpha, \beta) \cdot \gamma &= \frac{1}{2} [\Omega_\xi(\alpha), \Omega_\xi(\beta)] \cdot \gamma \\ &= \frac{1}{2} [\alpha \star_\xi (\beta \star_\xi \gamma) - \beta \star_\xi (\alpha \star_\xi \gamma)]. \end{aligned}$$

□

Lemma 2.1 can be rephrased as “ (TM, Ω) is a *Higgs pair*”. Higgs pairs and spectral covers originate from Hitchin’s famous paper [26], where M was a curve.

As for any Higgs pair, using the eigenvalues of Ω , the one at hand defines a *spectral cover*, that is, a subvariety $L \subset T^*M$ such that the projection

$$L \subset T^*M \xrightarrow{\pi} M$$

is a degree- N map. (For Higgs pairs, see [44], for Higgs pairs and spectral covers, see [16].)

Definition of the spectral cover. — As $\Omega \wedge \Omega = 0$, the morphism $\Omega : TM \rightarrow \text{End}(TM)$ extends to a morphism

$$\text{Sym}(TM) \longrightarrow \text{End}(TM)$$

by

$$(\xi, \alpha_1 \bullet \cdots \bullet \alpha_m) \cdot \beta = (\Omega_\xi(\alpha_1) \circ \cdots \circ \Omega_\xi(\alpha_m))(\beta) = \alpha_1 \star_\xi \cdots \star_\xi \alpha_m \star_\xi \beta.$$

Now, the sheaf of local sections of $\text{Sym}(TM)$ is nothing other than the sheaf $\pi_* \mathcal{O}_{T^*M}$ of regular functions on the cotangent bundle T^*M : a local section $\alpha_1 \bullet \cdots \bullet \alpha_m$ of $\text{Sym}(TM)$ defines a function on T^*M by

$$\alpha_1 \bullet \cdots \bullet \alpha_m(\xi, \mu) = \langle \mu, \alpha_1(\xi) \rangle \cdots \langle \mu, \alpha_m(\xi) \rangle$$

(ξ is a point of M , μ a linear form on $T_\xi M$ and the $\alpha_i(\xi)$ ’s are vectors in $T_\xi M$).

Thus the \mathcal{O}_M -module TM gets an \mathcal{O}_{T^*M} -module structure. Our spectral cover L is just the support of TM as an \mathcal{O}_{T^*M} -module.

Set-theoretically, this is to say that a point (ξ, μ) of T^*M is in L if and only if, for any (local) function $f : T^*M \rightarrow \mathbf{C}$ that annihilates local sections of TM , $f(\xi, \mu) = 0$.

An almost tautological property is the following (see *e.g.* [3]).

2.2 PROPOSITION. — *The ring sheaf of vector fields on M is the sheaf of regular functions on the spectral cover L associated with (TM, Ω) .* \square

Down-to-earth description. — Let α be a vector field defined on an open subset U of M . Associated with α is a section

$$P_\alpha : (\xi, \mu) \longmapsto \det(\Omega_\xi(\alpha) - \mu_\xi(\alpha) \text{Id}_\xi)$$

of $\pi^* \Lambda^N TM$ over $T^*U \subset T^*M$. Assuming that there is some local vector field α for which the minimal and characteristic polynomials coincide, the spectral cover can also be defined by the annulation of all the P_α 's.

Assume that at some point $\xi_0 \in M$, the algebra $(T_{\xi_0}M, \star_{\xi_0})$ is semi-simple, or, equivalently, that all the $\Omega_{\xi_0}(\alpha)$ for $\alpha \in T_{\xi_0}M$ are diagonalisable (recall that the $\Omega_\xi(\alpha)$ commute so that they can be diagonalised simultaneously).

Let us call such a point ξ_0 a *semi-simple point*. If moreover, there exists an α for which the eigenvalues of $\Omega_{\xi_0}(\alpha)$ are all distinct, we will call it a *regular semi-simple point*.

Then, over a neighbourhood \mathcal{U} of ξ_0 , the bundle $TM \rightarrow M$ then splits as a sum

$$(1) \quad TW = \bigoplus_{i=1}^N W_i$$

of line bundles that are the eigenline bundles of Ω . Equivalently, on \mathcal{U} , there is a basis

$$\xi \longmapsto (w_1(\xi), \dots, w_N(\xi))$$

of common eigenvectors of the multiplication \star_ξ .

To each W_i and each $\alpha \in T_\xi M$ corresponds an eigenvalue $\mu_i(\alpha, \xi)$ of $\Omega_\xi(\alpha)$. Such a μ_i can be considered as a local section of T^*M and altogether, the graphs of the μ_i 's form the N sheets of the spectral cover L .

Remark. — The spectral cover (and the projection onto M) may have singularities over the complement of the set of semi-simple points. However, according to Proposition 2.2, these singularities are relatively simple, since the structural sheaf of L is the sheaf of local sections of a vector bundle (namely, TM).

The Lagrange property. — Let us assume now that the tangent bundle of M is trivialisable and endowed with a distinguished trivialisaton (we will mainly use $M = \mathbf{C}^N$) so that the symbol $d\Omega$ is meaningful. (One could use the exterior covariant derivative of a flat connection ∇ in place of d .) The following property comes from [24, 23].

2.3 PROPOSITION. — *Assume $d\Omega = 0$. If there exists a semi-simple point in M , the spectral cover is a Lagrangian subvariety of T^*M .*

Proof. — To say that the spectral cover is Lagrangian is equivalent to say that it is Lagrangian over a non empty open subset of M . As there is some semi-simple point, the set of such points is open and dense. We will prove that the spectral cover is Lagrangian over neighbourhoods of the semi-simple points of M .

On a neighbourhood of such a point, we have vector fields (w_1, \dots, w_N) satisfying

$$w_i \star w_j = \delta_{i,j} w_i.$$

Differentiate this relation (for $i = j$) to get

$$2(dw_i) \star w_i = dw_i.$$

Write now the vector dw_i in the basis (w_1, \dots, w_N) :

$$dw_i = \sum_j a_i^j w_j$$

for some matrix (a_i^j) of 1-forms, so that, multiplying both sides by w_j for $j \neq i$:

$$\sum_j a_i^j w_j = dw_i = 2(dw_i) \star w_i = 2(\sum_j a_i^j w_j) \star w_i = 2a_i^i w_i$$

and thus $a_i^j = 0$ for all i and j , so that $dw_i = 0$. Write now

$$\Omega \cdot w_i = \mu_i w_i.$$

To say that L is Lagrangian is then just to say that all the 1-forms μ_i are closed. Once again, differentiate this relation and use the assumption $d\Omega = 0$:

$$(d\Omega) \cdot w_i + \Omega \cdot (dw_i) = (d\mu_i)w_i + \mu_i dw_i$$

and thus

$$(d\mu_i)w_i = 0.$$

□

Remarks.

- A more general framework would be that of a morphism

$$\Omega : TM \longrightarrow \text{End}(E)$$

for some vector bundle $E \rightarrow M$, endowed with a flat connection ∇ and such that $\Omega \wedge \Omega = 0$ (to define the spectral cover) and $d_{\nabla} \Omega = 0$ (to have the Lagrange property). Definitions, statements and proofs are completely analogous (see *e.g.* [3]).

- Proposition 2.3 is probably true without any semi-simplicity assumption. Notice that the proof given in [24] shows that all the branches of the spectral cover that correspond to simple eigenvalues are Lagrangian.

Canonical coordinates. — As the forms μ_i are closed, they are locally exact. Any family of local primitives (x_1, \dots, x_N) is then a system of local coordinates on M , that I will call *canonical⁴ coordinates*, as this is the standard terminology in the Frobenius world [17, 27].

In the associated coordinates on T^*M , the N branches of L are

$$L_i = \{(x_1, \dots, x_N; 0, \dots, 0, 1_i, 0, \dots, 0)\} \quad 1 \leq i \leq N.$$

Remarks.

1. Up to a rescaling, the eigenvectors w_i satisfy $w_i \star w_j = \delta_{i,j} w_i$ and $w_i = \partial/\partial x_i$: they are the vector fields associated with the canonical coordinates. As a consequence, in these coordinates, the unit vector field is

$$1 = \sum_{i=1}^N \frac{\partial}{\partial x_i}.$$

2. It is tempting to consider a vector field given by a formula such as $\sum x_i \frac{\partial}{\partial x_i}$ as an Euler vector field. More precisely, let M be a manifold whose sheaf of vector fields is endowed with a generically semi-simple ring structure. Assume that the spectral cover is Lagrangian. A globally defined vector field \mathcal{E} on M is called an Euler vector field if, in every open subset of M where canonical coordinates exist, these can be chosen in such a way that

$$\mathcal{E}(\xi) = \sum x_i \frac{\partial}{\partial x_i}.$$

Then, by definition, \mathcal{E} rescales the product \star_{ξ} in the sense that

$$[\mathcal{E}, \alpha \star \beta] - [\mathcal{E}, \alpha] \star \beta - \alpha \star [\mathcal{E}, \beta] = \alpha \star \beta.$$

3. Assuming that $M = \mathbf{C}^N$, the endomorphism valued 1-form Ω is a matrix of 1-forms. Assume as above that Ω is closed: all entries are closed 1-forms. On \mathbf{C}^N , closed 1-forms are exact, so that there exists a function

$$S : \mathbf{C}^N \longrightarrow \text{End}(\mathbf{C}^N)$$

satisfying $dS = \Omega$. This is where the potential of Frobenius manifolds comes from (see § 3.1).

2.2. Back to unfoldings

Associated to the multiplications $\star_{\tilde{\xi}}$ described in § 1.4 is a spectral cover, L_0 say, in $T^*\mathbf{C}^{\mu}$. The aim of this § is to check, that, up to the ξ_0 factor, this is the Lagrangian L defined by $F_{\tilde{\xi}}$.

Let us use previous notation (*i.e.* μ is the multiplicity and $\mu = m + 1$).

2.4 PROPOSITION. — *Let $\tilde{\xi} \in \mathbf{C}^{m+1}$ be a point such that the μ critical points $a_i \in \mathbf{C}^k$ ($1 \leq i \leq \mu$) of $F_{\tilde{\xi}}$ are distinct. The eigenvalues of the multiplication by the polynomial α in $Q_{F_{\tilde{\xi}}}$ are the values $\alpha(a_1), \dots, \alpha(a_{m+1})$ of α at the critical points.*

⁴although they are not canonical.

Proof. — Chose for all i a polynomial $\beta_i \in \mathbf{C}[z_1, \dots, z_k]$ such that

$$\beta_i(a_j) = \delta_{i,j} \quad (1 \leq j \leq k).$$

The classes of $\beta_1, \dots, \beta_\mu$ in Q_{F_ξ} are independent and thus form a basis of this vector space.

Fix a polynomial $\alpha \in \mathbf{C}[z_1, \dots, z_k]$. By the very definition of the product,

$$\alpha(z)\beta_i(z) = \left(\alpha \star_{\xi} \beta_i\right)(z) + \sum_{\ell=1}^k R_\ell(z) \frac{\partial F_\xi}{\partial z_\ell}(z).$$

Computing both sides at a_j , we get

$$\left(\alpha \star_{\xi} \beta_i\right)(a_j) = \delta_{i,j} \alpha(a_j)$$

so that $\alpha \star_{\xi} \beta_i$ is a scalar multiple of the class of β_i in Q_{F_ξ} (this is the place where we use that the μ critical points a_j are distinct). Thus β_i is an eigenvector of the multiplication by α and the computation above even gives its eigenvalue, $\alpha(a_i)$. \square

2.5 COROLLARY. — *The product \star_{ξ} is semi-simple for generic ξ .* \square

Over a point $\tilde{\xi}$ where the a_j 's are distinct, the spectral cover is described by the μ linear forms

$$\begin{array}{ccc} T_{\tilde{\xi}} \mathbf{C}^{m+1} & \longrightarrow & \mathbf{C} \\ \alpha & \longmapsto & \alpha(a_i) \end{array}$$

Consider now the Lagrangian L described by the unfolding F_ξ , and its “fattening” \tilde{L} by the ξ_0 factor:

$$\tilde{L} = \{((\xi_0, 1), (\xi, p)) \mid (\xi, p) \in L\} \subset \mathbf{C} \times \mathbf{C} \times T^* \mathbf{C}^m = T^* \mathbf{C}^{m+1}.$$

Of course we have:

2.6 PROPOSITION. — *The spectral cover of \mathbf{C}^μ associated with the family of products $\star_{\tilde{\xi}}$ defined by the versal unfolding $F_{\tilde{\xi}}$ of f is the Lagrange subvariety \tilde{L} described by the generating family $F_{\tilde{\xi}}$.*

Proof. — Let $\xi \in \mathbf{C}^m$ be a point such that the μ critical points of F_ξ are distinct. Let us check that, over $\tilde{\xi} = (\xi_0, \xi)$ (for any ξ_0), the spectral cover coincides with \tilde{L} — this is enough as these points are generic.

The functions F_ξ and $F_{\tilde{\xi}} = F_\xi + \xi_0$ have the same critical points a_1, \dots, a_μ . The manifold Λ that parametrises the Lagrangian L consists, over $\xi \in \mathbf{C}^m$, of the $m+1$ points $(a_1, \xi), \dots, (a_{m+1}, \xi)$. The “fattened” Lagrangian \tilde{L} is the image of

$$\begin{array}{ccc} \mathbf{C} \times \Lambda & \longrightarrow & T^* \mathbf{C}^{m+1} \\ (\xi_0, z, \xi) & \longmapsto & (\tilde{\xi}, d_{\tilde{\xi}} F). \end{array}$$

As the function $\tilde{\xi} \mapsto F_{\tilde{\xi}}$ is affine,

$$\left(d_{\tilde{\xi}} F\right)_{(\tilde{\xi}, z)}(\alpha) = \alpha(z)$$

hence the result. \square

Remark. — In this case, we did not need to prove that $d\Omega = 0$ and to use Proposition 2.3, since the spectral cover is automatically Lagrangian.

As I have explained it above (§ 2.1), the fact that the spectral cover is Lagrangian describes the existence of “canonical” coordinates (see [17] or [27]) on an open dense subset of the manifold under consideration. Here, this open subset is the product of the ξ_0 -axis and the complement of the caustic in \mathbf{C}^m . The linear forms $\alpha \mapsto \alpha(a_i)$ define locally μ differential forms. To say that \tilde{L} is Lagrangian amounts to say that these forms are closed. Their local primitives are a system of local coordinates, the canonical coordinates.

In our case, where the Lagrangian \tilde{L} is described by a generating family $F_{\tilde{\xi}}$, there exists a natural choice of primitives, namely $x_i = F_{\tilde{\xi}}(a_i)$, the *critical values* of $F_{\tilde{\xi}}$.

The vector fields defined by the eigenvalues, that is, the β_i 's in the proof of 2.4, are then the $\partial/\partial x_i$. For any polynomial $P \in \mathbf{C}[z_1, \dots, z_k]$,

$$P(z) \equiv \sum_{i=1}^{m+1} P(a_i)\beta_i(z) \pmod{J(F_{\tilde{\xi}})},$$

in other words, the vector field

$$P : \tilde{\xi} \longmapsto P \pmod{J(F_{\tilde{\xi}})}$$

on \mathbf{C}^{m+1} can be written, in canonical coordinates,

$$P = \sum_{i=1}^{m+1} P(a_i) \frac{\partial}{\partial x_i}.$$

In particular, the *unit vector field* ($P = 1$) is written

$$1 = \sum_{i=1}^{m+1} \frac{\partial}{\partial x_i}$$

and

$$F_{\tilde{\xi}} = \sum_{i=1}^{m+1} F_{\tilde{\xi}}(a_i) \frac{\partial}{\partial x_i} = \sum_{i=1}^{m+1} x_i \frac{\partial}{\partial x_i}$$

so that $F_{\tilde{\xi}}$, the unfolding itself, is an *Euler vector field* in the sense of § 2.1.

Remark. — Notice that, since L and \tilde{L} are smooth, the spectral cover considered here is smooth. Below I will give examples of spectral covers which do not satisfy this property (see *e.g.* § 2.3).

2.3. Other examples of spectral covers

Unfoldings give examples of spectral covers, as we have just seen it. Here are some other examples.

The spectral cover of a Cartesian product. — Suppose now that M_1 and M_2 are two varieties endowed with a ring structure on their sheaves of vector fields. The product $M_1 \times M_2$ is endowed with the same structure by:

$$\Omega : TM_1 \times TM_2 \xrightarrow{\Omega_1 \oplus \Omega_2} \text{End}(TM_1) \times \text{End}(TM_2) \subset \text{End}(TM_1 \times TM_2).$$

It is immediately checked that:

2.7 PROPOSITION. — *The spectral cover associated with $(M_1 \times M_2, \Omega)$ is the union*

$$L = L_1 \times M_2 \cup M_1 \times L_2 \subset T^*M_1 \times T^*M_2$$

where M_i is embedded in T^*M_i as the zero section. □

Remarks.

1. The intersection of L_i with the zero section M_i consists of the points of M_i where the multiplication has zero as an eigenvalue. But notice that, in a ring with unit, equality $\alpha \star v = 0$ for all α is only possible when $v = 0$, thus $L_i \cap M_i = \emptyset$ and

$$(L_1 \times M_1) \cap (M_1 \times L_2) \subset (L_1 \cap M_1) \times (L_2 \cap M_2) = \emptyset.$$

The two pieces we have used to construct L do not intersect and, in particular, the possible singularities of L may only come from singularities already present in L_1 or L_2 .

2. In the semi-simple case, the spectral cover is precisely of this form (obtained from the spectral covers in the 1-dimensional case by cartesian product).

Products on \mathbf{C}^2 . — Notice that the unfolding of z^3 gives an example with a double eigenvalue. Let us construct other examples, constructing a family of ring structures on \mathbf{C}^2 .

In general, given a ring structure on \mathbf{C}^2 , there exists two vectors u and v such that

$$\begin{cases} u \text{ is the identity} \\ v^2 = au. \end{cases}$$

The equation $(x_1u + x_2v)^2 = \lambda(x_1u + x_2v)$ has two distinct non trivial solutions when $a \neq 0$, in which case the algebra is semi-simple.

Now we want the product to depend on a point in \mathbf{C}^2 , so that we will assume that

$$u = \frac{\partial}{\partial \xi_0}, \quad v = \frac{\partial}{\partial \xi_1}$$

and that a is a function $\mathbf{C}^2 \rightarrow \mathbf{C}$. The product of two elements at the point ξ is described by the formula:

$$\left(x_0 \frac{\partial}{\partial \xi_0} + x_1 \frac{\partial}{\partial \xi_1} \right) \star_{\xi} \left(x'_0 \frac{\partial}{\partial \xi_0} + x'_1 \frac{\partial}{\partial \xi_1} \right) = (x_0 x'_0 + a(\xi) x_1 x'_1) \frac{\partial}{\partial \xi_0} + (x_1 x'_0 + x_0 x'_1) \frac{\partial}{\partial \xi_1}.$$

In other words, the form Ω is the matrix of 1-forms:

$$\Omega = \begin{pmatrix} d\xi_0 & a(\xi)d\xi_1 \\ d\xi_1 & d\xi_0 \end{pmatrix}.$$

Note that, in order that Ω be closed (and the spectral cover a Lagrangian), it is necessary and sufficient that a only depends on ξ_1 . Thus we assume that $a = a(\xi_1)$.

The spectral cover is the Lagrangian

$$L = \{(\xi_0, \xi_1, p_0, p_1) \mid p_0 = 1 \text{ and } p_1^2 = a(\xi_1)\} \subset T^*\mathbf{C}^2.$$

At points (ξ_0, ξ_1) where a vanishes ($a(\xi_1) = 0$), there is a double eigenvalue and it is impossible to find canonical coordinates. If, at a given point (ξ_0, ξ_1) , $\alpha(\xi_1)$ is a square root of $a(\xi_1)$, the two vectors

$$\pm\alpha(\xi_1)\frac{\partial}{\partial\xi_0} + \frac{\partial}{\partial\xi_1}$$

are eigenvectors, for the eigenvalues $d\xi_0 \pm \alpha(\xi_1)d\xi_1$ (respectively), for

$$\left(\eta_0\frac{\partial}{\partial\xi_0} + \eta_1\frac{\partial}{\partial\xi_1}\right) \star \left(\pm\alpha(\xi_1)\frac{\partial}{\partial\xi_0} + \frac{\partial}{\partial\xi_1}\right) = (\eta_0 \pm \alpha(\xi_1)\eta_1) \left(\pm\alpha(\xi_1)\frac{\partial}{\partial\xi_0} + \frac{\partial}{\partial\xi_1}\right).$$

Let $U \subset \mathbf{C}$ be an open subset on which there exists a function α with $\alpha(z)^2 = a(z)$. Choose a square root and let β be a primitive of α . Then $x_1 = \xi_0 + \beta(\xi_1) + a_1$ and $x_2 = \xi_0 - \beta(\xi_1) + a_2$ are canonical coordinates and

$$\frac{\partial}{\partial x_1} = \frac{1}{2} \left(\frac{\partial}{\partial \xi_0} + \frac{1}{\alpha} \frac{\partial}{\partial \xi_1} \right), \quad \frac{\partial}{\partial x_2} = \frac{1}{2} \left(\frac{\partial}{\partial \xi_0} - \frac{1}{\alpha} \frac{\partial}{\partial \xi_1} \right).$$

The unit vector field is

$$1 = \frac{\partial}{\partial \xi_0} = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.$$

Under the same assumptions, there is even an Euler vector field, since

$$\mathcal{E} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} = (\xi_0 + b_0) \frac{\partial}{\partial \xi_0} + \frac{\beta(\xi_1) + b_1}{\alpha(\xi_1)} \frac{\partial}{\partial \xi_1}.$$

Near a zero z_0 of a it is possible to choose β (with $\lim_{z \rightarrow z_0} \beta(z) = 0$) and $a_1 = a_2$ (so that $b_1 = 0$). In this way, β/α extends to a holomorphic function defined in a neighbourhood of z_0 .

Remarks.

- If α has an analytic continuation around a zero z_0 of a , the same is true for the eigenvalues and/or the canonical coordinates. These ones are then well defined, but not independent at z_0 : the spectral cover has two branches at z_0 . This is the case if $a(\xi_1) = \xi_1^{2m}$ for instance (see figure 3). Notice also that this gives examples of non smooth spectral covers.
- The case of the unfolding of z^3 corresponds to $a = -\xi_1/3$ (figure 2). There is no global choice of α in the complement of 0 in this case.
- Notice that, even when the canonical coordinates are defined only on some open subsets of the complement of the zero set of a , the Euler vector field is globally defined, as β/α in the formula above extends to a holomorphic function on \mathbf{C} .

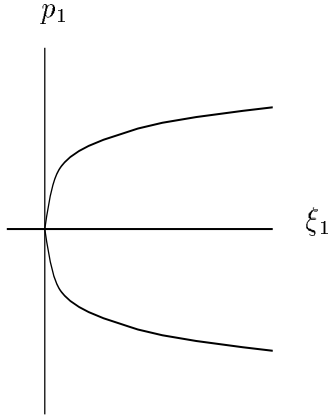


Figure 2: a fold

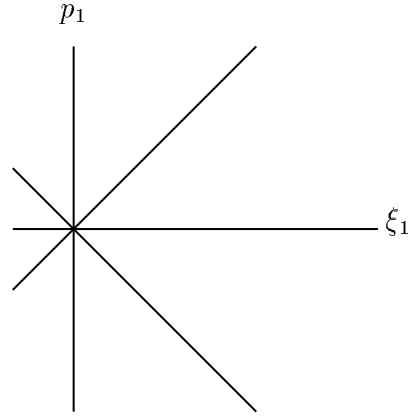


Figure 3: two branches

3. Frobenius manifolds, examples

3.1. Definition of a Frobenius manifold

Definition. — A Frobenius manifold is a complex analytic manifold M endowed with

- a commutative ring structure on the sheaf of vector fields, the product at ξ denoted by \star_ξ and the unit by 1_ξ ,
- a flat metric g , the Levi-Civita connection of which we will denote ∇ .
- a vector field \mathcal{E} , the Euler vector field

satisfying a couple of compatibility conditions, namely:

1. $g(\alpha \star \beta, \gamma) = g(\alpha, \beta \star \gamma)$.
2. $\nabla 1 = 0$.
3. If $c(\alpha, \beta, \gamma) = g(\alpha \star \beta, \gamma)$, the 4-tensor $(\nabla_\delta c)(\alpha, \beta, \gamma)$ is (completely) symmetric.
4. The vector field \mathcal{E} acts by conformal transformations of g and by rescalings of \star_ξ on $T_\xi M$, namely

$$\mathcal{E} \cdot g(\alpha, \beta) - g([\mathcal{E}, \alpha], \beta) - g(\alpha, [\mathcal{E}, \beta]) = Dg(\alpha, \beta)$$

and

$$[\mathcal{E}, \alpha \star \beta] - [\mathcal{E}, \alpha] \star \beta - \alpha \star [\mathcal{E}, \beta] = d_1 \alpha \star \beta$$

for some constants D, d_1 .

5. $\nabla(\nabla \mathcal{E}) = 0$.

Remarks.

1. Here, the term “metric” designs a (complex) non degenerate bilinear form.
2. $g(\alpha, 1)$ defines a 1-form θ on M , conversely, θ and \star give g by $g(\alpha, \beta) = \theta(\alpha \star \beta)$ (as we have shown in § 1.4).
3. It is usually assumed that the scaling constant d_1 is non zero (and thus that the Euler vector field is non trivial). It is then possible to assume that $d_1 = 1$ by rescaling \mathcal{E} .
4. Using our previous notation, let

$$\Omega : TM \longrightarrow \text{End}(TM)$$

be the 1-form defined by the multiplication \star_ξ . The compatibility condition between the metric and the multiplication implies that Ω has values in the *symmetric* (with respect to the metric g) endomorphisms of TM .

Spectral covers and canonical coordinates. — Let d_∇ be the exterior covariant derivative associated with ∇ . Then, compatibility conditions (1) and (3) imply:

3.1 PROPOSITION. $d_\nabla \Omega = 0$.

Proof. — This is a simple rephrasing of the symmetry of ∇c . The tensor c can be considered as a 1-form with values in $S^2(T^*M)$:

$$\begin{aligned} c : TM &\longrightarrow S^2(T^*M) \\ \alpha &\longmapsto ((\beta, \gamma) \mapsto c(\alpha, \beta, \gamma)). \end{aligned}$$

It is a section of $S^2(T^*M) \otimes T^*M$ and its covariant exterior derivative $d_\nabla c$ is a 2-form with values in $S^2(T^*M)$, that is, a section of $S^2(T^*M) \otimes \Lambda^2(T^*M)$.

On the other hand, ∇c is a section of $S^3(T^*M) \otimes T^*M$ and $d_\nabla c$ is obtained by antisymmetrisation, so that ∇c is symmetric if and only if $d_\nabla c = 0$.

Now, Ω is an avatar of c :

$$\begin{aligned} \Omega : TM &\longrightarrow \text{End}(TM) \\ \alpha &\longmapsto (\beta \mapsto \alpha \star \beta) \end{aligned}$$

takes its values in $\text{Sym}_g(TM)$, which is isomorphic to $S^2(T^*M)$ by

$$\begin{aligned} \Phi : \text{Sym}_g(TM) &\longrightarrow S^2(T^*M) \\ \varphi &\longmapsto ((\beta, \gamma) \mapsto g(\varphi(\beta), \gamma)) \end{aligned}$$

and $\Omega = \Phi^{-1} \circ c$, so that $d_\nabla \Omega = \Phi^{-1} \circ d_\nabla c = 0$. □

We are thus in the situation of Proposition 2.3, except that we must make a semi-simplicity assumption. A Frobenius manifold is called *massive* if it has an open subset of semi-simple points.

3.2 COROLLARY. — *Spectral covers of massive Frobenius manifolds are Lagrangian.* □

Remark. — General massive Frobenius manifolds have discriminants (the set of points ξ such that \star_ξ is not semi-simple). In the case of unfoldings, the discriminant coincides with the usual discriminant (see §§1.3 and 2.2).

Canonical coordinates are, as above (see §2.1), primitives of the eigenvalue forms. The vectors $\partial/\partial x_i$ are eigenvectors, chosen to satisfy

$$\frac{\partial}{\partial x_i} \star \frac{\partial}{\partial x_j} = \delta_{i,j} \frac{\partial}{\partial x_i}.$$

Notice that

$$g \left(\frac{\partial}{\partial x_i} \star \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right) = \delta_{i,j} g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right)$$

on the one hand, and that it is equal to

$$g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \star \frac{\partial}{\partial x_k} \right) = \delta_{j,k} g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right)$$

on the other hand, so that

$$g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k} \right) = 0 \text{ for } i \neq k :$$

canonical coordinates are orthogonal.

A remark on the Euler vector field. — According to the philosophy of §2.1, the Euler vector field is essentially determined by the canonical coordinates, that is, by the multiplication. The fact that it is conformal is a serious rigidification of the whole structure (compare the 2-dimensional examples in §§2.3 and 3.2, respectively without and with the “conformal” property).

The flat pencil of connections. — The connection ∇ and the 1-form Ω can be put together in a family $\nabla^t = \nabla + t\Omega$ of connections on M (the Dubrovin connection):

$$\nabla_\alpha^t(\beta) = \nabla_\alpha \beta + t\Omega(\alpha) \cdot \beta.$$

The commutativity of the products \star_α is obviously equivalent to the fact that ∇^t is torsionless. The flatness of ∇^t for all t 's is expressed by two equations, that correspond to the closedness of Ω (or the symmetry of ∇c , see Proposition 3.1) and the associativity of the product (respectively).

The potential and the WDVV equation. — Let us use now *flat* coordinates (t_0, \dots, t_{N-1}) , so that we assume that we are in \mathbf{C}^N , with a constant metric and $d_\nabla = d$. As we have noticed it in §2.1, Ω , being closed, has a primitive

$$S : \mathbf{C}^N \longrightarrow \text{End}(\mathbf{C}^N).$$

Since the 1-form Ω defines g -symmetric endomorphisms of \mathbf{C}^N , we can assume that the primitive S satisfies the same property:

$$S : \mathbf{C}^N \longrightarrow \text{Sym}_g(\mathbf{C}^N).$$

Such a mapping can be considered as the second derivative of a function

$$\Psi : \mathbf{C}^N \longrightarrow \mathbf{C}$$

in the sense that

$$\left(d^2\Psi\right)_\xi(\beta, \gamma) = g(S(\xi) \cdot \beta, \gamma).$$

By definition

$$\left(d^3\Psi\right)_\xi(\alpha, \beta, \gamma) = g\left((dS)_\xi(\alpha) \cdot \beta, \gamma\right) = g(\alpha \star_\xi \beta, \gamma)$$

or, in (flat) coordinates:

$$\frac{\partial^3\Psi}{\partial t_i \partial t_j \partial t_k} = g\left(\frac{\partial}{\partial t_i} \star \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\right)$$

or, equivalently

$$c = \sum \frac{\partial^3\Psi}{\partial t_i \partial t_j \partial t_k} dt_i dt_j dt_k.$$

Associativity of the products \star can now be expressed as a system of (non linear) partial differential equations on the function Ψ , the celebrated WDVV⁵ equation:

$$\sum_{k,n} \frac{\partial^3\Psi}{\partial t_i \partial t_j \partial t_k} g^{k,n} \frac{\partial^3\Psi}{\partial t_\ell \partial t_m \partial t_n} = \sum_{k,n} \frac{\partial^3\Psi}{\partial t_\ell \partial t_j \partial t_k} g^{k,n} \frac{\partial^3\Psi}{\partial t_i \partial t_m \partial t_n}$$

where the matrix $g^{k,n}$ is the inverse of the matrix $\left(g\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right)\right)_{i,j}$.

The “invariance” properties with respect to the Euler vector field can then be stated very simply in terms of the potential Ψ , namely

$$\mathcal{E} \cdot \Psi = d_\Psi \Psi + \text{terms of degree } \leq 2$$

for some degree d_Ψ .

Remark. — As the unit is parallel ($\nabla 1 = 0$), it is possible to chose the flat coordinates in such a way that

$$1 = \frac{\partial}{\partial t_0}.$$

If this is the case,

$$g = \sum \frac{\partial^3\Psi}{\partial t_0 \partial t_j \partial t_k} dt_j dt_k.$$

⁵for Witten, Dijkgraaf, Verlinde, Verlinde.

3.2. Frobenius manifolds of dimensions 2 and 3

Frobenius structures on \mathbf{C}^2 . — Let us come back to the family of ring structures on \mathbf{C}^2 we have described in §2.3 and try to add what is missing in order to get a Frobenius structure. Let us use the notation of §2.3.

We need a flat metric. Let us choose it so that the given coordinates (ξ_0, ξ_1) are flat (so that the second compatibility condition will be satisfied). With the first compatibility condition, this implies that

$$g = \gamma d\xi_0 d\xi_1$$

for some constant γ . Rescaling ξ_1 , we assume that $\gamma = 1$.

We know that Condition 3 amounts to the closedness of Ω and we have already imposed it, having assumed that $a = a(\xi_1)$. We have a candidate for the Euler vector field, namely

$$\mathcal{E} = \xi_0 \frac{\partial}{\partial \xi_0} + b(\xi_1) \frac{\partial}{\partial \xi_1}$$

where $b = \beta/\alpha$ in the notation of §2.3. This was chosen to rescale the product, let us now impose that it acts by conformal transformations, that is, that $\mathcal{L}_{\mathcal{E}}g$ is a multiple of g , or that

$$\mathcal{E} \cdot g(X, Y) - g([\mathcal{E}, X], Y) - g(X, [\mathcal{E}, Y]) = Dg$$

for some constant D (using Dubrovin's notation). With $X = \partial/\partial \xi_0$, $Y = \partial/\partial \xi_1$, this gives

$$D = 1 + b'(\xi_1)$$

so that we can assume, if $D \neq 1$, that $b(\xi_1) = C\xi_1$ for some constant C ($C = D - 1$).

Recall now that $b = \beta/\beta'$ with $(\beta')^2 = a$, so that we have to solve the differential equation

$$\frac{\beta}{\beta'} = C\xi_1,$$

which gives

$$\beta = A\xi_1^{\frac{1}{C}}, \quad \beta' = \alpha = \left(\frac{A}{C}\right) \xi_1^{\frac{1}{C}-1}, \quad a = \alpha^2 = \left(\frac{A}{C}\right)^2 \xi_1^{\frac{2}{C}-2}.$$

This is a very serious restriction on the function a defining the product. Notice also that, in order to have a structure that is well-defined at 0, we need that

$$m = 2 \left(\frac{1}{C} - 1 \right) \in \mathbf{N}.$$

We also know the third derivatives of the potential, so that we eventually get:

3.3 PROPOSITION. — *The metric $d\xi_0 d\xi_1$, together with the product \star_{ξ} for which $\partial/\partial \xi_0$ is the unit and*

$$\frac{\partial}{\partial \xi_1} \star_{\xi} \frac{\partial}{\partial \xi_1} = B\xi_1^m \frac{\partial}{\partial \xi_0}$$

and with the vector field

$$\mathcal{E} = \xi_0 \frac{\partial}{\partial \xi_0} + \left(\frac{m}{2} + 1 \right) \xi_1 \frac{\partial}{\partial \xi_1}$$

defines a Frobenius structure on \mathbf{C}^2 , whose potential has the form

$$\Psi(\xi_0, \xi_1) = \frac{1}{2} \xi_0^2 \xi_1 + C \xi_1^{m+3}$$

modulo degree- ≤ 2 terms. □

In the case where $D = 1$, b must be constant, so that $a(\xi_1) = A \exp \frac{2}{r} \xi_1$ for some $r \in \mathbf{C}$. In this case, the Euler vector field is

$$\mathcal{E} = \xi_0 \frac{\partial}{\partial \xi_0} + r \frac{\partial}{\partial \xi_1}$$

and the potential has the form

$$\Psi(\xi_0, \xi_1) = \frac{1}{2} \xi_0^2 \xi_1 + C \exp \frac{2}{r} \xi_1.$$

WDVV in dimension 3. — Notice that the associativity conditions (WDVV equations) are empty in dimension 2. Let us try a 3-dimensional example. As usual, $\partial/\partial \xi_0$ is the unit. Chose the flat metric

$$g = d\xi_1^2 + d\xi_0 d\xi_2$$

so that, assuming there is a potential Ψ ,

$$\frac{\partial}{\partial \xi_1} \star \frac{\partial}{\partial \xi_1} = \frac{\partial^3 \Psi}{\partial \xi_1^2 \partial \xi_2} \frac{\partial}{\partial \xi_0} + \frac{\partial^3 \Psi}{\partial \xi_1^3} \frac{\partial}{\partial \xi_1} + \frac{\partial}{\partial \xi_2}$$

$$\frac{\partial}{\partial \xi_1} \star \frac{\partial}{\partial \xi_2} = \frac{\partial^3 \Psi}{\partial \xi_1 \partial \xi_2^2} \frac{\partial}{\partial \xi_0} + \frac{\partial^3 \Psi}{\partial \xi_1^2 \partial \xi_2} \frac{\partial}{\partial \xi_1}$$

$$\frac{\partial}{\partial \xi_2} \star \frac{\partial}{\partial \xi_2} = \frac{\partial^3 \Psi}{\partial \xi_2^3} \frac{\partial}{\partial \xi_0} + \frac{\partial^3 \Psi}{\partial \xi_1 \partial \xi_2^2} \frac{\partial}{\partial \xi_1}$$

The associativity equation

$$\left(\frac{\partial}{\partial \xi_1} \star \frac{\partial}{\partial \xi_1} \right) \star \frac{\partial}{\partial \xi_2} = \frac{\partial}{\partial \xi_1} \star \left(\frac{\partial}{\partial \xi_1} \star \frac{\partial}{\partial \xi_2} \right)$$

gives the PDE

$$\frac{\partial^3 \Psi}{\partial \xi_2^3} = \left(\frac{\partial^3 \Psi}{\partial \xi_1^2 \partial \xi_2} \right)^2 - \frac{\partial^3 \Psi}{\partial \xi_1^3} \frac{\partial^3 \Psi}{\partial \xi_1 \partial \xi_2^2}.$$

This is related to Painlevé VI equation (see [17, 27, 35]). We shall meet examples of solutions of this equation later (in § 5.4).

Remark. — It can be shown (and the readers should check) that the PDE equation above is the only associativity equation in this dimension, namely that

$$\left(\frac{\partial}{\partial \xi_2} \star \frac{\partial}{\partial \xi_2} \right) \star \frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial \xi_2} \star \left(\frac{\partial}{\partial \xi_2} \star \frac{\partial}{\partial \xi_1} \right)$$

does not give anything new.

3.3. B- and A-models

Unfoldings. — Using unfoldings, we have constructed a product and a flat metric on \mathbf{C}^μ (in § 1.4). We have exhibited the canonical coordinates and a candidate to be the Euler vector field (in § 2.2).

These structures fit together into the structure of a Frobenius manifold. The only things which are left to check is that \mathcal{E} acts by conformal transformations of the metric and that $\nabla(\nabla\mathcal{E}) = 0$. These two properties are consequences of the way the Euler vector field is given in flat coordinates.

The A_n -example: the Euler vector field. — Let us do this in the A_n -case (using the notation of §§ 1.4, 2.2). To begin with, make the Euclidean division of F_ξ by F'_ξ , to express the Euler vector field in the coordinates $(\xi_0, \dots, \xi_{n-1})$:

$$F_\xi(z) - \frac{z}{n+1} F'_\xi(z) = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} \xi_j z^j,$$

in other words,

$$\mathcal{E}_\xi = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial}{\partial \xi_j}.$$

Let us now express this vector field in terms of the flat coordinates (t_0, \dots, t_{n-1}) . Recall that

$$t_i = -\xi_i + B_i(\xi_{i+1}, \dots, \xi_{n-1}) \quad 0 \leq i \leq n-1.$$

Give ξ_j the degree $n-j+1$ — so that $F_\xi(z)$ is homogeneous of degree $n+1$. The function B_i is then homogeneous of degree $n-i+1$. let us now compute:

$$\begin{aligned} \mathcal{E} \cdot t_i &= \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial t_i}{\partial \xi_j} \\ &= -\frac{n-i+1}{n+1} \xi_i + \sum_{j=i+1}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial B_i}{\partial \xi_j} \\ &= \frac{n-i+1}{n+1} t_i - \left(\frac{n-i+1}{n+1} B_i - \sum_{j=i+1}^{n-1} \frac{n-j+1}{n+1} \xi_j \frac{\partial B_i}{\partial \xi_j} \right) \\ &= \frac{n-i+1}{n+1} t_i \end{aligned}$$

according to the homogeneity of B_i . Thus \mathcal{E} has the expression

$$\mathcal{E} = \sum_{j=0}^{n-1} \frac{n-j+1}{n+1} t_j \frac{\partial}{\partial t_j}$$

in flat coordinates.

Notice that, if this gives the desired properties, this also implies that the potential of this Frobenius structure is a polynomial in the t_i 's.

Orbit spaces of Coxeter groups. — The space of degree- $(n+1)$ polynomials

$$F_\xi(z) = z^{n+1} + \xi_{n-1} z^{n-1} + \cdots + \xi_0$$

is also isomorphic to the quotient

$$\{(z_1, \dots, z_{n+1}) \in \mathbf{C}^{n+1} \mid \sum z_i = 0\} / \mathfrak{S}_{n+1}$$

by the map

$$(z_1, \dots, z_{n+1}) \longmapsto \prod_{i=1}^{n+1} (z - z_i).$$

In other words, our space of polynomials, as all spaces of parameters of versal unfoldings, is the space of orbits of a Coxeter group. These spaces have natural Frobenius structures (see [17]) and a conjecture of Dubrovin asserts that all polynomial solutions of the WDVV equations should be potentials of such structures.

Hurwitz spaces. — The polynomial z^{n+1} , the unfolding of which we have investigated in detail, generates another family of examples, that of Hurwitz spaces. Simply notice that a degree- $(n+1)$ polynomial in one variable is also an $(n+1)$ -fold covering of \mathbf{P}^1 with a single pole (of multiplicity $n+1$).

Recall that Hurwitz spaces are moduli spaces of coverings of \mathbf{P}^1 (or of meromorphic functions on curves) with prescribed degree. Dubrovin investigates, in Chapter V of [17], a variant of these spaces: a constraint on the structure of the pole divisor is added. Precisely, given non negative integers g, d_1, \dots, d_m , M_{g,d_1,\dots,d_m} is the set of isomorphism classes of data $(\Sigma, z_1, \dots, z_m, f)$ where Σ is a smooth genus- g curve, (z_1, \dots, z_m) an ordered set of m distinct points on Σ and f a meromorphic function on Σ with pole divisor

$$(f)_\infty = \sum d_i z_i.$$

For example, the parameter space of the unfolding of z^{n+1} is $M_{0,n+1}$. Dubrovin constructs a Frobenius structure on M_{g,d_1,\dots,d_m} generalising the one we have shown in the z^{n+1} case. As in §2.2, the critical values of the meromorphic functions are canonical coordinates.

Landau-Ginzburg vs field theory models. — In the two families of examples above, the structure of the Frobenius manifold M comes from a function F defined on some bundle over M :

- $M = \mathbf{C}^\mu$ and $F : \mathbf{C}^\mu \times \mathbf{C}^k \rightarrow \mathbf{C}$ is the unfolding itself,

- M is the Hurwitz space, $\mathcal{C} \rightarrow M$ the “universal curve” over M and $F : \mathcal{C} \rightarrow \mathbf{C}$ is the evaluation mapping $(\Sigma, z_1, \dots, z_m, f, z) \mapsto f(z)$.

The function F is a generating function for the spectral cover. These examples are called “Landau-Ginzburg models”, the function F is the “superpotential” of the model.

There are other examples of Frobenius manifolds where you start from the potential Ψ to describe the structure — these are the “field theory models”. This is the case of quantum cohomology.

Roughly speaking, in Landau-Ginzburg models, the easy part of the structure is the multiplication, while in field theory models, this is the metric.

Cohomology. — The next example is that of the (even part of the) cohomology of a compact manifold X : $M = H^*(X; \mathbf{C})$. The product is the cup-product \smile and the metric is defined by Poincaré duality. This is very simple, but unfortunately not very interesting, and this for two reasons

1. the product is constant (does not depend on any point of M),
2. it is far from being semi-simple, being nilpotent.

However, the manifold itself is very simple, being a complex vector space, the metric is obviously flat, being constant in the natural (linear) coordinates. Fortunately, there is another product (under certain assumptions on the manifold X) on the sheaf of vector fields of M , the quantum product \star_ξ , which is very often semi-simple. The idea is to define it through the *Gromov-Witten potential*:

$$\Psi(\xi) = \sum_{m \geq 3} \frac{1}{m!} \sum_A \Psi_A(\xi^{\otimes m}),$$

in which expression ξ is a point in $M = H^*(X; \mathbf{C})$, the summation is over all dimension 2 homology classes $A \in H_2(X; \mathbf{Z})$ and $\Psi_A(\xi^{\otimes m})$ designates a Gromov-Witten invariant, heuristically described as the number of certain rational curves of class A in X and rigorously described in the second chapter of these notes. We will see, for instance, that the potential found above in dimension 2 for $D = 1$ describes the Frobenius structure on the cohomology of \mathbf{P}^1 .

Chapter II

Stable maps and quantum cohomology

Let us turn now to the description of A -models. In this chapter, I will describe in some detail (and with examples) the moduli spaces of stable maps. I will then explain how to use them to construct the Gromov-Witten invariants and potential and show that this defines, indeed, a Frobenius structure. The last part is devoted to a special case of the products defined by the Gromov-Witten invariants, the so-called “small quantum product”, which is easier to handle and might contain enough information to solve some of the global problems of quantum cohomology.

4. Moduli spaces of stable maps, with examples

The moduli space invented by Kontsevich (see [31]) is the good moduli space to investigate curves in a given variety⁶.

4.1. Spaces of stable maps

Let X be a complex projective manifold (we will use the fact that the projectivity gives us a Kähler form on X when necessary).

The data. — A *stable map* is a package (Σ, \vec{z}, u) , where Σ is a genus- g complex curve with, at worse, ordinary double points (nodes), $\vec{z} = (z_1, \dots, z_m)$ is an (ordered) m -tuple of distinct points of Σ (marked points) and $u : \Sigma \rightarrow X$ is a holomorphic map. The genus here is the arithmetic genus. Here is a definition for topologists. Associate to Σ a graph Γ_Σ : any component of Σ gives a vertex, two vertices being connected by an edge when the two corresponding components intersect. The genus of Σ is, by definition, the sum of the genera of the components plus the dimension of $H_1(\Gamma_\Sigma)$.

We will impose a stability condition, in order to be able to quotient by isomorphisms. Let us first precise what the isomorphisms are.

Isomorphisms. — Two stable maps (Σ, \vec{z}, u) and (Σ', \vec{z}', u') are isomorphic if there exists an isomorphism $\varphi : \Sigma \rightarrow \Sigma'$ such that $\varphi(z_i) = z'_i$ for all i and $u' \circ \varphi = u$.

Stability requirements. — It is required that the data above have no infinitesimal automorphism. This amounts to requiring that the components of Σ on which u is constant be *stable curves* in the sense of algebraic geometry.

More concretely, if u is constant on a genus-0 component, there must be at least three special points (singular or marked) on this component, and if u is constant on a genus-1 component, it must contain at least one special point.

The “moduli space” of stable maps. — The set of isomorphism classes of genus- g curves with m marked points representing the class A in the manifold X will be denoted $\overline{\mathcal{M}}_{g,m}(X, A)$.

Remark. — We are not considering the image curve, but notice also that we are not really considering the mappings: quotienting by isomorphisms destroys the parametrisation.

When the isomorphism class of the marked curve (Σ, \vec{z}) cannot vary (*e.g.* $g = 0$ and $m \leq 3$), it is possible to take care of the parametrisation by looking at graphs (see Appendix B).

Before looking at examples, let us state and prove an elementary but crucial property.

4.1 PROPOSITION. — *Let ω be a Kähler form on X and A be a homology class. If A can be represented by a holomorphic map from a complex curve to X , $\langle \omega, A \rangle \geq 0$, with equality if and only if the holomorphic map is constant.*

In particular, the class $A = 0$ can only be represented by *constant* holomorphic maps.

⁶A good way to be convinced could be to think of the difference between elliptic curves and plane cubics. I explain this in Appendix A.

Proof. — Let $u : \Sigma \rightarrow X$ be a holomorphic map. Since its derivative is complex linear, for any $z \in \Sigma$, $\alpha \in T_z \Sigma$,

$$(u^* \omega)_z(\alpha, i\alpha) \geq 0.$$

Thus

$$\langle \omega, A \rangle = \langle u^* \omega, [\Sigma] \rangle \geq 0.$$

Moreover, if u is not constant, there is an open subset $U \subset \Sigma$ and a non zero vector field α on U such that $(u^* \omega)_z(\alpha, i\alpha) > 0$ for $z \in U$. Thus $\langle \omega, A \rangle > 0$. \square

First examples.

1. When X is a point, the map u is constant and we get the space $\overline{\mathcal{M}}_{g,m}$ of stable marked curves⁷. Notice that $\overline{\mathcal{M}}_{g,m}$ exists only if $m + 2g \geq 3$ (and that this corresponds to the stability condition in this case).
2. If $A = 0$, we have seen (Proposition 4.1) that the map u is constant and hence the moduli space $\overline{\mathcal{M}}_{g,m}(X, 0)$ is isomorphic to $\overline{\mathcal{M}}_{g,m} \times X$.
3. Another simple case is that of $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^1, L)$: we are dealing with degree-1 mappings

$$u : \mathbf{P}^1 \longrightarrow \mathbf{P}^1.$$

Up to isomorphism, there is only one such map, so that the set $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^1, L)$ is a point. See a generalisation of this example in Appendix B.

4. More generally, $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^1, dL)$ is a compactification of the Hurwitz space of degree- d polynomials.
5. A nice exercise, solved in the introduction of [20], is to describe the version $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^2, 2L)$ of the space of plane conics.

Topology on $\overline{\mathcal{M}}_{g,m}(X, A)$. — We shall see below (Theorem 4.6) that, under certain assumptions, $\overline{\mathcal{M}}_{g,m}(X, A)$ is a projective variety. In general, it is, at least, a topological space: it can be endowed, for example with a variant of the \mathcal{C}^k -topology (for any k). Here is a description of a basis of neighbourhoods of the class of a stable map (Σ, \bar{z}, u) . Fix a real number $\varepsilon > 0$, a neighbourhood U of the singular points of Σ , a neighbourhood V of the set of marked points and a metric μ on every component of Σ . The neighbourhood defined by these data consists of the stable maps (Σ', \bar{z}', u') such that:

- there exists a continuous map $\sigma : \Sigma' \rightarrow \Sigma$ which is a diffeomorphism outside singular points and such that the inverse image of every double point is either a double point of Σ' or an annulus of modulus $\geq 1/\varepsilon$, containing no point of \bar{z}' ,
- $\|u - u' \circ \sigma^{-1}\|_{\mathcal{C}^k} < \varepsilon$ outside U ,
- $\|J_\Sigma - \sigma_*^{-1} J_{\Sigma'}\|_{\mathcal{C}^k} < \varepsilon$,
- $\sigma(z'_i) \in V$.

⁷Except for $\overline{\mathcal{M}}_{1,1}$, I will only use $\overline{\mathcal{M}}_{0,m}$, the Deligne-Mumford-Knudsen compactification of the space of isomorphism classes of m ordered distinct points in \mathbf{P}^1 . For the properties of these spaces, see [30, 28].

Remarks.

1. Recall that a complex annulus is conformally equivalent to $S^1 \times]0, L[$ for a unique $L > 0$, its *modulus*: conformally, to be very long means to be very thin.
2. J_Σ is the complex structure of Σ , considered as an automorphism of $T\Sigma$, multiplication by i . The condition required on J_Σ and J'_Σ expresses the fact that Σ and Σ' shall be close to each other in the ad hoc moduli space. Norms are computed with the metric μ we have fixed on Σ and with the Kähler metric of the projective variety X .

The topology is such that:

4.2 PROPOSITION. — *Evaluation at the marked points*

$$\begin{aligned} \text{ev}_m : \overline{\mathcal{M}}_{g,m}(X, A) &\longrightarrow X^m \\ (\Sigma, \vec{z}, u) &\longmapsto (u(z_1), \dots, u(z_m)). \end{aligned}$$

is a continuous mapping. □

4.2. Relation with the space of stable marked curves

Contraction. — It is possible to forget the map u and to contract the unstable components of the curve Σ to get a mapping

$$\overline{\mathcal{M}}_{g,m}(X, A) \longrightarrow \overline{\mathcal{M}}_{g,m}$$

(for $m + 2g \geq 3$).

Compactness. — In principle, the topology on the space of stable maps is also designed to make the contraction mapping continuous.

4.3 PROPOSITION. — *The space $\overline{\mathcal{M}}_{g,m}(X, A)$ is compact.* □

Proof. — The contraction being continuous, this is a consequence of the compactness of the base space $\overline{\mathcal{M}}_{g,m}$ (see [30]) and of the compactness of the fibers. The latter is a consequence of Theorem 4.4 below and can be viewed as a variant, either of the completeness of the ad hoc Hilbert scheme, or of Gromov's compactness theorem. □

4.4 THEOREM. — *Let (X, ω) be a Kähler manifold and Σ be a complex curve. Let $u_n : \Sigma \rightarrow X$ be a sequence of holomorphic maps. Assume there exists a constant M such that*

$$\langle u_n^* \omega, [\Sigma] \rangle \leq M$$

for all n . There exists a finite subset $\Gamma \subset \Sigma$ such that

- a subsequence of the sequence of graphs $\tilde{u}_n : \Sigma \rightarrow \Sigma \times X$ converges to the graph \tilde{u} of a holomorphic map $u : \Sigma \rightarrow X$ outside Γ (for the \mathcal{C}^k topology).
- If for $z \in \Gamma$, $\tilde{u}_n(z)$ does not converge to $\tilde{u}(z)$, then there is a non trivial rational curve

$$\varphi_z : C_z \longrightarrow \{z\} \times X$$

passing through $u(z)$.

- For n large enough, $u_n^*[\Sigma] = u_*[\Sigma] + \sum_{z \in \Gamma} (\varphi_z)_* [C_z] \in H_2(X; \mathbf{Z})$.

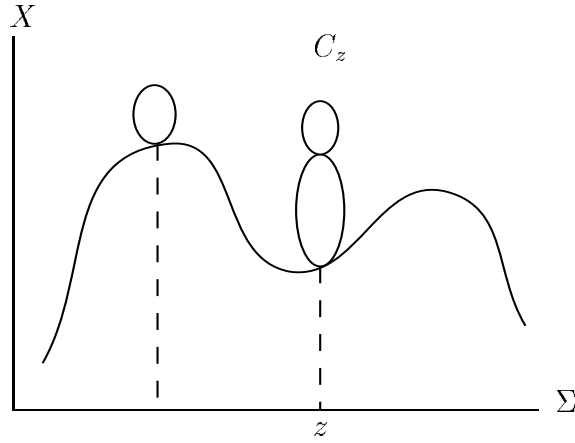


Figure 4

Remarks.

1. In other words, the limit of graphs is again a graph except, maybe, for a few vertical bubbles (see figure 4).
2. In the algebro-geometric context where we have stated Theorem 4.4, this is actually a theorem on families and not only a theorem on sequences.
3. The Kähler manifold X can be replaced by a Riemannian almost complex manifold here. Holomorphic maps must then be replaced by pseudo-holomorphic maps. Notice also that the used almost complex structures may depend on n (see [25, 38]). Even with these modifications, this is still a special case of Gromov's compactness theorem for the curve Σ is fixed.
4. The rational curves C_z may have nodes, as is shown by the following example, which was given to me by Jean-Claude Sikorav.

Let Σ_1 be the plane cubic of homogeneous equation $y^3 = x^3 - xz^2$ and $u_\alpha : \Sigma_1 \rightarrow \mathbf{P}^2$ be the map $[x, y, z] \mapsto [x, \alpha y, \alpha^2 z]$, whose image is the curve Σ_α of equation $\alpha y^3 = \alpha^6 x^3 - xz^2$.

Let α tend to 0. The limit curve $xz^2 = 0$ consists of two lines, one of them being double. The map u_α converges to the constant map $u([x, y, z]) = [1, 0, 0]$ outside the inflexion point $[0, 0, 1]$. Thus the finite set Γ consists of a single point $z = [0, 0, 1]$ and the corresponding rational curve C_z is reducible and consists of two components, one of them being a double cover.

Viewed from the point of view of Gromov's theorem, one of the bubbles comes from a disc and the other one from an annulus (figure 5).

Plane cubics. — Consider plane cubics parametrised by elliptic curves (see Appendix A), that is, consider the contraction

$$\overline{\mathcal{M}}_{1,1}(\mathbf{P}^2, 3L) \longrightarrow \overline{\mathcal{M}}_{1,1}.$$

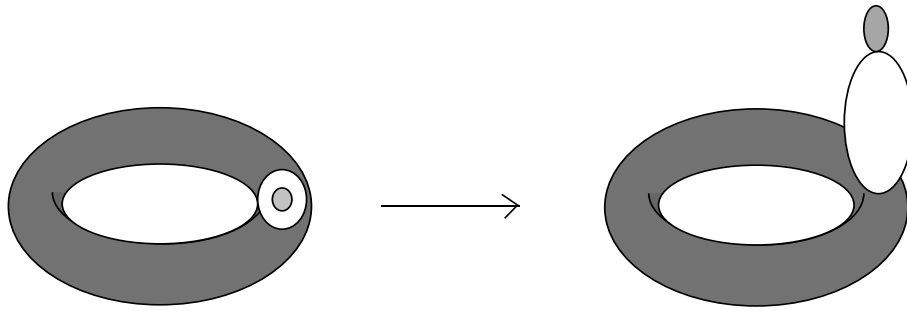


Figure 5

The figures below show some singular plane cubics. First a cusp curve and two lines, one being double, parametrised by reducible curves over an arbitrary elliptic curve in $\overline{\mathcal{M}}_{1,1}$ (figure 6), then various other singular cubics, parametrised by reducible curves over the point at infinity in $\overline{\mathcal{M}}_{1,1}$ (figure 7).

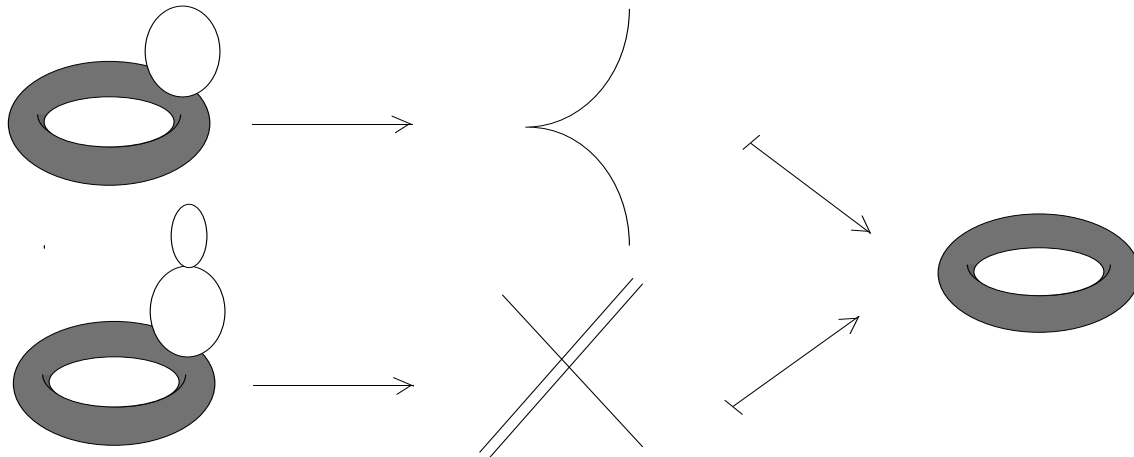


Figure 6: Some unstable singular cubics

Remarks.

- In the space of stable maps $\overline{\mathcal{M}}_{1,1}(\mathbf{P}^2, 3L)$, there are cusp curves over every points of the space $\overline{\mathcal{M}}_{1,1}$ (I explain in Appendix A that this is something one should expect). The same property is true for all “unstable cubics” (that is, for those who have other singularities than ordinary double points, as the cubic consisting of a conic and a tangent line, or of two lines, one of which is a double line, or of three lines through the same point).
- In contrast, for “semi-stable” singular cubics (with ordinary double points only), the j -invariant is infinite and all these curves are images of stable maps over the point at infinity in $\overline{\mathcal{M}}_{1,1}$.

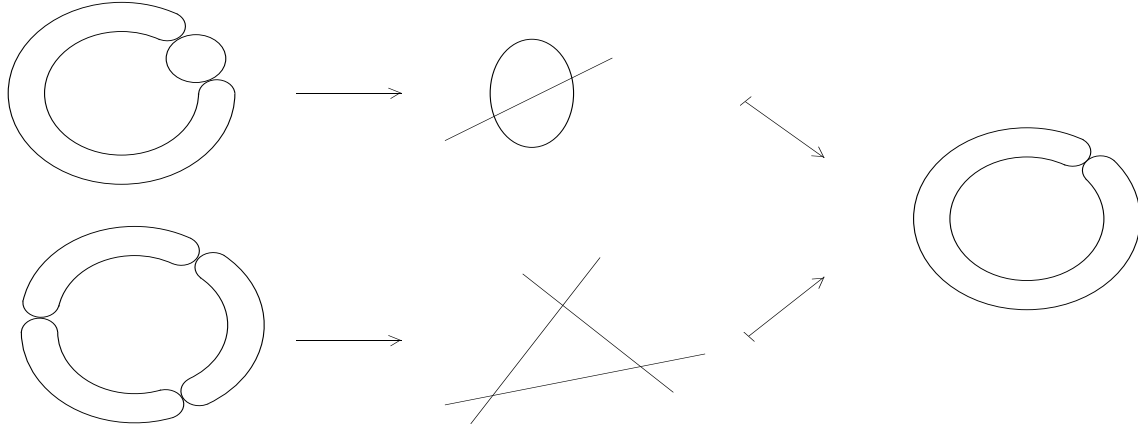


Figure 7: Some semi-stable singular cubics

There are explicit examples of sequences of smooth cubics converging to singular cubics in Véronique Lizan’s thesis [34].

I will describe further examples of moduli spaces of stable maps, related to algebraic loop spaces and configuration spaces in Appendix B.

Density of irreducible curves. — Notice also that Kontsevich’s space can often be considered as being too big in the sense that $\overline{\mathcal{M}}_{g,m}(X, A)$ is not, in general, the closure of the set of maps $\Sigma \rightarrow X$ where the curve Σ is irreducible.

This can be understood quite easily already in the case of plane genus-1 cubics. The space $\overline{\mathcal{M}}_{1,1}(\mathbf{P}^2, 3L)$ has dimension 11 (in general, $\dim \overline{\mathcal{M}}_{1,m}(\mathbf{P}^2, dL) = m + 3d + 1$). It contains all the maps $u : C \cup \mathbf{P}^1 \rightarrow \mathbf{P}^2$, where C has genus 1, C and \mathbf{P}^1 meet at some point, u is constant on C and has degree 3 on \mathbf{P}^1 . In other words, using the excedentary marked points to glue curves together, it contains a copy of $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{0,2}(\mathbf{P}^2, 3L)$. But this space, too, has dimension 11.

It is easy to understand that a map $u : C \cup \mathbf{P}^1 \rightarrow \mathbf{P}^2$ can only be in the closure of the set of maps $\Sigma \rightarrow \mathbf{P}^2$ (with Σ irreducible) if u has a singularity at the point where the two components meet: decreasing the genus adds singularities (as we have already mentioned it in the case of the cuspidal cubic).

This cannot happen for rational (genus 0) curves in *convex* manifolds (see below): here maps from irreducible curves are dense in $\overline{\mathcal{M}}_{0,m}(X, A)$.

4.3. Convex manifolds and smoothness properties

Fundamental class. — The spaces $\overline{\mathcal{M}}_{g,m}(X, A)$ are often projective varieties (see [20], [11]) and, as such, according to Borel and Haefliger [9], carry fundamental classes. However, we need much more, since we want to interpret cohomology computations in terms of intersection numbers (*e.g.* to get enumerative conclusions). That is, we need Poincaré duality (in other words a smoothness property) at least over \mathbf{Q} (an orbifold structure would be enough). Also, excedentary components could cause some difficulties dealing with enumerative questions.

For these reasons, we will now concentrate on rational curves (spaces $\overline{\mathcal{M}}_{0,m}(X, A)$). I will not try to obtain the best possible results and I will impose some restrictions on the projective manifold X (see below § 5.5 for generalisations and references).

A-convexity. — The manifold X is convex for the class A , or *A-convex*, if, for any decomposition

$$A = A_1 + \cdots + A_k$$

where each A_j may be represented by a rational curve, and for each $u : \mathbf{P}^1 \rightarrow X$ such that $u_*[\mathbf{P}^1] = A_j$ for some j ,

$$u^*TX = \bigoplus_{i=1}^n \mathcal{O}(m_i)$$

with $m_i \geq 0$ for all i .

The convexity condition ensures that the various strata of $\overline{\mathcal{M}}_{0,m}(X, A)$ are smooth: the positivity condition on the m_i 's implies that $H^1(\mathbf{P}^1, u^*TX) = 0$ for all maps u representing one of the classes A_j , or — in terms familiar to symplectic topologists — that the linearisation of the $\bar{\partial}$ operator associated with the complex structure on X is surjective (see *e.g.* §2.1 of [2]). But it also contains a positivity condition on the first Chern class, that is related to compactness properties (again, see *e.g.* §2.2 of [2]):

4.5 LEMMA. — *If X is A-convex, then for any decomposition $A = A_1 + \cdots + A_k$ where each A_j is non-zero and representable by a rational curve, $\langle c_1, A_j \rangle \geq 2$.*

Proof. — By definition of *A-convexity*, for any map $u : \mathbf{P}^1 \rightarrow X$ representing a class A_j , we have

$$u^*TX = \bigoplus \mathcal{O}(m_i) \text{ with } m_i \geq 0 \text{ for all } i.$$

Now, if A_j is non zero, u is non constant, hence its differential is injective (as a sheaf map) and carries a vector field with two zeroes on \mathbf{P}^1 to a non-zero section of u^*TX with at least two zeroes. Hence one of the m_i 's must be at least 2. \square

Now the *A-convexity* property gives everything we want:

4.6 THEOREM. — *If X is A-convex, $\overline{\mathcal{M}}_{0,m}(X, A)$ is a locally normal projective variety and locally a quotient of a nonsingular variety by a finite group. It has complex dimension $\dim X + m - 3 + \langle c_1, A \rangle$.*

There is a beautiful proof in [20]. Let us sketch it very briefly. The authors consider the case $X = \mathbf{P}^n$. Then they systematically add marked points to the curves in the space of stable maps, the points where the curve meet the coordinate hyperplanes, assuming these are distinct from the special (marked or singular) points already present on your curve. The choice of an ordering⁸ for the new marked points gives an abstract stable marked curve. Notice that the data of the intersection points of the curve with the coordinate hyperplanes allows us to reconstruct the stable map, up to the action of the big torus $(\mathbf{C}^*)^n$ on \mathbf{P}^n . In other words, we have established a close relation between some big open subset of $\overline{\mathcal{M}}_{0,m}(\mathbf{P}^n, A)$ and some subset of some $\overline{\mathcal{M}}_{0,N}$ — which is open in this case (genus 0).

Different coordinates in \mathbf{P}^n give a covering of $\overline{\mathcal{M}}_{0,m}(\mathbf{P}^n, A)$ with such open subsets. Then we can use the properties of $\overline{\mathcal{M}}_{0,N}$ to conclude the proof in the case of \mathbf{P}^n . The authors of [20] are then able to pass from \mathbf{P}^n to the projective variety X . \square

⁸This is a place where “quotients by a finite group” come from.

Remark. — It is also shown in [20] that $\overline{\mathcal{M}}_{0,m}(X, A)$ is a moduli space in the sense of algebraic geometry (namely, it represents the *ad hoc* functor). There is also a universal stable map⁹, that is, a space

$$\mathcal{U}_{0,m} \longrightarrow \overline{\mathcal{M}}_{0,m}(X, A)$$

which looks very much like $\overline{\mathcal{M}}_{0,m+1}(X, A)$ (the fiber at a point (Σ, \vec{z}, u) is the curve Σ itself and which is endowed with a “universal map” to X (the evaluation at the $(m+1)^{\text{st}}$ point)

$$(\Sigma, \vec{z}, u, z) \longmapsto u(z).$$

Convex manifolds. — A projective manifold X is *convex* if it is convex for all classes containing rational curves. Notice that this is equivalent to the following:

$$\forall u : \mathbf{P}^1 \rightarrow X, \quad H^1(\mathbf{P}^1; u^*TX) = 0.$$

This being true for *all* maps $u : \mathbf{P}^1 \rightarrow X$ implies that the degrees of the summands of u^*TX are ≥ 0 (if one of them was ≤ -1 , compose with a degree-2 map $\mathbf{P}^1 \rightarrow \mathbf{P}^1$ to get a degree- < -1 bundle and a contradiction).

Convex manifolds include homogeneous spaces, and in particular projective spaces, grassmannians and flag manifolds.

Non-convex manifolds. — There are very simple examples of non-convex manifolds. Consider for instance a Hirzebruch surface, which can be described as the total space of the bundle $\mathbf{P}(\mathcal{O}(k) \oplus \mathbf{1}) \rightarrow \mathbf{P}^1$. In this description, it is obvious that this surface contains an embedded rational curve C of self-intersection $-k$ (the section at infinity). Then, for the embedding $u : \mathbf{P}^1 \rightarrow \mathbf{P}(\mathcal{O}(k) \oplus \mathbf{1})$ whose image is C ,

$$\langle c_1(u^*TX), [\mathbf{P}^1] \rangle = -k + 2$$

so that, if $u^*TX = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$, $m_1 + m_2 = -k + 2$. If $k \geq 2$, at least one of the m_i 's is ≤ -2 . If $k = 1$, consider the composition of u with a self-map of degree-2 of \mathbf{P}^1 to get the same conclusion. We have thus proved:

4.7 PROPOSITION. — *The Hirzebruch surface $\mathbf{P}(\mathcal{O}(k) \oplus \mathbf{1})$ is convex if and only if $k = 0$. \square*

For instance, for $k = 1$, the Hirzebruch surface is $\widetilde{\mathbf{P}}^2$, the plane blown-up at a point. The homology class of the exceptional divisor is denoted E . It is represented by an isolated embedded rational curve. The composition with any self-map of degree 2 in \mathbf{P}^1 gives a rational curve in the class $2E$. Now, these degree-2 self-maps of \mathbf{P}^1 form a 5-dimensional space, so that

$$\dim \overline{\mathcal{M}}_{0,3}(\widetilde{\mathbf{P}}^2, 2E) \geq 5.$$

The dimension we could expect from Theorem 4.6 is 4: this confirms that $\widetilde{\mathbf{P}}^2$ is not convex.

⁹at least over the subspace consisting of stable maps that have no automorphism at all...

5. Gromov-Witten potential and quantum cohomology

We now have enough technical tools to define the Gromov-Witten invariants and the Gromov-Witten potential. We will then deduce the definition of the quantum cup-product and show that all this indeed defines a Frobenius manifold structure on the even part of the cohomology of certain projective manifolds.

5.1. Gromov-Witten invariants

Invariants. — Assume X to be A -convex. Given cohomology¹⁰ classes ξ_1, \dots, ξ_m in $H^*(X)$, consider

$$\left\langle (\text{ev}_m)^*(\xi_1 \otimes \cdots \otimes \xi_m), [\overline{\mathcal{M}}_{0,m}(X, A)] \right\rangle =: \Psi_A(\xi_1 \otimes \cdots \otimes \xi_m) \in \mathbf{Q}.$$

Notice that $\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m)$ is non zero only if

$$\frac{1}{2} \sum_{i=1}^m \deg \xi_i = \dim \overline{\mathcal{M}}_{0,m}(X, A) = \dim X + m - 3 + \langle c_1, A \rangle.$$

It is also obvious that the number $\Psi_A(\xi_1 \otimes \cdots \otimes \xi_m)$ does not depend on the ordering of the cohomology classes inside. The next proposition states an important first property of the invariants: they contain (and thus generalise) the cup-product on X .

5.1 PROPOSITION.

$$\Psi_A(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1) = 0 \text{ if } m \geq 4 \text{ or } A \neq 0,$$

$$\Psi_0(\xi_1 \otimes \cdots \otimes \xi_m) = \begin{cases} 0 & \text{for } m > 3 \\ \langle \xi_1 \smile \xi_2 \smile \xi_3, [X] \rangle & \text{for } m = 3. \end{cases}$$

Proof. — Assume $m \geq 3$ and consider the $A = 0$ case. Recall from § 4.1 that $\overline{\mathcal{M}}_{0,m}(X, 0) = \overline{\mathcal{M}}_{0,m} \times X$ and that in this case the evaluation mapping is

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,m} \times X & \longrightarrow & X^m \\ (\Sigma, \vec{z}, x) & \longmapsto & (x, x, \dots, x) \end{array}$$

therefore, if $p : \overline{\mathcal{M}}_{0,m} \times X \rightarrow X$ is the projection,

$$\begin{aligned} \left\langle (\text{ev}_m)^*(\xi_1 \otimes \cdots \otimes \xi_m), [\overline{\mathcal{M}}_{0,m} \times X] \right\rangle &= \left\langle p^*\xi_1 \smile \cdots \smile p^*\xi_m, [\overline{\mathcal{M}}_{0,m} \times X] \right\rangle \\ &= \begin{cases} 0 & \text{if } m > 3 \\ \langle \xi_1 \smile \xi_2 \smile \xi_3, [X] \rangle & \text{if } m = 3 \end{cases} \end{aligned}$$

as $\overline{\mathcal{M}}_{0,m}$ has positive dimension for $m > 3$.

Consider now the unit $1 \in H^0(X)$ of the cup-product. By definition,

$$\Psi_A(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1) = \left\langle (\text{ev}_m)^*(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1), [\overline{\mathcal{M}}_{0,m}(X, A)] \right\rangle.$$

¹⁰I use rational cohomology. In particular, $H^*(X^m) = (H^*(X))^{\otimes m}$.

Assume $m \geq 4$ or $A \neq 0$, so that there is a well-defined map $\theta : \overline{\mathcal{M}}_{0,m}(X, A) \rightarrow \overline{\mathcal{M}}_{0,m-1}(X, A)$, forgetting the last marked point (and contracting the components that became unstable). Consider the commutative diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,m}(X, A) & \xrightarrow{\text{ev}_m} & X^m \\ \downarrow \theta & & \downarrow p \\ \overline{\mathcal{M}}_{0,m-1}(X, A) & \xrightarrow{\text{ev}_{m-1}} & X^{m-1} \end{array}$$

where the projection p forgets the last factor. As $\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1 = p^*(\xi_1 \otimes \cdots \otimes \xi_{m-1})$,

$$\langle (\text{ev}_m)^*(\xi_1 \otimes \cdots \otimes \xi_{m-1} \otimes 1), [\overline{\mathcal{M}}_{0,m}(X, A)] \rangle = \langle (\theta^* \text{ev}_{m-1})^*(\xi_1 \otimes \cdots \otimes \xi_{m-1}), [\overline{\mathcal{M}}_{0,m-1}(X, A)] \rangle.$$

This vanishes as $\dim \overline{\mathcal{M}}_{0,m-1}(X, A) < \dim \overline{\mathcal{M}}_{0,m}(X, A)$. \square

Examples. — Let X be, once again, the complex projective space \mathbf{P}^n . Consider the space $\overline{\mathcal{M}}_{0,3}(\mathbf{P}^n, L)$. It has dimension $2n + 1$ and its generic points are parametrised lines in \mathbf{P}^n with three distinct marked points. It has an evaluation mapping to the $3n$ dimensional space $(\mathbf{P}^n)^3$. Let p be the generator of $H^2(\mathbf{P}^n; \mathbf{Z})$, so that $\Psi_L(p^n \otimes p^n \otimes p)$ is a well-defined Gromov-Witten invariant. From the enumerative point of view, this is the number of lines through two distinct points (p^n is the cohomology class dual to a point) meeting a hyperplane somewhere:

$$\Psi_L(p^n \otimes p^n \otimes p) = 1.$$

The next example is that of degree- d rational plane curves, namely that of the space $\overline{\mathcal{M}}_{0,m}(\mathbf{P}^2, dL)$, a space of dimension $m - 1 + 3d$. The evaluation map sends it to $(\mathbf{P}^2)^m$, a $2m$ dimensional space. When $m = 3d - 1$, the two dimensions coincide and the evaluation map

$$\overline{\mathcal{M}}_{0,3d-1}(\mathbf{P}^2, dL) \longrightarrow (\mathbf{P}^2)^{3d-1}$$

has a *degree*, traditionally denoted by N_d , which is nothing other than the Gromov-Witten invariant $\Psi_{dL}((p^2)^{\otimes(3d-1)})$ (recall that p is the generator of $H^2(\mathbf{P}^2; \mathbf{Z})$). From the enumerative point of view, N_d is the number of rational plane curves of degree d through $3d - 1$ general points.

Degree-2 classes. — It is in the nature of the Gromov-Witten invariants that degree-2 cohomology classes play a special role. Here is a property that we will need later and that illustrates this assertion.

5.2 PROPOSITION. — *Assume X is A -convex. Let ξ be a degree-2 cohomology class. Then, for any $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ in $H^*(X^{\otimes m})$,*

$$\Psi_A(\alpha \otimes \xi) = \langle \xi, A \rangle \Psi_A(\alpha).$$

Proof. — Consider the map

$$\begin{aligned} \overline{\mathcal{M}}_{0,m+1}(X, A) &\xrightarrow{\tilde{\theta}} \overline{\mathcal{M}}_{0,m}(X, A) \times X \\ (\Sigma, \vec{z}, u) &\longmapsto (\theta(\Sigma, \vec{z}, u), u(z_{m+1})) \end{aligned}$$

(as above, θ forgets the last point and contracts the unstable components). As the dimension of $\overline{\mathcal{M}}_{0,m+1}(X, A)$ is $\dim \overline{\mathcal{M}}_{0,m}(X, A) + 1$,

$$\tilde{\theta}_*[\overline{\mathcal{M}}_{0,m+1}(X, A)] = [\overline{\mathcal{M}}_{0,m}(X, A)] \otimes A' + b$$

where $A' \in H_2(X)$ and the components of b on $H_*(\overline{\mathcal{M}}_{0,m}(X, A))$ have dimension strictly less than the fundamental class. Obviously then,

$$\Psi_A(\alpha \otimes \xi) = \langle \xi, A' \rangle \Psi_A(\alpha),$$

so that we only need to check that $A' = A$. But this is obvious from the definition of $\tilde{\theta}$, for if $(\mathbf{P}^1, \vec{z}, u)$ represents an element of $\overline{\mathcal{M}}_{0,m+1}(X, A)$, $u_*[\mathbf{P}^1] = A$ and this is the class we see on the X factor of $\overline{\mathcal{M}}_{0,m}(X, A) \times X$. \square

The Gromov-Witten potential. — As we have announced it in §3.3, we want to define a function on $H^*(X; \mathbf{C})$, or at least some kind of formal series, by a formula like

$$\Psi(\xi) = \sum_{m \geq 3} \frac{1}{m!} \sum_A \Psi_A(\xi^{\otimes m}).$$

The discussion will be simpler if we rather try with

$$\Psi(\xi) = \sum_A \sum_{m \geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m}).$$

Indeed, fix a (homogeneous) class $\xi \in H^*(X; \mathbf{C})$. The class $A \in H_2(X; \mathbf{Z})$ being given, $\Psi_A(\xi^{\otimes m})$ can only be non zero if

$$m \deg \xi = 2(\dim X + m - 3) + \langle c_1, A \rangle$$

so that

- if $\deg \xi \neq 2$, $\Psi_A(\xi^{\otimes m}) \neq 0$ for only one value of m ,
- if $\deg \xi = 2$, using Proposition 5.2,

$$\sum_{m \geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m}) = \exp \langle \xi, A \rangle$$

(modulo quadratic terms).

In any case, the sum

$$\sum_{m \geq 3} \Psi_A(\xi^{\otimes m})$$

defines a complex number for any A . Taking sums of homogeneous elements, we get

5.3 LEMMA. — For any class $A \in H_2(X; \mathbf{Z})$, the formula

$$\Phi_A(\xi) = \sum_{m \geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m})$$

defines a function $\Phi_A : H^*(X; \mathbf{C}) \rightarrow \mathbf{C}$. □

There is still the problem raised by the summation over all homology classes A . There are two possibilities, either to impose a geometric condition on the manifold X in order to be sure that the sum is finite, or to make it a formal series in additional variables q .

Notation. — Assume that $H_2(X; \mathbf{Z})$ is torsion free — or denote by $H_2(X; \mathbf{Z})$ the torsion free part of the second homology group. Let Λ be its group ring, $\Lambda = \mathbf{Z}[H_2(X; \mathbf{Z})]$, and denote by q^A the (multiplicative) counterpart in Λ of $A \in H_2(X; \mathbf{Z})$. Thus q is a multi-variable of dimension $k = \text{rk } H_2(X; \mathbf{Z})$.

Put now

$$\Psi_q(\xi) = \sum_A \Phi_A(\xi) q^A = \sum_A \left(\sum_{m \geq 3} \frac{1}{m!} \Psi_A(\xi^{\otimes m}) \right) q^A.$$

This is a formal series in q with coefficients functions of ξ (involving only monomials and exponentials). To be more specific, let us choose a basis of $H^{2*}(X; \mathbf{Z})$ (considered modulo torsion) consisting of homogeneous elements. It will be convenient to use different names for the coordinates according to the degrees, ξ_0 in degree 0, ξ_1, \dots, ξ_m in degrees ≥ 4 and η_1, \dots, η_k in degree 2. The basis, in the same order, is of course, $\partial/\partial\xi_i$ ($0 \leq i \leq m$), $\partial/\partial\eta_j$ ($1 \leq j \leq k$). Let us write

$$N_A(k_1, \dots, k_m) = \Psi_A \left[\left(\frac{\partial}{\partial\xi_1} \right)^{\otimes k_1} \otimes \cdots \otimes \left(\frac{\partial}{\partial\xi_m} \right)^{\otimes k_m} \right]$$

so that

$$\Psi_q(\xi, \eta) = \sum_A \left(\sum_{k_1, \dots, k_m} N_A(k_1, \dots, k_m) \frac{\xi_1^{k_1} \cdots \xi_m^{k_m}}{k_1! \cdots k_m!} \right) \exp\langle \eta, A \rangle q^A.$$

The other possibility is to consider only manifolds for which the sums above are finite. Notice that, according to the dimension count above, given ξ and m , if $\Psi_A(\xi^{\otimes m}) \neq 0$, then

$$\langle c_1, A \rangle \leq (m-1) \dim X - m + 3.$$

Recall that, according to Lemma 4.5, $\langle c_1, A \rangle$ is bounded from below. Hence, if we assume

5.4 ASSUMPTIONS. — *The complex projective smooth variety X is convex and, for all K , the set*

$$\{A \in H_2(X; \mathbf{Z}) \mid A \text{ is represented by a rational curve and } \langle c_1, A \rangle \leq K\}$$

is finite.

the sum $\sum_A \Psi_A(\xi^{\otimes m})$ is finite for all ξ and m . It is not quite obvious why $\sum \frac{1}{m!} \sum_A \Psi_A(\xi^{\otimes m})$ would then be a convergent series.

Examples.

- Convex Fano manifolds satisfy the assumptions. This is a consequence of the compactness theorem 4.4 above: in the Fano case, the first Chern class is the class of a Kähler form (this is a definition of a Fano manifold).
- Manifolds without rational curves obviously satisfy all assumptions — all the Gromov-Witten invariants Ψ_A (for $A \neq 0$) vanish.
- If the set of homology classes containing rational curves is a cone generated by a finite number of classes, the finiteness assumption is also satisfied. There are classes, say, A_1, \dots, A_r , who generate the cone of all classes that are representable by rational curves. According to Lemma 4.5, $\langle c_1, A_i \rangle \geq 2$. If $u : \mathbf{P}^1 \rightarrow X$ represents a class A , $A = \sum d_i A_i$ for some $d_i \in \mathbf{N}$, and, for any given integer N , there are finitely many classes A such that $\langle c_1, A \rangle \leq N$.

5.2. Quantum cohomology

We hope now that the Gromov-Witten potential is the potential of a Frobenius structure on $H^*(X)$. We have a metric

$$g(\alpha, \beta) = \langle \alpha \smile \beta, [X] \rangle$$

which is constant in linear coordinates and thus flat. We still need products \star_ξ depending on $\xi \in H^*(X)$, and we hope that they can be defined by the third derivative of Ψ . We begin by noticing that

$$g(\alpha \star_\xi \beta, \gamma) = \left(d^3 \Psi \right)_\xi (\alpha, \beta, \gamma)$$

would define a commutative¹¹ (by symmetry) and unital (using Proposition 5.1) product... if there was no convergence problem. So, let us use the formal potential Ψ_q as above. Call $\hat{\Lambda}$ the *ad hoc* completion of Λ .

5.5 PROPOSITION. — *The formula*

$$g(\alpha \star_{\xi, q} \beta, \gamma) = \left(d^3 \Psi_q \right)_\xi (\alpha, \beta, \gamma)$$

defines a commutative product $\star_{\xi, q}$ on $H^(X; \mathbf{C}) \otimes \tilde{\Lambda}$, graded by the natural graduation of the cohomology and $\deg q^A = 2\langle c_1, A \rangle$, and having the same unit as the cup product.*

Proof. — The right hand side is indeed an element of $\tilde{\Lambda}$. We will then extend the product by $\tilde{\Lambda}$ -linearity. Notice that

$$\left(d^3 \Psi_q \right)_\xi (\alpha, \beta, \gamma) = \sum_A \left(\sum_{m \geq 0} \frac{1}{m!} \Psi_A(\xi^{\otimes m} \otimes \alpha \otimes \beta \otimes \gamma) \right) q^A.$$

¹¹Commutativity here is commutativity in the graded sense: it comes from the commutativity of the cup-product.

As above, the sum

$$\sum_{m \geq 0} \frac{1}{m!} \Psi_A(\xi^{\otimes m} \otimes \alpha \otimes \beta \otimes \gamma)$$

defines a complex number for all A, α, β, γ . The product $\star_{\xi, q}$ is obviously commutative, the assertion on the unit follows from Proposition 5.1. \square

Remark. — There are various definitions of “the quantum cup-product”. This is the most general version, as it depends on the point ξ , and, because of the variables q (and the use of the ring Λ), there is no convergence problem in the definition.

There are basically two ways to get a less general structure: to specialise, either at a value of ξ , or at a value of q , the latter leading to the mentioned convergence problems.

- To specialise at $\xi = 0$ would give the “usual” quantum product, a formal series in q , denoted $\alpha \star \beta$, a product on $QH^*(X) := H^*(X) \otimes \tilde{\Lambda}$.
- To specialise at $q = 1$, allowing ξ to be any cohomology class, but assuming the series defining $\alpha \star_{\xi, 1} \beta$, that we will denote $\alpha \star_{\xi} \beta$ to converge.

The former is the *small* quantum product, and the latter the *global* quantum product. As we shall see in § 6, these two apparently different specialisations are deeply related.

5.3. Composition rule and associativity

Let us prove now the non trivial result, related to what physicists call the “composition rule”:

5.6 THEOREM. — *The product $\star_{\xi, q}$ is associative.*

Proof. — The proof given here simply repeats that of [20]. Associativity amounts to the fact that Ψ_q satisfies the WDVV equation (§ 3.1). Writing (∂_i) for a basis of the cohomology of X and often forgetting \otimes ’s in the notation, this is:

$$\begin{aligned} & \sum \frac{1}{m_1! m_2!} \Psi_{A_1}(\xi^{m_1} \partial_i \partial_j \partial_k) g^{k, n} \Psi_{A_2}(\xi^{m_2} \partial_r \partial_s \partial_n) q^{A_1 + A_2} \\ & = \sum \frac{1}{m_1! m_2!} \Psi_{A_1}(\xi^{m_1} \partial_r \partial_j \partial_k) g^{k, n} \Psi_{A_2}(\xi^{m_2} \partial_i \partial_s \partial_n) q^{A_1 + A_2}. \end{aligned}$$

Equating coefficients of powers of q , we see that we are discussing the decomposition of a homology class A as a sum $A_1 + A_2$ and the ways to represent this decomposition by stable maps. We thus need to understand what happens — homologically in $\overline{\mathcal{M}}_{0, m}(X, A)$ — when the curves we are looking at split into two components, each of them containing some of the marked points. (See Figure 8.) Let us write $\{1, \dots, m\} = E_1 \cup E_2$ (and require $\#E_i \geq 2$).

Fixing the decompositions of A and $\{1, \dots, m\}$, we get an embedding

$$j : \overline{\mathcal{M}}_{0, E_1 \cup \{z_0\}}(X, A_1) \times_X \overline{\mathcal{M}}_{0, E_2 \cup \{z_0\}}(X, A_2) \longrightarrow \overline{\mathcal{M}}_{0, m}(X, A)$$

(using obvious notation). For simplicity, let us write $M_i = \overline{\mathcal{M}}_{0, E_i \cup \{z_0\}}(X, A_i)$, $M = \overline{\mathcal{M}}_{0, m}(X, A)$ and $D_{E_1, E_2}(A_1, A_2) = M_1 \times_X M_2$. As $\dim M_i = n + \#E_i - 2 + \langle c_1, A_i \rangle$, the dimension of

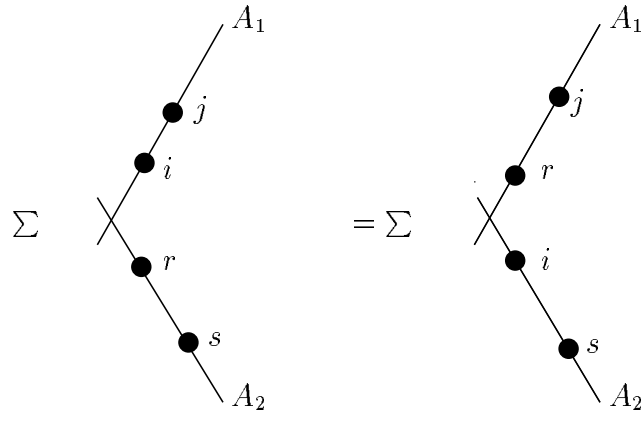


Figure 8

$D_{E_1, E_2}(A_1, A_2)$ is $\dim M - 1$, so that we are considering a divisor in our moduli space. Call i the inclusion $D_{E_1, E_2}(A_1, A_2) \subset M_1 \times M_2$ and, if a marked point is labelled by some element p , call e_p the evaluation mapping at this point.

5.7 LEMMA. — For any cohomology class $\eta = \eta_1 \otimes \cdots \otimes \eta_m$ in $H^*(X)$,

$$i_* j^*(\text{ev}_m)^*(\eta) = \sum_{k, n} g^{k, n} \left[\left(\prod_{p \in E_1} e_p^*(\eta_p) \right) e_{z_0}^*(\partial_k) \right] \times \left[\left(\prod_{q \in E_2} e_q^*(\eta_q) \right) e_{z_0}^*(\partial_n) \right].$$

Postponing the proof of the lemma, which is a direct computation, we get

$$(*) \quad \sum_{\substack{a, b \in E_1 \\ c, d \in E_2}} g^{k, n} \Psi_{A_1} \left(\left(\bigotimes_{p \in E_1} \eta_p \right) \otimes \partial_k \right) \Psi_{A_2} \left(\left(\bigotimes_{q \in E_2} \eta_q \right) \otimes \partial_n \right) \\ = \sum_{\substack{a, b \in E_1 \\ c, d \in E_2}} \langle \text{ev}_m^*(\eta), [D_{E_1, E_2}(A_1, A_2)] \rangle.$$

Put

$$\begin{cases} \eta_1 = \dots = \eta_{m-4} = \xi \\ \eta_{m-3} = \partial_i, \eta_{m-2} = \partial_j, \eta_{m-1} = \partial_r, \eta_m = \partial_s \\ a = m-3, b = m-2, c = m-1, d = m \end{cases}$$

in the LHS of (*) to get

$$\text{LHS} = \sum C(m_1, m_2) g^{k, n} \Psi_{A_1} (\xi^{m_1-2} \partial_i \partial_j \partial_k) \Psi_{A_2} (\xi^{m_2-2} \partial_r \partial_s \partial_n)$$

where $C(m_1, m_2)$ is the number of decompositions of $\{1, \dots, m\}$ as $E_1 \cup E_2$ with $\#E_i = m_i$ and the sum is over all decompositions with $m_1 \geq 2$ and $m_2 \geq 2$, so that

$$\text{LHS} = m! \sum_{\substack{m_1, m_2 \geq 0 \\ m_1 + m_2 = m-4}} \frac{1}{m_1! m_2!} g^{k, n} \Psi_{A_1} (\xi^{m_1} \partial_i \partial_j \partial_k) \Psi_{A_2} (\xi^{m_2} \partial_r \partial_s \partial_n).$$

5.8 LEMMA. — In $H_* (\overline{\mathcal{M}}_{0, m}(X, A))$, if a, b, c, d are four distinct elements of $\{1, \dots, m\}$,

$$\sum_{\substack{a, b \in E_1 \\ c, d \in E_2}} [D_{E_1, E_2}(A_1, A_2)] = \sum_{\substack{c, b \in E_1 \\ a, d \in E_2}} [D_{E_1, E_2}(A_1, A_2)].$$

This is precisely the symmetry argument needed to conclude. \square

Proof of Lemma 5.8. — This is the geometric heart and most beautiful part of the proof. The idea is that $\overline{\mathcal{M}}_{0,4}$ is isomorphic to \mathbf{P}^1 and that any two points of \mathbf{P}^1 represent the same homology class (actually they are linearly equivalent — and this gives a linear equivalence of divisors rather than an equality of homology classes).

Consider the moduli space $\overline{\mathcal{M}}_{0,4}$: four ordered distinct points in \mathbf{P}^1 up to isomorphism, this is a point in $\mathbf{P}^1 - \{0, 1, \infty\}$, their cross ratio. The compactification is, of course, \mathbf{P}^1 . The three added points $\{0, 1, \infty\}$ correspond to the reducible curves with two components, each carrying half of the marked points, that is, to the partitions of the marking set in two subsets of two elements.

Look now at the composed contraction:

$$\overline{\mathcal{M}}_{0,m}(X, A) \rightarrow \overline{\mathcal{M}}_{0,m} \rightarrow \overline{\mathcal{M}}_{0,\{a,b,c,d\}}.$$

The divisor $\sum_{\substack{a,b \in E_1 \\ c,d \in E_2}} D_{E_1,E_2}(A_1, A_2)$ is the inverse image of the point $(a, b \mid c, d)$, while the divisor $\sum_{\substack{c,b \in E_1 \\ a,d \in E_2}} D_{E_1,E_2}(A_1, A_2)$ is that of $(c, b \mid a, d)$. As the two points of \mathbf{P}^1 are linearly equivalent, so are the divisors in $\overline{\mathcal{M}}_{0,m}(X, A)$. \square

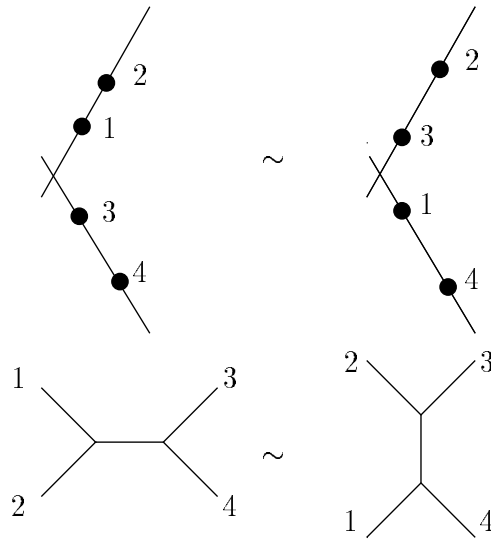


Figure 9

Remark. — This can be easily expressed by pictures that are familiar to physicists (see Figure 9).

Proof of Lemma 5.7. — This is a straightforward computation, starting from the commutative diagram

$$\begin{array}{ccccc}
 M & \xleftarrow{j} & D_{E_1,E_2}(A_1, A_2) & \xrightarrow{i} & M_1 \times M_2 \\
 \text{ev}_m \downarrow & & \varepsilon \downarrow & & \downarrow \text{ev}_{m_1+1} \times \text{ev}_{m_2+1} \\
 X^m & \xleftarrow{\text{proj}} & X^{m+1} & \xrightarrow{\Delta} & X^{m+2}
 \end{array}$$

in which it is seen that

$$\begin{aligned}
i_{\star} j^{\star}(\text{ev}_m)^{\star}(\eta) &= i_{\star} \varepsilon^{\star} \text{proj}^{\star}(\eta) \\
&= i_{\star} \varepsilon^{\star}(\eta \otimes D[X]) \\
&= (\text{ev}_{m_1+1} \times \text{ev}_{m_2+1})^{\star} \Delta_{\star}(\eta \otimes D[X]) \\
&= (\text{ev}_{m_1+1} \times \text{ev}_{m_2+1})^{\star}(\eta \otimes D[\Delta]).
\end{aligned}$$

This is the place where one has to realise that the Poincaré dual of the diagonal in $X \times X$ is

$$D[\Delta] = \sum_{k,n} g^{k,n} \partial_k \otimes \partial_n.$$

Then,

$$\begin{aligned}
i_{\star} j^{\star}(\text{ev}_m)^{\star}(\eta) &= \sum_{k,n} g^{k,n} (\text{ev}_{m_1+1} \times \text{ev}_{m_2+1})^{\star}(\eta \otimes \partial_k \otimes \partial_n) \\
&= \sum_{k,n} g^{k,n} \left[\left(\prod_{p \in E_1} e_p^{\star}(\eta_p) \right) e_{z_0}^{\star}(\partial_k) \right] \times \left[\left(\prod_{q \in E_2} e_q^{\star}(\eta_q) \right) e_{z_0}^{\star}(\partial_n) \right],
\end{aligned}$$

which is the relation we wanted to prove. \square

5.4. The Frobenius structure

In this §, we collect all the information above to show that the quantum products \star_{ξ} together with the flat metric defined by Poincaré duality, are pieces of a Frobenius structure on the cohomology of X . We still have three problems:

1. We must restrict ourselves to the even part of $H^{\star}(X)$ in order to have (strictly) commutative products (one could also consider, as in [32, 35], the whole cohomology, and get the structure of a Frobenius super-manifold).
2. The product \star_{ξ} and the Gromov-Witten potential are defined by formal series. We assume that they are convergent somewhere. Another possibility is to use, as in [35], *formal* Frobenius structures and the potential Ψ_q .
3. We still need an Euler vector field.

The Euler vector field. — Let us first use a basis of $H^{2\star}(X; \mathbf{Z})$ (considered modulo torsion) as above (§ 5.1) and express the first Chern class of X in this basis:

$$c_1 = \sum_{j=1}^k r_j \frac{\partial}{\partial \eta_j} \text{ for some integers } r_1, \dots, r_k.$$

Now we are able to define the vector field

$$\mathcal{E} = \sum_{i=0}^m (1 - \delta_i) \xi_i \frac{\partial}{\partial \xi_i} + \sum_{j=1}^k r_j \frac{\partial}{\partial \eta_j}.$$

5.9 THEOREM. — *Let X be a projective manifold satisfying the assumptions 5.4. Poincaré duality, the quantum product and the Euler vector field \mathcal{E} define the structure of a (formal) Frobenius manifold on the even part of the cohomology of X .*

Proof. — Using the same notation as in § 5.1,

$$\Psi(\xi, \eta) = \sum_A \exp\langle \eta, A \rangle \sum_{k_1, \dots, k_m} N_A(k_1, \dots, k_m) \frac{\xi_1^{k_1} \dots \xi_m^{k_m}}{k_1! \dots k_m!}.$$

Notice that $N_A(k_1, \dots, k_m)$ can be nonzero only if

$$k_1(\delta_1 - 1) + \dots + k_m(\delta_m - 1) = \dim X + \langle c_1, A \rangle - 3.$$

Now, the computation is straightforward:

$$\left(\sum r_j \frac{\partial}{\partial \eta_j} \Psi \right) (\xi, \eta) = \sum_A \langle c_1, A \rangle \exp\langle \eta, A \rangle \sum N_A(k_1, \dots, k_m) \frac{\xi_1^{k_1} \dots \xi_m^{k_m}}{k_1! \dots k_m!}$$

and

$$\left(\sum (1 - \delta_i) \xi_i \frac{\partial}{\partial \xi_i} \Psi \right) (\xi, \eta) = \sum_A \exp\langle \eta, A \rangle \sum_i (1 - \delta_i) k_i \sum N_A(k_1, \dots, k_m) \frac{\xi_1^{k_1} \dots \xi_m^{k_m}}{k_1! \dots k_m!}.$$

Thus,

$$(\mathcal{E} \cdot \Psi) (\xi, \eta) = \sum_A \exp\langle \eta, A \rangle \sum \left(\sum (1 - \delta_i) k_i + \langle c_1, A \rangle \right) N_A(k_1, \dots, k_m) \frac{\xi_1^{k_1} \dots \xi_m^{k_m}}{k_1! \dots k_m!}.$$

According to the remark on dimensions,

$$(\mathcal{E} \cdot \Psi) (\xi, \eta) = (3 - \dim X) \Psi(\xi, \eta)$$

so that Ψ is indeed \mathcal{E} -homogeneous of degree $(3 - \dim X)$. □

Remarks.

1. This is a place where 3-dimensional manifolds obviously play a special role.
2. That there are exponentials in the potential Ψ is related to the terms $r_j \partial / \partial \eta_j$ in the expression of the Euler vector field. The potentials of the Frobenius manifolds obtained this way seem to be rather different from the ones coming from unfoldings (see § 3.3).
3. The existence of the Gromov-Witten potential contains the symmetry of the tensor ∇c , or the closedness of the form Ω as in § 3.1. Hence the quantum products define a spectral cover. We will come back to this in § 6.1.
4. By integration along the fibers of the contraction map

$$\pi_A : \overline{\mathcal{M}}_{0, m+1}(X, A) \longrightarrow \overline{\mathcal{M}}_{0, m+1}$$

we get morphisms

$$\begin{array}{ccc} H^*(X)^{\otimes m+1} & \longrightarrow & H^*(\overline{\mathcal{M}}_{0, m+1}) \\ \xi & \longmapsto & (\pi_A)_* \text{ev}_{m+1}^*(\xi) \end{array}$$

the tree-level system of Gromov-Witten invariants of Kontsevich & Manin [32].

Note that, dualising these morphisms, we get a family of morphisms

$$I_m^A : H_*(\overline{\mathcal{M}}_{0,m+1}) \longrightarrow \text{Hom}\left(H^*(X)^{\otimes m}, H^*(X)\right)$$

and the associativity property (§ 5.3) says that this is a morphism of operads. According to Manin [35], this is the essence of a Frobenius structure.

5. A very interesting question is to know which projective varieties have a *massive* quantum cohomology. The question is whether there is a point ξ in $H^*(X; \mathbf{C})$ for which \star_ξ is the structure of a semi-simple ring. I will come back to this question in Appendix D.

The case of \mathbf{P}^1 . — Using the same notation as above, $H^*(\mathbf{P}^1; \mathbf{Z}) = \mathbf{Z}[p]/p^2$. Writing $\xi = \xi_0 + \xi_1 p$, one gets easily

$$p \star_{\xi,q} p = e^{\xi_1} q.$$

Specialising at $q = 1$ gives a Frobenius structure on \mathbf{C}^2 , which is everywhere semi-simple as e^{ξ_1} never vanishes, and that we have already met in § 3.2. The potential is

$$\Psi(\xi) = \frac{1}{2} \xi_0^2 \xi_1 + e^{\xi_1}$$

and the Euler vector field is indeed

$$\mathcal{E} = \xi_0 \frac{\partial}{\partial \xi_0} + 2 \frac{\partial}{\partial \xi_1}.$$

The potential for \mathbf{P}^2 and numbers of rational plane curves. — Write $\xi = \xi_0 + \xi_1 p + \xi_2 p^2$ for the elements of $H^*(\mathbf{P}^2)$, so that the Gromov-Witten potential

$$\Psi(\xi) = \sum_{m \geq 3} \frac{1}{m!} \sum_{d \geq 0} \Psi_{dL}(\xi^{\otimes m})$$

can be written

$$\Psi(\xi) = \frac{1}{2} (\xi_0^2 \xi_2 + \xi_0 \xi_1^2) + \sum_{d > 0} \sum_{m \geq 3} \frac{1}{m!} \Psi_{dL}(\xi^{\otimes m}).$$

The first term is usually called “classical” and the second is the “quantum” contribution. This can be rewritten, using the fact that p has degree 2 and Proposition 5.2,

$$\sum_{m \geq 3} \sum_{d > 0} \Psi_{dL}((p^2)^{\otimes m}) \frac{\xi_2^m}{m!} \exp\langle \xi_1 p, dL \rangle.$$

We have already noticed that the invariant $\Psi_{dL}((p^2)^{\otimes m})$ can be non zero only if $m = 3d - 1$, in which case it is the number N_d of rational degree- d plane curves through $3d - 1$ general points. Thus the Gromov-Witten potential for \mathbf{P}^2 is

$$\Psi(\xi) = \frac{1}{2} (\xi_0^2 \xi_2 + \xi_0 \xi_1^2) + \sum_{d \geq 1} N_d e^{d \xi_1} \frac{\xi_2^{3d-1}}{(3d-1)!}.$$

We have also noticed in §3.2 that, in dimension 3, there is only one associativity equation, namely

$$\frac{\partial^3 \Psi}{\partial \xi_2^3} = \left(\frac{\partial^3 \Psi}{\partial \xi_1^2 \partial \xi_2} \right)^2 - \frac{\partial^3 \Psi}{\partial \xi_1^3} \frac{\partial^3 \Psi}{\partial \xi_1 \partial \xi_2^2}.$$

Equating the coefficients of $e^{d\xi_1} \xi_2^{3d-4}$ on the two sides of this equation gives

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_1, d_2 \geq 1}} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2} = \sum_{\substack{d_1+d_2=d \\ d_1, d_2 \geq 1}} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2}.$$

This is of course the well-known recursion relation, due to Kontsevich [31], which theoretically allows you (or your computer) to compute N_d starting from $N_1 = N_2 = 1$.

5.5. Generalisations

There are Gromov-Witten invariants for all symplectic manifolds. They can be defined using “virtual fundamental classes”: the space $\overline{\mathcal{M}}_{g,m}(X, A)$ might be very bad, various authors (see *e.g.* [42, 18, 6]) are nevertheless able to show that it carries a homology class of the right (expected) dimension, which can be used as a fundamental class.

In this §, I explain briefly a very simple case of such a construction, that can be used to define Gromov-Witten invariants for certain (non convex) submanifolds of convex manifolds, including *e.g.* complete intersections in projective spaces¹². This generalisation comes from [31] and was used *e.g.* in [23]. I also mention, even more briefly, the relations between the Gromov-Witten invariants defined so far and other avatars available in the literature.

Convex vector bundles on convex manifolds. — Let X be, as above, a convex projective manifold and let $V \rightarrow X$ be a vector bundle. It is said to be *convex* if it is generated by its (global) sections, that is, if $H^1(X; V) = 0$. For instance, TX is a convex vector bundle.

Let us look now at the zero set Y of a holomorphic section of V transverse to the zero section: this is a submanifold of X , whose normal bundle is the pull-back of V on Y . There is no reason why Y should be convex. However, we aim to define some Gromov-Witten invariants for Y . The moduli space $\overline{\mathcal{M}}_{0,m}(Y, A)$ itself could be very bad, it might be non smooth, or might not have the expected dimension. Fortunately, there is a classical way in which topologists use to solve such problems: you find a vector bundle on some bigger space such that $\overline{\mathcal{M}}_{0,m}(Y, A)$ is the zero set of some section, and use the Euler class of this bundle (the zero set of a section *transverse to the zero section*) in place of your moduli space.

In the case at hand, this works as follows:

- Show that there exists a vector bundle \mathcal{V} on $\overline{\mathcal{M}}_{0,m}(X, j_* A)$ (j is of course the inclusion $Y \rightarrow X$) whose fiber at a point represented by a map $u : C \rightarrow X$ is $H^0(C, u^* V)$ (this can be achieved using the fact that $\overline{\mathcal{M}}_{0,m}(X, j_* A)$ is a moduli space in the sense of algebraic geometry, endowed with a universal stable map — see §4.3).

¹²This allows for instance to get the results of Beauville [5] on the quantum cohomology of Fano complete intersections using invariants defined by stable maps.

- For $\alpha \in H^*(X)^{\otimes m}$, define

$$\Psi_A((j^m)^*\alpha) = \left\langle (ev_m)^*\alpha \smile e(\mathcal{V}), [\overline{\mathcal{M}}_{0,m}(X, j_*A)] \right\rangle.$$

I insist that these invariants are only defined on cohomology classes coming from the ambient space X .

Remarks.

1. One easily checks that, if Y is A -convex, the definition coincides with the previous definition of $\Psi_A((j_m)^*\alpha)$.
2. One can also check that the result does not depend on the choice of the embedding in the convex manifold used.
3. Because the Euler class is multiplicative, it can be shown that these invariants satisfy the composition rule as in §5.3.

The example of the plane blown-up at a point. — Let Y be the manifold obtained by blowing up \mathbf{P}^2 at a point. This is the Hirzebruch surface $\mathbf{P}(\mathcal{O}(1) \oplus \mathbf{1})$ and, as we have mentioned it after Proposition 4.7, if E is the class of the exceptional divisor, Y is not $2E$ -convex.

Embed Y as the bidegree- $(1, 1)$ -hypersurface in $\mathbf{P}^1 \times \mathbf{P}^2$, so that it is the zero set of a section of the convex line bundle $V = \mathcal{O}(1) \otimes \mathcal{O}(1)$:

$$Y = \{(\ell, d) \in \mathbf{P}^1 \times \mathbf{P}^2 \mid d \subset \ell \oplus \mathbf{C}\} = \{([a, b], [x, y, z]) \mid ay - bx = 0\}.$$

As the cohomology of Y is generated by ambient classes, the recipe above allows to compute all the Gromov-Witten invariants of Y . Let us concentrate on Ψ_{2E} .

Notice first that $j_*E = F$, the class of a fiber $\mathbf{P}^1 \times \{\text{pt}\} \subset \mathbf{P}^1 \times \mathbf{P}^2$. We must thus look at the moduli space $\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1 \times \mathbf{P}^2, 2F)$, but

$$\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1 \times \mathbf{P}^2, 2F) \cong \overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L) \times \mathbf{P}^2,$$

maps in the class $2F$ having a constant projection on \mathbf{P}^2 . For the same reason, the bundle \mathcal{V} we are looking at has the form

$$\mathcal{V}' \otimes \mathcal{O}_{\mathbf{P}^2}(1) \longrightarrow \overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L) \times \mathbf{P}^2$$

where $\mathcal{V}' \rightarrow \overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L)$ is a bundle whose fiber at $u' : C \rightarrow \mathbf{P}^1$ is $H^0(C, u'^*\mathcal{O}(1))$. It has rank 3 and

$$e(\mathcal{V}) = c_3(\mathcal{V}') \otimes 1 + c_2(\mathcal{V}') \otimes t + c_1(\mathcal{V}') \otimes t^2 \in H^6(\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1 \times \mathbf{P}^2, 2F))$$

($t = c_1(\mathcal{O}_{\mathbf{P}^2}(1))$ is the generator of $H^2(\mathbf{P}^2)$ in this relation).

The next remark is that \mathcal{V}' has two everywhere independent sections, so that $c_3(\mathcal{V}') = 0$, $c_2(\mathcal{V}') = 0$ and $e(\mathcal{V}') = c_1(\mathcal{V}') \otimes t^2$. To show this, choose two independent elements s_1 and s_2 in the dimension-2 vector space $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$. Define sections \tilde{s}_1, \tilde{s}_2 of \mathcal{V}' by

$$\tilde{s}_i(C, \vec{z}, u) = s_i \circ u.$$

Now, if $\lambda\tilde{s}_1 + \mu\tilde{s}_2$ vanishes at some point of $\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L)$ represented by (C, \vec{z}, u) ,

$$(\lambda s_1 + \mu s_2) \circ u \equiv 0$$

so that the image of u must be contained in the zero set of $\lambda s_1 + \mu s_2$, a point of \mathbf{P}^1 . Thus, u must be constant, a contradiction since it has degree 2. Thus the two sections \tilde{s}_1 and \tilde{s}_2 are indeed independent everywhere.

From this remark, it is easy to deduce the following result, which will be useful in the computation of §6.2.

5.10 PROPOSITION. — *If $\alpha \in H^*((\mathbf{P}^1 \times \mathbf{P}^2)^m)$ is divisible by t , then $\Psi_{2E}((j^m)^* \alpha) = 0$.*

Proof. — The assumption on the cohomology class α means that, if one writes

$$\alpha = \sum_i \alpha_1^i \otimes \cdots \otimes \alpha_m^i,$$

then, for every i , one of the α_j^i 's has the form $a \otimes bt$ for some $a \in H^*(\mathbf{P}^1)$, $b \in H^*(\mathbf{P}^2)$. What we want to prove is that

$$\Psi_{2E}((j^m)^* (\alpha_1 \otimes \cdots \otimes \alpha_{m-1} \otimes (a \otimes bt))) = 0$$

for all $\alpha_i \in H^*(\mathbf{P}^1 \times \mathbf{P}^2)$, $a \in H^*(\mathbf{P}^1)$, $b \in H^*(\mathbf{P}^2)$. But the latter is obtained as

$$\langle \text{ev}_m^* (\alpha_1 \otimes \cdots \otimes \alpha_{m-1} \otimes (a \otimes bt)) \smile (c_1(\mathcal{V}') \otimes t^2), [\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L) \times \mathbf{P}^2] \rangle.$$

Now, the map

$$\text{ev}_m : \overline{\mathcal{M}}_{0,m}(\mathbf{P}^1 \times \mathbf{P}^2, 2F) \longrightarrow (\mathbf{P}^1 \times \mathbf{P}^2)^m \cong (\mathbf{P}^1)^m \times (\mathbf{P}^2)^m$$

coincides with

$$\text{ev}'_m \times \Delta : \overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L) \times \mathbf{P}^2 \longrightarrow (\mathbf{P}^1)^m \times (\mathbf{P}^2)^m$$

so that, decomposing the α_i 's, the invariant is a sum of terms of the form

$$\langle [\text{ev}'_m^* (a' \otimes a) \otimes (b' \smile bt)] \smile (c_1(\mathcal{V}') \otimes t^2), [\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, 2L)] \otimes [\mathbf{P}^2] \rangle$$

which are zero since $t^3 = 0$ in $H^*(\mathbf{P}^2)$. □

Relations with symplectic “Gromov-Witten invariants”. — Let us end this § by a few remarks on the other “Gromov-Witten invariants” the readers may have met in different contexts.

Gromov-Witten invariants come from symplectic topology, various descriptions and alternative definitions for a (restricted) class of symplectic manifolds can be found in *e.g.* [39, 36] (see also the survey [2]).

It seems now that there exist symplectic and algebraic definitions available in greater generality — some using spaces of stable pseudo-holomorphic maps. See *e.g.* [42, 18, 6]. Without entering in that generality, let us make a few remarks on the symplectic approach.

Remarks.

1. In the symplectic context, Gromov-Witten invariants are invariants of the deformation class of the symplectic structure. This is where their qualification as “invariants” come from.
2. The Ruan & Tian invariants were defined using moduli spaces of rational curves that are *almost* holomorphic with respect to some *almost* complex structure on the symplectic manifold X : the almost complex structure J defines a $\bar{\partial}$ -operator $\bar{\partial}_J$, holomorphic curves satisfy $\bar{\partial}_J u = 0$ and you perturb the equation as $\bar{\partial}_J u = \nu$. This allows you to use transversality arguments to prove *e.g.* smoothness properties (but of course, you may lose the enumerative interpretation of your invariants).

This approach by perturbations should be compared with the use of the Euler class above.

3. It seems to me that one should be able to use both approaches at the same time: the symplectic approach is more flexible — but, dealing with enumerative questions, one does not like to vary the complex structure.

6. Degree-2 classes and the small quantum product

Notation. — In this section, I will prefer to use the pairing $\langle \alpha, b \rangle$ between homology and cohomology and Poincaré duality as an isomorphism between homology and cohomology instead of the metric g . To save on notation, I will use greek characters for cohomology classes and latin for homology classes, Poincaré duality exchanging α and a etc. With this convention,

$$\langle \alpha, b \rangle = g(\alpha, \beta).$$

6.1. Quantum multiplication at a given point *vs* formal series

As we have noticed it above (see § 5.2), if ξ is a degree-2 class,

$$\langle \alpha \star_{\xi} \beta, c \rangle = \sum_A \Psi_A(\alpha \otimes \beta \otimes \gamma) \exp\langle \xi, A \rangle,$$

an expression which depends only of the class $[\xi]$ of ξ in $H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$. Recall from § 5.2 that the small quantum product is given by

$$\langle \alpha \star \beta, c \rangle = \sum_A \Psi_A(\alpha \otimes \beta \otimes \gamma) q^A$$

which is exactly the same thing (write $q^A = \exp\langle \xi, A \rangle$). In other words, to compute $\alpha \star_{\xi} \beta$ in $H^*(X; \mathbf{C})$ amounts to the same as to compute $\alpha \star \beta$ in $H^*(X) \otimes \Lambda$ and to specialise it at the corresponding value of q .

Remarks.

- One can define in the same mood a variant of the global quantum product: in the formula for $\star_{\xi, q}$, replace q^A by $\exp\langle \omega, A \rangle$ for some fixed class ω , usually the cohomology class of the symplectic form.
- We do not really need that the manifold X be convex: we will use the generalised invariants defined in § 5.5. See for instance the example of the plane blown-up at a point in § 6.2.

Products. — The Gromov-Witten potential of a product $X_1 \times X_2$ is not easily expressible in terms of the potentials of the factors (see [33]). However, the small quantum cohomology ring behaves very simply with respect to products¹³.

6.1 PROPOSITION. $QH^*(X_1 \times X_2) \cong QH^*(X_1) \otimes QH^*(X_2)$.

Proof. — In general, although to give a map $u : C \rightarrow X_1 \times X_2$ is equivalent to giving its components (u_1, u_2) , because of the marked points and the stability requirement, there is no obvious relation between spaces $\overline{\mathcal{M}}_{0,m}(X_1 \times X_2, A)$ and spaces of stable maps into the factors. However, there is a map

$$\overline{\mathcal{M}}_{0,m}(X_1 \times X_2, A_1 \otimes 1 + 1 \otimes A_2) \longrightarrow \overline{\mathcal{M}}_{0,m}(X_1, A_1) \times \overline{\mathcal{M}}_{0,m}(X_2, A_2)$$

taking $(C, \vec{z}, u = (u_1, u_2))$ to the pair of stable maps obtained by contraction of unstable components in the factors. It is obviously not onto, but its image contains the set of pairs $((C_1, \vec{z}_1, u_1), (C_2, \vec{z}_2, u_2))$ such that (C_2, \vec{z}_2) is isomorphic to (C_1, \vec{z}_1) . On this subspace, it is even possible to define an inverse map.

Now, when the number of marked points is 3, our image contains the open set where C_1 and C_2 are irreducible curves, so that there is a birational map

$$\overline{\mathcal{M}}_{0,3}(X_1, A_1) \times \overline{\mathcal{M}}_{0,3}(X_2, A_2) \longrightarrow \overline{\mathcal{M}}_{0,3}(X_1 \times X_2, A_1 \otimes 1 + 1 \otimes A_2).$$

It is defined on the subset of $\overline{\mathcal{M}}_{0,3}(X_1, A_1) \times \overline{\mathcal{M}}_{0,3}(X_2, A_2)$ consisting of pairs of stable maps $((C_1, \vec{z}_1, u_1), (C_2, \vec{z}_2, u_2))$ such that (C_2, \vec{z}_2) is isomorphic to (C_1, \vec{z}_1) , which contains the open set where C_1 and C_2 are irreducible curves. It allows to relate fundamental classes and to prove that

$$\Psi_{A_1 \otimes 1 + 1 \otimes A_2}((\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2) \otimes (\gamma_1 \otimes \gamma_2)) = \Psi_{A_1}(\alpha_1 \otimes \beta_1 \otimes \gamma_1) \Psi_{A_2}(\alpha_2 \otimes \beta_2 \otimes \gamma_2).$$

Thus

$$\langle (\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2), c_1 \otimes c_2 \rangle = \left(\sum \Psi_{A_1}(\alpha_1 \otimes \beta_1 \otimes \gamma_1) q_1^{A_1} \right) \left(\sum \Psi_{A_2}(\alpha_2 \otimes \beta_2 \otimes \gamma_2) q_2^{A_2} \right)$$

and the proposition follows. \square

Symplectic reduction. — Consider the submanifold

$$B = H^2(X; \mathbf{C}) \xrightarrow{j} M = \oplus H^{2k}(X; \mathbf{C}).$$

We want to show that, in some cases, the small quantum product defines a Lagrangian subvariety of T^*B . This will be achieved by a very general symplectic reduction process.

Assume more generally that $j : B \rightarrow M$ is the inclusion of a submanifold, so that j^*T^*M is a co-isotropic subvariety of T^*M , and that the tangent sheaf to M is endowed with a product \star_ξ such that the spectral cover is a Lagrangian subvariety of T^*M . At least at semi-simple points, the spectral cover is locally the graph of a (closed) 1-form and thus is transversal to j^*T^*M .

It is a simple form of the symplectic reduction process that the intersection $L \cap (j^*T^*M)$ projects to a Lagrangian subvariety $L_B \subset T^*B$. Of course, we have:

6.2 PROPOSITION. — *The Lagrangian subvariety L_B is the spectral Lagrangian in T^*B associated with the morphism $j^*\Omega : TB \rightarrow \text{End}(j^*TM)$ over B .*

¹³This is a good place to mention that there is no obvious functoriality property in this theory.

Proof. — The spectral cover for

$$j^*\Omega : TB \longrightarrow \text{End}(j^*E)$$

is defined by the eigenvalues $\mu_i(\alpha, \xi)$ of $\Omega_\xi(\alpha)$ for $\xi \in B$, $\alpha \in T_\xi B \subset T_\xi M$. In other words, one considers the section μ_i of T^*M as a section of T^*B by restriction and this is the definition of the symplectic reduction explained above. \square

In the specific situation considered here: to look at j^*T^*M is the same thing as to look at the product \star_ξ for $\xi \in H^2$, that is, the small quantum product. The intersection $L \cap j^*T^*M$ thus describes the relations in the quantum cohomology ring $QH^*(X)$.

Let us make now a crucial assumption on the manifold X . We assume that the (classical) cohomology algebra $H^*(X)$ is generated by $H^2(X)$, its degree-2 part. Denoting by $S^*(X) = S[H^2(X; \mathbf{C})]$ the symmetric algebra on $H^2(X; \mathbf{C})$, this means that the natural ring morphism

$$S^*(X) \longrightarrow H^*(X)$$

is onto. This is the case, *e.g.* for projective spaces and more generally toric manifolds, complete flag manifolds...

Using the notation of [24], call (p_1, \dots, p_k) a basis of $H^2(X)$. The small quantum cohomology ring $QH^*(X) = (H^*(X; \mathbf{C}) \otimes \Lambda, \star)$ then consists of polynomials in p and q , with some relations. In other words, we have a surjective homomorphism:

$$S^*(H^2(X; \mathbf{C})) \otimes \mathbf{C}[H_2(X; \mathbf{Z})] \longrightarrow QH^*(X),$$

the kernel of which will be denoted by \mathcal{J} . The ring in the LHS is nothing other than the ring of regular functions on the cotangent bundle $T^*\mathcal{B}$ of the torus $\mathcal{B} = H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$. Notice that, with this remarkable notation, the symplectic form on $T^*\mathcal{B}$ is $\sum dp_i \wedge \frac{dq_i}{q_i}$.

As the reduction mod $2i\pi\mathbf{Z}$ is a covering map $B \rightarrow \mathcal{B}$ we get:

6.3 COROLLARY (Givental & Kim [24]). — *Assume X is a projective Fano manifold whose cohomology ring is generated by degree-2 classes. Assume that, for some value of q , the quantum product gives $H^*(X)$ the structure of a semi-simple ring. Then $QH^*(X)$ is the ring of functions on a Lagrangian subvariety of the cotangent bundle $T^*\mathcal{B}$ of the torus $\mathcal{B} = H^2(X; \mathbf{C}/2i\pi\mathbf{Z})$. \square*

Remark. — In the context of degree-2 classes, Givental & Kim [24] have noticed that the homogeneity property with respect to the Euler vector field implies that the Lagrangian is quasi-homogeneous — and that the first Chern class can be considered as a primitive of the Liouville form (see also [3]).

6.2. Examples of quantum cohomology rings

Here are a few examples. The varieties considered are Fano. Moreover, their cohomology ring is generated by degree-2 classes. The first example here is very well known, but difficult to avoid.

The complex projective space. — This is of course a convex manifold. Let p be the generator of the cohomology of \mathbf{P}^n , so that

$$H^*(\mathbf{P}^n; \mathbf{Z}) = \mathbf{Z}[p]/p^{n+1} \text{ and } c_1 = (n+1)p.$$

We must add an invertible variable q of degree $2(n+1)$. Then

$$QH^*(\mathbf{P}^n) = \mathbf{C}[p]/p^{n+1} \otimes \mathbf{C}[q, q^{-1}]$$

as a $\mathbf{C}[q, q^{-1}]$ -module and the quantum multiplication is $\mathbf{C}[q, q^{-1}]$ -linear. As the degree of q is $2(n+1)$, $p^{*k} = p^k$ for $k \leq n$ and we need only to compute $p^{*(n+1)} = (p^n) \star p$. But

$$\langle p^n \star p, c \rangle = \sum_d \Psi_{dL}(p^n \otimes p \otimes \gamma) q^{dL}.$$

Recall that

$$\Psi_{dL}(p^n \otimes p \otimes \gamma) = \left\langle \text{ev}_3^*(p^n \otimes p \otimes \gamma), [\overline{\mathcal{M}}_{0,3}(\mathbf{P}^n, dL)] \right\rangle$$

so that it can only be non zero when $\deg \gamma = (n+1)d - 1$. As $0 \leq \deg \gamma \leq n$, this allows only $d = 1$ and $\deg \gamma = n$. Now we are looking at $\overline{\mathcal{M}}_{0,3}(\mathbf{P}^n, L)$ and at $\Psi_L(p^n \otimes p \otimes p^n)$. But we have already noticed in §5.2 that this is the number of lines in \mathbf{P}^n through two general points that meet a general hyperplane somewhere, that is, 1. In conclusion:

6.4 PROPOSITION. — *The small quantum cohomology ring $QH^*(\mathbf{P}^n(\mathbf{C}))$ is isomorphic to the ring $\mathbf{Z}[p, q, q^{-1}]/(p^{n+1} - q)$.*

Remark. — In this example the multiplication is obviously generically semi-simple and the Lagrangian is the curve $p^{n+1} = q$ in $\mathbf{C} \times \mathbf{C}^*$.

The plane blown up at a point. — Let us come back to the manifold Y obtained by blowing up a point in the complex projective plane. This is one of the examples investigated in [13, 39, 2] for instance. Figure 10 shows Y as the \mathbf{P}^1 -bundle $\mathbf{P}(\mathcal{O}(1) \oplus \mathbf{1})$ over \mathbf{P}^1 . All properties of homology and cohomology classes I will write are easily derived from this description. Recall from Proposition 4.7 that Y is *not* a convex manifold.

Let A_1 be the class of the exceptional divisor (it was called E in §5.5) and A_2 be the class of a line through the blown-up point, or of a fiber (see Figure 10). The two classes A_1 and A_2 form a basis of $H_2(Y; \mathbf{Z})$ with

$$A_1 \cdot A_1 = -1, \quad A_1 \cdot A_2 = 1, \quad A_2 \cdot A_2 = 0.$$

Let (p_1, p_2) be the basis of $H^2(Y; \mathbf{Z})$ dual to (A_1, A_2) (that is, such that $\langle p_i, A_j \rangle = \delta_{i,j}$), so that, denoting again Poincaré duality by D ,

$$Dp_1 = A_2, \quad Dp_2 = A_1 + A_2$$

(as $(Dp_i) \cdot A_j = \delta_{i,j}$). Also,

$$\langle p_i \smile p_j, [Y] \rangle = (Dp_i) \cdot (Dp_j)$$

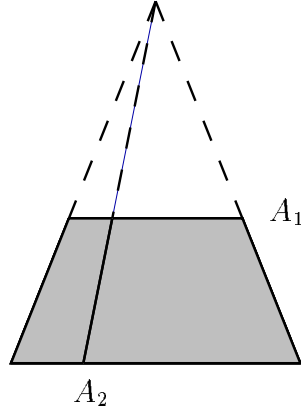


Figure 10: the plane blown up at a point

so that $p_1^2 = 0$, $\langle p_1 \smile p_2, [Y] \rangle = \langle p_2 \smile p_2, [Y] \rangle = 1$. The (classical) cohomology ring is

$$H^*(Y; \mathbf{Z}) = \mathbf{Z}[p_1, p_2] / \langle p_1^2, p_2^2 - p_1 p_2 \rangle.$$

The first Chern class is $p_1 + 2p_2$. Let us add two new variables q_1 and q_2 of respective degrees 2 and 4.

To determine the structure of the quantum cohomology ring, we need only to calculate the products $p_i \star p_j$. For this, we need all the invariants $\Psi_A(p_i \otimes p_j \otimes \gamma)$, for all homology classes A containing holomorphic curves and all cohomology classes γ such that $2\langle c_1, A \rangle = \deg \gamma$. This implies in particular that $0 \leq \langle c_1, A \rangle \leq 2$. Moreover, if a non zero class A can be represented by a rational map $\mathbf{P}^1 \rightarrow Y$, it is a multiple of a class A' that can be represented by a somewhere injective rational map. But these must satisfy the adjunction formula, namely

$$A' \cdot A' - \langle c_1, A' \rangle + 2 \geq 0.$$

Writing $A' = m'_1 A_1 + m'_2 A_2$, this writes

$$(1 - m'_1)(m'_1 - 2m'_2 + 2) \geq 0.$$

From these restrictions, it is seen that the only invariants that we need to compute are Ψ_0 , Ψ_{A_1} , Ψ_{A_2} and Ψ_{2A_1} (this is the reason why I have chosen this basis of $H_2(Y; \mathbf{Z})$).

Recall that the class $2A_1$ is the one which prevents Y to be convex (see the proof of Proposition 4.7). However, a consequence of Proposition 5.10 is:

6.5 LEMMA. $\Psi_{2A_1}(p_i \otimes p_j \otimes \gamma) = 0$ for all i, j, γ .

Proof. — As p_i and p_j have degree 2 and $\langle c_1, 2A_1 \rangle = 2$, the invariant $\Psi_{2A_1}(p_i \otimes p_j \otimes \gamma)$ can only be non zero when γ has degree 4. Now the only non zero classes in $H^4(Y)$ have the form $j^*(a \otimes t)$ for some $a \in H^2(\mathbf{P}^1)$ and we can apply Proposition 5.10. \square

We are left with the computation of Ψ_{A_1} and Ψ_{A_2} .

- Consider first $\Psi_{A_1}(p_i \otimes p_j \otimes \gamma)$. We know that this is zero unless $\deg \gamma = 2$ (so that all the classes involved have degree 2) and that there is only one curve in the class A_1 . Thus

$$\Psi_{A_1}(p_i \otimes p_j \otimes \gamma) = \delta_{i,1} \delta_{j,1} \langle \gamma, A_1 \rangle.$$

- Look now at $\Psi_{A_2}(p_i \otimes p_j \otimes \gamma)$. It vanishes unless $\deg \gamma = 4$, so that we must only consider the case where γ is Poincaré dual to a point. Now, there is exactly one curve of the class A_2 through any given point of Y and thus

$$\Psi_{A_2}(p_i \otimes p_j \otimes D[\text{pt}]) = \delta_{i,2} \delta_{j,2}.$$

We can now calculate our monomials $p_i \star p_j$. By definition

$$\langle p_i \star p_j, c \rangle = \langle p_i \smile p_j, c \rangle + \Psi_{A_1}(p_i \otimes p_j \otimes \gamma)q_1 + \Psi_{A_2}(p_i \otimes p_j \otimes \gamma)q_2.$$

- $\langle p_1 \star p_1, c \rangle = (c \cdot A_1)q_1 = \langle DA_1, c \rangle q_1$ and thus

$$p_1 \star p_1 = (p_2 - p_1)q_1,$$

- $\langle p_1 \star p_2, c \rangle = \langle p_1 \smile p_2, c \rangle$, so that

$$p_1 \star p_2 = p_1 \smile p_2,$$

- $\langle p_2 \star p_2, c \rangle = \langle p_2 \smile p_2, c \rangle + \Psi_{A_2}(p_2 \otimes p_2 \otimes \gamma)q_2$ and

$$p_2 \star p_2 = p_2 \smile p_2 + q_2.$$

This gives the desired result:

6.6 PROPOSITION. — *The small quantum cohomology ring of the plane blown up at a point is isomorphic to the ring*

$$\mathbf{Z}[p_1, p_2, q_1, q_1^{-1}, q_2, q_2^{-1}] / \langle p_1^2 - (p_2 - p_1)q_1, p_2^2 - p_1p_2 - q_2 \rangle.$$

□

Remarks.

- That the ideal generated by our two relations $f = p_1^2 - (p_2 - p_1)q_1$ and $g = p_2^2 - p_1p_2 - q_2$ defines a Lagrangian subvariety of $T^*(\mathbf{C}^*)^2$ is easily checked, as $\{f, g\} = q_1g$ is in the ideal.
- Proposition 6.6 appears in [13] (where the basis used is not symplectic) and in [39] (in a different basis¹⁴).
- There is also a proof of Proposition 6.6 using (a less convenient basis and) symplectic invariants in [2]. There the class $2A_1$ was avoided using spaces of “somewhere injective” holomorphic curves as in [36], which gives in this case the same invariants as the “almost” holomorphic curves of [39].

Appendix

A. Elliptic curves and plane cubics

In this appendix, I try to motivate the introduction of the spaces of stable maps by a description of the difference between (abstract) elliptic curves and plane cubics.

¹⁴and with a wrong sign.

Elliptic curves. — An (abstract) elliptic curve is, by definition, a smooth genus-1 complex curve C with a preferred point P on it. To say that the genus is one is to say that the dimension of the complex vector space $H^0(\Omega_C^1)$ of holomorphic 1-forms is one. According to the Abel-Jacobi theorem, there is an isomorphism

$$\begin{aligned} C &\longrightarrow H^0(\Omega_C^1)^*/\Lambda \\ Q &\longmapsto \left(\omega \mapsto \int_P^Q \omega \right) \end{aligned}$$

where Λ is the period lattice, *i.e.* the isomorphic image of $H_1(C; \mathbf{Z}) \cong \mathbf{Z}^2$ in $H^0(\Omega_C^1)^* \cong \mathbf{C}$. The marked point P is sent to 0.

The space of isomorphism classes of elliptic curves can then be described as an appropriate quotient of the space of lattices in \mathbf{C} .

Plane cubics. — To any such lattice is associated a Weierstrass \wp -function. Together with its derivative, it defines an embedding

$$\begin{aligned} \mathbf{C}/\Lambda &\longrightarrow \mathbf{P}^2(\mathbf{C}) \\ z &\longmapsto [\wp(z), \wp'(z), 1]. \end{aligned}$$

Due to the differential equation satisfied by \wp , our elliptic curve is a plane cubic, with (affine) equation

$$C' : \quad y^2 = 4x^3 - g_2x - g_3$$

where g_2 and g_3 are complex constants defined by Λ , satisfying $g_2^3 - 27g_3^2 \neq 0$.

Recall that \wp has a double pole at 0, so that the marked point P , alias 0, is sent to $[0, 1, 0]$, the point at ∞ on the curve C' .

The moduli space of elliptic curves. — It turns out that, up to isomorphism, the elliptic curve (C, P) is determined by a single numerical invariant, which can be easily expressed in terms of the constants g_2 and g_3 above:

$$j = \frac{1728g_2^3}{g_2^3 - 27g_3^2}.$$

For instance, t being fixed and ε varying, all the curves of equations

$$\Gamma_\varepsilon^t : \quad y^2 = 4x^3 - \varepsilon^2x - \varepsilon^3t$$

should be isomorphic, as they have the same j -invariant (namely $1728/(1 - 27t^2)$), and they actually are, since $(x, y) \mapsto (\alpha^2x, \alpha^3y)$ extends to an isomorphism $\Gamma_1^t \rightarrow \Gamma_{\alpha^2}^t$.

Singular cubics. — There are of course a lot of different singular plane cubic curves: a conic plus a line, three lines, and so on. Among the cubics having an affine equation of the form

$$y^2 = 4x^3 - g_2x - g_3$$

there are curves with an ordinary double point ($g_2^3 - 27g_3^2 \neq 0$, g_2 and $g_3 \neq 0$), or with a cusp ($g_2 = g_3 = 0$). Notice that these are rational curves, being parametrised by the set of lines through the singular point.

Compactification of the moduli space. — Once we have accepted that the moduli space of elliptic curves is isomorphic¹⁵ to \mathbf{C} (via j), it seems to be easy to compactify: just add a point at infinity, allowing $j = \infty$.

This way, you certainly add the curve with a double point. . . but you cannot “add the curve with a cusp point”: because of the families Γ_ε^t , all converging to $y^2 = 4x^3$ when ε goes to 0, this curve would be arbitrarily close to *all* elliptic curves.

Notice that $j = \infty$ for curves with a double point, but also that the rational function of g_2 and g_3 has an “indeterminate form” $0/0$ for the cusp curve, and that this is why we were able to get any j :

$$j(\Gamma_\varepsilon^t) = \frac{1728}{1 - 27t^2} \text{ for } \varepsilon \neq 0$$

thus

$$\lim_{\varepsilon \rightarrow 0} j(\Gamma_\varepsilon^t) = j(\Gamma_1^t) = \frac{1728}{1 - 27t^2}.$$

Notice that Theorem 4.4 describes exactly what happens in the family Γ_ε^t when $\varepsilon \rightarrow 0$: the genus tends to be concentrated at the singular point. In other words, the map u mentioned in Theorem 4.4 is constant, its graph is horizontal, and there is a bubble whose image is the whole rational cuspidal cubic.

The idea of stable maps is to construct, starting from the space of *maps* of genus-1 curves to the complex projective plane, a space that contains parametrisations of cusp curves taking into account the phenomenon just described.

B. Algebraic loop spaces and configuration spaces

In this appendix, I will give examples of spaces of stable maps from a *fixed* curve to X . For simplicity, I will consider only the case of rational curves.

The examples discussed here are not only beautiful examples, they are very useful in Givental’s approach to the mirror conjecture [21, 22, 23]. For they provide a compactification of the algebraic loop space of a projective manifold.

We start with *algebraic loops* in X , represented as maps $\mathbf{P}^1 \rightarrow X$. . . in other words as parametrised rational curves.

We then fix the homology class and consider the space $L_A X$ of maps $\mathbf{P}^1 \rightarrow X$ in the class A . This space looks very much like $\mathcal{M}_{0,0}(X, A)$ — except that, in $\mathcal{M}_{0,0}(X, A)$, we have quotiented by isomorphisms, thus losing part of the parametrisation. To keep track of the parametrisation, it is of course sufficient to consider the graph of the maps, that is, to look at the embedding

$$L_A X \subset \overline{\mathcal{M}}_{0,0}(\mathbf{P}^1 \times X, L \oplus A).$$

We see that the moduli spaces of stable maps include compactifications of the spaces of algebraic loops.

¹⁵There should be some structure to give a sense to this word. The whole space $\overline{\mathcal{M}}_{1,1} = \mathbf{C} \cup \{\infty\}$ can be given an algebraic structure. It looks like \mathbf{P}^1 but has orbifold-type singularities at points corresponding to elliptic curves which have a non trivial group of automorphisms, namely for $j = 0, 1728$ and ∞ .

Example. — Take $X = \mathbf{P}^n(\mathbf{C})$ and $A = dL$, so that we are looking at degree- d algebraic loops in the projective space. An element in $L_d\mathbf{P}^n$ can be represented by $n + 1$ degree- d polynomials without common root (modulo multiplication by a scalar). Therefore our space $L_d\mathbf{P}^n$ has an obvious, “naïve”, compactification as a projective space $\mathbf{P}^{(n+1)(d-1)-1}$. We thus have two distinct compactifications:

- The naïve one, $L'_A\mathbf{P}^n = \mathbf{P}^{(n+1)(d-1)-1}$ ($A = dL$), where we are simply allowed to consider polynomials of degree $\leq d$.
- The clever one, $\overline{L}_A\mathbf{P}^n = \overline{\mathcal{M}}_{0,0}(\mathbf{P}^1 \times X, L \oplus A)$, where we allow graphs of degree- $\leq d$ maps $\mathbf{P}^1 \rightarrow \mathbf{P}^n$ with vertical components, their total degree being d .

Notice that there exists a map

$$\mu : \overline{L}_A\mathbf{P}^n \longrightarrow L'_A\mathbf{P}^n,$$

used by Givental in [23], that is a morphism, whose set-theoretical definition is the following. Let $u : \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^n$ be a bidegree- $(1, d)$ stable map (representing an element of $\overline{L}_A\mathbf{P}^n$). All components of Σ are copies of \mathbf{P}^1 . One of them, C_0 , say, which is sent by u onto the graph of a degree- d' map ($d' \leq d$) from \mathbf{P}^1 to \mathbf{P}^n . It can be represented by an $(n + 1)$ -tuple (f_0, \dots, f_n) of degree- d' polynomials. The other components, C_1, \dots, C_r are bubbles, that is, C_i is a rational curve, u has bidegree $(0, d_i)$ on C_i and sends C_i to $z_i \times \mathbf{P}^n$ for some point z_i in \mathbf{P}^1 ($d_1 + \dots + d_r = d - d'$).

Consider a degree- $(d - d')$ polynomial g , that vanishes at each z_i with multiplicity d_i and associate to the stable map u the $(n + 1)$ -tuple of polynomials $(f_0g, \dots, f_n g)$, that is an element of $L'_A\mathbf{P}^n$.

Configuration spaces. — In this situation, there is an analogue of the contraction morphism considered in § 4.2, the natural map

$$\overline{\mathcal{M}}_{0,m}(\mathbf{P}^1 \times X, L \oplus A) \longrightarrow \overline{\mathcal{M}}_{0,m}(\mathbf{P}^1, L).$$

We have already met $\overline{\mathcal{M}}_{0,0}(\mathbf{P}^1, L)$ and noticed that this was a single point. In the same way, for each degree-1 map

$$u : \Sigma \longrightarrow \mathbf{P}^1,$$

there exists a distinguished component C_0 of Σ . This is the only component on which u is non constant, and where the map u is an isomorphism $C_0 \rightarrow \mathbf{P}^1$. Using an isomorphism of Σ , we can assume that $u = \text{Id}$ on C_0 , so that we are left with a compactification of the configuration space of m distinct points on \mathbf{P}^1 (without quotient).

Such a compactification is considered in greater generality in [19] for a general projective variety Y instead of \mathbf{P}^1 . In the \mathbf{P}^1 -case, it is easily seen that the compactification as moduli space of stable maps and the Fulton-MacPherson compactification coincide (the description by “screens” in [19] actually coincides with that of reducible curves we are using here).

C. Poisson commuting relations

The Lagrangian subvariety defined by an unfolding has very specific properties:

C.1 PROPOSITION. — *Let L be the Lagrangian subvariety of $T^*\mathbf{C}^N$ defined by the polynomial generating family F_ξ . This is a smooth complete intersection. Moreover, there exists a system of generators (H_1, \dots, H_N) of the ideal $I(L)$ of polynomials vanishing on L which is a completely integrable system.*

Proof. — Call $(z_1, \dots, z_k, z_{k+1}, \dots, z_N)$ the coordinates dual to (ξ_1, \dots, ξ_N) , so that, choosing the generators of Q_f as in § 1.2, the Lagrangian L is described by the N equations

$$\begin{cases} \frac{\partial F_\xi}{\partial z_i}(z, \xi) = 0 & 1 \leq i \leq k \\ z_j - g_j(z) = 0 & k+1 \leq j \leq N \end{cases}$$

(here z denotes the k -uple (z_1, \dots, z_k)). Thus L is a complete intersection.

Call these functions H_i ($1 \leq i \leq N$), that is,

$$H_i(z, \xi) = \frac{\partial F_\xi}{\partial z_i}(z, \xi) = \frac{\partial f}{\partial z_i}(z) + \sum_{\ell=k+1}^N \xi_\ell \frac{\partial g_\ell}{\partial z_i}(z)$$

for $1 \leq i \leq k$, and

$$H_{k+j}(z, \xi) = z_{k+j} - g_{k+j}(z)$$

for $k+1 \leq k+j \leq N$. A straightforward computation shows that the Poisson bracket of any two of them vanishes. \square

We have already met (*e.g.* in § 2.3) examples of non smooth spectral covers. The “Liouville integrability” property has the flavour of the Givental & Kim theorem on the quantum cohomology of flag manifolds¹⁶. Recall that they have found that, in this case, the Lagrangian subvariety was a (singular) common level set of first integrals of the Toda lattice. Notice that Corollary 6.3 says that the ideal \mathcal{J} is stable under Poisson bracket:

$$f, g \in \mathcal{J} \Rightarrow \{f, g\} \in \mathcal{J}.$$

In the case of flag manifolds, the property they get is much stronger: there exists a system (f_1, \dots, f_k) of generators of \mathcal{J} which Poisson commute.

This is not the case in general: one can check for instance that the ideal defining the quantum cohomology of the plane blown-up at a point cannot be generated by two Poisson commuting elements (this is easily done using Proposition 6.6 and the computation of the Poisson bracket $\{f, g\}$ in the remark following it).

Moreover, this property of being defined by an integrable system would imply that L_B is a complete intersection and it is quite easy to find examples in which L_B is not, *e.g.* \mathbf{P}^2 blown-up at two points. This is left as an exercise for the readers: one does not need to compute the quantum cohomology ring completely, but only to check that the ideal defining L_B needs five generators, although L_B has codimension 3 in $T^*(\mathbf{C}^*)^3$.

¹⁶Corollary 6.3 was proved by Givental & Kim as a comment of this theorem.

D. The semi-simplicity assumption

One of the reasons why massive Frobenius manifolds were interesting for Dubrovin is that they parametrise isomonodromic deformations (see [17, 27, 40, 35]).

One cannot expect the quantum cohomology of a projective manifold to be massive in general: if the manifold contains no rational curve (think of a $K3$ surface), all its quantum products are nilpotent. Better candidates for semi-simplicity are the Fano manifolds, as they contain many rational curves. The semi-simplicity of the quantum cohomology of Fano manifolds is conjectured by Tian [45] and Manin [35].

Recall (see §3.1) that the semi-simplicity the ring $(T_\xi M, \star_\xi)$ implies the existence of *canonical coordinates* on a neighbourhood of the point ξ in the Frobenius manifold M . In these coordinates, the product is split and the Euler vector field can be written

$$\mathcal{E}(\xi) = \sum_i x_i \frac{\partial}{\partial x_i}.$$

Notice that, by definition, the x_i 's are then the eigenvalues of the multiplication by $\mathcal{E}(\xi)$ at ξ . All the data being analytic, this implies:

D.1 PROPOSITION. — *For a Frobenius manifold, the following properties are equivalent*

1. *The ring $(T_\xi M, \star_\xi)$ is semi-simple for generic ξ .*
2. *There exists a point ξ in M such that the ring $(T_\xi M, \star_\xi)$ is semi-simple.*
3. *There exists a point ξ in M and a vector α in $T_\xi M$ such that all the eigenvalues of the multiplication by α are simple.*
4. *Multiplication by $\mathcal{E}(\xi)$ at ξ has only simple eigenvalues for generic ξ .* □

Notice also that semi-simplicity of the small quantum cohomology ring (for generic q) would imply semi-simplicity of global quantum cohomology. For instance:

D.2 PROPOSITION. — *The quantum cohomologies of \mathbf{P}^n , $\widetilde{\mathbf{P}}^2$ are massive Frobenius manifolds.*

Proof. — It is easily checked that the small quantum cohomology rings computed in §6.2 are semi-simple for generic q . The minimal polynomial of multiplication by p in $QH^*(\mathbf{P}^n)$ is $z^{n+1} - q$, that of multiplication by p_1 in $QH^*(\widetilde{\mathbf{P}}^2)$ is $z^4 + q_1 z^3 + q_1^2 q_2$. Both coincide with the characteristic polynomial and both are split for generic q . □

But there are examples for which the small quantum cohomology ring is not sufficient to check the semi-simplicity.

D.3 PROPOSITION. — *Let $X^n \subset \mathbf{P}^{n+r}$ be a smooth complete intersection of dimension $n \geq 3$ and of degree (d_1, \dots, d_r) . Assume*

$$\sum_{i=1}^r (d_i - 1) \leq \frac{n+1}{2}.$$

There is a value of q for which the small quantum cohomology ring of X is semi-simple if and only if X is a linear subspace or a quadric (that is, $\sum (d_i - 1) = 0$ or 1).

Proof. — This is based on Beauville’s computation. According to the Lefschetz theorem, the complex¹⁷ cohomology of X is generated by the class dual to the hyperplane section, $\eta \in H^2(X; \mathbf{C})$ and the primitive cohomology $H_0^n(X; \mathbf{C})$. The first Chern class is $k\eta$, ($k = n + 1 - \sum(d_i - 1)$). According to [5], η satisfies a relation

$$\eta^{n+1} - aq\eta^{n+1-k} = 0$$

for some (non zero) integer a .

In the classical cohomology ring, the minimal polynomial of $\eta \smile \cdot$ is z^{n+1} , so that this shows that the minimal polynomial of $\eta \star \cdot$ is $z^{n+1} - aqz^{n+1-k}$. This polynomial has a multiple root for all q as soon as $n + 1 - k \geq 2$ — that is, $\sum(d_i - 1) \geq 2$. In this case, $\eta \star \cdot$ is *not* diagonalisable... and this prevents $QH^*(X)$ to be semi-simple (for all values of q).

When $\sum(d_i - 1) = 0$, $X = \mathbf{P}^n(\mathbf{C})$ and we have already mentioned above that $QH^*(X)$ was (generically) semi-simple. We are left with the case of quadrics.

If n is odd, the primitive cohomology is zero, so that $z^{n+1} - aqz$ is also the minimal polynomial and all the eigenvalues of $\eta \star \cdot$ are simple for $q \neq 0$.

If n is even, $H_0^n(X)$ is 1-dimensional, generated by the dual to the generator of $H_n(X_{\text{aff}})$ (affine quadric). Still using [5], we know that, for $\alpha \in H_0^n(X)$,

$$\begin{cases} \eta \star \alpha = 0 & \text{as for } \eta \smile \alpha \\ \alpha \star \alpha = b(\eta^n - aq) & \text{for some non-zero } b \text{ if } \alpha \neq 0. \end{cases}$$

Thus $\eta \star \cdot$ is diagonalisable (its minimal polynomial is split) but it has 0 as a double eigenvalue. The corresponding eigenspace (kernel) is generated by $\eta^n - aq$ and a generator α of $H_0^n(X)$. However, quantum multiplication by α , restricted to this kernel, has two distinct eigenvalues for $q \neq 0$, since

$$\begin{cases} \alpha \star (\eta^n - aq) = -aq\alpha \\ \alpha \star \alpha = b(\eta^n - aq). \end{cases}$$

□

Remarks.

- The result for \mathbf{P}^n and quadrics also holds of course for $n = 2$ as well.
- The proposition gives *e.g.* the semi-simplicity of the small quantum cohomology ring of $G_2(\mathbf{C}^4)$ (using Plücker, this is a quadric in \mathbf{P}^5). See [8, 43, 50, 20] for the computation of the small quantum cohomology ring of Grassmannians, [24, 10] for that of flag manifolds.

Thus, there is no hope that the small quantum cohomology ring be semi-simple in general, even for Fano complete intersections. However, for the global quantum cohomology of the same complete intersections, one can prove:

D.4 PROPOSITION. — *Let X be a smooth complete intersection of odd dimension $n \geq 3$. Let $\eta \in H^2(X; \mathbf{Z})$ be the generator and assume that the degree of X is such that the first Chern class $k\eta$ of X satisfies*

$$\frac{n+1}{2} \leq k \leq n+1.$$

There is a point ξ in $H^(X)$ such that all the eigenvalues of $\eta \star_{\xi} \cdot$ are simple.* □

¹⁷rational is enough

As a consequence (recall we consider only the even part of the cohomology), the quantum cohomology of such an odd dimensional complete intersection is massive. The proof is based on a simple approximation argument that I hope to make more explicit elsewhere. Notice that the massivity of the quantum cohomology for these examples can also be deduced from the computation of [46].

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