Foliations of surfaces and semi-Markovian subsets of subshifts of finite type.

Athanase Papadopoulos

Foliations of surfaces and semi-Markovian subsets of subshifts of finite type

Athanase Papadopoulos

School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540 USA

and

Institut de Recherche Mathématique Avancée *
CNRS et Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex France.

Abstract.— Let S be a closed surface of genus $g \geq 2$. In this paper, we consider a space, which we call $\mathcal{F}$, of equivalence classes of measured foliations of S, defined as the quotient of Thurston's measured foliation space where one forgets the transverse measure associated to a measured foliation. We give a presentation, in the sense of symbolic dynamics, of the action of a pseudo-Anosov mapping class of M in the neighborhood of its attracting fixed point in $\mathcal{F}$. The action is semi-Markovian. The elements of the combinatorics associated to the presentation consist in an invariant train track with a marking on its set of vertices and a certain number of elementary moves on it.

AMS Classification: 58F03, 54H20 57N05, 57R30

Keywords: Bernoulli shift, foliation, pseudo-Anosov, mapping class group, measured foliation, semi-Markovian subset, subshift of finite type, symbolic dynamics, train track.

---

* Permanent Address

§0.—Introduction

In this paper, we show that the theory of RLS expansions for measured foliations, as described in the paper [8] by R. Penner and the author, provides the elements of a semi-Markovian presentation (in the sense which Gromov uses in [3]) for an action of a pseudo-Anosov mapping class of a surface. This action takes place in a (large) neighborhood of the attracting fixed point in the space $\mathcal{F}$ of equivalence classes of measured foliations of the surface, where one forgets the transverse measure. (In other words, $\mathcal{F}$ is the quotient space of Thurston's measured foliation space $\mathcal{MF}$ by the equivalence relation which identifies two elements of $\mathcal{MF}$ if they can be represented by measured foliations on the surface which agree as foliations, regardless of the transverse measure.) This gives a natural combinatorial description of the action of the pseudo-Anosov mapping class in the neighborhood of its attracting fixed point. The elements of the combinatorics are objects which are naturally associated to the map: an invariant train track with a marking on its set of vertices and a certain number of elementary moves on it. It is true that the action that we consider in this paper takes place in our space $\mathcal{F}$ and not in the usual space $\mathcal{MF}$, but as we explain at the end of the paper, the space $\mathcal{F}$ is, in a certain sense, equal to the space $\mathcal{MF}$ up to a subset of measure zero, and it is in any case an interesting space to study.

§1.—Background (foliations and subshifts)

Let S be a closed surface of genus $g \geq 2$ and let $\mathcal{MF}$ be Thurston's space of equivalence classes of measured foliations on S. We refer the reader to [2] for background material and basic facts concerning the space $\mathcal{MF}$. We recall that a measured foliation is a foliation with a measure defined on each transverse arc which is equivalent to Lebesgue measure, such that the local holonomy maps are measure preserving. Such a foliation is allowed to have singular points of the types described in Figure 1.

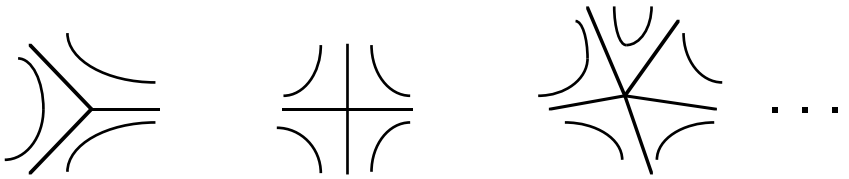

Figure 1

An element of $\mathcal{MF}$ is an equivalence class of measured foliations, the equivalence relation being generated by isotopy and Whitehead moves (that is, collapsing to a point a segment joining two singularities; see Figure 2).

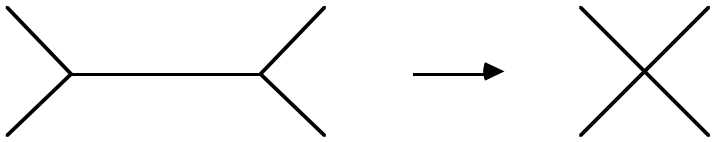

Figure 2

We introduce now another space, which we call $\mathcal{F}$. It is the quotient of $\mathcal{MF}$ by the following equivalence relation (denoted by $\sim$): two elements of $\mathcal{MF}$ are equivalent if they can be represented by measured foliations on S which are identical as foliations, regardless of the transverse measure. Let $\mathcal{PMF}$ be the space of projective classes of measured foliations on S, that is, the quotient of $\mathcal{MF}$ by the natural action of the group $\mathbb{R}^*$ of positive real numbers (this action consists of multiplying the transverse measure of a foliation by a positive scalar). The space $\mathcal{F}$ is a quotient of $\mathcal{PMF}$, and we denote by

$$\mathcal{MF} \stackrel{p}{\longrightarrow} \mathcal{PMF} \stackrel{f}{\longrightarrow} \mathcal{F}$$

the sequence of quotient maps.

There is a natural topology on $\mathcal{MF}$, which we shall review in section 2 below, and which makes it homeomorphic to a Euclidean space of dimension $6g-6$ which is punctured at the origin. The spaces $\mathcal{PMF}$ and $\mathcal{F}$ are equipped with the quotient topologies. Let us note right away that the equivalence relation $\sim$ is not closed, and $\mathcal{F}$ is not a Hausdorff space. Even so (and perhaps because of its complicated nature), it is an interesting space to study. The result which we prove in this paper concerns the space $\mathcal{F}$, and the reason for which we began by considering $\mathcal{MF}$ is that $\mathcal{F}$ is naturally defined as a quotient of $\mathcal{MF}$. As we shall recall below, there is a well-known canonical measure on $\mathcal{MF}$, which defines canonical measure classes on the spaces $\mathcal{PMF}$ and $\mathcal{F}$. ¿From this measure-class point of view, the two spaces $\mathcal{PMF}$ and $\mathcal{F}$ are equivalent. Indeed, there is a subset of full measure of $\mathcal{PMF}$ (consisting of the equivalence classes of uniquely ergodic measured foliations) on which the restriction of the map f is one-to-one, and such that the measure class on $\mathcal{F}$ is defined by the push-forward of the measure class on $\mathcal{PMF}$ by this map.

Let Γ be the mapping class group of S, that is, the group of isotopy classes of orientation preserving homeomorphisms of this surface. Γ acts naturally on the three spaces $\mathcal{MF}$, $\mathcal{PMF}$ and $\mathcal{F}$. According to Thurston's classification (see [10]), an element of Γ is either of finite order or reducible or pseudo-Anosov. We are interested here in the dynamics of the action of a pseudo-Anosov element on the space $\mathcal{F}$. Let us note that the action of an element of finite order is not interesting from the dynamical point of view. The action of a reducible element is itself "reducible", in the sense that an iterate of it can be decomposed into pseudo-Anosov and finite order actions on subspaces of $\mathcal{F}$, and then one can study how these components are combined. If an element $\gamma \in \Gamma$ is of pseudo-Anosov type, then it has two fixed points, F^s and F^u on $\mathcal{PMF}$, respectively the projective classes of the stable and unstable measured foliations. For every element F in $\mathcal{PMF}$ which is distinct from F^s, the sequence $\gamma^n F$ converges to F^u, and thus the dynamics of the action of γ on $\mathcal{PMF}$ is very simple. The same property holds for the action of γ on $\mathcal{F}$, where we

denote by $[F^s]$ and $[F^u]$ the equivalence classes of F^s and F^u in $\mathcal{F}$. Despite the simplicity of this action, one can ask for a symbolic coding of it, that is, a combinatorial description of this action using a subshift of finite type where the set of symbols arises from the action of the mapping class on the surface itself. This is the subject matter of this paper.

Let us note that one essential feature of a subshift of finite type is that it is defined by means of a finite set of symbols, and it is always interesting to find subshifts of finite type arising in different sorts of mathematical contexts. We recall now the precise definition:

We start with a finite set $\mathcal{S}$, whose elements are called symbols. The associated *one-sided Bernoulli shift*, denoted by $\Sigma = \Sigma(\mathbb{N}, \mathcal{S})$, is defined as the set of all maps from $\mathbb{N}$ to $\mathcal{S}$, where $\mathbb{N}$ is the set of nonnegative integers. Thus, elements of Σ are infinite sequences of elements in $\mathcal{S}$, indexed by $\mathbb{N}$. The set Σ is equipped with the product topology. The *shift map* $\sigma\colon \Sigma \to \Sigma$ is the map from Σ to itself defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, x_3, \ldots).$$

A *subshift* $\Sigma_0 \subset \Sigma$ is a closed subset which is invariant by the shift map. It has therefore an induced shift map, which we denote also by $\sigma : \Sigma_0 \to \Sigma_0$. A subshift $\Sigma_0 \subset \Sigma$ is said to be *of finite type* if it can be defined in the following manner:

There exists an integer $n \geq 1$ and a finite set $W \subset \mathcal{S}^n$ such that an element $x = (x_0, x_1, x_2, \ldots)$ of Σ belongs to Σ_0 if and only if for every $i = 0, 1, 2, \ldots$, the element $(x_i, \ldots, x_{i+n-1})$ of $\mathcal{S}^n$ belongs to W.

The elements of W are called the *admissible* words of length n.

A subset $C \subset \Sigma$ is called a *cylinder* if there exists a finite set of integers $F \subset \mathbb{N}$ and a set A of maps from F to $\mathcal{S}$ such that $C = \{x \in \Sigma \mid x_{|F} \in A\}$.

Finally, a *semi-Markovian* subset of Σ is the intersection of a cylinder with a subshift of finite type. This last notion is introduced in [3], section 8.5.H where it appears as a basic notion in the theory of symbolic dynamics applied to the study of compact metric spaces (see also [1], chapter 6, where this notion is further developed). Let us note that a semi-Markovian subset is, in general, not invariant by the shift map. As we shall see in §4, the semi-Markovian set that we produce will have an induced action (which will be a section of the shift map), and thus one can talk about a *semi-Markovian action.*

§2.—Train Tracks and RLS Sequences

A *train track* on S is a graph imbedded in this surface whose vertices are all trivalent and such that at each vertex, the edges meet smoothly according to the local model given in Figure 3. In other words, at each vertex, there is a well-defined notion of a side where two edges abut together and a side where only one edges abuts. For natural reasons, a vertex in a train track is called a *switch.* We refer the reader to [9] and [7] for the basics of the

theory of train tracks. We note however that the notion of train track as defined in these two references is more general than the one we are using here in the sense that it allows local models with n-valent switches, with $n \geq 3$. But for our purposes here, it is sufficent to consider trivalent switches only. On the other hand, it is in general necessary (and this is usually made part of the definition) to exclude some special kinds of configurations for the connected components of $S - \tau$ (the surface S cut open along τ), and we deal now with this matter.

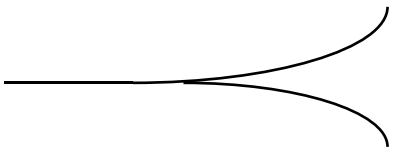

Figure 3

Lte τ be a train track, and consider $S - \tau$, which consists in a collection of surfaces with boundary and with cusps on their boundary. It is sufficient for our purpose here to consider only the case where each connected component of $S - \tau$ is a disk with three cusps on its boundary (see Figure 4).

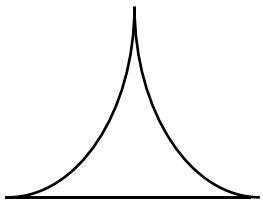

Figure 4

A train track with this property is usually called *maximal*.

There are three basic types of operations which are performed in a small neighborhood of an edge of a train track. They are represented in Figure 5 (r), (ℓ) and (s) and are called the *elementary moves of type R, L and S*, respectively. There is some (visual) advantage in thinking of an elementary move as being associated to one of the two switches bordering that edge. The first two types of moves are imagined as an unzipping operation, and the third one as sliding one of the switches above (or below) the other one.

Without going too deeply into the details of the theory, let us review for the convenience of the reader how a train track can be used to describe a parameter space for a subset of $\mathcal{MF}$ (and we denote by $\mathcal{MF}(\tau)$ this subset) as well as the effect of an elementary move performed on τ on the associated space $\mathcal{MF}(\tau)$.

Let τ be a train track on S, and let N be the number of edges of τ. We define a convex cone, $E(\tau)$, in the vector space $\mathbb{R}^N$, as the set of nonnegative real weights on

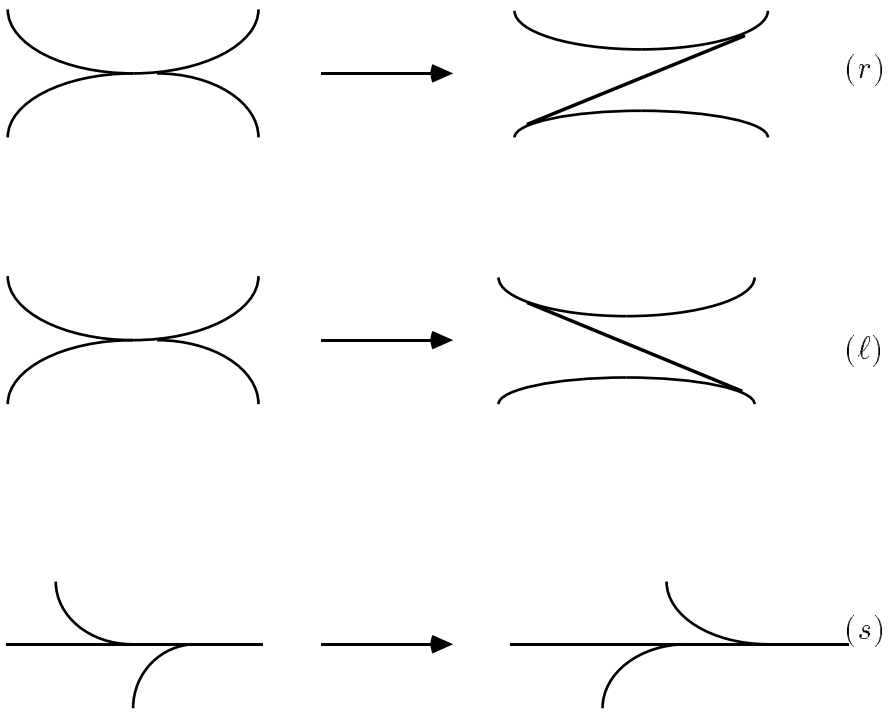

Figure 5

the edges of τ satisfying the switch conditions (the sum of the weights on the two edges incoming from the same side is equal to the weight of the remaining edge). If $E(\tau)$ contains a point whose coordinates are all positive, then the train track τ is said to be *recurrent*.

There is a natural map $\varphi_\tau : E(\tau) \to \mathcal{MF}$ which is defined in the following manner: Let X be an element of $E(\tau)$, and suppose that X has at least one non-zero coordinate (otherwise, we define $\varphi_\tau(X)$ to be the "empty foliation"). To define an associated measured foliation, we begin by replacing each edge of τ by a rectangle foliated horizontally in a standard manner, and we equip this foliation with a transverse measure such that the total transverse measure of each "vertical" side of the rectangle is equal to the weight of the given edge. We then glue together all these foliated rectangles along their vertical sides by a measure preserving map, the gluing pattern being prescribed by the incidence of the edges themselves in the train track. We obtain in this way a *partial measured foliation* on S, that is to say, a measured foliation supported on a subsurface with boundary in S, which has cusps on its boundary (see Figure 6).

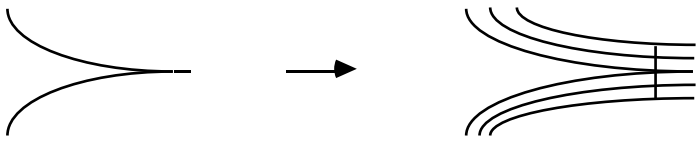

Figure 6

We then collapse each of the nonfoliated regions to a spine and obtain a measured

foliation on S, whose equivalence class depends only on the element $X \in E(\tau)$. This defines the element $\varphi_\tau(X) \in \mathcal{MF}$. For further details, we refer to [6], [7] and [9]. In the case where τ is maximal and recurrent, the image $\mathcal{MF}(\tau) = \varphi_\tau(E(\tau))$ has nonempty interior in $\mathcal{MF}$, and φ_τ is a homeomorphism between E_τ and its image. We note finally (although we shall not use this fact) that the maximal recurrent train tracks on S define the basis of a natural PL structure for $\mathcal{MF}$.

Let τ and τ' be two train tracks on S, with τ' being obtained from τ by an elementary move. We have then a natural injective map $i : \mathcal{MF}(\tau') \to \mathcal{MF}(\tau)$. In the case of a move of type S, this map is also surjective. In the case of a move of type R or L, it is generally a strict injection.

One of the main ideas in this kind of study is that an infinite sequence of successive elementary moves performed on a train track (and we call such a sequence an *RLS sequence*) defines a sequence of nested sets in $\mathcal{MF}$. The intersection of these nested sets in general is not equal to a point, but under suitable conditions on the sequence, it is equal to a cone of dimension at most $3g - 3$, and in fact, in a certain generic sense (which we describe below), the intersection is indeed equal to a point in $\mathcal{MF}$. For our purposes, we shall rather work in the space $\mathcal{F}$ and study the effect of an RLS sequence on subsets in $\mathcal{F}$.

§3.—More about the space $\mathcal{F}$

In the same manner as for the space $\mathcal{MF}$, maximal recurrent train tracks can be used to give parameters for open subsets in the space $\mathcal{F}$. Indeed, let us consider such a train track τ and let $\overset{\circ}{E}(\tau)$ denote the interior of the convex cone $E(\tau)$, defined in §2 above. The train track τ possesses a regular neighborhood $V(\tau)$ in S, which is naturally fibered over τ, each fiber being homeomorphic to a closed interval. The fibers are called *ties*, and τ can be viewed as the quotient of $V(\tau)$ by the relation which identifies each tie to a point. The open set $\overset{\circ}{\mathcal{MF}}(\tau)$ in $\mathcal{MF}$ is the image $\varphi_\tau(\overset{\circ}{E}(\tau))$. An element of $\overset{\circ}{\mathcal{MF}}(\tau)$ is in $\mathcal{MF}(\tau)$ if and only if it can be represented by a partial measured foliation whose support is contained in a regular neighborhood $V(\tau)$ of τ, which is transverse to the ties and such that the intersection of each tie with the foliation is nonempty. The same property can be used to describe open sets in $\mathcal{F}$. Indeed, let us first note that the open set $\overset{\circ}{\mathcal{MF}}(\tau)$ is saturated for the equivalence relation $\sim$ on $\mathcal{MF}$. It follows that a maximal recurrent train track τ defines an open set $\overset{\circ}{\mathcal{F}}(\tau)$ of $\mathcal{F}$, which is the interior of the image $\mathcal{F}(\tau)$ of $\mathcal{MF}(\tau)$ under the composed map $f \circ p$. Elements in the space

$\overset{\circ}{\mathcal{F}}(\tau)$ are characterized by the fact that they can be represented by partial measured foliations with support in $V(\tau)$ and which intersect transversely and nontrivially each tie.

Let τ be a train track with n vertices and suppose that we have a fixed *marking* on the vertices, that is, a one-to-one map from the set of vertices to the set $\{1, \ldots, n\}$. If we perform an elementary move on τ, then the marking on the vertices of τ induces naturally

a marking on the vertices of the resulting train track. (This is clear when we think of the move as an unzipping or a sliding operation, as explained above). The following is a basic result:

Proposition 1.— *Consider an infinite sequence of elementary moves which are performed starting from the marked train track τ. At each stage, we suppose that the vertices of the train track are equipped with the marking induced from that on the vertices of the preceding one in the sequence. Suppose the following condition holds:*

(3.1) *For every $i = 1, \ldots, n$, there are infinitely many elementary moves performed on the vertex marked i.*

Let us call $\tau_0 = \tau, \tau_1, \tau_2, \ldots$ the sequence of train tracks successively obtained from τ by these moves, and consider the infinite nested sequence of subspaces of $\mathcal{F}$,

$$\mathcal{F}(\tau_0) \supset \mathcal{F}(\tau_1) \supset \mathcal{F}(\tau_2) \supset \cdots$$

Then the intersection $\bigcap_{i=0}^{\infty} \mathcal{F}(\tau_i)$ consists of a finite number of points in $\mathcal{F}$, whose cardinality is bounded by 2^{3g-3}. Furthermore, if this intersection contains an element of $\mathcal{F}$ which is represented by a foliation of S which has no leaves connecting singular points, then it is a singleton.

Proof: We give only a sketch of the proof, since it uses the same ideas and follows the same outline as the proof of Theorem 1 of [8].

Using condition (3.1), one shows that if F_1 and F_2 are any two elements of the set $\bigcap_{i=0}^{\infty} \mathcal{MF}(\tau_i)$, their geometric intersection number $i(F_1, F_2)$ is zero. In [8], instead of condition (3.1), there is a weaker condition which says only that there are infinitely many elementary moves of type R or L in the sequence. This is sufficient in the setting of [8] where we consider only foliations in which every leaf is dense. In the present setting, we need condition (3.1), which is used in the following argument. Start with an arbitrary nonnegative weight on τ which represents a measured foliation in the intersection $\bigcap_{i=0}^{\infty} \mathcal{MF}(\tau_i)$ and consider the sequence of induced weights on the sequence of train tracks τ_i $(i = 1, 2, \ldots)$. When an elementary move of type R or L is performed on a vertex of the train track τ_i, then there is an edge in the train track τ_{i+1} (which results from τ_i after the move) whose weight decreases by a controlled amount (it decreases with respect to the weight of the corresponding edge in τ_i), and the weights of the other edges remain unchanged. Condition (3.1) implies that there are infinitely many moves of type R or L involved in the sequence, and furthermore that these moves are not confined to some proper part of the train track, since the index of each vertex appears infinitely often in the sequence. We can see easily that this implies that the sequence of weights on the edges of τ_i tends uniformly to zero as i tedns to infinity, and this is the main point needed in the proof of $i(F_1, F_2) = 0$, as explained in [8].

Given now a set $\{F_i\}_{i \in I} \subset \mathcal{MF}$ satisfying $i(F_i, F_j) = 0$ for every i and j in I, we can find a measured foliation F on S such that each F_i is represented, as a topological

foliation, by a union of components of F, that is, by a partial foliation on S defined as a closed saturated subset of the foliation F. Since any measured foliation on S has at most $3g-3$ components, the set $\{F_i\}_{i\in I} \subset \mathcal{MF}$ gives rise to at most 2^{3g-3} elements of $\mathcal{F}$. Finally, if F is a measured foliation on S which has no leaves connecting singular points, then it cannot be equivalent to a closed saturated proper subset of another foliation. ¿From that we deduce the proposition. ∎

§4.—The semi-Markovian action

We explain now the construction which gives the set of symbols and the subshift that we seek to define. The main technical points in this construction consist in some improvements of results proved in [8], and instead of doing the whole theory again, we shall use the results of [8] and give explicitly the necessary modifications. Let $\gamma \in \Gamma$ be a pseudo-Anosov mapping class. (We shall denote by the same letter, γ, the mapping class and a homeomorphism of the surface representing this mapping class.) There exists a recurrent train track τ with the property that $\gamma(\tau)$ *is carried by* γ (see [6], Chapter 2, Proposition 1, or [8], Theorem 4.1). Let us recall that this means that $\gamma(\tau)$ is, up to isotopy, contained in a regular neighborhood $V(\tau)$ of τ transversely to the ties. We represent the fact that $\gamma(\tau)$ is carried by τ by the relation $\gamma(\tau) < \tau$. Furthermore, it follows from the proof of the result which we have just cited that, up to replacing γ by one of its powers, say γ^q, we can assume that τ is maximal. (In fact, this property holds if the integer q is chosen in such a way that γ^q preserves each singular point of the stable or unstable foliation and each sector at such a point.)

By [8], the relation $\gamma(\tau) < \tau$ is equivalent to the fact that there is a finite sequence of elementary moves which produce the train track $\gamma(\tau)$ from τ. Let A be such a sequence. Without loss of generality, we can suppose that the sequence A satisfies a condition which we formulate now.

Let n be the number of vertices of τ and let us fix a marking on the set of vertices. For each elementary move, we have an induced marking on the set of vertices of the resulting train track. The condition which we impose on the sequence A of moves is the following:

(*) For any $i \in \{1, \ldots, n\}$, there is at least one elementary move in A which uses a vertex whose index is i.

By examining the construction of the train track τ in [6], Chapter 2, Proposition 1 or in [8], Theorem 4.1, we can see that if we take q large enough, then we can choose A with the required property. The main point that is being used here is that the pseudo-Anosov map γ stretches by a definite amount the leaves of the unstable foliation F^u, and τ is defined in an appropriate manner as a quotient of that foliation. Now by taking a sufficiently large power q, the way the train track $\gamma^q(\tau)$ sits naturally in a neighborhood $V(\tau)$ of τ (as explained in the results we are referring to) insures that each edge of $\gamma(\tau)$ intersects each of the ties of $V(\tau)$ in a large number of points. This clearly suffices to find

a sequence A of moves which satisfies property (*).

The semi-Markovian subset Σ_0 which we are going to define will be the intersection of a cylinder C with a subshift of finite type Σ_0^* in a Bernoulli shift. Let us first describe the Bernoulli shift.

We take as the set $\mathcal{S}$ of symbols the set of pairs (c, s) where:

— c is an element of the set of all conjugacy classes of maximal recurrent train tracks on S together with a marking on the set of vertices, and

— s belongs to the set $\{R_1, \ldots, R_n, L_1, \ldots, L_n, S_1, \ldots, S_n\}$, where n is as above the number of vertices of τ.

Let us note that given a marked train track τ and a a vertex marked i, it is not true that we can apply on τ any one of the moves R_i, L_i and S_i. This fact will be taken into account in the definition below (conditions (ii) and (iii)).

Let $\Sigma = \Sigma(\mathbb{N}, \mathcal{S})$ denote the one-sided Bernoulli shift on $\mathcal{S}$. The subshift $\Sigma_0^* \subset \Sigma$ is defined by the following conditions (i), (ii) and (iii).

(i) A symbol (c, s) is admissible only if it is possible to perform an elementary move s on a marked train track in the class c.

(ii) A sequence of two symbols (c, s) and (c', s') is admissible only if the result of applying the elementary move s to a marked train track in the class c produces a marked train track in the class c'.

Before formulating condition (iii), we begin by fixing once and for all a marking on the vertices of τ, which is chosen in the following manner:

Suppose that τ is equipped with a certain marking $\mathcal{N}$: {vertices} $\to \{1, \ldots, n\}$. Consider the sequence A of elementary moves which transforms τ to $\gamma^q(\tau)$. This sequence induces naturally a marking $\mathcal{N}'$ on the vertices of $\gamma^q(\tau)$. This train track $\gamma^q(\tau)$ is of course conjugate to τ, but the conjugacy does not necessarily carry the marking $\mathcal{N}$ of τ to the marking $\mathcal{N}'$. However, up to replacing the mapping class γ^q by one of its powers, we can find a marking $\mathcal{N}$ which has the property that $\mathcal{N}'$ coincides with the marking $\gamma^q(\mathcal{N})$. We fix once and for all such a power (which we denote also by γ^q), with a marking $\mathcal{N}$ having the required property. Let $(a_0, \ldots, a_k)$ be the sequence of symbols represented by the word A. Each a_i is therefore a pair (c_i, s_i), and the last symbol $a_k = (c_k, s_k)$ is such that the elementary move s_k on a marked train track in the conjugacy class c_k produces a marked train track in the class $\gamma^q(\tau)$, that is to say, in the class of τ.

We can state now condition (iii):

(iii) For every subword w of length n (= the number of vertices of τ) in an element of Σ_0^*, one of the following two properties hold:
— either each one of the integers $\{1, \ldots, n\}$ appears at least once as an index value for an elementary move involved (*i.e.*, as a second coordinate of a symbol) in the word, or:

— if w is preceded by a word w' whose length is equal to length(A), then the word A appears as a subword of $w'w$.

Let Σ_0^* be the subshift defined by conditions (i), (ii), and (iii). The following lemma is clear :

Lemma 2.— *The subshift Σ_0^* is of finite type.* ∎

Let C be the cylinder in Σ defined by the property that the first symbol x_0 in the sequence $(x_0, x_1, x_2, \ldots)$ representing an element of Σ is the conjugacy class of the marked train track τ.

Let Σ_0 be the semi-Markovian subset of Σ defined by

$$\Sigma_0 = C \cap \Sigma_0^* \qquad (4.1)$$

.

We define now a multivalued map

$$m : \Sigma_0 \to \mathcal{F} \qquad (4.2)$$

in the following way: given an element $x = (x_0, x_1, x_2, \ldots,)$ in Σ_0, perform the sequence of elementary moves starting with the train track τ together with its marking $\mathcal{N}$. Let $\tau_0 = \tau, \tau_1, \tau_2, \ldots$ be the resulting sequence of train tracks. The image $m(x)$ is the intersection $\bigcap_{i=0}^{\infty} \mathcal{F}(\tau_i)$. We note that condition (iii) insures that in the infinite sequence of elementary moves that we are considering, the index of each vertex appears infinitely often. (This uses condition $(*)$ on the sequence A that was stated above. Therefore, condition (3.1) in Proposition 1 is satisfied. The cardinality of the set $\bigcap_{i=0}^{\infty} \mathcal{MF}(\tau_i)$ is therefore bounded by 2^{3g-3}.

Before stating our final result, we need to make one more definition :

Let Σ be a Bernoulli shift and $\Sigma_0 \subset \Sigma$ a subshift of finite type or a semi-Markovian subset, and let $A_0 = (a_0, \ldots, a_k)$ be a finite sequence of symbols with the following property:

For every element $x = (x_0, x_1, x_2 \ldots)$ of Σ_0, the element

$$A_0 x = (a_0, a_1, \ldots, a_k, x_0, x_1, x_2, \ldots)$$

is also in Σ_0.

We define then a map $\sigma_{A_0} \colon \Sigma_0 \to \Sigma_0$ by the rule $x \mapsto A_0 x$, and we call it (by abuse of language) a *section* of the the restriction of the shift to Σ_0. (In fact, σ_{A_0} is a section of the $(n+1)$-th power σ^{n+1} of the restriction of the shift.)

To state the theorem, we consider again our pseudo-Anosov mapping class γ, the semi-Markovian subset Σ_0 defined by (4.1), and the map m defined by (4.2), and we let A

be the finite sequence of symbols which we have defined above, obtained from the relation $\sigma(\tau) < \tau$. We can now state the following

Theorem 3.—*The map $\sigma_A : \Sigma_0 \to \Sigma_0$ is a section of the shift restricted to the semi-Markovian subset σ_0, and there exists an integer $q \geq 1$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \Sigma_0 & \xrightarrow{\sigma_A} & \Sigma_0 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ V & \xrightarrow{\gamma^q} & V \end{array}$$

Furthermore, for every $x \in \Sigma_0$, the number of points in the image $m(x)$ is bounded by 2^{3g-3}, and if $\mathcal{F}_0 \subset \mathcal{F}$ is the subset of elements which can be represented by foliations which have no leaves connecting singular points, then the restriction of m to the set $m^{-1}(\mathcal{F}_0)$ is a genuine map (i.e., it is not multivalued).

Proof: The proof follows from the construction. Let us only note that the fact that the map σ_A is a section of the shift is also a consequence of condition (iii) above and that the commutativity of the diagram follows from the relation $\gamma^q(\tau) < \tau$ and from the definition of the word A. ∎

Finally, let us make the following remarks which can help explaining the technical side of the theorem :

1. The equivalence relation $\sim$ on $\mathcal{MF}$ is not closed and the space $\mathcal{F}$ is not Hausdorff. Indeed, let F be an element of $\mathcal{MF}$ which can be represented by a measured foliation F_0 on S which has a closed leaf separating S into two subsurfaces with boundary. The foliation F_0 induces two measured foliations on these subsurfaces, which we denote by F_0' and F_0''. Each of the foliations F_0' and F_0'' has a well-defined equivalence class as a measured foliation on S, and we denote these classes respectively by F' and $F'' \in \mathcal{MF}$. For every positive real numbers x_1 and x_2, the expression $x_1F' + x_2F''$ defines naturally an element in $\mathcal{MF}$. These elements are distinct if the x_i's are distinct, but all such elements are $\sim$-equivalent to F. Let x_n be now a sequence of positive real numbers converging to 0. The measured foliation class $x_nF' + F''$ converges to F'' in the topology of $\mathcal{MF}$, whereas F'' is not $\sim$-equivalent to F. This proves that the topology of $\mathcal{F}$ is not Hausdorff. In fact, the set of elements such as F, F' or F'' above, which either can be decomposed into the sum of two disjoint measured foliations on the surface, or can be represented by a component of such a decomposition, is exactly the "bad set" of the space $\mathcal{F}$, that is, the set of points which the topology does not separate. The set of such elements is exactly the complement of $\mathcal{F}_0$ in $\mathcal{F}$, and it is not surprising that the map m, whose domain is Σ_0, which is Hausdorff, is precisely multivalued when it takes values in the set $\mathcal{F}_0 \subset \mathcal{F}$.

2. In a certain sense, the space $\mathcal{F}_0$ of "good points" in $\mathcal{F}$ is equivalent, as a measure space, to the space $\mathcal{PMF}$. Let us make this statement more precise. For this, recall that $\mathcal{MF}$ is equipped with a canonical measure which is invariant by the action of the mapping class group. This measure, which has been defined by Thurston in [9], coincides with Lebesgue measure in the coordinate charts of $\mathcal{MF}$ associated to maximal recurrent train tracks (see section 2 above). The projection map $p : \mathcal{MF} \to \mathcal{PMF}$ induces a well-defined

measure class on $\mathcal{PMF}$ (a set $E \subset \mathcal{PMF}$ has measure zero if and only if its inverse image $p^{-1}(E) \subset \mathcal{MF}$ has measure zero). Let $\mathcal{E} \subset \mathcal{F}$ be the set of equivalence classes of uniquely ergodic foliations and with no leaves connecting singular points. The set $\mathcal{E}$ has full measure in $\mathcal{MF}$ (cf. [4], [5] or [11]). The image $p(\mathcal{E})$ has also full measure in $\mathcal{PMF}$. It is easy to see, from the definitions, that the restriction of the quotient map f to $p(\mathcal{E})$ is a bijection between $p(\mathcal{E})$ and its image. In this way we define a measure class on $f \circ p(\mathcal{E})$ and extend trivially to a measure class on $\mathcal{F}$.

Note. After a first preprint version of this paper, Lee Mosher informed the author that he is preparing a paper which contains related ideas. Finally, the author would like to thank the referee for a detailed list of corrections.

References

[1] M. Coornaert & A. Papadopoulos, "Symbolic dynamics and hyperbolic groups", Lecture Notes in Mathematics **1539**, Springer Verlag (1993).

[2] A. Fathi, F. Laudenbach, V. Poénaru, " Travaux de Thurston sur les surfaces", Astérisque **66-67** (1979).

[3] M. Gromov, "Hyperbolic groups", MSRI Publications **8**, Springer Verlag (1987).

[4] S. Kerckhoff, "Simplicial systems for interval exchange maps and measured foliations", Ergod. Th. & Dynam. Sys. **5** (1985) p. 257-271.

[5] H. Masur, "Interval exchange transformations and measured foliations", Ann. Math. **115** (1982) p. 169-200.

[6] A. Papadopoulos, "Réseaux ferroviaires, difféomorphismes pseudo-Anosov et automorphismes symplectiques de l'homologie d'une surface", Publ. Math. d'Orsay No. **83-03** (1983).

[7] R. C. Penner (with J. Harer), "Combinatorics of train tracks", Ann. of Math. Studies N. **125**, Princeton University Press (1992).

[8] A. Papadopoulos & R. C. Penner, "A characterization of pseudo-Anosov foliations", Pacific J. of Math. Vol. **130**, No. 2 (1987) p. 359-377.

[9] W. P. Thurston, "The geometry and topology of 3-manifolds", Lecture notes, Princeton University (1976).

[10] W. P. Thurston, "On the geometry and dynamics of diffeomorphisms of surfaces", Bull. of the AMS, Vol. **19**, No. 2 (1988) p. 417-432.

[11] W. Veech, "Gauss measures for transformations on the space of interval exchange maps", Ann. Math. **115** (1982) p. 201-242.