Stability of Liquid Rope Coiling

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(Dated: July 18, 2006)

Abstract

A thin 'rope' of viscous fluid falling from a sufficient height coils as it approaches a rigid surface. Here we perform a linear stability analysis of steady coiling, with particular attention to the 'inertio-gravitational' regime in which multiple states with different frequencies exist at a fixed fall height. The basic states analyzed are numerical solutions of asymptotic 'thin-rope' equations that describe steady coiling. To analyze their stability, we first derive in detail a set of more general equations for the arbitrary time-dependent motion of a thin viscous rope. Linearization of these equations about the steady coiling solutions yields a boundary-eigenvalue problem of order twenty-one which we solve numerically to determine the complex growth rate. The multivalued portion of the curve of steady coiling frequency vs. height comprises alternating stable and unstable segments whose distribution agrees closely with high-resolution laboratory experiments. The dominant balance of (perturbation) forces in the instability is between gravity and the viscous resistance to bending of the rope; inertia is not essential, although it significantly influences the growth rate.

PACS numbers:

I. INTRODUCTION

The coiling of a thin 'rope' of viscous fluid falling from a sufficient height onto a surface is a common fluid dynamical instability that occurs in situations ranging from food processing to lava flows. The history of its investigation spans nearly fifty years, and includes laboratory experiments[1–10], linear stability theory for incipient coiling[11,12], finite-amplitude scaling analysis for high-frequency coiling[13], and direct numerical simulation[10,14]. Fig. 1 shows the configuration considered in most of these studies, in which fluid with constant density ρ, viscosity ν and surface tension coefficient γ is injected at a volumetric rate Q from a hole of diameter $d \equiv 2a_0$ and then falls a distance H onto a solid surface.

The present study is motivated by our recent numerical and experimental results[9,10,14], which are summarized in Figs. 2 and 3. Fig. 2a shows a schematic view of the experimental apparatus, in which a thin rope of silicone oil is extruded downward from a syringe pump driven by a stepper motor. In a typical experiment, the fluid was injected continuously at a constant rate Q while the fall height H was varied over a range of discrete values, sufficient time being allowed at each height to measure the coiling frequency. Anticipating the possibility of hysteresis, we made measurements both with height increasing and decreasing, and in a few cases we varied the height randomly. For fall heights within a certain range, we observed two or three different steady coiling states with different frequencies, each of which persisted for a time before changing spontaneously into one of the others. Fig. 2b and c show the low- and high-frequency coiling states observed for $\nu = 5000$ cm^2 s^{-1}, $d = 0.15$ cm, $Q = 0.0066$ cm^3 s^{-1}, and $H = 20$ cm. The coexistence of two states at the same fall height reflects the multivalued character of the curve of frequency vs. height, which is illustrated in more detail in Fig. 3. The symbols show coiling frequencies measured in an experiment performed using viscous silicone oil ($\rho = 0.97$ g cm^{-3}, $\nu = 1000$ cm^2 s^{-1}, $\gamma = 21.5$ dyne cm^{-1}) with $d = 0.068$ cm and $Q = 0.00215$ cm^3 s^{-1}, and the solid line shows the curve of frequency vs. height predicted numerically for the same parameters using the method of Ribe[14]. As the fall height H increases, the coiling traverses four distinct dynamical regimes. For small heights $H < 0.7$ cm, both gravity and inertia are negligible in the rope, and coiling occurs in a viscous (V) regime with a frequency[14]

$$\Omega \sim \frac{Q}{H a_1^2} \equiv \Omega_V, \tag{1}$$

where a_1 is the radius of the 'coil' portion of the rope (Fig. 1). The frequency decreases

strongly with height, and is independent of viscosity because the fluid velocity is determined kinematically by the injection speed. For 1 cm$\leq H \leq$ 5 cm, coiling occurs in a gravitational (G) regime in which the viscous forces that resist bending in the coil are balanced by gravity. The corresponding coiling frequency is[14]

$$\Omega \sim \left(\frac{gQ^3}{\nu a_1^8}\right)^{\frac{1}{4}} \equiv \Omega_G. \tag{2}$$

For 7 cm $\leq H \leq$ 15 cm, a complex inertio-gravitational regime (IG) is observed in which viscous, gravitational, and inertial forces are all significant. The curve of frequency vs. height is now multivalued, with up to seven different frequencies at a given height. The (rightward- and downward-facing) peaks in the curve correspond to resonant oscillations of the 'tail' portion of the rope with frequencies equal to the eigenfrequencies of a whirling viscous string[10]. The scaling law for all these frequencies is

$$\Omega \sim \left(\frac{g}{H}\right)^{1/2} \equiv \Omega_{IG}, \tag{3}$$

with constants of proportionality that depend weakly on the dimensionless parameter $gd^2H^2/\nu Q^{10}$. As the height increases further, the amplitude of the oscillations in $\Omega(H)$ gradually decreases until the curve becomes smooth again at $H \approx 18$ cm. Viscous forces in the coil are now balanced almost entirely by inertia, giving rise to inertial (I) coiling with a frequency[13]

$$\Omega \sim \left(\frac{Q^4}{\nu a_1^{10}}\right)^{\frac{1}{3}} \equiv \Omega_I. \tag{4}$$

The existence of the four regimes just described has now been confirmed experimentally by Mahadevan *et al.*[8] for the inertial regime, by Maleki *et al.*[9] for the viscous, gravitational, and inertial regimes, and by Ribe *et al.*[10] for the inertio-gravitational regime. The experimental observations in the IG regime are of particular interest. As shown in Fig. 3, the observed frequencies in this regime are concentrated along the roughly horizontal 'steps' of the $\Omega(H)$ curve, leaving the steeper portions with negative slope ('switchbacks') empty. The absence of observed steady coiling states along the switchbacks suggests that such states may be unstable to small perturbations. Here we investigate this question by means of a formal linear stability analysis.

II. GOVERNING EQUATIONS FOR AN UNSTEADY ROPE

The starting point of our analysis is a set of equations governing the unsteady motion of a thin viscous rope, i.e.., one whose 'slenderness' $\epsilon \equiv a_0/L \ll 1$, where a_0 is a characteristic value of the rope radius and L is the characteristic length scale for the variations of the flow variables along the rope. Equations for a thin viscous rope have been derived by Entov and Yarin[15], who described the geometry of the rope's axis using the standard triad of basis vectors from differential geometry (the unit tangent, the principal normal, and the binormal). However, such a description can lead to numerical instability when the total axial curvature is small, as it is over most of the length of a coiling liquid rope. Here we present an alternative formulation in which the basis vectors normal to the rope's axis are material vectors that are convected with the fluid. Because our goal is to perform a linear stability analysis of steady coiling, we write the equations in a reference frame that rotates with angular velocity $\Omega \boldsymbol{e}_3$ relative to a fixed laboratory frame. The Einstein summation convention over repeated indices or subscript/superscript pairs is assumed. Greek indices range over the values 1 and 2 only. Latin indices range over the values 1, 2, and 3 except for the Euler parameters q_i, in which case $i = 0$, 1, 2, and 3. The quantity ϵ_{ijk} is the usual alternating tensor. Finally, derivatives with respect to arclength along the rope axis are denoted by primes.

A. Geometry

Fig. 4 shows the geometry of an element of a thin viscous rope. Let $\boldsymbol{x}(s,t)$ be the Cartesian coordinates of a point on the rope's axis, where s is the arc length along it and t is time, and let $\boldsymbol{d}_i(s,t)$ be a triad of orthogonal unit vectors defined at each point on the axis. The tangent vector to the axis is $\boldsymbol{d}_3$, and $\boldsymbol{d}_1$ and $\boldsymbol{d}_2 \equiv \boldsymbol{d}_3 \times \boldsymbol{d}_1$ are material vectors normal to the axis that follow the rotation of the fluid. The rope's cross-section is assumed to be circular, with radius $a(s,t)$, area $A = \pi a^2$, and moment of inertia

$$I = \frac{\pi a^4}{4}. \tag{5}$$

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The rate of change of the axial coordinates $\boldsymbol{x}$ as a function of arclength is

$$\boldsymbol{x}' = \boldsymbol{d}_3. \tag{6}$$

The rates of change of the local basis vectors $\boldsymbol{d}_i$ are in turn given by the generalized Frenet relations

$$\boldsymbol{d}_i' = \boldsymbol{\kappa} \times \boldsymbol{d}_i, \tag{7}$$

where $\boldsymbol{\kappa} \equiv \kappa_i \boldsymbol{d}_i$ is the generalized curvature vector. The components κ_i are related to the total curvature κ and the torsion τ of the axis by

$$\kappa = \left(\kappa_1^2 + \kappa_2^2\right)^{1/2}, \qquad \tau = \kappa_3 + \kappa^{-2}\left(\kappa_1 \kappa_2' - \kappa_2 \kappa_1'\right). \tag{8}$$

To avoid the polar singularities associated with the traditional Eulerian angles, it is convenient to describe the orientation of the basis $\boldsymbol{d}_i$ using four 'Euler parameters' q_i ($i = 0, 1, 2, 3$), which are related to the direction cosines $d_{ij}(s) \equiv \boldsymbol{d}_i(s) \cdot \boldsymbol{e}_j$ by[16]

$$d_{ij} = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1 q_2 + q_0 q_3) & 2(q_1 q_3 - q_0 q_2) \\ 2(q_1 q_2 - q_0 q_3) & q_0^2 + q_2^2 - q_1^2 - q_3^2 & 2(q_2 q_3 + q_0 q_1) \\ 2(q_1 q_3 + q_0 q_2) & 2(q_2 q_3 - q_0 q_1) & q_3^2 + q_0^2 - q_1^2 - q_2^2 \end{bmatrix}, \tag{9}$$

The Euler parameters satisfy identically the relation $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, so only three of them are independent. The inverse Frenet relations for the curvatures κ_i in terms of the Euler parameters and their derivatives are

$$\kappa_1 = 2(q_0 q_1' + q_3 q_2' - q_2 q_3' - q_1 q_0') \tag{10a}$$

$$\kappa_2 = 2(-q_3 q_1' + q_0 q_2' + q_1 q_3' - q_2 q_0') \tag{10b}$$

$$\kappa_3 = 2(q_2 q_1' - q_1 q_2' + q_0 q_3' - q_3 q_0') \tag{10c}$$

We now turn from the geometry of the axis to that of the rope as a whole. Given the basis vectors $\boldsymbol{d}_i$, the Cartesian coordinates $\boldsymbol{X}$ of an arbitrary point within the rope can be written

$$\begin{aligned} \boldsymbol{X}(y_1, y_2, y_3, t) &= \boldsymbol{x}(y_3, t) + y_1 \boldsymbol{d}_1(y_3, t) + y_2 \boldsymbol{d}_2(y_3, t) \\ &\equiv \boldsymbol{x} + \boldsymbol{y}, \end{aligned} \tag{11}$$

where y_1 and y_2 are coordinates normal to the rope axis and $y_3 \equiv s$ is an alternate notation for the arclength along the axis. In the following derivation, we shall make frequent use of the notation $\partial_i = \partial / \partial y_i$ ($i = 1, 2, 3$).

The variable transformation (11) defines a set of (covariant) basis vectors $\boldsymbol{g}_i = \partial_i \boldsymbol{X}$:

$$\boldsymbol{g}_1 = \boldsymbol{d}_1, \qquad \boldsymbol{g}_2 = \boldsymbol{d}_2,$$

$$\boldsymbol{g}_3 = h\boldsymbol{d}_3 - \kappa_3(y_2\boldsymbol{d}_1 - y_1\boldsymbol{d}_2) \tag{12}$$

where

$$h \equiv (\boldsymbol{g}_1 \times \boldsymbol{g}_2)\cdot\boldsymbol{g}_3 = 1 - \kappa_2 y_1 + \kappa_1 y_2. \tag{13}$$

Note that away from the rope axis, the vectors $\boldsymbol{g}_i$ are not orthogonal if $\kappa_3 \neq 0$. It is therefore necessary also to use a set of contravariant (reciprocal) base vectors $\boldsymbol{g}^i$ which satisfy $\boldsymbol{g}^i\cdot\boldsymbol{g}_j = \delta^i_j$, whence

$$\boldsymbol{g}^1 = \boldsymbol{d}_1 + h^{-1}\kappa_3 y_2 \boldsymbol{d}_3, \quad \boldsymbol{g}^2 = \boldsymbol{d}_2 - h^{-1}\kappa_3 y_1 \boldsymbol{d}_3,$$

$$\boldsymbol{g}^3 = h^{-1}\boldsymbol{d}_3. \tag{14}$$

The covariant and contravariant components of the metric tensor for the rope are then

$$g_{ij} = \boldsymbol{g}_i\cdot\boldsymbol{g}_j, \qquad g^{ij} = \boldsymbol{g}^i\cdot\boldsymbol{g}^j. \tag{15}$$

Note that there is no distinction between covariant and contravariant basis vectors on the rope axis itself, because $\boldsymbol{g}_i = \boldsymbol{g}^i = \boldsymbol{d}_i = \boldsymbol{d}^i$ when $y_1 = y_2 = 0$.

In what follows it will sometimes be useful to replace the coordinates y_1 and y_2 by the polar coordinates

$$r \equiv \left(y_1^2 + y_2^2\right)^{1/2}, \qquad \theta = \tan^{-1}\left(\frac{y_2}{y_1}\right). \tag{16}$$

The covariant and contravariant base vectors for these coordinates are

$$\boldsymbol{g}_r = \boldsymbol{g}^r = \cos\theta\boldsymbol{d}_1 + \sin\theta\boldsymbol{d}_2,$$

$$\boldsymbol{g}_\theta = r(\cos\theta\boldsymbol{d}_2 - \sin\theta\boldsymbol{d}_1), \qquad \boldsymbol{g}^\theta = r^{-2}\boldsymbol{g}_\theta - \kappa_3\boldsymbol{g}^3. \tag{17}$$

A final geometric parameter of interest is the mean curvature $\mathcal{H}$ of the rope's surface, which determines the pressure associated with surface tension. In the limit $a' \ll 1$, $aa'' \ll 1$,

$$2\mathcal{H} \approx -\frac{1}{a} + \kappa_2\cos\theta - \kappa_1\sin\theta. \tag{18}$$

B. Kinematics

Because the axis $y_1 = y_2 = 0$ of the rope is defined geometrically, it is not precisely a material line. Accordingly, two different velocity fields on the axis must be distinguished. The first is simply the true fluid velocity $\boldsymbol{u}$ evaluated on the axis, viz. $\boldsymbol{u}(0,0,y_3,t) \equiv \boldsymbol{U}(y_3,t)$. A second velocity field is

$$\boldsymbol{V}(y_3,t) = \mathcal{D}_t \boldsymbol{x}, \tag{19}$$

where

$$\mathcal{D}_t = \partial_t + W\partial_3, \tag{20}$$

$$W(y_3,t) = W(0,t) + \int_0^{y_3} \Delta(s,t)\, ds, \tag{21}$$

$\Delta(y_3,t)$ is the (yet to be determined) stretching rate of a material element that is aligned with the axis at time t, and $W(0,t)$ is the velocity at which the rope is injected at $y_3 = 0$. The convective derivative $\mathcal{D}_t$ is the rate of change measured by an observer traveling at a speed (W) equal to the sum of the injection speed and an additional velocity increment due to distributed stretching of the rope along its length[17]. Unlike the usual convective derivative, $\mathcal{D}_t$ applies only to field variables defined on the rope axis.

Although $\boldsymbol{U} \neq \boldsymbol{V}$ in principle, $\boldsymbol{V} - \boldsymbol{U} \sim \epsilon^2 \boldsymbol{U}$ in the slender rope limit $\epsilon \to 0$[18]. Accordingly, we shall ignore the distinction between $\boldsymbol{V}$ and $\boldsymbol{U}$ from now on, and use the symbol $\boldsymbol{U}$ for both. The near-equality of $\boldsymbol{U}$ and $\boldsymbol{V}$ implies that the axis of the rope is very nearly a material line. The velocity W defined by (21) can therefore be regarded (with negligible error) as the rate of change of the arclength coordinate y_3 of a material point.

Expressions for the stretching rate Δ and the convective rate of change of $\boldsymbol{d}_3$ are obtained by differentiating (19) with respect to y_3 and using the generalized Frenet relations. We thereby obtain

$$\Delta = U_3' - \kappa_2 U_1 + \kappa_1 U_2, \tag{22}$$

$$\mathcal{D}_t \boldsymbol{d}_3 = \omega_2 \boldsymbol{d}_1 - \omega_1 \boldsymbol{d}_2, \tag{23}$$

where

$$\omega_1 = -U_2' - \kappa_3 U_1 + \kappa_1 U_3, \tag{24a}$$

$$\omega_2 = U_1' - \kappa_3 U_2 + \kappa_2 U_3. \tag{24b}$$

Now because the lateral unit vectors $\boldsymbol{d}_\alpha$ are material, their angular velocity about the rope axis is just the rate of rotation ω_3 of the fluid (note that ω_3 is a primitive variable, unlike

ω_1 and ω_2 which are defined in terms of the velocity and geometry of the rope axis.) In certain situations, moreover, one must allow the unit vectors $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ at the end $y_3 = 0$ of the rope to rotate relative to the fluid with an additional spin ω_0. For steady coiling with angular frequency Ω, for example, $\omega_0 \equiv -\Omega$ is the additional spin required to make the base vectors $\boldsymbol{d}_i(s)$ independent of time along the whole length of the rope (see Ribe[14] for further discussion.) The generalization of (23) is therefore

$$\mathcal{D}_t \boldsymbol{d}_i = (\boldsymbol{\omega} + \boldsymbol{\omega}_0) \times \boldsymbol{d}_i, \qquad \boldsymbol{\omega} = \omega_i \boldsymbol{d}_i, \qquad \boldsymbol{\omega}_0 = \omega_0 \boldsymbol{d}_3. \tag{25}$$

As noted in § II A, the orientation of the basis vectors $\boldsymbol{d}_i$ can be described by the Euler parameters q_i. The evolution equations for these parameters that correspond to (25) are

$$\mathcal{D}_t q_0 = \frac{1}{2}\left[-\omega_1 q_1 - \omega_2 q_2 - (\omega_3 + \omega_0) q_3\right] \tag{26a}$$

$$\mathcal{D}_t q_1 = \frac{1}{2}\left[\omega_1 q_0 - \omega_2 q_3 + (\omega_3 + \omega_0) q_2\right] \tag{26b}$$

$$\mathcal{D}_t q_2 = \frac{1}{2}\left[\omega_1 q_3 + \omega_2 q_0 - (\omega_3 + \omega_0) q_1\right] \tag{26c}$$

$$\mathcal{D}_t q_3 = \frac{1}{2}\left[-\omega_1 q_2 + \omega_2 q_1 + (\omega_3 + \omega_0) q_0\right] \tag{26d}$$

The final kinematic equation needed describes the evolution of the rope's thickness. Consider a material element of the rope with (infinitesimal) length $l(t)$, and let the arclength coordinate of its center be $s(t)$. The incompressibility of the fluid requires that the volume $V \equiv A(s(t), t) l(t)$ of this element be constant. Setting $\mathrm{d}V/\mathrm{d}t = 0$, we find

$$(\partial_t A + \dot{s} \partial_s A)\, l + A \dot{l} = 0, \tag{27}$$

where dots denote total time derivatives. Now $\dot{s} = W$ and $\dot{l}/l = \Delta$, where W is the rate of change of the arclength (21) and Δ is the stretching rate (22). Eqn. (27) then becomes

$$\mathcal{D}_t A = -A\Delta. \tag{28}$$

The convective rate of thinning of the rope is proportional to the rate of stretching of a material line that lies along the axis.

C. Local dynamical equations

We turn now to the equations of conservation of mass and momentum satisfied at each point in the rope. The strain rate tensor relative to general nonorthogonal coordinates is

$$e_{ij} = \frac{1}{2}\left(\boldsymbol{g}_i \cdot \partial_j \boldsymbol{u} + \boldsymbol{g}_j \cdot \partial_i \boldsymbol{u}\right). \tag{29}$$

Incompressibility of the fluid requires $g^{ij}e_{ij} = 0$, or

$$\partial_\alpha \left(h u_\alpha\right) + \partial_3 u_3 + \kappa_3 \left(y_2 \partial_1 u_3 - y_1 \partial_2 u_3\right) = 0. \tag{30}$$

Turning now to the momentum equations, we note first that the stress tensor relative to the local basis vectors $\boldsymbol{g}_i$ and per unit local surface area is

$$\tau^{ij} = -p g^{ij} + 2\mu g^{ik} g^{il} e_{kl}, \tag{31}$$

where p is the pressure. However, it is more convenient to work with the modified (nonsymmetric) stress tensor

$$\sigma^{ij} = \sigma_{ij} = h \tau^{ik} \boldsymbol{g}_k \cdot \boldsymbol{d}^j, \tag{32}$$

which represents the stresses relative to the axial basis vectors $\boldsymbol{d}_i$ and per unit area of a reference surface at the axis. Note that $\sigma^{ij} = \sigma_{ij}$ because $\boldsymbol{d}^i = \boldsymbol{d}_i$. Unlike τ^{ij}, σ_{ij} can meaningfully be integrated over cross-sections, because the basis vectors and the surface to which it is referred do not vary across the rope. The equations of equilibrium in terms of σ_{ij} are (Green and Zerna[19], p. 150)

$$\rho h \ddot{\boldsymbol{X}} = \partial_i \left(\sigma_{ij} \boldsymbol{d}_j\right) + \rho h \boldsymbol{g}, \tag{33}$$

where $\ddot{\boldsymbol{X}}$ is the acceleration of a fluid particle and $g_i \boldsymbol{d}_i \equiv \boldsymbol{g}$ is the gravitational acceleration. Although the standard notation for gravity is similar to that for the covariant and contravariant basis vectors in § II A, the different numbers of subscripts and superscripts used in the two cases prevents confusion. For later use, we define the stress vector

$$\boldsymbol{\sigma}_i = \sigma_{ij} \boldsymbol{d}_j. \tag{34}$$

that acts on a surface whose normal is parallel to $\boldsymbol{d}_i$.

We now calculate explicitly the acceleration $\ddot{\boldsymbol{X}}$. To first order in the lateral coordinates y_1 and y_2, the velocity field within the rope as measured in the rotating reference frame is

$$\boldsymbol{u} = \boldsymbol{U} - \frac{1}{2}\boldsymbol{y}\Delta + \boldsymbol{\omega} \times \boldsymbol{y}, \tag{35}$$

where $\boldsymbol{y} = y_1\boldsymbol{d}_1 + y_2\boldsymbol{d}_2$. The total velocity $\boldsymbol{u}$ is the sum of the axial velocity $\boldsymbol{U}$, a radial inflow associated with stretching at a rate Δ, and a velocity $\boldsymbol{\omega}\times\boldsymbol{y}$ associated with bending in two mutually perpendicular planes (at rates ω_1 and ω_2) and twisting (at a rate ω_3). The acceleration corresponding to (35), measured now relative to the fixed laboratory frame, is

$$\begin{aligned}\ddot{\boldsymbol{X}} = {}& \boldsymbol{\Omega}\times[\boldsymbol{\Omega}\times(\boldsymbol{x}+\boldsymbol{y})] + 2\boldsymbol{\Omega}\times\boldsymbol{u} + \mathcal{D}_t\boldsymbol{U} \\ & + (y_2\mathcal{D}_t\omega_1 - y_1\mathcal{D}_t\omega_2)\,\boldsymbol{d}_3 + \mathcal{D}_t\omega_3(\boldsymbol{d}_3\times\boldsymbol{y}) + (\boldsymbol{\omega}\cdot\boldsymbol{y})(\omega_3\boldsymbol{d}_3 - \boldsymbol{\omega}_0) \\ & + [(\boldsymbol{\omega}\times\boldsymbol{y})\cdot\boldsymbol{d}_3](\boldsymbol{\omega}\times\boldsymbol{d}_3) - \Delta(\boldsymbol{\omega}\times\boldsymbol{y}) \\ & - \left(\frac{1}{2}\mathcal{D}_t\Delta - \frac{\Delta^2}{4} + \omega_3^2\right)\boldsymbol{y}. \end{aligned} \tag{36}$$

The first two terms on the right side of (36) are the additional centrifugal and Coriolis accelerations associated with the angular velocity Ω of the rotating frame relative to the fixed laboratory frame. Note that the vectors $\boldsymbol{u}$, $\boldsymbol{U}$ and $\boldsymbol{\omega}$ that appear in (36) are measured relative to the *rotating* frame.

D. Global force and torque balance

The equations of global force balance are obtained by integrating the momentum equations (33) together with (36) over a cross-section S of the rope with area A, yielding

$$\rho A\boldsymbol{J} = \boldsymbol{N}' + \boldsymbol{\mathcal{P}} \tag{37}$$

where

$$\boldsymbol{N} = \int_S \boldsymbol{\sigma}_3\, dS, \tag{38}$$

is the stress resultant vector,

$$\boldsymbol{J} = \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) + 2\boldsymbol{\Omega}\times\boldsymbol{U} + \mathcal{D}_t\boldsymbol{U}, \tag{39}$$

is the acceleration averaged over the cross-section,

$$\boldsymbol{\mathcal{P}} = \rho A\boldsymbol{g} + \oint_C [(\boldsymbol{g}_r\cdot\boldsymbol{d}_\alpha)\boldsymbol{\sigma}_\alpha - a'\boldsymbol{\sigma}_3]\,\mathrm{d}l, \tag{40}$$

is the applied load vector, and C is the (circular) contour around the cross-section.

The equations of global torque balance are obtained by applying the operator $\boldsymbol{y}\times$ to (33) and then integrating over the cross-section. This yields

$$\rho I\boldsymbol{K} = \boldsymbol{M}' + \boldsymbol{d}_3\times\boldsymbol{N} + \boldsymbol{\mathcal{M}}, \tag{41}$$

where I is the moment of inertia (5) of the cross-section,

$$\boldsymbol{M} = \int_S \boldsymbol{y} \times \boldsymbol{\sigma}_3 \, dS, \tag{42}$$

is the bending/twisting moment vector,

$$\begin{aligned} K_\alpha = {} & \mathcal{D}_t \omega_\alpha + \kappa_\alpha \left(\mathcal{D}_t U_3 + \epsilon_{\beta\gamma 3} \omega_\beta U_\gamma \right) \\ & + \Omega^2 \left(-\kappa_\alpha d_{3\beta} x_\beta + \epsilon_{\alpha\beta 3} d_{\beta 3} d_{33} \right) \\ & + \Omega \left[2 d_{\beta 3} \left(\epsilon_{\alpha\beta 3} \omega_3 + \epsilon_{\beta\gamma 3} \kappa_\alpha U_\gamma \right) - d_{\alpha 3} \Delta \right] \\ & - \omega_\alpha \Delta + \epsilon_{\alpha\beta 3} \omega_\beta (\omega_3 - \omega_0), \end{aligned} \tag{43a}$$

and

$$\begin{aligned} K_3 = {} & 2 \mathcal{D}_t \omega_3 - \kappa_\alpha \mathcal{D}_t U_\alpha \\ & + \Omega^2 d_{\alpha\beta} \kappa_\alpha x_\beta \\ & + 2\Omega \left\{ \epsilon_{\alpha\beta 3} \left[d_{\alpha 3} (\omega_\beta + \kappa_\beta U_3) + d_{33} \kappa_\alpha U_\beta \right] - d_{33} \Delta \right\} \\ & - 2 \omega_3 \Delta + \epsilon_{\alpha\beta 3} \kappa_\alpha \left[(\omega_3 + \omega_0) U_\beta - \omega_\beta U_3 \right], \end{aligned} \tag{43b}$$

are the components of the average moment of the acceleration, and

$$\begin{aligned} \boldsymbol{\mathcal{M}} = {} & \rho I \left[(\boldsymbol{g} \times \boldsymbol{d}_3) \boldsymbol{\kappa} - (\boldsymbol{\kappa} \times \boldsymbol{g}) \boldsymbol{d}_3 \right] \\ & + \oint_C \boldsymbol{y} \times \left[(\boldsymbol{g}_r \cdot \boldsymbol{d}_\alpha) \boldsymbol{\sigma}_\alpha - a' \boldsymbol{\sigma}_3 \right] \mathrm{d}l \end{aligned} \tag{44}$$

is the applied moment vector.

E. Applied loads and moments

We now determine simplified expressions for the applied load and moment vectors (40) and (44), assuming that the outer surface $r = a$ of the rope is acted upon by surface tension but is otherwise stress-free. Accordingly, the stress vector there is

$$\tau^{ij} n_i \boldsymbol{g}_j |_{r=a} = 2\gamma \mathcal{H} \boldsymbol{n} \tag{45}$$

where $\mathcal{H}$ is the mean curvature (18),

$$\boldsymbol{n} \equiv n_i \boldsymbol{g}^i = \left[h_c^2 + a'^2 \right]^{-1/2} \left(h_c \boldsymbol{g}_r - a' \boldsymbol{d}_3 \right) \tag{46}$$

is the unit vector normal to the surface and

$$h_c \equiv h|_{r=a} = 1 + a(\kappa_1 \sin\theta - \kappa_2 \cos\theta). \tag{47}$$

By projecting (45) onto the base vector $\boldsymbol{d}_k$ and rewriting the result in terms of σ_{ij} using (32), we obtain

$$[(\boldsymbol{g}_r \cdot \boldsymbol{d}_\alpha)\boldsymbol{\sigma}_\alpha - a'\boldsymbol{\sigma}_3]_{r=a} = 2\gamma\mathcal{H}(h_c \boldsymbol{g}_r \cdot \boldsymbol{d}_i - \delta_{i3}a')\boldsymbol{d}_i. \tag{48}$$

Eqn. (48) together with (17), (18), and (47) permits evaluation of the line integrals in (40) and (44), which yields

$$\boldsymbol{\mathcal{P}} = \rho A \boldsymbol{g} + 2\pi\gamma(a\boldsymbol{\kappa} \times \boldsymbol{d}_3 + a'\boldsymbol{d}_3), \tag{49}$$

$$\mathcal{M}_\alpha = \rho I \kappa_\alpha g_3 + \gamma A \kappa_\alpha a', \qquad \mathcal{M}_3 = -\rho I \kappa_\alpha g_\alpha. \tag{50}$$

F. Constitutive relations

The dynamical equations are completed by constitutive relations for the stress resultant N_3, the bending moments M_1 and M_2, and the twisting moment M_3. These can be derived by asymptotic expansion of the governing equations in powers of the slenderness $\epsilon = a_0/L \ll 1$, following a procedure similar to that of Ribe[18]. To facilitate the derivation, define dimensionless variables $\hat{y}_\alpha = y_\alpha/a_0$, $\hat{a} = a/a_0$, $\hat{y}_3 = y_3/L$, and $\hat{\kappa}_i = L\kappa_i$, and the dimensionless derivative $\hat{\partial}_3 = \partial/\partial\hat{y}_3$.

To determine the constitutive relation for the axial stress resultant N_3, we consider slow (inertia-free) deformations dominated by stretching. Suppose for definiteness that gravity is the primary force responsible for the deformation of the rope, as is the case e.g. for a vertical liquid rope stretching under its own weight. The scales for the velocity and pressure within the rope are then

$$\boldsymbol{u} \sim \frac{\rho g L^2}{\mu}, \qquad p \sim \rho g L. \tag{51}$$

We also suppose that the magnitude of the surface tension coefficient γ is such that

$$\frac{\gamma}{\epsilon \rho g L^2} \equiv \hat{B}^{-1} = O(1), \tag{52}$$

where $\hat{B}$ is a modified inverse Bond number.

The scales (51) suggest that the velocity and pressure fields can be represented by asymptotic expansions of the form

$$u_i = \frac{\rho g L^2}{\mu} \sum_{j=0} \sum_{m=0} \sum_{n=0} \epsilon^j \hat{y}_1^m \hat{y}_2^n u_i^{(jmn)}(\hat{y}_3), \tag{53a}$$

$$p = \rho g L \sum_{j=0} \sum_{m=0} \sum_{n=0} \epsilon^j \hat{y}_1^m \hat{y}_2^n p^{(jmn)}(\hat{y}_3), \tag{53b}$$

where the coefficients $u_i^{(jmn)}$ and $p^{(jmn)}$ are dimensionless functions of the arclength $\hat{y}_3$. To simplify the notation, let

$$\Delta^{(ijk)} = \hat{\partial}_3 u_3^{(ijk)} + \hat{\kappa}_1 u_2^{(ijk)} - \hat{\kappa}_2 u_1^{(ijk)}, \tag{54}$$

We now substitute the expansions (53) into the continuity equation (30), the momentum equations (33) with the inertial term on the left-hand side neglected, and the boundary condition (45), and then require terms proportional to the same powers of ϵ, $\hat{y}_1$, and $\hat{y}_2$ in each equation to vanish separately. This yields a set of coupled linear algebraic equations for the coefficients $u_i^{(jmn)}$ and $p^{(jmn)}$ that can be solved sequentially. A Mathematica[20] script that implements this solution procedure is available upon request from the first author. The leading-order expression for the (dimensional) stress resultant N_3 is

$$\frac{N_3}{\rho g L A} = -p^{(000)} + 2\Delta^{(000)}, \tag{55}$$

and the sequential solution procedure described above gives

$$p^{(000)} = -\Delta^{(000)} + \hat{B}^{-1}. \tag{56}$$

Substituting (56) into (55) and redimensionalizing using $\Delta \approx \rho g L \Delta^{(000)}/\mu$, we find

$$N_3 = 3\mu A \Delta - \pi \gamma a. \tag{57}$$

To determine the constitutive relations for M_1, M_2 and M_3, we consider slow deformations dominated by bending and twisting. The velocity and pressure then scale as

$$\boldsymbol{u} \sim \frac{\rho g L^2}{\epsilon^2 \mu}, \qquad p \sim \frac{\rho g L}{\epsilon}, \tag{58}$$

and the appropriate asymptotic expansions are

$$u_i = \frac{\rho g L^2}{\epsilon^2 \mu} \sum_{j=0} \sum_{m=0} \sum_{n=0} \epsilon^j \hat{y}_1^m \hat{y}_2^n u_i^{(jmn)}(\hat{y}_3), \tag{59a}$$

$$p = \frac{\rho g L}{\epsilon} \sum_{j=0} \sum_{m=0} \sum_{n=0} \epsilon^j \hat{y}_1^m \hat{y}_2^n p^{(jmn)}(\hat{y}_3), \tag{59b}$$

Let

$$\omega_1^{(ijk)} = -\hat{\partial}_3 u_2^{(ijk)} - \hat{\kappa}_3 u_1^{(ijk)} + \hat{\kappa}_1 u_3^{(ijk)}, \tag{60a}$$

$$\omega_2^{(ijk)} = \hat{\partial}_3 u_1^{(ijk)} - \hat{\kappa}_3 u_2^{(ijk)} + \hat{\kappa}_2 u_3^{(ijk)}. \tag{60b}$$

The leading-order expressions for the (dimensional) moments M_i can then be written

$$\frac{M_1}{\rho g I} = -p^{(001)} + 2\Delta^{(101)} + 2\hat{\kappa}_3 u_3^{(110)}, \tag{61a}$$

$$\frac{M_2}{\rho g I} = p^{(010)} - 2\Delta^{(110)} + 2\hat{\kappa}_3 u_3^{(101)}, \tag{61b}$$

$$\frac{M_3}{\rho g I} = -\omega_1^{(110)} - \omega_2^{(101)} + \hat{\kappa}_1 \omega_2^{(000)} - \hat{\kappa}_2 \omega_1^{(000)}. \tag{61c}$$

The sequential solution procedure described previously yields

$$\begin{gathered}
\Delta^{(101)} = \hat{\partial}_3 \omega_1^{(000)} + \hat{\kappa}_2 u_2^{(110)}, \\
\Delta^{(110)} = -\hat{\partial}_3 \omega_2^{(000)} + \hat{\kappa}_1 u_2^{(110)}, \\
u_3^{(110)} = -\omega_2^{(000)}, \quad u_3^{(101)} = \omega_1^{(000)}, \\
\omega_1^{(110)} = -\hat{\kappa}_1 \omega_2^{(000)} - \hat{\partial}_3 u_2^{(110)}, \\
\omega_2^{(101)} = \hat{\kappa}_2 \omega_1^{(000)} - \hat{\partial}_3 u_2^{(110)}, \\
p^{(001)} = -\hat{\partial}_3 \omega_1^{(000)} + \hat{\kappa}_3 \omega_2^{(000)} - \hat{\kappa}_2 u_2^{(110)}, \\
p^{(010)} = \hat{\partial}_3 \omega_2^{(000)} + \hat{\kappa}_3 \omega_1^{(000)} - \hat{\kappa}_1 u_2^{(110)}.
\end{gathered} \tag{62}$$

Substituting (62) into (61) and redimensionalizing using the relations

$$\{\omega_\alpha, \omega_3\} \approx \frac{\rho g L}{\epsilon^2 \mu} \left\{ \omega_\alpha^{(000)}, u_2^{(110)} \right\}, \tag{63}$$

we obtain

$$M_1 = 3\mu I \left(\omega_1' + \kappa_2 \omega_3 - \kappa_3 \omega_2 \right) \tag{64a}$$

$$M_2 = 3\mu I \left(\omega_2' + \kappa_3 \omega_1 - \kappa_1 \omega_3 \right) \tag{64b}$$

$$M_3 = 2\mu I (\omega_3' + \kappa_1 \omega_2 - \kappa_2 \omega_1). \tag{64c}$$

G. Summary

The unsteady flow of a liquid rope is described by the twenty-one variables A, x_1, x_2, x_3, q_0, q_1, q_2, q_3, U_1, U_2, U_3, W, ω_1, ω_2, ω_3, N_1, N_2, N_3, M_1, M_2, and M_3. The twenty-one differential equations they satisfy are (6), (21) in the form $W' = \Delta$, (24), (26), (28), (37), (41), (57), and (64). The auxiliary definitions required to close the system are (9), (22), (39), (43), (49), and (50).

III. LINEARIZED EQUATIONS FOR STABILITY OF STEADY COILING

A. Basic state

The basic states whose stability we shall analyze are numerical solutions for the steady coiling of a viscous rope, obtained using the continuation method described in Ribe[14]. The equations governing steady coiling are obtained from the full unsteady equations in § II by setting $\partial_t = U_1 = U_2 = 0$, $U_3 = W \equiv U$, and $\omega_\alpha = \kappa_\alpha U$, where $U(s)$ is the velocity parallel to the axis of the rope. The result is a seventeenth order two-point boundary-value problem for the variables $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{q}_0$, $\bar{q}_1$, $\bar{q}_2$, $\bar{q}_3$, $\bar{U}$, $\bar{\kappa}_1$, $\bar{\kappa}_2$, $\bar{\omega}_3$, $\bar{N}_1$, $\bar{N}_2$, $\bar{N}_3$, $\bar{M}_1$, $\bar{M}_2$, and $\bar{M}_3$, where the overbars have been added to distinguish the variables of the basic state from the perturbation variables to be introduced in a moment. Because the flow is steady, each barred variable is a function of the arclength s only. Examples of the geometry of steady coiling are shown in Figs. 1 and 2, and the mathematical structure of the solutions is discussed in more detail in Ribe[14].

B. Perturbation expansion

The next step is to write each of the twenty-one unsteady dependent variables as the sum of a steady (barred) value and an exponentially growing perturbation. Denoting the spatially varying parts of the perturbation variables by hats, we have

$$
\begin{gathered}
A = Q/\bar{U} + \hat{A}E, \qquad x_i = \bar{x}_i + \hat{x}_i E, \qquad q_j = \bar{q}_j + \hat{q}_j E, \\
U_\alpha = \hat{U}_\alpha E, \qquad U_3 = \bar{U} + \hat{U}_3 E, \qquad W = \bar{U} + \hat{W} E, \\
\omega_\alpha = \bar{\kappa}_\alpha \bar{U} + \hat{\omega}_\alpha E, \qquad \omega_3 = \bar{\omega}_3 + \hat{\omega}_3 E, \\
N_i = \bar{N}_i + \hat{N}_i E, \qquad M_i = \bar{M}_i + \hat{M}_i E,
\end{gathered}
\tag{65}
$$

where $E = \exp(\sigma t)$ and σ is the growth rate. By substituting (65) into the equations derived in § II and linearizing in the usual way, we obtain a set of twenty-one coupled linear ODEs for the perturbation variables $\hat{A} \equiv 2\pi\bar{a}\hat{a}$, $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, $\hat{q}_0$, $\hat{q}_1$, $\hat{q}_2$, $\hat{q}_3$, $\hat{U}_1$, $\hat{U}_2$, $\hat{U}_3$, $\hat{W}$, $\hat{\omega}_1$, $\hat{\omega}_2$, $\hat{\omega}_3$, $\hat{N}_1$, $\hat{N}_2$, $\hat{N}_3$, $\hat{M}_1$, $\hat{M}_2$, and $\hat{M}_3$. These equations are given explicitly in the Appendix.

C. Boundary conditions

The boundary conditions satisfied by the dependent variables are most conveniently obtained by working in a fixed laboratory reference frame. The velocity and angular velocity vectors in the laboratory frame are related to those in the corotating frame by

$$\boldsymbol{V}_{\rm lab} = \boldsymbol{V} + \Omega \boldsymbol{e}_3 \times \boldsymbol{x}, \qquad \boldsymbol{\omega}_{\rm lab} = \boldsymbol{\omega} + \Omega \boldsymbol{e}_3. \tag{66}$$

Consider first the boundary conditions at the injection point $s = 0$. The cross-sectional area of the rope, the Cartesian coordinates of its axis, and the advection velocity W are all fixed there, requiring

$$A(0) - A_0 = x_1(0) = x_2(0) = x_3(0) = W(0) - U_0 = 0, \tag{67}$$

where $A_0 = \pi a_0^2$ and $U_0 = Q/A_0$. With no loss of generality, we stipulate that the local basis vectors $\boldsymbol{d}_i(0)$ at the injection point are constant in the rotating frame, which implies

$$d_{ij}(0) = \begin{pmatrix} \bar{q}_1(0)^2 - \bar{q}_2(0)^2 & 2\bar{q}_1(0)\bar{q}_2(0) & 0 \\ 2\bar{q}_1(0)\bar{q}_2(0) & \bar{q}_2(0)^2 - \bar{q}_1(0)^2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{68}$$

where $\bar{q}_1(0)$ and $\bar{q}_2(0)$ are the Euler parameters for the steady coiling solution. The fluid velocity at $s = 0$ is equal to the imposed injection velocity, or

$$\boldsymbol{V}(0) = -U_0 \boldsymbol{e}_3. \tag{69}$$

The vanishing of the angular velocity in the laboratory frame requires

$$\boldsymbol{\omega}(0) + \Omega \boldsymbol{e}_3 = 0. \tag{70}$$

Turning now to the contact point $s = \ell$, we note that the vertical coordinate of the rope's axis there is just (minus) the total fall height less a small correction for the finite radius of the rope, or

$$x_3(\ell) = -H + a(\ell). \tag{71}$$

Because fluid typically piles up beneath the coiling rope in laboratory experiments (Fig. 2), the fall height H appearing in (71) should be interpreted as an effective value, i.e., the total fall height less the height of the fluid pile. The mobility of the contact point requires that

the rope's axis there must be horizontal and have zero curvature about a horizontal axis normal to the rope, or

$$d_{33}(\ell) = d'_{33}(\ell) = 0. \tag{72}$$

The no-slip condition requires the velocity of the rope axis at the contact point to be zero in the laboratory frame, except for a small vertical velocity proportional to the rate of change of the rope's radius. Thus

$$\boldsymbol{V}(\ell) + \Omega \boldsymbol{e}_3 \times \boldsymbol{x}(\ell) = \boldsymbol{e}_3 \frac{d}{dt} a(\ell), \tag{73}$$

Finally, the vanishing of the angular velocity vector in the laboratory frame requires

$$\boldsymbol{\omega}(\ell) + \Omega \boldsymbol{e}_3 = 0. \tag{74}$$

All the above boundary conditions are valid for an arbitrary time-dependent motion of the rope. To determine the boundary conditions satisfied by the perturbation (hatted) variables, we begin by recalling the boundary conditions for steady coiling, which are[14]

$$\begin{aligned} 0 = \bar{A}(0) - A_0 = \bar{x}_1(0) = \bar{x}_2(0) = \bar{x}_3(0) \\ = \bar{q}_0(0) = \bar{q}_3(0) = \bar{V}_3(0) - U_0 = \bar{W}(0) - U_0 \\ = \bar{\omega}_1(0) = \bar{\omega}_2(0) = \bar{\omega}_3(0) - \Omega \end{aligned} \tag{75}$$

at the injection point $s = 0$ and

$$\begin{aligned} = \bar{x}_1(\bar{\ell}) - \Omega^{-1}\bar{U}(\bar{\ell}) = \bar{x}_2(\bar{\ell}) = \bar{x}_3(\bar{\ell}) + H - \bar{a}(\bar{\ell}) \\ = \bar{q}_0(\bar{\ell}) = \bar{q}_1(\bar{\ell}) = \bar{q}_2(\bar{\ell}) - 2^{-1/2} = \bar{q}_3(\bar{\ell}) + 2^{-1/2} \\ = \bar{\omega}_1(\bar{\ell}) = \bar{\omega}_2(\bar{\ell}) - \Omega = \bar{\omega}_3(\bar{\ell}). \end{aligned} \tag{76}$$

at the contact point $s = \bar{\ell}$, where $\bar{U}(s) = Q/\bar{A}(s)$. The boundary conditions satisfied by the perturbation (hatted) variables are obtained by linearizing the boundary conditions (67) through (74) about the steady boundary conditions (75) and (76). At the injection point, the resulting conditions are

$$\begin{aligned} 0 = \hat{A}(0) = \hat{x}_1(0) = \hat{x}_2(0) = \hat{x}_3(0) \\ = \hat{q}_0(0) = \hat{q}_1(0) = \hat{q}_3(0) = \hat{V}_1(0) = \hat{V}_2(0) = \hat{V}_3(0) \end{aligned}$$

$$= \hat{W}(0) = \hat{\omega}_1(0) = \hat{\omega}_2(0) = \hat{\omega}_3(0). \tag{77}$$

The derivation of the boundary conditions at the contact point $s = \ell$ is somewhat more complicated, because both the rope length $\ell \equiv \bar{\ell}+\hat{\ell}$ and the base vectors $\boldsymbol{d}_i(\ell) \equiv \bar{\boldsymbol{d}}_i(\ell)+\hat{\boldsymbol{d}}_i(\ell)$ change with time. To first order in the perturbation quantities, therefore, the expansion of a generic scalar or vector variable $\phi(\ell)$ at the contact point is

$$\phi(\ell) \approx \bar{\phi}(\bar{\ell}) + \hat{\ell}\bar{\phi}'(\bar{\ell}) + \hat{\phi}(\bar{\ell}). \tag{78}$$

Linearizing the boundary conditions (71) through (74) about the steady conditions (76) with the help of (78), we obtain the following boundary conditions for the perturbation variables at $s = \bar{\ell}$:

$$\begin{aligned}
0 = \hat{x}_3(\bar{\ell}) - \hat{\ell}\bar{a}'(\bar{\ell}) - \hat{a}(\bar{\ell}) &= \hat{q}_2(\bar{\ell}) + \hat{q}_3(\bar{\ell}) \\
&= \hat{V}_1(\bar{\ell}) - \Omega\left[\bar{x}_1(\bar{\ell})\hat{d}_{31}(\bar{\ell}) - \hat{x}_2(\bar{\ell})\right] \\
&= \hat{V}_2(\bar{\ell}) + \sigma\left[\hat{\ell}\bar{a}'(\bar{\ell}) + \hat{a}(\bar{\ell})\right] \\
= \hat{V}_3(\bar{\ell}) - \Omega\left[\bar{x}_1(\bar{\ell})\hat{d}_{32}(\bar{\ell}) + \hat{x}_1(\bar{\ell})\right] &+ \hat{\ell}\bar{\Delta}(\bar{\ell}) \\
&= \hat{\omega}_1(\bar{\ell}) + \hat{\ell}\frac{\bar{M}_1(\bar{\ell})}{3\mu\bar{I}(\bar{\ell})} - \Omega\hat{d}_{21}(\bar{\ell}) \\
&= \hat{\omega}_2(\bar{\ell}) + \hat{\ell}\frac{\bar{M}_2(\bar{\ell})}{3\mu\bar{I}(\bar{\ell})} - \Omega\hat{d}_{23}(\bar{\ell}) \\
&= \hat{\omega}_3(\bar{\ell}) + \hat{\ell}\frac{\bar{M}_3(\bar{\ell})}{2\mu\bar{I}(\bar{\ell})} - \Omega\hat{d}_{22}(\bar{\ell}).
\end{aligned} \tag{79}$$

In (79) and henceforth, $\hat{d}_{ij}(s)$ are the perturbations of the direction cosines (9), e.g. $\hat{d}_{12} = 2(\bar{q}_1\hat{q}_2 + \bar{q}_2\hat{q}_1 + \bar{q}_0\hat{q}_3 + \bar{q}_3\hat{q}_0)$. The constitutive relations (64) have been used to eliminate $\bar{\omega}_i'(\bar{\ell})$ from the boundary conditions on $\hat{\omega}_i(\bar{\ell})$ in (79).

In summary, (77) and (79) are the twenty-two boundary conditions required to constrain the twenty-one perturbation variables and the unknown perturbation $\hat{\ell}$ of the rope length.

IV. NUMERICAL STABILITY ANALYSIS

The equations (A.1) together with the boundary conditions (77) and (79) constitute a linear two-point boundary-value problem of order 21 that has nontrivial solutions only for particular values of the growth rate σ. We solve this eigenvalue problem numerically using

a continuation method implemented by the software package AUTO 97[21,22] (freely available at http://indy.cs.concordia.ca/auto/). The basic idea (e.g., Keller[23], p. 235) is to introduce into the boundary conditions three new adjustable real parameters β_i $(i = 1, 2, 3)$ which are then varied gradually to refine an initial guess for the (possibly complex) eigenvalue σ. In particular, we introduce a new boundary condition

$$\hat{M}_1(\bar{\ell}) = \beta_1 + i\beta_2, \tag{80}$$

and modify the boundary condition on $\hat{\omega}_1(\bar{\ell})$ from (79) to

$$\hat{\omega}_1(\bar{\ell}) = -\hat{\ell}\frac{\bar{M}_1(\bar{\ell})}{3\mu\bar{I}(\bar{\ell})} + \Omega\hat{d}_{21}(\bar{\ell}) + \beta_3. \tag{81}$$

The problem is initialized by setting $\beta_1 = \beta_2 = \beta_3 = 0$ and making an initial guess for the growth rate σ. The solution procedure then comprises two steps. First, we 'pull' β_3 away from 0 to some finite value (e.g., 1) with σ fixed, letting β_1 and β_2 float freely. Then β_3 is 'pushed' gradually back to 0 with β_1 and β_2 fixed, leaving the real and imaginary parts of σ free to float. At the end of this process, one has both an eigenvalue σ and the full set of associated complex eigenfunctions for the twenty-one perturbation variables. High accuracy is ensured by solving the equations for the steady basic state simultaneously in the same program, on the same numerical grid as the perturbation equations. The resulting system is of order 59 (17 steady variables plus the real and imaginary parts of 21 perturbation variables).

Here we present the results of stability analyses for three of the laboratory experiments reported by Ribe *et al.*[10], in each of which the coiling frequency Ω is measured as the fall height H is varied for fixed values of the hole diameter d, the flow rate Q, and the fluid properties ρ, ν, and γ. Each experiment is therefore defined by particular values of the dimensionless groups

$$\Pi_1 = \left(\frac{\nu^5}{gQ^3}\right)^{1/5}, \quad \Pi_2 = \left(\frac{\nu Q}{gd^4}\right)^{1/4}, \quad \Pi_3 = \frac{\gamma d^2}{\rho\nu Q}. \tag{82}$$

To carry out the stability analysis for a given experiment, we first calculate numerically the dimensionless frequency $\Omega(\nu/g^2)^{1/3} \equiv \tilde{\Omega}$ of steady coiling as a function of the dimensionless height $H(g/\nu^2)^{1/3} \equiv \tilde{H}$. This yields a curve similar to that shown (in dimensional form) in Fig. 3. Next, we choose a trial value of $\tilde{H}$, and use the 'pull/push' procedure described above to search for unstable modes having $\Re(\sigma) > 0$. We then continue any such modes

in both directions along the curve $\tilde{\Omega}(\tilde{H})$, monitoring σ to identify the fall heights at which $\Re(\sigma)$ becomes zero, i.e. at which the mode in question becomes stable. By repeating this procedure for different trial values of $\tilde{H}$ along the curve $\tilde{\Omega}(\tilde{H})$, we determine the portions of the curve that represent unstable steady states.

The results of this procedure are shown in Figs. 5 - 7 for the parameters (Π_1, Π_2, Π_3) corresponding to the three laboratory experiments referred to above. In each figure, the symbols indicate experimental measurements obtained in series with H increasing (squares), decreasing (circles), and varied randomly (triangles.) The continuous curve in each figure shows the numerically calculated curve $\hat{\Omega}(\hat{H})$ for steady coiling, and its solid and dashed portions indicate stable and unstable steady states, respectively. Overall, the agreement between the numerical calculations and the experiments is very close: the observed steady states are concentrated along the stable portions of the calculated curves, leaving the unstable portions almost entirely 'unpopulated'. The only significant exceptions are the three measurements with the highest frequencies in Fig. 5, which lie close to an unstable segment of the calculated curve. However, the growth rate of the instability along this portion of the curve is very small ($\sigma \approx 0.02\Omega$), implying that the coiling rope executes $\Omega/2\pi\sigma \approx 8$ revolutions during the time required for a perturbation to grow by a factor e. This may explain why apparently steady states such as those in Fig. 5 are observed despite their instability *sensu stricto*.

V. DISCUSSION

A more detailed examination of our numerical solutions helps to understand the mechanism by which steady coiling becomes unstable. As an illustration, we consider the case $\Pi_1 = 3690$, $\Pi_2 = 2.19$, $\Pi_3 = 0$, and $\tilde{H} = 0.894$, for which the steady coiling frequency is $\tilde{\Omega} = 1.401$ (solid black circle in Fig. 6). Setting $\Pi_3 = 0$ eliminates the uninteresting effect of surface tension, which increases the steady coiling frequency by only 5%.

Fig. 8 shows the lateral displacement $\bar{x}_1(s)$ of the rope's axis (in the plane containing the injection point and the contact point with the plate), the bending moment $\bar{M}_1(s)$, and the viscous, gravitational, and inertial forces per unit rope length in the $\boldsymbol{d}_2$-direction. These forces are defined (using an obvious notation) as

$$\left\{\bar{f}_V, \bar{f}_G, \bar{f}_I\right\} = \{\boldsymbol{N}', \rho A \boldsymbol{g}, -\rho A \boldsymbol{J}\} \cdot \boldsymbol{d}_2, \tag{83}$$

where $\boldsymbol{J}$ is given by (39). Because $\gamma = 0$ for our illustrative example, (37) implies that $\bar{f}_V + \bar{f}_G + \bar{f}_I = 0$.

The steady coiling solution comprises an interior region in which bending is negligible ($\bar{M}_1 \approx 0$), and two boundary layers near the injection and contact points where significant bending is concentrated (Fig. 8b). In the interior, the rope behaves essentially as a 'whirling viscous string'[10]: the lateral deflection increases smoothly downward (Fig. 8a), and the gravitational force is balanced about equally by the viscous force associated with axial stretching and by (centrifugal) inertia (Fig. 8c). In the lower (and more dynamically significant) boundary layer, the gravitational force is balanced almost entirely by the viscous force associated with bending, with inertia playing a subsidiary role.

The most unstable eigenmode of the steady solution shown in Fig. 8 has a real eigenvalue $\sigma = 0.625\Omega$, where Ω is the steady coiling frequency. The structure of this eigenmode is shown in Fig. 9, using the same variables (lateral deflection, bending moment, and forces per unit length) as for the steady solution. The perturbation forces $\hat{f}_V$, $\hat{f}_G$, and $\hat{f}_I$ are given respectively by the viscous, gravitational, and inertial terms of (A.1l) with $\gamma = 0$ and $\alpha = 2$, and satisfy $\hat{f}_V + \hat{f}_G + \hat{f}_I = 0$. The structural features of the eigenmode are concentrated in the lower boundary layer, where the gravitational force is balanced primarily by viscous forces (Fig. 9c). The mechanism of the instability therefore involves a balance between gravity and the viscous resistance of the rope to bending, with inertia playing a secondary role. This conclusion can be verified by 'turning off' all the inertial terms in the perturbation equations (A.1) while holding constant all the other parameters in the numerical code. The instability still occurs; but the growth rate $\sigma = 1.42\Omega$ is now more than double the 'true' growth rate $\sigma = 0.625\Omega$ predicted by the full numerical model with all inertial terms retained. This demonstrates that inertia is not essential to the instability, but that it nevertheless significantly influences the growth rate.

A comparison of Figs. 5-7 raises a further question: how does the number N_s of stable segments of the curve $\Omega(H)$ depend on the experimental parameters? The stable segments are confined for the most part to the roughly horizontal portions ('steps') of the $\Omega(H)$ curve. Ribe *et al.*[10] showed that the total number N of (stable and unstable) steps in the curve scales as $N \sim \Pi_1^{5/32}$ in the limit when $\Pi_1 \to \infty$ and gravitational stretching of the rope is strong ($a_1 \ll a_0$). Figs. 5-7 suggest that N_s also increases with Π_1: $N_s = 2$ for $\Pi_1 = 1220$, and $N_s = 3$ for $\Pi_1 = 3690$ and 10050. Moreover, the fourth step in Fig. 7 is only slightly

unstable ($\sigma \approx 0.004\Omega$), suggesting that $\Pi_1 = 10050$ may be just below the value above which $N_s = 4$. Unfortunately, numerical convergence becomes difficult to achieve when Π_1 and/or Ω is too large, and we were therefore not able to determine a scaling law for N_s. For now, we can only speculate that it scales in the same way as the total number of steps, viz., $N_s \sim \Pi_1^{5/32}$.

In conclusion, our linear stability analysis shows that steady coiling in the multivalued 'inertio-gravitational' (IG) regime is stable only along discrete segments of the frequency vs. height curve, the distribution of which agrees very well with high-resolution laboratory measurements. The stability analysis further shows that coiling is stable at all heights in the three remaining regimes (viscous, gravitational, and inertial), in agreement with the experiments of Maleki *et al.*[9]. Analytical theory, numerical analysis, and laboratory experiments thus come together to offer a consistent portrait of steady coiling over the whole range of fall heights and frequencies at which the phenomenon occurs.

Acknowledgments

We are grateful to A. Boudaoud, A. Davaille, and J. Lister for helpful discussions, and to J. Lister for suggesting the method used to determine the eigenvalues of the linear stability problem. We also thank two anonymous referees for comments that led to significant improvements in the presentation. Symbolic mathematical manipulations were performed with the aid of Mathematica[20]. This is IPGP contribution number 2150.

APPENDIX: PERTURBATION EQUATIONS

The twenty-one first-order ODEs satisfied by the perturbation variables are given below. Here $\langle P \rangle \equiv \hat{P}$ is an alternate notation for the perturbation of the enclosed quantity, and $\|P, Q\| = \bar{P}\hat{Q} - \bar{Q}\hat{P}$. In addition, $\mathcal{N}_3 = N_3 + \pi\gamma a$.

$$\bar{U}\hat{a}' = -\sigma\hat{a} + \frac{\|\mathcal{N}_3, aW\|}{6\mu\bar{A}\bar{U}}, \tag{A.1a}$$

$$\hat{x}_i' = \hat{d}_{3i}, \tag{A.1b}$$

$$2\bar{U}\hat{q}_0' = -2\sigma\hat{q}_0 - \bar{q}_1\hat{\omega}_1 - \bar{q}_2\hat{\omega}_2 - \bar{q}_3\hat{\omega}_3$$

$$+\bar{\kappa}_1\|q_1,W\|+\bar{\kappa}_2\|q_2,W\|+\bar{\kappa}_3\|q_3,W\| \tag{A.1c}$$

$$2\bar{U}\hat{q}_1' = -2\sigma\hat{q}_1+\bar{q}_0\hat{\omega}_1-\bar{q}_3\hat{\omega}_2+\bar{q}_2\hat{\omega}_3$$

$$-\bar{\kappa}_1\|q_0,W\|+\bar{\kappa}_2\|q_3,W\|-\bar{\kappa}_3\|q_2,W\| \tag{A.1d}$$

$$2\bar{U}\hat{q}_2' = -2\sigma\hat{q}_2+\bar{q}_3\hat{\omega}_1+\bar{q}_0\hat{\omega}_2-\bar{q}_1\hat{\omega}_3$$

$$-\bar{\kappa}_1\|q_3,W\|-\bar{\kappa}_2\|q_0,W\|+\bar{\kappa}_3\|q_1,W\| \tag{A.1e}$$

$$2\bar{U}\hat{q}_3' = -2\sigma\hat{q}_3-\bar{q}_2\hat{\omega}_1+\bar{q}_1\hat{\omega}_2+\bar{q}_0\hat{\omega}_3$$

$$+\bar{\kappa}_1\|q_2,W\|-\bar{\kappa}_2\|q_1,W\|-\bar{\kappa}_3\|q_0,W\| \tag{A.1f}$$

$$\hat{U}_\alpha' = \epsilon_{\alpha\beta3}\left(\hat{\omega}_\beta-\bar{U}\hat{\kappa}_\beta-\bar{\kappa}_\beta\hat{U}_3+\bar{\kappa}_3\hat{U}_\beta\right) \tag{A.1g}$$

$$\hat{U}_3' = \frac{\|A,\mathcal{N}_3\|}{3\mu\bar{A}^2}+\bar{\kappa}_2\hat{U}_1-\bar{\kappa}_1\hat{U}_2, \tag{A.1h}$$

$$\hat{W}' = \frac{\|A,\mathcal{N}_3\|}{3\mu\bar{A}^2}, \tag{A.1i}$$

$$\hat{\omega}_\alpha' = \frac{\|I,M_\alpha\|}{3\mu\bar{I}^2}+\epsilon_{\alpha ij}\bar{\kappa}_i\left(\bar{U}\hat{\kappa}_j-\hat{\omega}_j\right)-\epsilon_{\alpha i3}\Omega\hat{\kappa}_i, \tag{A.1j}$$

$$\hat{\omega}_3' = \frac{\|I,M_3\|}{2\mu\bar{I}^2}+\epsilon_{3\alpha\beta}\bar{\kappa}_\alpha\left(\bar{U}\hat{\kappa}_\beta-\hat{\omega}_\beta\right), \tag{A.1k}$$

$$\hat{N}_i' = \epsilon_{ijk}\langle N_j\kappa_k\rangle+\rho\langle AJ_i\rangle+\rho g\langle Ad_{i3}\rangle$$
$$-\gamma\left(\hat{A}'\delta_{i3}+2\pi\epsilon_{ij3}\langle a\kappa_j\rangle\right) \tag{A.1l}$$

$$\hat{M}_\alpha' = \epsilon_{\alpha jk}\langle M_j\kappa_k\rangle+\epsilon_{\alpha j3}\hat{N}_j+\rho\langle IK_\alpha\rangle$$
$$+\rho g\langle I\kappa_\alpha d_{33}\rangle-\gamma\langle A\kappa_\alpha a'\rangle, \tag{A.1m}$$

$$\hat{M}_3' = \epsilon_{\alpha\beta3}\langle M_\alpha\kappa_\beta\rangle+\rho\langle IK_3\rangle-\rho g\langle I\kappa_\beta d_{\beta3}\rangle. \tag{A.1n}$$

The perturbation curvatures $\hat{\kappa}_i$ are eliminated from the above equations using the auxiliary relations

$$\bar{U}\hat{\kappa}_1 = \hat{\omega}_1-\bar{\kappa}_1\hat{W}+2\sigma\left(\|q_1,q_0\|+\|q_2,q_3\|\right), \tag{A.2a}$$

$$\bar{U}\hat{\kappa}_2 = \hat{\omega}_2-\bar{\kappa}_2\hat{W}+2\sigma\left(\|q_2,q_0\|+\|q_3,q_1\|\right), \tag{A.2b}$$

$$\bar{U}\hat{\kappa}_3 = \hat{\omega}_3-\bar{\kappa}_3\hat{W}+2\sigma\left(\|q_3,q_0\|+\|q_1,q_2\|\right), \tag{A.2c}$$

which are themselves obtained by combining the perturbation forms of (10) and (26).

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FIG. 1: Steady coiling of viscous corn syrup (photograph by N. Ribe.) Fluid with density ρ, viscosity ν and surface tension coefficient γ is injected at volumetric rate Q through a hole of diameter $d \equiv 2a_0$ and falls a distance H onto a solid surface. The angular coiling frequency is Ω, the radius of the 'coil' portion of the rope is R, and the radius of the rope at the base of the coil is a_1.

FIG. 2: (a) Schematic diagram of the experimental apparatus used for the coiling experiments. Silicone oil is pumped from a syringe using a stepper motor, and falls onto a table of adjustable height. Observations are recorded using a CCD camera operating at 25 frames s^{-1}. (b) and (c): Coiling of silicone oil with viscosity $\nu = 5000$ cm^2 s^{-1}, injected from a hole of diameter $d = 0.15$ cm at a rate $Q = 0.0066$ cm^3 s^{-1} and falling a distance $H = 20$ cm. The low-frequency and high-frequency states observed for these parameters are shown in panels (b) and (c), respectively.

FIG. 4: Geometry of a thin viscous rope. The Cartesian coordinates of the rope's axis relative to an arbitrary origin O are $\boldsymbol{x}(s,t)$, where s is the arclength along the axis and t is time. The rope's radius is $a(s,t)$. The unit tangent vector to the axis is $\boldsymbol{d}_3(s,t) \equiv \boldsymbol{x}'$, and $\boldsymbol{d}_1(s,t)$ and $\boldsymbol{d}_2(s,t) \equiv \boldsymbol{d}_3 \times \boldsymbol{d}_1$ are material unit vectors in the plane of the rope's cross-section. The Cartesian unit vectors $\boldsymbol{e}_i$ are fixed in the reference frame rotating with an angular velocity equal to the angular frequency Ω of steady coiling.

FIG. 3: Regimes of liquid rope coiling. The symbols show experimental observations of the coiling frequency Ω as a function of the fall height H for an experiment performed using viscous silicone oil ($\rho = 0.97$ g cm^{-3}, $\nu = 1000$ cm^2 s^{-1}, $\gamma = 21.5$ dyne cm^{-1}) with $d = 0.068$ cm and $Q = 0.00215$ cm^3 s^{-1}[10]. The solid line is the numerically predicted curve of frequency vs. height for the same parameters. Portions of the curve representing the different coiling regimes are labeled: viscous (V), gravitational (G), inertio-gravitational (IG), and inertial (I).

FIG. 5: Stability of steady coiling with $\Pi_1 = 1220$, $\Pi_2 = 2.09$, and $\Pi_3 = 0.019$. The continuous curve shows the numerically calculated frequency of steady coiling as a function of height. The solid and dashed portions of the curve indicate stable and unstable steady states, respectively, as predicted using the numerical stability analysis described in the text. Symbols indicate experimental measurements[10] obtained in series with H increasing (squares), decreasing (circles), and varied randomly (triangles.)

FIG. 6: Same as Fig. 5, but for $\Pi_1 = 3690$, $\Pi_2 = 2.19$, and $\Pi_3 = 0.044$. The black dot indicates the coiling frequency at $H(g/\nu^2)^{1/3} = 0.894$ with the same values of Π_1 and Π_2 but with surface tension neglected ($\Pi_3 = 0$).

FIG. 7: Same as Fig. 5, but for $\Pi_1 = 10050$, $\Pi_2 = 3.18$, and $\Pi_3 = 0.048$. The parameters for this experiment are identical to those used in Fig. 3.

FIG. 8: Structure of the steady coiling solution for the case $\Pi_1 = 3690$, $\Pi_2 = 2.19$, $\Pi_3 = 0$, and $\tilde{H} = 0.894$ (Fig. 6). In this figure and in Fig. 9, the arclength s increases along the rope from the injection point $s = 0$ to the contact point $s = \ell$. (a) Lateral displacement $\bar{x}_1$. (b) Bending moment $\bar{M}_1$. (c) Forces per unit rope length in the $\boldsymbol{d}_2$-direction: viscous (heavy dashed line), gravitational (heavy solid line), and inertial (light solid line). The vertical light dashed line indicates $\bar{f} = 0$. $A_0 \equiv \pi a_0^2$ and $I_0 \equiv \pi a_0^4/4$ are the area and moment of inertia, respectively, of the injection hole.

FIG. 9: Structure of the most unstable eigenmode of the steady coiling solution shown in Fig. 8. The variables displayed in each panel are the perturbations of the steady variables in the corresponding panels of Fig. 8, and are all normalized to unit amplitude.