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To cite this version:

Fouad Elzein. Deligne-Hodge-DeRham theory with coefficients. 2007. hal-00128886

HAL Id: hal-00128886

https://hal.science/hal-00128886

Preprint submitted on 3 Feb 2007

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Deligne-Hodge-DeRham theory with coefficients

Fouad Elzein

Abstract

Let $\mathcal{L}$ be a variation of Hodge structures on the complement X^* of a normal crossing divisor (NCD) Y in a smooth analytic variety X and let $j : X^* = X - Y \to X$ denotes the open embedding. The purpose of this paper is to describe the weight filtration W on a combinatorial logarithmic complex computing the (higher) direct image $\mathbf{j}_*\mathcal{L}$, underlying a mixed Hodge complex when X is proper, proving in this way the results in the note [14] generalizing the constant coefficients case. When a morphism $f : X \to D$ to a complex disc is given with $Y = f^{-1}(0)$, the weight filtration on the complex of nearby cocycles $\Psi_f(\mathcal{L})$ on Y can be described by these logarithmic techniques and a comparison theorem shows that the filtration coincides with the weight defined by the logarithm of the monodromy which provides the link with various results on the subject.

Mathematics Subject Classification(2000): 32S35 (primary), 32G20, 14D07 (secondary).

hal-00128886, version 1 - 3 Feb 2007

1§. Introduction

The subject of this article is to construct a mixed Hodge structure MHS on the cohomology of a local system $\mathcal{L}$ underlying a polarised variation of Hodge structures (VHS) on the complement $X^* = X - Y$ of a normal crossing divisor (NCD) Y in a smooth proper complex algbraic or analytic variety X. Let $j : X^* \to X$ denotes the open embedding. Technically, we need to define a structure of mixed Hodge complex MHC on the higher direct image $\mathbf{j}_*\mathcal{L}$.
The work consists then in two parts, first to define the rational weight filtration W and second to construct the complex weight W and Hodge filtration F. Although we will use the same letter $\mathcal{L}$ for the rational as well the complex local system, sometimes when we need to stress the difference we denote by $\mathcal{L}^r$ the rational and by $\mathcal{L}^c$ the complex local system.
In the rational case, we don't have a particular representative of $\mathbf{j}_*\mathcal{L}$ by a distinguished complex, so the method is to use the theory of perverse sheaves to describe W.
While in the complex case we need to construct a bi-filtered complex and we use the logarithmic complex with coefficients in *Deligne's analytic extension with regular singular connection* $\mathcal{L}_X$ the bundle extension of $\mathcal{L} \otimes \mathcal{O}_{X^*}$ since by Deligne's theorem

$$\mathbf{j}_*\mathcal{L} \cong \Omega^*_X(LogY) \otimes \mathcal{L}_X \tag{1}$$

By the subsequent work of Schmid, Cattani and Kaplan, Kashiwara and Kawai, the Hodge filtration F extends by sub-bundles. In this article we describe a bi-filtered complex

$$(\Omega^*\mathcal{L}, \mathcal{W}, F) \tag{2}$$

constructed as a sum of a combinatorial complex constantly equal to the logarithmic complex with coefficients and which underly the structure of MHC we are looking for. Although the existence of such bi-filtered complex is important in the general theory, the basic results on $Gr^{\mathcal{W}}\mathcal{L}$ can be stated more easily so to reflect the topological and geometrical properties of the variety and the local system. Let us fix the hypothesis and the notations for the rest of the article.

Hypothesis . *Let $\mathcal{L}$ be a polarised local system defined over $\mathbb{Q}$, on the complement of the normal crossing divisor (NCD) Y in a smooth analytic variety X, $(\mathcal{L}_X, \nabla)$ the canonical extension of $\mathcal{L}_{X^*} = \mathcal{L} \otimes \mathcal{O}_{X^*}$ [6] with a meromorphic connection ∇ on X having a regular singularity along Y in X and $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ the associated DeRham logarithmic complex defined by ∇, moreover $\mathcal{L}^{\mathbb{C}} = \mathcal{L} \otimes \mathbb{C}$ underlies a polarised variation of Hodge structures (VHS).*
In the text we write $\mathcal{L}$ for the rational $\mathcal{L}^r$ as well complex local system $\mathcal{L}^c$, $\mathcal{L}_X$ for the analytic extension and to simplify the exposition, we suppose $\mathcal{L}$ locally unipotent along Y.

Notation. *We suppose the NCD, $Y = \cup_{i \in I} Y_i$ equal to the union of irreducible and smooth components Y_i for i in I. For all subset K of I, let $Y_K = \cap_{i \in K} Y_i$, $Y^*_K = Y_K - \cup_{i \notin K} Y_i$, and $j^K : Y^*_K \to Y_K$ the locally closed embedding, then $Y_K - Y^*_K$ is a NCD in Y_K and the open subsets Y^*_K of Y_K form with X^* a natural stratification of X. All extensions of $\mathcal{L}$ considered are constructible with respect to this stratification and even perverse.*
We write $\mathcal{L}_{Y_K}$ for the restriction of $\mathcal{L}_X$ to Y_K, $\mathcal{N}_i$ for the nilpotent endomorphisms of the restriction $\mathcal{L}_{Y_K}$ logarithm of the unipotent part of the monodromy, and

$$\mathcal{W}^K_{Y_K} = \mathcal{W}(\Sigma_{i \in K} \mathcal{N}_i) \tag{3}$$

for the filtration by sub-bundles defined on $\mathcal{L}_{Y_K}$ by $\Sigma_{i \in K} \mathcal{N}_i$.
Let $i_K : Y^*_K \to X$, we introduce the local systems

$$\mathcal{L}^K = i^*_K R^{|K|} \mathbf{j}_* \mathcal{L}, \quad \mathcal{L}'^K = i^*_K R^0 \mathbf{j}_* \mathcal{L}$$

and the bundles

$$\mathcal{L}^K_X = \mathcal{L}_{Y_K} / (\Sigma_{i \in K} \mathcal{N}_i \mathcal{L}_{Y_K}), \quad \mathcal{L}'^K_X = \cap_{i \in K} (ker \mathcal{N}_i : \mathcal{L}_{Y_K} \to \mathcal{L}_{Y_K})$$

Throughout this work we prove the following results

Theorem 1. *i) $\mathcal{L}^K_X$ (resp. $\mathcal{L}'^K_X$) is a flat bundle with flat sections isomorphic to the local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$); precisely they are resp. Deligne's extension of the complex local system.*
ii) The filtration $\mathcal{W}^K_{Y_K}$ (3) induces a filtration by flat sub-bundles of $\mathcal{L}^K_X$ (resp $\mathcal{L}'^K_X$), hence induces a filtration by complex sub-local systems $\mathcal{W}^K$ of $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).
iii) The filtration $\mathcal{W}^K$ is defined on the rational local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).
iv) Let $\mathcal{L}^K_r := Gr^{\mathcal{W}^K}_{r-|K|} \mathcal{L}^K$ and $\mathcal{L}^K_{X,r} := Gr^{\mathcal{W}^K_{Y_K}}_{r-|K|} \mathcal{L}^K_X$ for $r > 0$, resp.
$\mathcal{L}^K_r := Gr^{\mathcal{W}^K}_{r+|K|} \mathcal{L}'^K$ and $\mathcal{L}^K_{X,r} := Gr^{\mathcal{W}^K_{Y_K}}_{r+|K|} \mathcal{L}'^K_X$ for $r < 0$,
then the system defined by $(\mathcal{L}^K_r, \mathcal{L}^K_{X,r}, F)$ where F is the Hodge filtration induced from $\mathcal{L}_X$, is a polarised variation of Hodge structures VHS.
v) The following decomposition property of $(\Omega^* \mathcal{L}, \mathcal{W})$ (2) into intermediate extensions of polarised VHS is satisfied

$$(Gr^{\mathcal{W}}_r \Omega^* \mathcal{L}, F) \cong \oplus_{K \subset I} \mathbf{j}^K_{!*} \mathcal{L}^K_r [-|K|], F[-|K|]), \text{ for } r > 0$$
$$(Gr^{\mathcal{W}}_0 \Omega^* \mathcal{L}, F) \cong 0$$
$$(Gr^{\mathcal{W}}_r \Omega^* \mathcal{L}, F) \cong \oplus_{K \subset I} j^K_{!*} \mathcal{L}^K_r [1 - |K|], F), \text{ for } r < 0$$

Theorem 2. *There exists a rational weight filtration $(\mathbf{j}_* \mathcal{L}, \mathcal{W})$ and a quasi-isomorphism $(\mathbf{j}_* \mathcal{L}, \mathcal{W}) \otimes_{\mathbb{Q}} \mathbb{C} \cong (\Omega^* \mathcal{L}, \mathcal{W})$, such that $\mathcal{W}_0 \cong \mathbf{j}_{!*} \mathcal{L}$.*

Theorem 3. *For X proper of dim.n, there exists a natural MHS on various cohomology groups with coefficients in $\mathcal{L}$ underlying a polarised VHS of weight m on $X - Y$, as follows*

i) The bi-filtered complex

$$(\Omega^*\mathcal{L}[n], W, F) \text{ where } W = \mathcal{W}[m+n] \text{ (that is } W_{i+m+n} = \mathcal{W}_i) \text{ for } i \geq 0 \text{ and } 0 \text{ for } i < 0$$

underlies a MHC isomorphic to $\mathbf{j}_*\mathcal{L}[n]$ *s.t.* $H^i(X-Y, \mathcal{L}[n])$ *is of weight* $\geq i+m+n$. *The filtration* W *is by perverse sheaves.*
Dually $H^i_c(X-Y, \mathcal{L}[n])$ *is of weight* $\leq i+m+n$.

ii) *Let* $i_Y : Y \to X$ *denotes the embedding. The quotient complex* $i_Y^*(\Omega^*\mathcal{L}[n]/W_{m+n})$ *with the induced filtrations is a MHC quasi-isomorphic to* $\mathbf{i}_Y^!\mathbf{j}_{!*}\mathcal{L}[n+1]$ *s.t.* $H^i_Y(X, \mathbf{j}_{!*}\mathcal{L}[n])$ *is of weight* $\geq i+m+n$.

iii) *The bi-filtered complex*

$$(i_Y^*\mathcal{W}_0\Omega^*\mathcal{L}[n], W, F) \text{ where } W = \mathcal{W}[m+n+1], \ (W_{i+m+n+1} = \mathcal{W}_i) \text{ for } i < 0$$

underlies a MHC isomorphic to $i_Y^*\mathbf{j}_{!*}\mathcal{L}[n]$ *s.t.* $H^i(Y, i_Y^*\mathbf{j}_{!*}\mathcal{L}[n])$ *is of weight* $\leq i+m+n$.

As we see the Intersection cohomology [17] is the fundamental ingredient which provides the new class of Hodge complexes not merely defined by complete non singular projective varieties. In a different direction, the theory has been successfully related to the theory of differential modules [29]. We give now a more detailed discussion of the contents.

Weight filtration. The defining property of such filtration can be understood after a digression on the sheaves of nearby cycles. Under the additional hypothesis of the existence of a local equation f of Y, defining a morphism on an open subset $f : U \to D$ to a complex disc, such that $Y \cap U = f^{-1}(0)$, *let* $\mathcal{N} = Log\mathcal{T}^u$ *denotes the logarithm of the unipotent part of the monodromy. The filtration* $W(\mathcal{N})$ *on* Ψ_f^u *is defined by the nilpotent endomorphism* $\mathcal{N}$ *in the abelian category of rational (resp. complex) perverse sheaves.* The isomorphism in the abelian category of perverse sheaves [1,2]

$$\mathbf{j}_*\mathcal{L}[n]/\mathbf{j}_{!*}\mathcal{L}[n] \ \simeq \ Coker\mathcal{N}\colon \Psi_f^u(\mathcal{L})[n-1] \to \Psi_f^u(\mathcal{L})[n-1] \tag{4}$$

suggest to start the weight filtration with $\mathbf{j}_{!*}\mathcal{L}[n]$ and continue with $W(\mathcal{N})$ induced on $Coker\mathcal{N}$, then the main problem is to show, that *the various weights defined locally on* Y *(for different local equations) glue together on* $\mathbf{j}_*\mathcal{L}[n]/\mathbf{j}_{!*}\mathcal{L}[n]$.
For each local equation f, Ψ_f^u can be defined by a section of Verdier's monodromic specialization sheaf on the normal cone to Y in X [11]. For a different equation $g = uf$ with u invertible Ψ_g^u and Ψ_f^u are isomorphic but not canonically, so we cannot define a global complex Ψ_Y^u on Y and $W(\mathcal{N})$ on it, however we prove that the isomorphism becomes canonical on $Coker\mathcal{N}$ hence the induced $W(\mathcal{N})$ is globally defined on $\mathbf{j}_*\mathcal{L}[n]/\mathbf{j}_{!*}\mathcal{L}[n]$.
Here we solve the problem in the case of NCD by a different approach. We define first the complex weight filtration and then prove it is rationnally defined. The detailed study of the complex $\Omega^*\mathcal{L}[n]$, necessary for Hodge theory, leads to such approach of the rational weight.

Theorem 4. *For each local equation* f *of* Y *defined on an open set* U *of* X, $\mathcal{N}$ *acts on* $\Psi_f^u\mathcal{L}[n-1]$*; then the induced filtration by* $\mathcal{W}(\mathcal{N})$ *on Cokernel of* $\mathcal{N}$ *(the right term of (4)) coincides with the induced filtration by* $\mathcal{W}$ *on* $\mathbf{j}_{!*}\mathcal{L}[n]/\mathbf{j}_*\mathcal{L}[n]$ *(the left term).*

This problem has been solved successfully for any divisor using the deformation to the normal cone and will appear later in a joint work with Lê D.T. and Migliorini L.

Let $p : C_YX - Y \to Y$ *denotes the projection of the punctured normal cone onto* Y. *There exists a canonical filtration* $\mathcal{W}$ *on* $\mathbf{j}_*\mathcal{K}$ *defined as* $\mathcal{W}_0 = j_{!*}\mathcal{K}$ *and for* $i > 0$
$\mathcal{W}_i(\mathbf{j}_*\mathcal{K}/\mathbf{j}_{!*}\mathcal{K}) \ \simeq \ \mathcal{W}_{i-1}{}^p\mathcal{H}^0(\mathbf{p}_*(\Psi_Y\mathcal{K})) := Ker[{}^p\mathcal{H}^0(\mathbf{p}_*(\Psi_Y\mathcal{K}) \to {}^p\mathcal{H}^0(\mathbf{p}_*(\Psi_Y\mathcal{K}/\mathcal{W}_i(\mathcal{N}|\Psi_Y\mathcal{K})]$

s.t. for each local equation $f=0$ of Y defined on an open set U_f, we have on $Y_f=U_f\cap Y$ $\mathcal{W}_i(\mathbf{j}_*\mathcal{K}/\mathbf{j}_{!*}\mathcal{K})_{|Y_f}\ \simeq\ \mathcal{W}_{i-1}(Coker\mathcal{N}|(\Psi_f^u\mathcal{K}[-1])$.
Dually on $i^*\mathbf{j}_{!*}\mathcal{K}[-1]\simeq {}^p\mathcal{H}^{-1}(\mathbf{p}_*(\Psi_Y\mathcal{K})$ *the weight is defined as*
$(\mathcal{W}_i\, i^*\mathbf{j}_{!*}\mathcal{K})[-1]\simeq {}^p\mathcal{H}^{-1}(\mathbf{p}_*\mathcal{W}_{i+1}(\mathcal{N}|\Psi_Y\mathcal{K}))$ ========== *Purity and decomposition (local results).* Working with the complex local system $\mathcal{L}^c$ we need to exhibit a bifiltered complex $(\Omega^*\mathcal{L},\mathcal{W},F)$ underlying a mixed Hodge complex (MHC). Since the local system itself results in the geometric case from singularities of morphisms, it is natural to search for a combinatorial logarithmic complex for such bi-filtered complex. Its construction is suggested by an algebraic formula of the Intersection complex given by Kashiwara and Kawai in [26]. We explain now the main basic local results.
1- If we consider a point $y\in Y_M^*$, a variation of Hodge structures VHS on $\mathcal{L}$ of weight m defines a nilpotent orbit L with a set of nilpotent endomorphisms $N_i, i\in M$. The nilpotent orbit theorem [4], [24] states that the VHS degenerates along Y_M^* into a variation of MHS with weight filtration $W^M=W(\Sigma_{i\in M}N_i)$ shifted by m.
However the difficulty in the construction of the weight is to understand what happens at the intersection of Y_M and Y_K for two subsets M and K of I. This difficulty couldn't be explained for the $\mathbb{Q}-$ structure until the discovery of perverse sheaves. In order to prove the decomposition of $Gr^{\mathcal{W}}\Omega^*\mathcal{L}$ into intermediate direct image of various local systems on the components of $Y=\cup_{i\in I}Y_i$, we introduce in (3.3) for $K\subset M$, the complex C_r^KL constructed out of the nilpotent orbit $(L,N_i)_{i\in K}$ defined by $\mathcal{L}$ at a point $z\in Y_K^*$ and $W^K=W(\Sigma_{i\in K}N_i)$. We prove that C_r^KL has a unique non vanishing cohomology isomorphic to $Gr^{W^K}_{r-|K|}(L/\Sigma_{i\in K}N_iL)$, the fiber at z of the rationally defined local system $\mathcal{L}_r^K$ previously introduced for $r>0$.
Now Y_M is a subset of Y_K and we are interested in the fiber of $j_{!*}^K\mathcal{L}_r^K$ at $y\in Y_M^*$, so we introduce the complex $C_r^{KM}L$ in (3.3), which is quasi-isomorphic to this fiber and will appear as a component of the decomposition (3.8) of the graded part of the weight filtration.
These main results in the open case form the content of the second and third sections.
2- The second major technical result (5.3) in the case of nearby co-cycles, shows that if f is a local equation of Y, C_r^KL is isomorphic to the primitive part of the fiber ν_r^KL at y of a local system on Y_K^* component of the decomposition of $Gr_r^{W(\mathcal{N})}\psi_f^u\mathcal{L}$, which proves that the complex weight filtration is compatible with our description of the rational weight.
3- The key result that enables us to give most of the proofs is the existence, for a nilpotent orbit $(L,N_i,i\in K)$, of a natural decomposition (2.4)

$$Gr_r^{W^K}L=\oplus Gr^{W(N_{i_n})}_{m_{i_n}}\cdots Gr^{W(N_{i_1})}_{m_{i_1}}L\ :\ \Sigma_{i_j\in K}m_{i_j}=r$$

As a consequence the local systems on Y_K^* defined for $r>0$ by C_r^KL decompose into a direct sum of elementary components

$$\oplus Gr^{W(N_{i_k})}_{m_{i_k}}\cdots Gr^{W(N_{i_1})}_{m_{i_1}}(L/(N_{i_1}L+\cdots+N_{i_k}L))\ :\ \Sigma_{i_j\in K}m_{i_j}=r+|K|, m_{i_j}\geq 2$$

the corresponding elementary complexes are introduced in (3.5) and are key ingredients in the proof.
The above results explain the subtle relation between the filtration $\mathcal{W}(\mathcal{N})$ on $\Psi_f^u(\mathcal{L})$ which is hard to compute and the various local monodromy at points of Y.

We use the terminology of perverse sheaves but we leave to the reader the choice in the shift in degrees, generally by $-n$.
Finally the local definition of the weight is in (3.2), purity in (3.6) and decomposition in (3.8). The global definition of the weight is in (4.1) and the decomposition in (4.3). The weight of the

nilpotent action on Ψ_f^u is in (5.1) and the comparison in (5.3). Finally we suggest strongly to the reader to follow the proofs on an example, sometimes on the surface case as in the example in (3.2); this example will be again useful for $\Psi_f\mathcal{L}$ in (5.3). For X a line and $Y=0$ a point, the fiber at 0 of $\mathbf{j}_*\mathcal{L}$ is a complex $L \xrightarrow{N} L$ where (L,N) is a nilpotent orbit of weight m and the weight $\mathcal{W}[m]$ on the complex is $(\mathcal{W}[m])_{r+m} = \mathcal{W}_r$ defined by the sub-complex $(W_{r+1}L \xrightarrow{N} W_{r-1}L)$.

2§. Local invariants of $\mathcal{L}$

We need a precise description in terms of the local invariants of the local system $\mathcal{L}$. We recall some preliminaries on $\mathbf{j}_*\mathcal{L}$ and we give a basic local decomposition of weight filtrations defined by the local monodromy on nilpotent orbits.

I. Preliminaries

2.1 Local and global description of $\mathbf{j}_*\mathcal{L}$. In the neighbourhood of a point y in Y, we can suppose $X \simeq D^{n+k}$ and $X^* = X - Y \simeq (D^*)^n \times D^k$ where D is a complex disc, denoted with a star when the origin is deleted. The fundamental group $\Pi_1(X^*)$ is a free abelian group generated by n elements representing classes of closed paths around the origin, one for each D^* in the various axis with one dimensional coordinate z_i (the hypersurface Y_i is defined locally by the equation $z_i = 0$). Then the local system $\mathcal{L}$ corresponds to a representation of $\Pi_1(X^*)$ in a vector space L defined by the action of commuting automorphisms T_i for $i \in [1,n]$ indexed by the local components Y_i of Y and called monodromy action around Y_i. The automorphisms T_i decomposes as a product of commuting automorphisms, semi-simple and unipotent $T_i = T_i^s T_i^u$. Classically L is viewed as the fiber of $\mathcal{L}$ at the reference point for the fundamental group $\Pi_1(X^*)$, however since we will need to extend the Hodge filtration on Deligne's extended bundle, it is important to view L as the vector space of multiform sections of $\mathcal{L}$ (that is the sections of the inverse of $\mathcal{L}$ on a universal covering of X^*).

Given a $\mathbb{Q}-$local system, locally unipotent along Y (to simplify the exposition) we consider $\mathcal{L}_{X^*} := \mathcal{L} \otimes_{\mathbb{Q}} \mathcal{O}_{X^*}$ and its Deligne's bundle extension $\mathcal{L}_X$ which has a nice description as the subsheaf of $\mathbf{j}_*\mathcal{L}_{X*}$ generated locally at a point y in Y by sections associated to multiform sections of $\mathcal{L}$ as follows.

The logarithm of the unipotent monodromy, $N_i := LogT_i^u = -\Sigma_{k\geq 1}(1/k)(I-T_i^u)^k$ is defined as the sum of nilpotent endomorphisms $(I-T_i^u)$ so that this sum is finite.

A multiform section v corresponds to a germ $\tilde{v} \in \mathbf{j}_*\mathcal{L}_{X*}$ with an explicit description of the action of the connection by the formulas

$$\tilde{\mathrm{v}}(z) = (exp(-\frac{1}{2i\pi}\Sigma_{j\in J}(logz_j)N_j)).v, \quad \nabla\tilde{v} = -\frac{1}{2i\pi}\Sigma_{j\in J}\widetilde{N_j.v} \otimes \frac{dz_j}{z_j} \tag{5}$$

a basis of L is sent on a basis of $\mathcal{L}_{X,y}$.

The residue of the connection ∇ along each Y_j defines an endomorphism $\mathcal{N}_j$ on the restriction $\mathcal{L}_{Y_j}$ of $\mathcal{L}_X$.

The fiber at the origin of the complex $\Omega_X^*(LogY) \otimes \mathcal{L}_X$ is quasi-isomorphic to a Koszul complex as follows. We associate to $(L,N_i), i \in [1,n]$ a strict simplicial vector space such that for all sequences $(i.) = (i_1 < \cdots < i_p)$

$$L(i.) = L \qquad , \qquad N_{i_j}: L(i. - i_j) \to L(i.)$$

Definition. *The simple complex defined by the simplicial vector space above is the Koszul complex (or the exterior algebra) defined by* (L, N_i) *and denoted by* $\Omega(L, N.)$. *A general notation is* $s(L(J), N.)_{J\subset[1,n]}$ *where* J *is identified with the strictly increasing sequence of its elements and where* $L(J) = L$.

It is quasi-isomorphic to the Koszul complex $\Omega(L, Id-T.)$ defined by $(L, Id-T_i), i \in [1, n]$. This local setting compares to the global case via Grothendieck and Deligne DeRham cohomology results.

Lemma. *For* $M \subset I$ *and* $y \in Y^*_M$, *the above correspondence* $v \mapsto \tilde{v}$, *from* L *to* $\mathcal{L}_{X,y}$, *extended from* $L(i_1, \ldots, i_j)$ *to* $(\Omega^*_X(LogY) \otimes \mathcal{L}_X)_y$ *by* $v \mapsto \tilde{v}\frac{dz_{i_1}}{z_{i_1}} \wedge \ldots \wedge \frac{dz_{i_j}}{z_{i_j}}$, *induces quasi-isomorphisms*

$$(\Omega^*_X(LogY) \otimes \mathcal{L}_X)_y \cong \Omega(L, N_j, j \in M) \cong s(L(J), N.)_{J\subset[1,n]} \tag{6}$$

hence $(\mathbf{j}_*\mathcal{L}^c)_y \cong \Omega(L, N_j, j \in M)$.

This description of $(\mathbf{j}_*\mathcal{L})_y$ is the model for the description of the next various perverse sheaves.

2.2 The intermediate extension $\mathbf{j}_{!*}\mathcal{L}$ Let $N_J = \Pi_{j\in J}N_j$ denotes a composition of endomorphisms of L, we consider the strict simplicial sub-complex of the DeRham logarithmic complex defined by ImN_J in $L(J) = L$.

Definition. *The simple complex defined by the above simplicial sub-vector space is denoted by*

$$IC(L) := s(N_JL, N.)_{J\subset M},\ N_JL := N_{j_1}N_{j_2}\ldots N_{j_p}L, j_i \in J$$

Locally the germ of the intermediate extension $\mathbf{j}_{!*}\mathcal{L}$ of $\mathcal{L}$ at a point $y \in Y^*_M$ is quasi-isomorphic to the above complex [24 (3)]

$$\mathbf{j}_{!*}(\mathcal{L})_y \simeq IC(L) \simeq s(N_JL, N.)_{J\subset M}$$

The corresponding global DeRham description is given as a sub-complex $IC(X, \mathcal{L})$ of $\Omega^*_X(LogY)\otimes \mathcal{L}_X$. In terms of a set of coordinates $z_i, i \in M$, defining Y_M in a neighbourhood of $y \in Y^*_M$, $IC^j(X, \mathcal{L})$ is the subanalytic complex of $\Omega^*_X(LogY) \otimes \mathcal{L}_X$ with fiber at y generated, as an $\Omega^*_{X,y}$ sub-module, by the sections $\tilde{v} \wedge_{j\in J} \frac{dz_j}{z_j}$ for $v \in N_JL$. This formula is independent of the choice of coordinates, since if we choose a different coordinate $z'_i = fz_i$ instead of z_i, with f invertible holomorphic at y, the difference $\frac{dz'_i}{z'_i} - \frac{dz_i}{z_i} = \frac{df}{f}$ is holomorphic at y, hence the difference of the sections $\tilde{v} \wedge_{j\in J} \frac{dz'_j}{z'_j} - \tilde{v} \wedge_{j\in J} \frac{dz_j}{z_j}$ is still a section of the sub-complex $IC(X, \mathcal{L})$; moreover the restriction of the section is still defined in the sub-complex near y, since $N_JL \subset N_{J-i}L$ for all $i \in J$.

Lemma. We have a quasi-isomorphism $\mathbf{j}_{!*}\mathcal{L}^c \cong IC(X, \mathcal{L})$.

2.3 Hodge filtration and Nilpotent orbits *Variation of Hodge structures (VHS).* Consider the flat bundle $(\mathcal{L}_X, \nabla)$ in the previous hypothesis and suppose now that $\mathcal{L}_{X^*}$ underlies a VHS that is a polarised filtration by sub-bundles F of weight m satisfying Griffith's conditions [19].

The *nilpotent and the SL_2 orbit* theorems [19], [4], [24], [25] show that F extends to a filtration by sub-bundles F of $\mathcal{L}_X$ such that the restrictions to open intersections Y^*_M of components of Y underly *locally* a variation of mixed Hodge structures $VMHS$ where the weight filtration

is defined by the nilpotent endomorphism $\mathcal{N}_M$, residue of the connection, (there is no flat bundle defined globally on Y_M^*, if $z_{i_1}, \ldots, z_{i_n}$ for $i_j \in M$ are local equations at $y \in Y_M^*$, then $\Psi_{z_{i_n}} \circ \ldots \circ \Psi_{z_{i_1}} \mathcal{L}$ is the underlying local system nar y).

Local version. Near a point $y \in Y_M^*$ with $|M| = n$ a neighbourhood of y in the fiber of the normal bundle looks like a disc D^n and the above hypothesis reduces to

Local Hypothesis : Nilpotent orbits [4]. Let

$$(L, N_i, F, P, m, i \in M = [1, n]) \tag{7}$$

be defined by the VHS, that is a $\mathbb{Q}-$vector space L with endomorphisms N_i viewed as defined by the multiform horizontal (zero) sections of the connection on $(D^*)^n$ (hence sections on the inverse image on the universal covering), a Hodge structure F on $L^{\mathbb{C}} = L \otimes_{\mathbb{Q}} \mathbb{C}$ viewed as the fiber of the vector bundle $\mathcal{L}_X$ at y (here $y = 0$), a natural integer m the weight and the polarization P.

The main theorem in [4] states that for all $N = \Sigma_{i \in M} \lambda_i N_i$ with $\lambda_i > 0$ in $\mathbf{R}$ the filtration $W(N)$ (with center 0) is independent of N when λ_i vary and $W(N)[m]$ is the weight filtration of a graded polarised MHS called the limit MHS of weight m ($L, F, W(N)[m]$).

Remark: $W(N)[m]$ is $W(N)$ with indices shifted by m to the right: $(W(N)[m])_r := W_{r-m}(N)$, the convention being a shift to left for a decreasing filtration and to right for an increasing filtration.

It is important to notice that the orbits depend on the point z near y considered, in particular $F_z \neq F_y$. In this case when we restrict the orbit to $J \subset M$, we should write

$$(L, N_i, F(J), P, m, i \in J \subset M)$$

We write W^J for $W(N_J)$ where $N_J = \Sigma_{i \in J} N_i$. We will need the following result [4 p 505]:

Let $I, J \subset M$ then $W^{I \cup J}$ is the weight filtration of N_J relative to $W(N_I)$

$$\forall j, i \geq 0, N_J^i : Gr_{i+j}^{W^{I \cup J}} Gr_j^{W^I} \xrightarrow{\sim} Gr_{j-i}^{W^{I \cup J}} Gr_j^{W^I}.$$

II. Properties of the relative weight filtrations

Given a nilpotent orbit we may consider various filtrations $W^J = W(\Sigma_{i \in J} N_i)$ for various $J \subset M$. They are centered at 0 (that is we suppose here the weight of the nilpotent orbit equal to zero, otherwise the true weight of the MHS is defined up to a shift), preserved by N_i for $i \in M$ and shifted by -2 for $i \in J$: $N_i W_r^J \subset N_i W_{r-2}^J$. We need to know more about the action of N_i which is compatible with $W(N_j)$. The starting point of this study is the definition of the relative weight filtration by Deligne [10]

Let (L, W) be a finite dimensional vector space L endowed with an increasing filtration W and N a nilpotent endomorphism compatible with W. There may exists at most a unique filtration $M = M(N, W)$ satisfying

1) $N : M_j \simeq M_{j-2}$

2) $N^j : Gr_{k+j}^M Gr_k^W L \simeq Gr_{k-j}^M Gr_k^W L$,

A main result in [4] shows

The filtrations W^J defined by a polarised nilpotent orbit satisfy

1) For a subset $J \subset [1, n]$, $\forall j \in J, \forall \lambda_j > 0$, $N_J = \sum_{j \in J} \lambda_j N_j$, the filtration $W^J = W(N_J)$ is independent of $\lambda_j > 0$

2) For subsets J and J' in $[1, n]$, $A = J \cup J'$ we have for all $j \in \mathbb{N}, k \in \mathbb{Z}$:

$N_J^j : Gr_{k+j}^{W^A} Gr_k^{W^{J'}} L \simeq Gr_{k-j}^{W^A} Gr_k^{W^{J'}} L$,

that is W^A is the relative weight filtration of N_J acting on $(L, W^{J'})$.
We remark also that for all $J \subseteq B \subseteq A$, N_B and N_J induce the same morphism on $Gr_k^{W^{J'}} L$.
Finally we need the following result of Kashiwara ([25, thm 3.2.9, p 1002])
Let (L, N, W) consists of a vector space endowed with an increasing filtration W preserved by a nilpotent endomorphism N on L and suppose that the relative filtration $M = M(N, W)$ exists, then there exists a canonical decomposition:

$$Gr_l^M L = \oplus_k Gr_l^M Gr_k^W L$$

Precisely, Kashiwara exhibits a splitting of the exact sequence:
$0 \to W_{k-1} Gr_l^M L \to W_k Gr_l^M L \to Gr_k^W Gr_l^M L \to 0$.
by constructing a natural section of $Gr_k^W Gr_l^M L$ into $W_k Gr_l^M$. We will need later more precise relations between these filtrations that we discuss now.

2.4 Key lemma *(Decomposition of the relative weight filtrations) : Let $(L, N_i, i \in [1, n], F)$ be a polarised nilpotent orbit and for $A \subseteq [1, n]$ let $W^A := W(\sum_{i \in A} N_i)$ (all weights centered at 0), then :*
i) For all $i \in A$, the filtration W^A induces a trivial filtration on $Gr_k^{W^i} Gr_{k'}^{W^{(A-i)}} L$ of weight $k + k'$
ii) For $A = \{i_1, \ldots, i_j\} \subset [1, n]$, of length $|A| = j$ we have a natural decomposition

$$Gr_r^{W^A} L \simeq \oplus_{m_{i.} \in X_r^A} Gr_{m_{i_j}}^{W^{i_j}} \cdots Gr_{m_{i_1}}^{W^{i_1}} L \text{ where } X_r^A = \{m_{i.} \in \mathbb{Z}^j : \Sigma_{i_l \in A} m_{i_l} = r\}$$

more precisely

$$Gr_r^{W^A}(\cap_{i_s \in A} W_{a_{i_s}}^{i_s} L) \simeq \oplus_{\{m_{i.} \in X_r^A, m_{i_s} \leq a_{i_s}\}} Gr_{m_{i_j}}^{W^{i_j}} \cdots Gr_{m_{i_1}}^{W^{i_1}} L \tag{8}$$

iii) Let $A = B \cup C$, N_i' denotes the restriction of N_i to $Gr_c^{W^C}$ and $N_B' = \Sigma_{i \in B} N_i'$, then W_b^B induces $W_b(N_B')$ on $Gr_c^{W^C}$, that is

$$Gr_{b+c}^{W^A} Gr_c^{W^C} L \simeq Gr_b^{W^B} Gr_c^{W^C} L \simeq Gr_{b+c}^{W^A} Gr_b^{W^B} L \simeq Gr_c^{W^C} Gr_b^{W^B} L$$

iv) The repeated graded objects in i) do not depend on the order of the elements in A.

Remark: This result give relations between various weight filtrations in terms of the elementary ones $W^i := W(N_i)$ and will be extremely useful in the study later of the properties of the weight filtration on the mixed Hodge complex.

Proof . To stress the properties of commutativity of the graded operation for the filtrations, we prove first

Sublemma: For all subsets $[1, n] \supset A \supset \{B, C\}$, the isomorphism of Zassenhaus $Gr_b^{W^B} Gr_c^{W^C} L \simeq Gr_c^{W^C} Gr_b^{W^B} L$ is an isomorphism of MHS with weight filtration (up to a shift) $W = W^A$ and Hodge filtration $F = F_A$, hence compatible with the third filtration W^A or F_A.

Proof of the sub-lemma: Recall that both spaces $Gr_b^{W^B} Gr_c^{W^C}$ and $Gr_c^{W^C} Gr_b^{W^B}$ are isomorphic to $W_b^B \cap W_c^C$ modulo $W_c^C \cap W_{b-1}^B + W_b^B \cap W_{c-1}^C$. In this isomorphism a third filtration like F_A (resp. W^A) is induced on one side by $F_k' = (F_A^k \cap W_c^C) + W_{c-1}^C$ (resp. $W_k' = (W_k^A \cap W_c^C) + W_{c-1}^C$) and on the second side by $F_k'' = (F_A^k \cap W_b^B) + W_{b-1}^B$ (resp. $W_k'' = W_k^A \cap W_b^B) + W_{b-1}^B$). We introduce the third filtration $F_k''' = F_A^k \cap W_b^B \cap W_c^C$ (resp. $W_k''' = W_k^A \cap W_b^B \cap W_c^C$) and we

notice that all these spaces are in the category of MHS, hence the isomorphism of Zassenhaus is strict and compatible with the third filtrations induced by F_A (resp W^A).

Proof of the key lemma i). Let $A \subset [1,n]$ and $i \in A$, then W^A exists on L and coincides with the relative weight filtration for N_i with respect to $W^{(A-i)}$ by a result of Cattani and Kaplan. Then we have by Kashiwara's result $Gr_l^{W^A} L \simeq \oplus_k Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$. Let us attach to each point (k,l) in the plane the space $Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ and let $M_j = \oplus_l Gr_l^{W^A} Gr_{l-j}^{W^{(A-i)}} L$ be the direct sum along indices in the plane (k,l) on a parallel to the diagonal $(l = k+j)$. Then we have for $j > 0$

$$(N_i)^j : Gr_{k+j}^{W^A} Gr_k^{W^{(A-i)}} L \simeq Gr_{k-j}^{W^A} Gr_k^{W^{(A-i)}} L,\ (N_i)^j : M_j \simeq M_{-j}.$$

This property leads us to introduce the space $V = \oplus_l Gr_l^{W^A} L \simeq \oplus_{l,k} Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$, then N_i on L extends to a nilpotent endomorphism on V, $N_i : V \to V$ inducing $N_i : Gr_l^{W^A} L \to Gr_{l-2}^{W^A} L$ on each $l-$component of V. We consider on V two increasing filtrations $W'_s := \oplus_{l-k \leq s} Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ and $W''_s := \oplus_l W_s^i Gr_l^{W^A} L$. Then N_i shift these filtrations by -2. In fact $N_i : W'_s \to W'_{s-2}$ sends $Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ to $Gr_{l-2}^{W^A} Gr_k^{W^{(A-i)}} L$ and $(N_i)^j$ induces an isomorphism $Gr_j^{W'} V \simeq Gr_{-j}^{W'} V$. As well we have an isomorphism $Gr_j^{W''} V \simeq Gr_{-j}^{W''} V$, since $(N_i)^j : (Gr_j^{W^i} L, W^A, F_A) \simeq (Gr_{-j}^{W^i} L, W^A, F_A)$ is an isomorphism of MHS up to a shift in indices, hence strict on W^A and F_A and induces an isomorphism $Gr_j^{W^i} Gr_l^{W^A} \simeq Gr_{-j}^{W^i} Gr_{l-2j}^{W^A}$. Then these two filtrations W'_s and W''_s are equal by uniqueness of the weight filtration of N_i on V, that is

$$W''_s = \oplus_{k,l} W_s^i Gr_l^{W^A} Gr_k^{W^{(A-i)}} L = W'_s = \oplus_{l-k \leq s} Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$$

that is $W_s^i Gr_l^{W^A} Gr_k^{W^{(A-i)}} L = Gr_l^{W^A} Gr_k^{W^{(A-i)}} L$ if $l - k \leq s$ and $W_s^i Gr_l^{W^A} Gr_k^{W^{(A-i)}} L = 0$ otherwise, hence

$$Gr_l^{W^A} Gr_j^{W^i} L \simeq Gr_l^{W^A} Gr_j^{W^i} Gr_{l-j}^{W^{A-i}} L, \text{and for all } l \neq k+k',\ Gr_l^{W^A} Gr_k^{W^i} Gr_{k'}^{W^{(A-i)}} L \simeq 0.$$

which ends the proof of (i).

ii) Since W^A induces a trivial filtration on $Gr_k^{W^i} Gr_{k'}^{W^{(A-i)}} L$ of weight $k+k'$ we have

$$Gr_l^{W^A} L \simeq \oplus_k Gr_l^{W^A} Gr_k^{W^{A-i}} L \simeq \oplus_k Gr_l^{W^A} Gr_k^{W^{A-i}} Gr_{l-k}^{W^i} L \simeq \oplus_k Gr_k^{W^{(A-i)}} Gr_{l-k}^{W^i} L.$$

Now if we suppose by induction on length of A, the decomposition true for $A - i$, we deduce easily the decomposition for A from the above result.

iii) We restate here the property of the relative monodromy for W^A with respect to W^C and we apply ii).

iv)In the proof above we can start with any i in A, hence the decomposition is symmetric in elements in A. It follows that the graded objects of the filtrations $W^i, W^r, W^{\{i,r,j\}}$ commute and since W^j can be expressed using these filtrations, we deduce that W^i, W^r, W^j also commute, for example: $Gr_{a+b+c}^{W^{\{i,r,j\}}} Gr_{a+b}^{W^{\{i,j\}}} Gr_a^{W^i} \simeq Gr_c^{W^r} Gr_b^{W^j} Gr_a^{W^i}$ is symmetric in i, j, r.

3§. The weight filtration and main theorems in the local case

To describe the weight filtration, we introduce a category $S(I) = S$ attached to a set I already used by Kashiwara and Kawai [26] for the intersection complex. We start with a local study,

that is to say with the hypothesis of a polarised nilpotent orbit and we describe the weight filtration W on a combinatorial complex quasi-isomorphic to the DeRham complex $\Omega(L, N.)$. The features of the purity theory will appear relatively quickly. First we ask the reader to take some time to get acquainted with the new category $S(I)$ whose objects are indices for the combinatorial complex. The weights zero or -1 describe a complex $IC(L)$ quasi- isomorphic to the fiber of the intermediate extension of $\mathcal{L}$ and for the other weights we need to introduce the complexes $C_r^{KM}L$ for $K \subset M \subset I$ (3.3) which describe the purity theory (3.6) and the geometry of the decomposition theorem (4.3). A basic technique in the proof is the decomposition into elementary complexes (3.5), reflecting relations between the weight filtrations of the various N_i.

I. Construction of the weight filtration

3.1 Complexes with indices in the category $S(I)$. The techniques are similar to the simplicial techniques in Deligne's paper. Here the singularities may come from the coefficients as well as the NCD in X. We introduce a category $S(I)$ attached to a set I, whose objects consist of sequences of increasing subsets of I of the following form:

$$(s_.) = (I = s_1 \supsetneqq s_2 \dots \supsetneqq s_p \neq \varnothing),\ (p > 0)$$

Subtracting a subset s_i from a sequence $s_.$ defines a morphism $\delta_i(s.) : (s. - s_i) \to s.$ and more generally $Hom(s.', s.)$ is equal to one element iff $(s.') \leq (s.)$ that is $(s.')$ is obtained from $(s.)$ by deleting some subsets. We write $s. \in S(I)$ and define its degree or length $|s.|$ as the number of subsets s_i in $(s_.)$.

Correspondence with an open simplex. If $I = \{1, \dots, n\}$ is finite, $S(I)$ can be realised as a barycentric subdivision of the open simplex Δ_{n-1} of dimension $n-1$. A subset K corresponds to the barycenter of the vertices in K and a sequence of subsets to an oriented simplex defined by the vertices associated to the subsets.
For example, for $I = \{1, 2\}$, $S(I)$ consists of the barycenter $\{3/2\}$ of $]1, 2[$ defined by $\{1, 2\}$, and the open simplices $]1, 3/2[$, $]3/2, 2[$ defined resp. by the sequences $\{1, 2\} \supset \{1\}$ and $\{1, 2\} \supset \{2\}$. Since all sequences contain I, all corresponding simplices must have the barycenter defined by I as vertex, that is a sub-simplex contained in the open simplex $\Delta^*_{n-1} = \Delta_{n-1} - \partial\Delta_{n-1}$. In this way we define an incidence relation $\varepsilon(s., s.')$ between two adjacent sequences equal to $+1$ or -1 according to orientation. Incidence relations $\varepsilon(\Delta_{n-1}, s.)$ are defined as well between Δ_{n-1} and the simplices corresponding to maximal sequences.

Combinatorial objects of an abelian category with indices in $S(I)$, that is functors, are thus defined, as well as complexes of such objects.
We need essentially the following construction. An algebraic or analytic variety over a fixed variety X with indices in $S(I)$ denoted by Π is a covariant functor defined by $\Pi(s.) : X_{s.} \to X$ and morphisms $\Pi(s.' \leq s.) : X_{s.'} \to X_{s.}$ over X for $s. \in S$. An abelian sheaf over Π (resp. complex of abelian sheaves) $\mathcal{F}$ is a contravariant functor of abelian sheaves (resp. complex of abelian sheaves) $\mathcal{F}_{s.}$ over $X_{s.}$ (with functorial morphisms $\varphi(s.' \leq s.) : \Pi^*\mathcal{F}_{s.} \to \mathcal{F}_{s.'}$ for $(s.') \leq (s.)$).
The direct image of an abelian sheaf over Π (resp. complex of sheaves) denoted $\Pi_*\mathcal{F}$ or preferably $s(\mathcal{F}_{s.})_{s.\in S}$ is the simple complex (resp.simple complex associated to a double complex) on X:

$$s(\mathcal{F}_{s.})_{s.\in S} := \Pi_*\mathcal{F} := \oplus_{s.\in S}(\Pi_*\mathcal{F}_{s.})[|s.| - |I|],\ d = \Sigma_{i\in[1,|s.|]}(-1)^{\varepsilon(s., \delta_i(s.))}\varphi(\delta_i(s.) \leq s.)$$

Example. The variety X defines the constant variety $X_{s.} = X$. The constant sheaf $\mathbb{Z}$ lifts to a sheaf on $X_{s.}$ such that the "diagonal morphism" : $\mathbb{Z}_X \to \oplus_{|(s.)|=|I|}\mathbb{Z}_{X_{s.}}$ (that is : $n \in \mathbb{Z} \to (\ldots, \varepsilon(\Delta_{n-1}, s.)n_{s.}, \ldots) \in \oplus_{|(s.)|=|I|}\mathbb{Z}$ defines a quasi-isomorphism $\mathbb{Z}_X \cong \Pi_*\Pi^*(\mathbb{Z}_X)$. This is true since $S(I)$ is isomorphic to the category defined by the barycentric subdivision of an open simplex of dimension $|I| - 1$.

3.2 Local definition of the weight filtration Our hypothesis here consists again of the polarised nilpotent orbit $(L, (N_i)_{i\in M}, F, m)$ of weight m and the corresponding filtrations $(W^J)_{J\subset M}$ where $W^J = W(\sum_{i\in J} N_i)$.
We will use the category $S(M)$ attached to M whose objects consist of sequences of decreasing subsets of M of the form $(s_.) = (M = s_1 \supsetneqq s_2 \ldots \supsetneqq s_p \neq \varnothing)$, $p > 0$.
In this construction we will need double complexes, more precisely complexes of the previously defined exterior complexes, so we introduce *the category $M_.^+$ whose objects are the subsets $J \subset M$ including the empty set* so that the DeRham complex $\Omega(L, N.)$ *is written now as* $s(L_J)_{J\subset M}$ and we consider objects with indices in the category $M_.^+ \times S(M)$.
Geometrically M corresponds to a normal section to Y_M^* in X and J to $\wedge_{i\in J} dz_i$ in the exterior DeRham complex on the normal section to Y_M^*. The decomposition $M_.^+ \simeq (M - K)_.^+ \times K_.^+$ corresponds to the isomorphism $\mathbb{C}^M \simeq \mathbb{C}^{(M-K)} \times \mathbb{C}^K$.

Notations. For each $s. \in S(M)$ let $W^{s_\lambda} = W(\sum_{i\in s_\lambda} N_i)$ centered at 0, for $J \subset M$ and an integer r, we define $a_{s_\lambda}(J, r) = |s_\lambda| - 2|s_\lambda \cap J| + r$, and for all $(J, s.) \in M_.^+ \times S(M)$ the functorial vector spaces
$W_r(J, s.)L := \bigcap_{s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)}L, \quad F^r(J, s.) := F^{r-|J|}L, \quad W^{s_\lambda} = W(\sum_{i\in s_\lambda} N_i)$,
then we consider for each $(s.)$ a DeRham complex $\Omega(L, N.)$

Definition. *The weight $\mathcal{W}$ (centered at zero) and Hodge F filtrations on the combinatorial DeRham complex $\Omega^*L = s(\Omega(L, N.))_{s.\in S(M)}$ are defined by "summing" over J and s.*

$$(\Omega^*L, \mathcal{W}, F) \tag{9}$$

where $\mathcal{W}_r(\Omega^*L) := s(\bigcap_{s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)}L)_{(J,s.)\in M_.^+\times S(M)}$, $a_{s_\lambda}(J, r) = |s_\lambda| - 2|s_\lambda \cap J| + r$
and $F^r(\Omega^*L) := s(F^{r-|J|}L)_{(J,s.)\in M_.^+\times S(M)}$.
The filtrations can be constructed in two times, first by summing over J to get the sub-complexes $W_r(s.) = s(W_r(J, s.))_{J\subset M}$ (weight) and $F^r(s.) := s(F^r(J, s.))_{J\subset M}$ (Hodge).
Example in dimension 2.
Let $W^{1,2} = W(N_1 + N_2), W^1 = W(N_1)$ and $W^2 = W(N_2)$, the weight $\mathcal{W}_r$ is a double complex:
$\mathcal{W}_r(\{1, 2\} \supsetneqq 1) \oplus \mathcal{W}_r(\{1, 2\} \supsetneqq 2) \to \mathcal{W}_r(\{1, 2\})$
where the first line is the direct sum of:

$$\mathcal{W}_r(\{1, 2\} \supsetneqq 1) = (W^{1,2}_{r+2} \cap W^1_{r+1} \xrightarrow{(N_1,\, N_2)} W^{1,2}_r \cap W^1_{r-1} \oplus W^{1,2}_r \cap W^1_{r+1} \xrightarrow{(-N_2,\, N_1)} W^{1,2}_{r-2} \cap W^1_{r-1})$$

and

$$\mathcal{W}_r(\{1, 2\} \supsetneqq 2) = (W^{1,2}_{r+2} \cap W^2_{r+1} \xrightarrow{(N_1,\, N_2)} W^{1,2}_r \cap W^2_{r+1} \oplus W^{1,2}_r \cap W^2_{r-1} \xrightarrow{(-N_2,\, N_1)} W^{1,2}_{r-2} \cap W^2_{r-1})$$

The second line for $\{1, 2\}$ is

$$\mathcal{W}_r(\{1, 2\}) = (W^{1,2}_{r+2} \xrightarrow{(N_1,\, N_2)} W^{1,2}_r \oplus W^{1,2}_r \xrightarrow{(-N_2,\, N_1)} W^{1,2}_{r-2}).$$

which reduces to the formula in [26] for $r = -1$.

3.3 The Complexes $C_r^{KM}L$ and $C_r^K L$. To study the graded part of the weight, we need to introduce the following subcategories:

For each subset $K \subset M$, let $S_K(M) = \{s. \in S(M) : K \in s.\}$ (that is $\exists \lambda : K = s_\lambda$). The isomorphism of categories:

$$S(K) \times S(M-K) \xrightarrow{\sim} S_K(M),\ (s., s.') \to (K \cup s.', s.)$$

will be of important use later. We consider the vector spaces with indices $(J, s.) \in M.^+ \times S_K M$, $C_r^{KM} L(J, s.) := \bigcap_{K \neq s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L$ and for each $(s.)$ the associated complex obtained by summing over J (resp. over $(s.)$):
$C_r^{KM} L(s.) := s(C_r L(J, s.))_{J \in M.^+}$, $C_r^{KM} L := s(C_r^{KM} L(s.))_{(s.) \in S_K M.}$
We write $C_r^K L(J, s.)$, $C_r^K L(s.)$ and $C_r^K L$ when $K = M$.

Definition. *For $K \subseteq M$ the complex $C_r^{KM} L$ is defined by summing over J and $(s.)$*

$$C_r^{KM} L := s\,(\bigcap_{K \neq s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L)_{(J,s.) \in M.^+ \times S_K(M)} \tag{10}$$

In the case $K = M$ we write $C_r^K L$

$$C_r^K L = s(C_r^K L(J, s.))_{(J,s.) \in K.^+ \times S(K)} = s((\cap_{K \supsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} Gr^{W^K}_{a_{K(J,r)}} L)_{(J,s.) \in K.^+ \times S(K)} \tag{11}$$

II. Purity of the cohomology of the complex $C_r^K L$

In this subsection we aim to prove that the filtration $\mathcal{W}$ will lead to the weight of what would be in the proper case a mixed Hodge complex in Deligne's terminology, that is the induced filtration by F on the graded parts $Gr^{\mathcal{W}} \Omega^* L$ is a Hodge filtration. For this we need to decompose the complex as a direct sum of intermediate extensions of variations of Hodge structures (which has a meaning locally) whose cohomologies are pure Hodge structures [5] and [24] in the proper case. The decomposition itself is in the next section. Here we prove the purity of the complex $C_r^K L$. Its unique non vanishing cohomology will the fiber of the variations of Hodge structures needed in the decomposition of $Gr^{\mathcal{W}} \Omega^* L$. The result here is a fundamental step in the general proof. The key lemma proved earlier provides what seems to be the elementary property at the level of a nilpotent orbit that leads to establish the purity and decomposition results. The proof of the theorem below will occupy the whole subsection. First we present a set of elementary complexes. Second we prove the purity result on the complexes $C_r^K L$ which behave as a direct sum of elementary complexes.

Let L be a polarised nilpotent orbit, then the complexes $C_r^K L$ satisfy the following properties

3.4 Proposition *(Purity). The cohomology of the complex $C_r^K L$, concentrated in a unique degree, underlies a polarised HS.*

The proof of the proposition will occupy the whole subsection and is divided in two parts. Precise information can be found in the proposition below.

3.5 Elementary complexes. We suppose K of length $|K| = n$ and we identify K with the set of integers $[1, n]$, then the elementary complexes are defined by the following simplicial vector spaces. For $J \subset [1, n]$, let

$$K((m_1, \cdots, m_n), J) L = Gr^{W^n}_{m_n - 2|\{n\} \cap J|} \cdots Gr^{W^i}_{m_i - 2|\{i\} \cap J|} \cdots Gr^{W^1}_{m_1 - 2|\{1\} \cap J|} L \tag{12}$$

The endomorphism N_i induces a morphism denoted also
$N_i : K((m_1, \cdots, m_n), J)L \to K((m_1, \cdots, m_n), J \cup i)L$ trivial for $i \in J$.
Remark. Instead of L, we can consider such formulas for various natural spaces derived from L such as $L/N_J L$, $N_J L$ or $\cap_{s_\lambda \in s.} W^{s_\lambda}_{a_\lambda} L$ for a sequence $s.$ of subsets of $[1, m]$ containing $[1, n] \subset [1, m]$.

Definition. *The elementary complexes are the simple complexes associated to the simplicial vector spaces (12) by summing over $J \subseteq [1, n]$*

$$K(m_1, \cdots, m_n)L := s(K((m_1, \cdots, m_n), J)L, N_i)_{J \subseteq [1,n]} \tag{13}$$

Proposition: *For any $((m_1, \cdots, m_n) \in \mathbb{Z}^n$ let $J(m.) = \{i \in [1, n] : m_i > 1\}$.*
The cohomology of an elementary complex $K(m_1, \cdots, m_n)L$ is isomorphic to a sub-quotient of the vector space $K((m_1, \cdots, m_n), J(m.))L$ concentrated in degree $|J(m.)|$. Moreover it vanishes if there exists at least one $m_i = 1$.
More precisely, the cohomology is isomorphic to
$K((m_1, \cdots, m_n), J(m.))[(\cap_{i \notin J(m.)}(kerN_i : L/(\Sigma_{j \in J(m.)} N_j L) \to L/(\Sigma_{j \in J(m.)} N_j L)] \simeq$
$Gr^{W^n}_{m_n - 2|\{n\} \cap J(m.)|} \cdots Gr^{W^1}_{m_1 - 2|\{1\} \cap J(m.)|}[(\cap_{i \notin J(m.)}(kerN_i : L/(\Sigma_{j \in J(m.)} N_j L) \to L/(\Sigma_{j \in J(m.)} N_j L))]$.
The proof by induction on n is based on the fact that given an index i, we can view $K(m_1, \cdots, m_n)L$ as the cone over
$N_i : K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i} L) \stackrel{N_i}{\to} K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i - 2} L)$.
It is enough to notice that $N_i : Gr^{W^i}_{m_i} L \to Gr^{W^i}_{m_i - 2} L$ is injective if $m_i > 0$, surjective if $m_i < 2$ (bijective for $m_i = 1$). The associated morphism on the complex $K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)$ will have the same property since the constituent vector spaces respect exact sequences by strictness of MHS.
Hence if $m_i > 0$ (resp. $m_i < 2$), N_i is injective on $Gr^{W^i}_{m_i} L$ (resp. surjective onto $Gr^{W^i}_{m_i - 2} L$) and
$K(m_1, \cdots, m_i, \cdots, m_n)L \cong K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i - 2}(L/N_i L))[-1]$
(resp. $K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)(Gr^{W^i}_{m_i}(kerN_i : L \to L))$
where $K(m_1, \cdots, \widehat{m_i}, \cdots, m_n)$ is applied to the polarised nilpotent orbit $Gr^{W^i}_{m_i - 2}(L/N_i L)$ (resp.$Gr^{W^i}_{m_i}(kerN_i : L \to L)$) with the nilpotent endomorphisms N'_j induced by N_j for $j \neq i$.
Remark. The cohomology space is symmetric in the operations kernel and cokernel and is isomorphic to
$K((m_1, \cdots, m_n), J(m.))[(\cap_{\{i : m_i < 1\}} kerN_i)/(\Sigma_{\{j : m_j > 1\}} N_j (\cap_{\{i : m_i < 1\}} kerN_i))]$.
that is at each process of taking $Gr^{W^i}_{m_i}$ we apply the functor *ker* if $m_i \notin J(m.)$ and *coker* if $m_i \in J(m.)$

3.6 Purity of $C^K_r L$. *Decomposition into combinatorial elementary complexes.*
By the natural decomposition of the relative filtrations in the Key lemma, we have isomorphisms, functorial for the differentials of $C^K_r L$

$$Gr^{W^K}_{a_{K(J,r)}}(\cap_{K \supsetneq s_\lambda \in s.} W^{s_\lambda}_{a_\lambda(J, r-1)})L \simeq \oplus_{m. \in X(J, s., r)} Gr^{W^n}_{m_n} \cdots Gr^{W^i}_{m_i} \cdots Gr^{W^1}_{m_1} L,$$

where for all $(J, s.) \in K^+ \times S(K)$,
$X(J, s., r) = \{m. \in \mathbb{Z}^n : \Sigma_{i \in K} m_i = a_K(J, r)$ and $\forall s_\lambda \in s., s_\lambda \neq K, \Sigma_{i \in s_\lambda} m_i \leq a_\lambda(J, r-1)\}$
In particular, if we define for $J = \varnothing, X(s., r) = X(\varnothing, s., r)$ as

$$X(s., r) = \{m. \in \mathbb{Z}^n : \Sigma_{i \in K} m_i = |K| + r \text{ and } \forall s_\lambda \in s., s_\lambda \neq K, \Sigma_{i \in s_\lambda} m_i \leq |s_\lambda| + r - 1\}$$

the complex $C_r^K L(s.)$ splits as a direct sum of elementary complexes
$C_r^K L(s.) \simeq \oplus_{X(s.,r)} K(m_1, \ldots, m_n)L$.
The combinatorial elementary complex. For each $(m_1, \ldots, m_n) \in \mathbb{Z}^n$ we define a complex with indices in $s. \in S(K)$ as follows:
$K(m_1, \ldots, m_n; r)L(s.) = K(m_1, \ldots, m_n)L$ if $(m_1, \ldots, m_n) \in X(s., r)$ and 0 otherwise.
Definition. *The combinatorial elementary complex is defined by summing over s.*

$$\tilde{K}(m_1, \ldots, m_n; r)L = s[K(m_1, \ldots, m_n; r)L(s.)]_{s. \in S(K)}$$

Lemma. *Define $X(r) = \{m. \in \mathbb{Z}^n : \Sigma_{i \in K} m_i = |K| + r\}$, then we have the decomposition:*

$$C_r^K L \simeq \oplus_{m. \in X(r)} \tilde{K}(m_1, \ldots, m_n; r). \tag{14}$$

Example. For $K = \{1, 2\}$, $s.$ is one of the 3 elements $s.' = \{1, 2\} \supset \{1\}$, $s.'' = \{1, 2\} \supset \{1\}$ or $K = \{1, 2\}$. Then $\tilde{K}(m_1, \ldots, m_n; r)L$ is defined for fixed r by the combinatorial complex
$K(m_1, m_2; r)L(s.') \simeq K(m_1, m_2)L$ if $m_1 + m_2 = 2 + r$ and $m_1 \leq r$ and 0 otherwise
$K(m_1, m_2; r)L(s.'') \simeq K(m_1, m_2)L$ if $m_1 + m_2 = 2 + r$ and $m_2 \leq r$ and 0 otherwise
$K(m_1, m_2; r)L(K) \simeq K(m_1, m_2)L$ if $m_1 + m_2 = 2 + r$ and 0 otherwise.
Notice $\tilde{K}(m_1, m_2; r)(W_1^1 L) \cong 0$ for $r \geq 0$ that is the case $m_1 < 2$. In fact suppose $r > 0$ in the example and $m_1 < 2$, then $m_2 = 2 + r - m_1 > r$, hence $K(m_1, m_2; r)L(s.'') \simeq 0$. Moreover $K(m_1, m_2; r)L(s.') \simeq K(m_1, m_2; r)L(K)$ so that $\tilde{K}(m_1, m_2; r)(W_1^1 L) \cong 0$.
This is a main point that we prove in a more general setting in the coming basic lemma.
The elementary sub-complexes supporting the cohomology.
For $r > 0$ we define
$T(r) = \{(m_1, \cdots, m_n) \in \mathbf{N}^n : \forall i \in K, m_i \geq 2 \text{ and } \Sigma_{i \in K} m_i = |K| + r\}$
(for $r = 0$, $T(0) = \varnothing$) so to introduce the complex

$$C(T(r))L \simeq \oplus_{(m_1, \cdots, m_n) \in T(r)} K(m_1, \ldots, m_n)L$$

Dually, for $r < 0$ we define
$T'(r) = \{(m_1, \cdots, m_n) \in \mathbf{Z}^n : \forall i \in K, m_i \leq 0 \text{ and } \Sigma_{j \in K} m_j = |K| + r\}$
so to introduce the complex

$$C(T'(r))L \simeq \oplus_{(m_1, \cdots, m_n) \in T'(r)} K(m_1, \ldots, m_n)L$$

Lemma. *The complex $C(T(r))L$ embeds diagonally into the direct sum of $C_r(s.)L$ for all sequences $(s.)$ of maximal length $|K|$ so to define a morphism of complexes $C(T(r))L \to C_r^K L$. Dually, the complex $C(T'(r))$ embeds in $C_r(s.)L$ for $s. = K$ consisting of one subset K, so to define a morphism of complexes $C(T'(r))L[1 - |K|] \to C_r^K L$.*
Proof. We check the conditions defined by all $s_\lambda \in (s.)$, $s_\lambda \neq K$, namely $\Sigma_{i \in s_\lambda} m_i \leq |s_\lambda| + r - 1$ for all $(m.) \in T(r)$ by induction on $|s_\lambda|$. We start with the condition $\Sigma_{j \in K} m_j = |K| + r$ defined by K, which is satisfied by definition of $T(r)$. Let $k \in K$, then $\Sigma_{j \in K - \{k\}} m_j \leq |K - \{k\}| + r - 1$ by subtracting $m_k > 1$, which proves the assertion for $s_\lambda = K - \{k\}$, hence for all s_λ such $|s_\lambda| = |K| - 1$. Let $A \subset K, A \neq K$ and suppose $\Sigma_{j \in A} m_j \leq |A| + r - 1$ true, by induction on $|A|$, then for $k \in A, B = A - \{k\}$ we deduce $\Sigma_{j \in B} m_j \leq |B| + r - 2$ by subtracting $m_k > 1$.
Dually, for $(s.) = K$ there is no additional conditions, so the statement is clear.

3.7 Proposition i)*For $r > 0$, the canonical embedding of $C(T(r))L$ into $C_r^K L$ induces an isomorphism on the cohomology.*
In particular the cohomology of $C_r^K L$, concentrated in degree $|K|$, is isomorphic to
$$H^{|K|}(C_r^K L) \simeq Gr^{W^K}_{r-|K|}[L/(\Sigma_{i\in K} N_i L)] \simeq \oplus_{(m.)\in T(r)} Gr^{W^n}_{m_n-2} \cdots Gr^{W^1}_{m_1-2}[L/(\Sigma_{i\in K} N_i L)]$$
it is a polarised HS of weight $r+m-|K|$ with the weight filtration induced by W^K shifted by m and Hodge filtration induced by F^K.
ii) If $r = 0$, the complex $C_r^K L$ is acyclic.
iii)*Dually, for $r < 0$, the canonical embedding of $C(T'(r))L[1-|K|]$ into $C_r^K L$ induces an isomorphism on the cohomology.*
In particular the cohomology of $C_r^K L$, concentrated in degree $|K|-1$, is isomorphic to
$H^{|K|-1}(C_r^K L) \simeq Gr^{W^K}_{r+|K|}(\cap_{i\in K}(kerN_i : L \to L)) \simeq \oplus_{(m.)\in T'(r)} Gr^{W^n}_{m_n} \cdots Gr^{W^1}_{m_1}[\cap_{i\in K}(kerN_i : L \to L)]$
it is a polarised HS of weight $r+m+|K|$ with the weight filtration induced by W^K shifted by m and Hodge filtration induced by F^K.

Remark: If $r \in [1, |K|-1]$, $T(r)$ is empty and $C_r^K L$ is acyclic. If $r \in [-|K|+1, 0]$, $T'(r)$ is empty and $C_r^K L$ is acyclic. In all cases $C_r^K L$ appears in $Gr^{\mathcal{W}}\Omega^* L$.

The principal ingredient in the proof is based on:
Lemma (basic). *i) For $r \geq 0$, the complex $\tilde{K}(m_1, \ldots, m_n; r)L$ is acyclic whenever at least one $m_i < 2$. Equivalently, for each $i \in K$, the complex $C_r^K(W_1^i L)$ is acyclic.*
ii) Dually, for $r \leq 0$, the complex $\tilde{K}(m_1, \ldots, m_n; r)L$ is acyclic whenever at least one $m_i \geq 2$. Equivalently, for each $i \in K$, the complex $C_r^K(L/W_0^i L)$ is acyclic.

The equivalences follow from the decompositions
$C_r^K(W_1^i L) \simeq \oplus_{m.\in X(r), m_i<2} \tilde{K}(m_1, \ldots, m_n)L$
$C_r^K(L/W_1^i L) \simeq \oplus_{m.\in X(r), m_i\geq 2} \tilde{K}(m_1, \ldots, m_n, r)L.$
We note that $\tilde{K}(m_1, \ldots, m_n, r)L \cong 0$ if at least one $m_i = 1$.

Proof of the lemma. The result is based on the following elementary remark. Consider in $s.$ a sequence $s_{a+2} = s_{a+1} \cup \{i\} \supset s_{a+1}$ with $i \notin s_{a+1}$, then the condition on $m. \in X(s., r)$ associated to $s.$ defined by $s_{a+1} \cup \{i\}$ is $m_i + \Sigma_{j\in s_{a+1}} m_j \leq |s_{a+1} \cup \{i\}| + r - 1$ to compare with the condition $\Sigma_{j\in s'_{a+1}} m_j \leq |s'_{a+1}| + r - 1$ defined by s_{a+1}. Precisely when $m_i < 2$ (that is in $W_1^i L$) the condition for $s_{a+1} \cup \{i\}$ follows from the condition for s_{a+1}, hence the conditions defined by the subsequence $s_{a+1} \cup \{i\} \supset s'a + 1$ in $s.$ is the same as the condition defined by s_{a+1} in $d_{s_{a+2}}(s.)$ where $d_{s_{a+2}}$ is the differential consisting in the removal of s_{a+2}, so that in the sum over all $s.$ this couple is quasi-isomorphic to zero. The following rigorous proof consists on filtering the complex by carefully choosing subsets in $S(K)$. All constructions below are compatible with the decompositions and apply to each complex $\tilde{K}(m_1, \ldots, m_n, r)L$.
i) We construct a filtration of $C_r^K(W_1^i L)$ with acyclic sub-complexes. We define the i-length $|s.|_i$ of a sequence $s.$ as the number of subset s_λ not containing i. Let S_a be the full subcategory whose objects satisfy $|s.|_i \leq a$, hence an object $s.$ in S_a is written as $s.'' \cup \{i\} \supset s.'$ with $|s.'| \leq a$ with no subset in $s.'$ containing i. Deleting a subset of $s.$ in S_a gives another object in S_a, hence $C(S_a)L := s[C_r^K L(s.)]_{s.\in S_a}$ is a sub-complex of $C_r^K L$ where $C_r^K L(s.)$ is obtained by summing over over $J \subset K$ for a fixed $(s.)$.
1) To start we write $C(S_0)L$ as a cone over a morphism inducing a quasi-isomorphism on $C(S_0)(W_1^i L)$. We divide the objects of S_0 in two families : S_0' whose objects are defined by the sequences $(s.)$ starting with the subset $\{i\}$ and S_0'' whose objects are defined by the sequences

$(s.)$ whose elements s_λ contain i but are different from $\{i\}$. Then we consider the two complexes $C(S'_0)L := s[C_r^K L(s.)]_{s.\in S'_0}$ and $C(S''_0)L := s[C_r^K L(s.)]_{s.\in S''_0}$.
Let $(s.) = K \supset \ldots s_\lambda \supset \ldots \supset \{i\}$ in S'_0; deleting $s_\lambda \neq \{i\}$ is a morphism in S'_0 but deleting $\{i\}$ gives an element in S''_0 . It is easy to check that deleting $\{i\}$ defines a morphism of complexes $I_{\{i\}} : C(S'_0)L \to C(S''_0)L$ defined by embedding $C_r^K L(s.)$ into $C_r^K L(d_{\{i\}}(s.))$ where $d_{\{i\}}(s.) = (s.) - \{i\}$. The cone over this $I_{\{i\}}$ is isomorphic to $C(S_0)L[1]$.
We show now that if we reduce the construction to $W_1^i L$ instead of L, the morphism $I_{\{i\}}$: $C(S'_0)(W_1^i L) \to C(S''_0)(W_1^i L)$ is an isomorphism. Let $s. = (K \supset \ldots s_\lambda \supset \ldots \supset \{i\})$ in S'_0. It is enough to notice that the condition on $m. \in X(s., r)$ associated to $\{i\}$ defined by $m_i \leq |\{i\}| + r - 1 = r$ is irrelevant since $m_i \leq 1$ and $r > 0$ (since for $m_i = 1$ the elementary complexes are acyclic, we could also suppose $m_i < 1$ and $r = 0$) . It follows that $C(S_0)(W_1^i L)$ is acyclic.

2) *We extend the proof from S_0 to S_a.*
Suppose by induction that $C(S_a)(W_1^i L)$ is acyclic for $a \geq 0$, we prove $C(S_{a+1})(W_1^i L)$ is also acyclic. It is enough to prove that the quotient $G_a L := C(S_{a+1})L/C(S_a)L$ is acyclic.
Let $s. = s.'' \cup \{i\} \supset s.'$ with $|s.'| \leq a+1$ ($s.'$ not containing i). Deleting $s.'' \cup \{i\}$ is a morphism in S_{a+1} but deleting $s.'$ gives an element in S_a, hence defines a differential zero in $G_a L$.
We divide the objects of S_{a+1} in two families containing S_a according to the subsequence $s''_{a+2} \cup \{i\} \supset s'_{a+1}$, the family S'_{a+1} (and respectively S''_{a+1}) whose objects are defined by the sequences satisfying $s''_{a+2} = s'_{a+1}$ (resp. $s''_{a+2} \supsetneq s'_{a+1}$). Deleting $s''_\lambda \cup \{i\}$ for $s''_\lambda \neq s''_{a+2}$ is a morphism in S'_{a+1} but for $s''_\lambda = s''_{a+2}$ it defines a functor $d_{s_{a+2}} : S'_{a+1} \to S''_{a+1}$. If we consider the complexes $C(S'_{a+1})L/C(S_a)L$ and $C(S''_{a+1})L/C(S_a)L$, we deduce a morphism of complexes $I_{s_{a+2}} : C(S'_{a+1})L/C(S_a)L \to C(S''_{a+1})L/C(S_a)L$ which consists in an embedding of $C_r^K L(s.)$ into $C_r^K L(d_{s_{a+2}}(s.))$ where $d_{s_{a+2}}(s.) = (s.) - s_{a+2}$. It is easy to check that the cone over this $I_{s_{a+2}}$ is isomorphic to $G_a L[1]$.
We show now that if we reduce the construction to $W_1^i L$ the morphism $I_{s_{a+2}}$ is an isomorphism. The condition on $m. \in X(s., r)$ associated to $s.$ defined by $s''_{a+2} \cup \{i\}$ when $s''_{a+2} = s'_{a+1}$ is $m_i + \Sigma_{j\in s'_{a+1}} m_j \leq |s'_{a+1} \cup \{i\}| + r - 1$ to compare with the condition $\Sigma_{j\in s'_{a+1}} m_j \leq |s'_{a+1}| + r - 1$ defined by s'_{a+1}). Precisely when $m_i < 2$ (that is in $W_1^i L$) the condition for $s'_{a+1} \cup \{i\}$ follows from the condition for s'_{a+1}, hence the conditions defined by the subsequence $s'_{a+1} \cup \{i\} \supset s'a+1$ in $s.$ is the same as the condition defined by s'_{a+1} in $d_{s_{a+2}}(s.)$. This proves that $I_{s_{a+2}}$ is an isomorphism for $W_1^i L$, hence $G_a(W_1^i L)$ is acyclic and i) follows by induction.

ii) *Dual proof* (to be skipped). We construct a dual filtration of $C_r^K(L/W_0^i L)$ with acyclic sub-complexes. To simplify notations we denote a sequence by $s^\cdot$ and define its i-colength $|s^\cdot|^i$ as the number of subsets s_λ containing i. Let S^a be the full subcategory whose objects satisfy $|s^\cdot|^i \leq a$, hence an object $s^\cdot$ in S^a is written as $s.'' \cup \{i\} \supset s'^\cdot$ with $|s''^\cdot| \leq a$ and $s'^\cdot$ not containing i. Deleting a subset of $s^\cdot$ in S^a gives another object in S^a, hence $C(S^a)L := s[C_r^K(s^\cdot)L]_{s^\cdot\in S^a}$ is a sub-complex of $C_r^K L$.
To start with $C(S^0)L = 0$ since S^0 is empty.
Suppose by induction that $C(S^a)(L/W_0^i L)$ is acyclic for $a \geq 0$, we prove $C(S^{a+1})(L/W_0^i L)$ is also acyclic. It is enough to prove that the quotient $G^a L := C(S^{a+1})L/C(S^a)L$ is acyclic.
Let $s^\cdot = s''^\cdot \cup \{i\} \supset s'^\cdot$ with $|s''^\cdot| \leq a+1$ ($s''^\cdot$ not containing i). Deleting $s'^\cdot$ is a morphism in S^{a+1} but deleting $s''^\cdot \cup \{i\}$ gives an element in S^a, hence defines a differential zero in $G^a L$.
We divide the objects of S^{a+1} in two families containing S^a according to the subsequence $s''^{a+1} \cup \{i\} \supset s'^{a+2}$: S_1^{a+1} (resp. S_2^{a+1}) whose objects are defined by the sequences $s''^{a+1} = s'^{a+2}$ (resp. $s''^{a+1} \supsetneq s'a+2$). Deleting s'^λ for $s'^\lambda \neq s'^{a+2}$ is a morphism in S_1^{a+1} but for $s'^\lambda = s'^{a+2}$

it defines a functor $d_{s^{a+2}} : S_1^{a+1} \to S_2^{a+1}$. If we consider the complexes $C(S_1^{a+1})L/C(S^a)L$ and $C(S_2^{a+1})L/C(S^a)L$, we deduce a morphism of complexes $I_{s^{a+2}} : C(S_1^{a+1})L/C(S^a)L \to C(S_2^{a+1})L/C(S^a)L$ by embedding $C_r^K L(s^\cdot)$ into $C_r^K L(d_{s^{a+2}}(s^\cdot))$. It is easy to check that the cone over this $I_{s^{a+2}}$ is isomorphic to $G^a L[1]$.

We show now that if we reduce the construction to $L/W_0^i L$ the morphism $I_{s^{a+2}}$ is an isomorphism. The condition on $m. \in X(s^\cdot, r)$ defined by $s''^{a+1} \cup \{i\}$ when $s''^{a+1} = s'^{a+2}$ (resp. by s'^{a+2}) is $m_i + \Sigma_{j \in s'^{a+2}} m_j \leq |s'^{a+2} \cup \{i\}| + r - 1$ (resp. $\Sigma_{j \in s'^{a+2}} m_j \leq |s'^{a+2}| + r - 1$), but precisely when $m_i > 0$ (that is in $L/W_1^i L$) the condition for s'^{a+2} follows from the condition for $s'^{a+2} \cup \{i\}$, hence the conditions defined by the subsequence $s'^{a+2} \cup \{i\} \supset s'^{a+2}$ in $s^\cdot$ is the same as the condition defined by $s'^{a+2} \cup \{i\}$ in $d_{s^{a+2}}(s^\cdot)$. This proves that $I_{s^{a+2}}$ is an isomorphism for $L/W_0^i L$, hence $G^a(L/W_0^i L)$ is acyclic.

This shows by induction that $C(S^{|K|-1})(L/W_0^i)L$ is acyclic. At the last step, we show that $C_r^K(L/W_0^i L)$ is quasi-isomorphic to $C(S^{|K|})(L/W_0^i)L = C(S^{|K|-1})(L/W_0^i)L$. In colength $|K|$ there is only one sequence not in $S^{|K|-1}$ that is the full sequence starting with $\{i\}$ which imposes the condition $m_i \leq r$ which is impossible since $r \leq 0$ and $m_i \geq 1$. This ends the proof of the lemma. The proposition follows immediately from

Corollary. *i) For $r > 0$, the complex $C(T(r))$ is contained in each $C_r^K L(s.)$ that is $T(r) \subset X(s., r)$ and the complex $\tilde{C}(T(r)) = s(C(T(r))_{s. \in S(K)}$ is contained in and quasi-isomorphic to $C_r^K L$.*

ii) Dually, the complex $C(T'(r))$ is contained only in $C_r^K L(s.)$ for $s. = K$ and is quasi-isomorphic to $C_r^K L[|K| - 1]$.

We did check that the complex $C(T(r))$ is contained in each $C_r^K(s.)$ for $s.$ of maximal length, hence for all $(s.)$. The lemma shows that $\tilde{K}(m_1, \ldots, m_n, r)L \cong 0$ whenever at least one $m_i < 2$, hence i) follows.

Dually, the condition for K, $\Sigma_{j \in K} m_j = |K| + r \Rightarrow \forall A = K - k \subset K, \Sigma_{j \in A} m_j > |A| + r - 1$ by subtracting $m_k < 1$. If this is true for all $A : |A| = a$ then $\forall B = (A - k) \subset A, \Sigma_{j \in B} m_j > |B| + r - 1$ as well. The shift in degree corresponds to the shift for $s. = K$ in the total complex $C_r^K L$.

III. Local decomposition.

Since the purity result is established, we can easily prove now the decomposition theorem after a careful study of the category of indices $S(I)$.

3.8 Theorem *(decomposition). For a nilpotent orbit $L, N_i, i \in M$, $|M| = n$ and for all subsets $K \subset M$ there exist canonical morphisms of $C_r^{KM} L$ in $Gr_r^{\mathcal{W}} \Omega^* L$ inducing a quasi-isomorphism (decomposition as a direct sum)*

$$Gr_r^{\mathcal{W}}(\Omega^* L) \cong \oplus_{K \subset M} C_r^{KM} L.$$

Moreover $Gr_0^{\mathcal{W}} \Omega^* L \cong 0$ is acyclic.

For $n = 1$, K and M reduces to one element 1 and the theorem reduces to

$$Gr_r^{\mathcal{W}}(\Omega^* L) \simeq C_r^1 L \text{ is the complex } Gr_{r+1}^{W^1} L \xrightarrow{N_1} Gr_{r-1}^{W^1} L$$

By the elementary properties of the weight filtration of N_1, it is quasi-isomorphic to $Gr_{r-1}^{W^1}(L/N_1 L)[-1]$ if $r > 0$, $Gr_{r+1}^{W^1}(ker N_1 : L \to L)$ if $r < 0$ and $C_0^1 L \simeq 0$.

The proof is by induction on n; we use only the property $Gr_0^{\mathcal{W}}(\Omega^* L) \simeq 0$ in $n-1$ variables to get the decomposition for n, then we use the fact that $C_0^K L$ for all K is acyclic to get again $Gr_0^{\mathcal{W}}(\Omega^* L) \cong 0$ for n variables so to complete the induction step.
The proof of the decomposition is carried in the three lemmas below.

For each $i \in \mathbf{N}$ we define a map into the subsets of M
$\varphi_i : S(M) \to \mathcal{P}(M) : \varphi_i(s.) = Sup\{s_\lambda : |s_\lambda| \leq i\}$. For each $(J, s.) \in M.^+ \times S(M)$, we consider the subspaces of L with indices i and t
$W_t(i, J, s.) := (\bigcap_{\varphi_i(s.) \subsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}) \cap (\bigcap_{s_\lambda \subset \varphi_i(s.), s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)$
We define
$G_i(J, s.)(L) = Gr_0^{W(i,J,s.)} L = W_0(i, J, s.)/W_{-1}(i, J, s.)$ and the complexes

$$G_i(s.)L = s(G_i(J, s.)L)_{J \subset M}, \quad G_i^{SM} L = s(G_i(s.)L)_{s. \in S(M)}$$

In particular, $\varphi_0(s.) = \varnothing$, and $\varphi_{|M|} = M$ so that

$$G_0(s.)L = Gr_r^{W(s.)} L, \; G_0^{SM} L = Gr_r^{\mathcal{W}} \Omega^* L, \quad G_{|M|}^{SM} L = C_r^M L$$

The proof of the theorem by induction on i, starting with $i = 0$, is based on

Lemma 1.

$$G_i^{SM} L \cong G_{i+1}^{SM} L \oplus (\oplus_{K \subset M, |K| = i+1} C_r^{KM} L)$$

To relate $G_i^{SM} L$ and $G_{i+1}^{SM} L$, we define $S^i(M) = \{s. \in S(M) : |\varphi_i(s.)| \; = i\}$ in $S(M)$ and consider the subcategory $S(M) - S^{i+1}(M)$. The restrictions of the simplicial vector spaces G_i and G_{i+1} to $S(M) - S^{i+1}(M)$, define two sub-complexes:
$G_i' L = s(G_i(s.)L)_{s. \in S(M) - S^{i+1}(M)}$ embedded in $G_i^{SM} L$ and
$G_{i+1}'' L = s(G_{i+1}(s.)L)_{s. \in S(M) - S^{i+1}(M)}$ embedded in $G_{i+1}^{SM} L$
since deleting an object $s. \in S(M) - S^{i+1}(M)$ gives an object in the same subcategory. We have $G_{i+1}'' L = G_i' L$ since $\varphi_i = \varphi_{i+1}$ on $S(M) - S^{i+1}(M)$; hence we are reduced to relate the the quotient complexes: $G_i^{SM} L / G_i' L$ and $G_{i+1}^{SM} L / G_{i+1}'' L$ which are obtained by summing over $S^{i+1}(M)$.
We remark that $S^{i+1}(M) \simeq \cup_{|K| = i+1} S_K(M)$ is a disjoint union of $S_K(M)$ where $\varphi_{i+1}(s.) = K$ with $|K| = i + 1$.
Definition of $B_r^{KM} L$.
For K fixed in M, we introduce the subspaces with index t
$W_t(K, J, s.)L := (\bigcap_{K \subsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}) \cap (\bigcap_{s_\lambda \subset K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)$
$Gr_0^{W(K,J,s.)} L = W_0(K, J, s.)L / W_{-1}(K, J, s.)L$
$B_r^{KM} L = s[Gr_0^{W(K,J,s.)} L]_{(J,s.) \in M.^+ \times S_K(M)}$
It follows by construction
$G_{i+1}^{S^{i+1}(M)} L := s(G_{i+1}(s.)L)_{s. \in S^{i+1}(M)} \simeq \oplus_{|K| = i+1} B_r^{KM} L$,
and a triangle

$$G_{i+1}'' L \to G_{i+1}^{SM} L \to \oplus_{|K| = i+1} B_r^{KM} L$$

Definition of $A_r^{KM} L$.
$A_r^{KM}(J, s.)L = [(\bigcap_{K \subset s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r)} L) \cap (\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)] \; / \; [(\bigcap_{s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)} L)]$
$A_r^{KM} L = s(A_r^{KM}(J, s.))_{(J,s.) \in M.^+ \times S_K(M)}$
We have by construction: $s(G_i(s.)L)_{s. \in S_{KM}} = A_r^{KM} L$. Suppose $|K| = i + 1$, then

$G_i^{S^{i+1}(M)}L = s(G_i(s.)L)_{s.\in S^{i+1}(M)} \simeq \oplus_{|K|=i+1}A_r^{KM}L.$
Considering the triangle

$$G_i'L \to G_i^{SM}L \to \oplus_{|K|=i+1}A_r^{KM}L$$

and the morphism of triangles: $G_{i+1}^{SM}L \to G_i^{SM}L$, $B_r^{KM}L \to A_r^{KM}L$, $G_{i+1}''L \simeq G_i'L$, the relation in the lemma follows from a relation between $C_r^{KM}L$ and $A_r^{KM}L$

Lemma 2: *We have a quasi-isomorphism in the derived category:*

$$C_r^{KM}L \oplus B_r^{KM}L \cong A_r^{KM}L$$

The proof is based on the following elementary remark:

Remark. *Let W^i for $i = 1, 2$ be two increasing filtrations on an object V of an abelian category and a_i two integers, then we have an exact sequence:*

$$0 \to W^2_{a_2-1}Gr^{W^1}_{a_1} \oplus W^1_{a_1-1}Gr^{W^2}_{a_2} \to W^1_{a_1} \cap W^2_{a_2}/W^1_{a_1-1} \cap W^2_{a_2-1} \to Gr^{W^1}_{a_1}Gr^{W^2}_{a_2} \to 0$$

We apply this remark to the space
$V = V(J, s.) = \bigcap_{s_\lambda \subsetneq K, s_\lambda \in (s.)} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L,$
filtered by:
$W^1_t(K, J, s.)L = W^K_{a_K(J,r+t)}L \cap V(J, s.)),\ \ W^2_t(K, J, s.)L = (\bigcap_{K \subsetneq s_\lambda \in (s.)} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)})L \cap V(J, s.)$
so that
$W^1_t(K, J, s.)L \cap W^2_t(K, J, s.)L = (\bigcap_{K \subset s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)})L) \cap (\bigcap_{s_\lambda \subsetneq K, s_\lambda \in (s.)} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L).$
Let $a_1 = a_2 = t = 0.$, then we deduce from the above sequence an exact sequence of vector spaces

$$0 \to W^2_{-1}(J, s.)Gr^{W^K}_{a_K(J,r)}(V(J, s.)) \oplus W^K_{a_K(J,r-1)}Gr^{W^2}_0(V(J, s.))$$
$$\to W^K_{a_K(J,r)} \cap W^2_0(V(J, s.))/W^K_{a_K(J,r-1)} \cap W^2_{-1}(V(J, s.)) \to Gr^{W^K}_{a_K(J,r)}Gr^{W^2}_0(V(J, s.)) \to 0$$

By summing over $(J, s.)$ in $M.^+ \times S_K(M)$ we get an exact sequence of complexes

$$0 \to C_r^{KM}L \oplus B_r^{KM}L \to A_r^{KM}L \to D_r^{KM}L \to 0$$

where by definition
$D_r^{KM}L = s[Gr_0^{W^2(J,s.)}Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)]_{(J,s.)\in M.^+ \times S_K(M)}$
Lemma 2 follows if we prove

Lemma 3. $D_r^{KM}L \cong 0.$

Proof. The idea of the proof is to consider C_r^KL as the fiber of a local system on Y_K^* and form the filtered complex $(\Omega^*(C_r^KL), \mathcal{W})$ for the polarised nilpotent orbit $(C_r^KL, N_i, i \in M - K$ to which we can apply the theorem on the lower dimensional space Y_K^*. It happens that D_r^{KM} is quasi-isomorphic to $Gr_0^{\mathcal{W}}(\Omega^*(C_r^KL))$ hence acyclic by induction on dimension. We can either use that C_r^KL is reduced to its unique non zero cohomology or as well prove the acyclicity for each term in C_r^KL, as we do now.

We consider the filtration induced by W^2 on $Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$
$W^2_t(K, J, s.) := \bigcap_{K \subsetneq s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r+t)}Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda \subsetneq K, s_\lambda \in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L).$
We want to rewrite the complex $D_r^{KM}L$ as a sum in two times over $(J, s.) \in K.^+ \times S(K)$ and $(J', s.') \in (M - K).^+ \times S(M - K))$ corresponding to $((J \cap J'), (K \cup s.', s.)) \in M.^+ \times S(M)$.
Using the expression of sequences in $S_K(M)$ as $s. = (K \cup s.', s.)$ and the result on relative weight

filtrations with respect to $Gr^{W^K}_{a_K(J,r)}$ we rewrite W^2_t as
$W^2_t(K,(J,J'),(K\cup s.',s.)) = \bigcap_{s'_\alpha\in s.'} W^{K\cup s'_\alpha}_{a_{s'_\alpha}(J',a_K(J,r)+t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda\subsetneq K, s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}((J,J'),r-1)}L)$
$= \bigcap_{s'_\alpha\in s.'} W^{s'_\alpha}_{a_{s'_\alpha}(J',t)} Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda\subsetneq K, s_\lambda\in s.} W^{s_\lambda}_{a_\lambda(J,r-1)}L)$
For a fixed $(J,s.)\in K.^+\times S(K)$ we introduce the filtration $W'^2_t = \bigcap_{s'_\alpha\in s.'} W^{s'_\alpha}_{a_{s'_\alpha}(J',t)}$ on the space
$L(r,J,s.) := Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda\subsetneq K, s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)$
and the complex
$D(M-K)(L(r,J,s.) := s[Gr^{W'^2}_0(L(r,J,s.))]_{(J',s.')\in(M-K).^+\times S(M-K)}$
We have by construction

$$D^{KM}_r L = s[D(M-K)(L(r,J,s.)]_{(J,s.)\in K.^+\times S(K)}$$

We prove by induction on n that each complex

$$D(M-K)(L(r,J,s.) := s[Gr^{W'^2}_0(Gr^{W^K}_{a_K(J,r)}(\bigcap_{s_\lambda\subsetneq K, s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L)]_{(J',s.')\in(M-K).^+\times S(M-K)} \cong 0$$

is acyclic. Fixing $(J,s.)$, we decompose $L(r,J,s.)$ into a direct sum of
$L(i.) = Gr^{W^{i_s}}_{m_{i_s}-2|i_s\cap J|}\cdots Gr^{W^{i_1}}_{m_{i_1}-2|i_1\cap J|}L$ for $\{i_1,\dots,i_s\}=K$ and $\sum_{i_p\in K} m_{i_p} = a_K(J,r)$.
We reduce the proof to $D(M-K)(L(i.))\cong 0$, then we introduce the weight filtration $\mathcal{W}$ on the combinatorial DeRham complex $\Omega^*(L(i.))$ for the nilpotent orbit $L(i.)$ of dimension strictly less then n and weight $a_K(J,r)$ and we notice that $D(M-K)(L(i.))\simeq Gr^{\mathcal{W}}_0(\Omega^*(Gr^{W^K}_{a_K(J,r)}(L(i.)))$ is acyclic by the inductive hypothesis in dimension $< n$. This ends the proof of the lemma.
The relation between $C^K_r L$ and $C^{KM}_r L$
The following result describes $C^{KM}_r L$ as the fiber of an intersection complex of the local system defined by $C^K_r L$.
Proposition. *Let $H = H^*(C^K_r L)$ be considered as a nilpotent orbit with indices $i\in M-K$, then we have a quasi-isomorphism: $C^{KM}_r L\cong \mathcal{W}_{-1}\Omega^*(H)$.*
The proof is based on a decomposition as above: $(K^+_. \times S(K)\times (M-K)^+_.\times S(M-K))\simeq M^+_.\times S_K(M)$, with the correspondence $(J,s.,J',s.')\to((J,J'),(s'_.\cup K,s.))$ then using the relations:
$a_{s_\lambda}((J,J'),r-1) = a_{s_\lambda}(J,r-1)$ when $s_\lambda\subset K$
$a_{s'_\lambda}((J,J'),r-1) = a_{s'_\lambda}(J',-1)+a_K(J,r)$ and
$W^{s'_\lambda\cup K}_{a_{s'_\lambda\cup K}((J,J'),r-1)}(Gr^{W^K}_{a_K(J,r)}) = W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(Gr^{W^K}_{a_K(J,r)})$ since $W^{s'_\lambda\cup K}$ is relative to W^K
we find
$C^{KM}_r L := s(\cap_{K\neq s_\lambda\in(s.'\cup K,s.)} W^{s_\lambda}_{a_{s_\lambda}((J,J'),r-1)} Gr^{W^K}_{a_K((J,J'),r)}L)_{((J,J'),(s'_.\cup K,s.))\in M^+_.\times S_K M}\simeq$
$s[(\cap_{s'_\lambda\in s.'} W^{K\cup s'_\lambda}_{a_{s'_\lambda}(J',a_K(J,r-1)}(s[Gr^{W^K}_{a_K(J,r)}(\cap_{K\supsetneq s_\lambda\in s.} W^{s_\lambda}_{a_{s_\lambda}(J,r-1)}L]_{(J,s.)\in K^+_.\times S(K)})]_{(J',s.')\in(M-K)^+_.\times S(M-K)}$
$\simeq s[\cap_{s'_\lambda\in s.'} W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(C^K_r L)]_{(J',s.')\in(M-K)^+_.\times S(M-K)}$
where we used $\cap_{s'_\lambda\in s.'} W^{s'_\lambda}_{a_{s'_\lambda}(J',-1)}(C^K_r L)\simeq C^K_r(\cap_{s'_\lambda\in s.'} W^{K\cup s'_\lambda}_{a_{s'_\lambda}(J',-1)}L)$ is defined as above for each subset of L.
This formula is precisely the weight $\mathcal{W}_{-1}$ of the terms of $C^K_r L$. It shows that $C^{KM}_r L$ is constructed in two times, once as C^K over $K^+_.\times S(K)$ (that is a space normal to Y_K) and once as a weight filtration over $(M-K)^+_.\times S(M-K)$ (that is the space Y_K).

4§. Global construction of the weight filtration.

In this section we construct a global bi-filtered combinatorial logarithmic complex and prove a global decomposition of the graded weight into intermediate extensions of polarised VHS on the various intersections of components of Y. We use a formula of the intersection complex announced by Kashiwara and Kawai [26] that we prove since we have no reference for its proof.

Let Y be a NCD in X with smooth irreducible components $Y_{i\in I}$ with indices in the set I. The direct image of the complex local system $\mathcal{L}^c$ is computed globally via the logarithmic complex with coefficients in Deligne's analytic extension $\Omega^*_X(LogY)\otimes\mathcal{L}_X$. It is on a quasi-isomorphic constant combinatorial complex with indices $s.\in S(I)$

$$\Omega^*\mathcal{L}=s(\Omega^*_{X_{s.}}(LogY)\otimes\mathcal{L}_{X_{s.}})_{s.\in S(I)}$$

that we can define the two filtrations $\mathcal{W}$ and F.

I. Comparison with the local definition

Lemma. *Let $M\subset I$, $y\in Y^*_M$ and $L\simeq\mathcal{L}_X(y)$ the space of multivalued sections of $\mathcal{L}$ at y, then the correspondence from $v\in L$ to $\tilde{v}\in\mathcal{L}_{X,y}$ extends to a quasi-isomorphism*

$$\Omega^*L\cong(\Omega^*\mathcal{L})_y$$

Proof. The quasi-isomorphism $\Omega(L,N_j,j\in M)\cong(\Omega^*_X(LogY)\otimes\mathcal{L}_X)_y$ (2.1 (6)) is compatible with the differentials when defined with indices $s.\in S(I)$.

The weight $\mathcal{W}$.

1- For each $(s.)\in S(I)$ we deduce, from the weight filtration by sub-complexes $W_r(s.)L=s(W_r(J,s.)L)_{J\subset M}$ (3.2) of the locally defined DeRham complex $\Omega(L,N.)$, a corresponding global filtration by sub-complexes

$\mathcal{W}_r(X,\mathcal{L})(s.)$ in $\Omega^*_X(LogY)\otimes\mathcal{L}_X$.

In terms of a set of coordinates $z_i,i\in M$ defining Y_M in a neighbourhood of $y\in Y^*_M$, the fiber at y in $(\Omega^*_X(LogY)\otimes\mathcal{L}_X)_y$ is defined as follows

Definition. *$\mathcal{W}_r(X,\mathcal{L})_y(s.)$ is generated as an $\Omega^*_{X,y}-$ sub-module by the germs of the sections $\wedge_{j\in J}\frac{dz_j}{z_j}\otimes\tilde{v}$ for $v\in W_r(J,s.)L$.*

This formula is independent of the choice of coordinates, since if we choose a different coordinate $z'_i=fz_i$ instead of z_i with f invertible holomorphic at y, the difference $\frac{dz'_i}{z'_i}-\frac{dz_i}{z_i}=\frac{df}{f}$ is holomorphic at y, hence the difference of the sections $\wedge_{j\in J}\frac{dz'_j}{z'_j}\otimes\tilde{v}-\wedge_{j\in J}\frac{dz_j}{z_j}\otimes\tilde{v}$ is still a section of the sub-complex $\mathcal{W}_r(X,\mathcal{L})(s.)$.

Moreover the restriction of the section is still defined in the sub-complex near y, since $W_r(J,s.)L\subset W_r(J-i,s.)L$ for all $i\in J$, then we have a quasi-isomorphism

Lemma. *We have an induced quasi-isomorphism $W_r(s.)L\cong\mathcal{W}_r(X,\mathcal{L})_y(s.)$, functorial in $(s.)$.*

4.1 Definition 1 *(weight and Hodge filtrations). The weight filtration is defined on the combinatorial logarithmic complex with indices $s.\in S(I)$*

$$\Omega^*\mathcal{L}=s(\Omega^*_{X_{s.}}(LogY)\otimes\mathcal{L}_{X_{s.}})_{s.\in S(I)} \tag{15}$$

as follows:

$$\mathcal{W}_r(X,\mathcal{L}):=s(\mathcal{W}_r(X,\mathcal{L})(s.))_{s.\in S(I)}$$

The Hodge filtration F is constant in $(s.)$ and deduced from Schmid's extension to $\mathcal{L}_X$

$$F^p(s.) = 0 \to F^p\mathcal{L}_X \ldots \to \Omega^i_{X_{s.}}(LogY) \otimes F^{p-i}\mathcal{L}_{X_{s.}} \to \ldots; \quad F^p = s(F^p(s.))_{s.\in S}$$

The fiber of Deligne's bundle $\mathcal{L}_X(y)$ at the point y is identified with the space L of multivalued sections L.

Definition 2 *(weight). With the same notations, let $M \subset I, |M| = p$ and $y \in Y^*_M$, then in terms of a set of n coordinates $z_i, i \in [1,n]$ where we identify M with $[1,p]$, we write a section $f = (f^{s.})_{s.\in S}$ with $f^{s.} = \Sigma_{J\subset M, J'\cap M=\varnothing}\frac{dz_J}{z_J} \wedge dz_{J'} \otimes f^{s.}_{J,J'}$ s.t.$f^{s.}_{J,J'}$ is not divisible by $y_j, j \in J$, then f is in $(\mathcal{W}_r\Omega^*\mathcal{L})_y$ if and only if $f^{s.}_{J,J'}(y)$ in $\mathcal{L}_X(y) = L$ satisfy*

$$\forall J \subset M,\ f^{s.}_{J,J'}(y) \in \bigcap_{s_\lambda\in s., s_\lambda\subset M} W^{s_\lambda}_{a_\lambda(J,r)}L$$

3- *Definition with residue.* Let W^Y denotes the weight along Y on $\Omega^*_X(LogY)$; choose an order on I and an integer m, then the residue morphism Res_m of order m is defined on $W^Y_m(\Omega^p_X(LogY)) \otimes \mathcal{L}_X$ with value in $\Omega^{p-m}_{Y^m} \otimes \mathcal{L}_{Y^m}$ on the disjoint union of intersections of m components of Y (the residue does not commute with differentials). For $M \subset I$ s.t. $Y_M \neq \varnothing$ and $|M| = m$ we deduce the residue $Res_M : W^Y_m(\Omega^p_X(LogY)) \otimes \mathcal{L}_X \to \Omega^{p-m}_{Y_M} \otimes \mathcal{L}_{Y_M}$ by composition of the residue morphism Res_m with the obvious projection. At a point $y \in Y_M$, the morphism induced on the fiber with value in $(\Omega^{p-m}_{Y^m} \otimes \mathcal{L}_{Y^m})(y)$ is denoted by $Res_{M,y}$.

Definition 3. *With the same notations, let $M \subset I, |M| = p$ and $y \in Y^*_M$, then the fiber of the sub-analytic sheaf $(\mathcal{W}_r\Omega^*\mathcal{L})_y$ at y is defined successively on its intersection with $(W^Y_i(\Omega^p_{X_{s.}}(LogY) \otimes \mathcal{L}_{X_{s.}})_y$ by the following formula:*
a section $f \in (W^Y_i\Omega^p_X(LogY) \otimes \mathcal{L}_X)_y$ is in $(\mathcal{W}_r\Omega^*\mathcal{L})_y$ if and only if setting $\mathcal{L}_{Y^m}(y) = L$

$$\forall J, s_\lambda \subset M :\ |J| = i,\ Res_{J,y}(f) \in \Omega^{p-m}(y) \otimes W^{s_\lambda}_{a_\lambda(J,r)}\mathcal{L}_{Y^m}(y)$$

Remark. By construction, for all integers r, $\mathcal{W}_r/X-Y = \Omega^*(\mathcal{L})/X-Y$, so that $\mathcal{W}_r$ is exhaustive for r big enough, and equal to the extension by zero for r small enough. It is a filtration by sub-complexes of analytic sub-sheaves globally defined on X.

Proposition (*Comparison with the local definition). Let $L = \mathcal{L}_X(y)$ denotes the space of multi-valued sections of $\mathcal{L}$ at a point $y \in Y^*_M$, then we have a bi-filtered quasi-isomorphism*

$$(\Omega^*L, \mathcal{W}, F) \cong (\Omega^*\mathcal{L}, \mathcal{W}, F)_y$$

At left we sum on $S(M)$ and at right on S(I). The statement asserts that on Y^*_M, we still have a quasi-isomorphism. For a fixed subset M in I, the definition of the weight filtration near a point in Y^*_M involves only the subsets $s_\lambda \subset M$, hence if we consider the correspondence $s.' \in S(M)$ with the family $(M \cup s.'', s.') \in S_M(I)$ where $s.'' \in S(I-M)$, the diagonal embedding of the restrictions to Y^*_M, $\Omega^*_{X_{(s.')}}(LogY) \otimes \mathcal{L}_{X_{(s.')}}, \mathcal{W}, F)_{|Y^*_M}$ into

$$s((\Omega^*_{X_{(M\cup(s.''),(s.'))}}(LogY) \otimes \mathcal{L}_{X_{(M\cup(s.''),(s.'))}}, \mathcal{W}, F)_{|Y^*_M})_{(s.'')\in S(I-M)}$$

is a bi-filtered quasi-isomorphism, hence the local study at points of Y^*_M of $(\Omega^*L, \mathcal{W}, F)$ reduces to $s(\Omega^*_{X_{(s.')}}(LogY) \otimes \mathcal{L}_{X_{(s.')}}, \mathcal{W}, F)_{|Y^*_M})_{s.'\in S(M)}$.

The variation of Hodge structures $(\mathcal{L}^K_r, F)$
Let $i_K : Y^*_K \to X$. Recall the definitions in the introduction:
$\mathcal{L}^K = i^*_K R^{|K|}\mathbf{j}_*\mathcal{L}, \quad \mathcal{L}'^K = i^*_K R^0\mathbf{j}_*\mathcal{L}$
$\mathcal{L}^K_X = \mathcal{L}_{Y_K}/(\Sigma_{i\in K}\mathcal{N}_i\mathcal{L}_{Y_K}), \quad \mathcal{L}'^K_X = \cap_{i\in K}(ker\mathcal{N}_i : \mathcal{L}_{Y_K} \to \mathcal{L}_{Y_K})$
and $\mathcal{W}^K_{Y_K} = \mathcal{W}(\Sigma_{i\in K}\mathcal{N}_i)$ for the filtration by sub-bundles defined on $\mathcal{L}_{Y_K}$ by $\Sigma_{i\in K}\mathcal{N}_i$.

4.2 Proposition. *i) $\mathcal{L}_X^K$ (resp. $\mathcal{L}_X'^K$) induces a flat bundle on Y_K^*, with flat sections isomorphic to the local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$); precisely they are respectively Deligne's extension of the corresponding complex local system.*

ii) The filtration $\mathcal{W}_{Y_K}^K$ induces a filtration by flat sub-bundles of $\mathcal{L}_X^K$ (resp $\mathcal{L}_X'^K$) on Y_K^*, hence induces a filtration by complex sub-local systems $\mathcal{W}^K$ of $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).

iii) The filtration $\mathcal{W}^K$ is rationally defined on the rational local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$).

iv) Let $\mathcal{L}_r^K := Gr_{r-|K|}^{\mathcal{W}^K}\mathcal{L}^K$ and $\mathcal{L}_{X,r}^K := Gr_{r-|K|}^{\mathcal{W}_{Y_K}^K}\mathcal{L}_X^K$ for $r > 0$, resp.

$\mathcal{L}_r^K := Gr_{r+|K|}^{\mathcal{W}^K}\mathcal{L}'^K$ and $\mathcal{L}_{X,r}^K := Gr_{r+|K|}^{\mathcal{W}_{Y_K}^K}\mathcal{L}_X'^K$ for $r < 0$,

then the system defined on Y_K^* by $(\mathcal{L}_r^K, (\mathcal{L}_{X,r}^K)_{|Y_K}, F)$ where F is the Hodge filtration induced from $\mathcal{L}_X$, is a polarised variation of Hodge structures VHS of weights $r - |K| + m$ for $r > 0$ and $r + |K| + m$ for $r < 0$.

Proof. We deduce from the comparison propositions with local definitions at each point $y \in Y_K^*$ the following complexes quasi-isomorphic to $\mathbf{j}_*\mathcal{L}_y$

$$(\Omega_X^*(LogY) \otimes \mathcal{L}_X)_y \cong \Omega(L, N_j, j \in K) \cong s(L(J), N.)_{J\subset K},\ (\Omega^*L, \mathcal{W}, F) \cong (\Omega^*\mathcal{L}, \mathcal{W}, F)_y$$

hence near each point $y \in Y_K^*$, $\mathcal{L}^K$ and $\mathcal{L}'^K$ are (locally) constant. The local system $\mathcal{L}^K$ (resp. $\mathcal{L}'^K$) is defined by the flat sections of the bundles $\mathcal{L}_X^K$ (resp. $\mathcal{L}_X'^K$) whose connection has logarithmic singularity since it is induced by the connection on $\mathcal{L}_X$ which proves (i).
The same argument apply to the filtration $\mathcal{W}_{Y_K}^K$, which proves ii).

iii) Let us denote by $\mathcal{L}^{K,rat}$ the rational local system underlying the complex $\mathcal{L}^K$. The intersection $\mathcal{W}^K \cap \mathcal{L}^{K,rat}$ defines a rational filtration underlying the complex one. This can be checked locally as the graded vector space $Gr_r^{W^K}L$ has a rational structure at each point y.

iv) The sheaf $\mathcal{L}_r^K$ is locally constant and isomorphic to the cohomology of the complex $C_r^K L$ for $L = \mathcal{L}_X(y)$ (Prop. 3.6) which shows that the local system $\mathcal{L}_r^K$ is defined by the flat sections of the bundle $\mathcal{L}_{X,r}^K$, then (iv) follows.

Remark. Given the VHS $\mathcal{L}_r^K$, we can construct a corresponding complex $(\Omega^*\mathcal{L}_r^K, \mathcal{W}, F)$, then for each point $y \in Y_M^*$, $K \subset M$, we have a quasi-isomorphism $C_r^{KM}L \cong \mathcal{W}_{-1}(\mathcal{L}_r^K[-|K|])_y$ for $r > 0$ and $C_r^{KM}L \cong \mathcal{W}_{-1}(\mathcal{L}_r^K[1 - |K|])_y$ for $r < 0$ (recall $C_0^{KM}L \cong 0$). That is the cohomology of the various complexes $C_r^{KM}L$ is globally defined on Y_M^*. In fact we will see in the next result it is the restriction of the intermediate extension of $\mathcal{L}_r^K$ to Y_M^*.
it will follow from the next proof that we could define $\mathcal{L}_r^K$ as
$\mathcal{L}_r^K := i_K^*\mathcal{H}^{|K|}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}), r > 0$ (*resp.* $\mathcal{L}_r'^K := i_K^*\mathcal{H}^{|K|-1}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}), r < 0$)

4.3 Theorem. *Let $\mathcal{L}$ be a local system with locally unipotent monodromy, underlying a variation of polarised Hodge structures of weight m on $X - Y$ of dim n and let $j^K : Y_K^* \to Y_K$, $i_K : Y_K^* \to X$, then the bi-filtered complex*

$$(\Omega^*\mathcal{L}, \mathcal{W}, F)$$

is filtered quasi-isomorphic to $(\Omega_X^*(LogY) \otimes \mathcal{L}_X, F)$.

i) The restriction to Y_K^*, $i_K^*\mathcal{H}^{|K|}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L})$ for $r > 0$ (resp. $i_K^*\mathcal{H}^{|K|-1}(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L})$ for $r < 0$) is a complex local system isomorphic to $\mathcal{L}_r^K$, moreover the following decomposition property into intermediate extensions is satisfied

$$(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}, F) \cong \oplus_{K\subset I}\mathbf{j}_{!*}^K\mathcal{L}_r^K[-|K|], F[-|K|]),\ \textit{for } r > 0$$
$$(Gr_0^{\mathcal{W}}\Omega^*\mathcal{L}, F) \cong 0$$

$$(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}, F) \cong \oplus_{K\subset I} j^K_{!*}\mathcal{L}^K_r[1-|K|], F), \text{ for } r<0$$

ii) (Kashiwara and Kawai's formula [26]) The sub-complex $\mathcal{W}_{-1}\Omega^*(\mathcal{L}[2n])$ is quasi-isomorphic to the intermediate extension $\mathbf{j}_{!*}\mathcal{L}[2n]$ of $\mathcal{L}[2n]$ and $(\mathcal{W}_{-1}\Omega^*(\mathcal{L}[2n]), F)$ is a Hodge complex of weight $2n+m$.

Proof. i) The decomposition of $(Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}, F)$ reduces near a point $y \in Y^*_M$ to the local decomposition for the nilpotent orbit L defined at the point y by the local system $Gr_r^{\mathcal{W}}\Omega^*L \cong \oplus_{K\subset M} C_r^{KM}L$. The global decomposition that follows is $Gr_r^{\mathcal{W}}\Omega^*\mathcal{L} \cong \oplus_{K\subset I}\mathcal{W}_{-1}\Omega^*(\mathcal{L}^K_r)$ that is the complex Ω^* considered for the polarised local system $\mathcal{L}^K_r$, as it follows from the local formula $C_r^{KM}L \cong \mathcal{W}_{-1}L^K_r)$ where $L^K_r \simeq H^*(C^K_r L)$ which has been checked. The fact that $C_r^{KM}L$ is precisely the fiber of $j^K_{!*}\mathcal{L}^K_r[-|K|]$ for $r>0$ (resp. $j^K_{!*}\mathcal{L}^K_r[1-|K|]$ for $r<0$ will follow from (ii) by induction on the dimension.

The count of weight and the shift in F take into account for $r>0$ the residue in the isomorphism with L that shifts W and F but also the shift in degrees, while for $r<0$ there is no residue (since the cohomology is in degree 0 of the logarithmic complex with index $s. = K \in S(K)$ but only a shift in degrees $|K|-1$ in the combinatorial complex, the rule being as follows:

Let (K, W, F) be a mixed Hodge complex then for all $m, h \in \mathbb{Z}$, $(K', W', F') = (K[m], W[m-2h], F[h])$ is also a mixed Hodge complex.

The same proof apply for $r=0$.

ii) The proof is based on the decomposition of $S_K M$ as a product in (3.3) and follows by induction on the dimension n, from the local decomposition of the graded parts of the weight filtration above in i).

The proof is true in dimension 1 and if we suppose the result true in dimension strictly less than n, we can apply the result for $Gr_r^{\mathcal{W}}\Omega^*\mathcal{L}$ that is for local systems defined on open subsets of the closed sets Y_K, namely the local system $\mathcal{L}^K_r[-|K|]$ for $r>0$ (resp.$\mathcal{L}^K_r[1-|K|]$ for $r<0$) whose fiber at each point $y \in Y^*_K$ is quasi-isomorphic to $C^K_r L$. Let $j^K : Y^*_K \to Y_K$ be the open embedding in Y_K and consider the associated DeRham complex $\Omega^*(\mathcal{L}^K_r)$ on Y_K whose weight filtration will be denoted locally near a point in Y^*_M by $\mathcal{W}^{M-K}$ for $K \subset M$; then by the inductive hypothesis we have at the point y: $\mathcal{W}^{M-K}_{-1}\Omega^*L^K_r \simeq \mathcal{W}^{M-K}_0\Omega^*L^K_r$ is also quasi-isomorphic to the fiber of the intermediate extension of $\mathcal{L}$, that is

$$\forall r>0, C_r^{KM}(L) \simeq (j^K_{!*}\mathcal{L}^K_r[-|K|])_y \simeq \mathcal{W}^{M-K}_{-1}\Omega^*L^K_r$$

and similarly for $r<0$.

In order to check the result for $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$, we use the following criteria characterising intermediate extension [17] where the degree shift is by $2n$:

Consider the stratification defined by Y on X and the middle perversity $p(2k) = k-1$ associated to the closed subset $Y^{2k} = \cup_{|K|=k} Y_K$ of real codimension $2k$. We let $Y^{2k-1} = Y^{2k}$ and $p(2k-1) = k-1$. For any complex of sheaves $S.$ on X which is constructible with respect to the stratification, let $S.^{2k} = S.^{2k-1} = S.|X - Y^{2k}$ and consider the four properties:

a) Normalization: $S.|X - Y^2 \cong \mathcal{L}[2n]$

b) Lower bound: $\mathbf{H}^i(S.) = 0$ for all $i < -2n$

c) vanishing condition: $\mathbf{H}^m(S.^{2(k+1)}) = \mathbf{H}^m(S.^{2k+1}) = 0$ for all $m > k-2n$

d) dual condition: $\mathbf{H}^m(j^!_{2k}S.^{2(k+1)}) = 0$ for all $k \geq 1$ and all $m > k-2n$ where $j_{2k} : (Y^{2k} - Y^{2(k+1)}) \to (X - Y^{2(k+1)})$ is the closed embedding,

then we can conclude that $S.$ is the intermediate extension of $\mathcal{L}[2n]$.

In order to prove the result in dim. n we check the above four properties for $\mathcal{W}_0\Omega^*(\mathcal{L}[2n])$. The first two are clear and we use the exact sequences

$$0 \to \mathcal{W}_{r-1}\Omega^*(\mathcal{L}[2n]) \to \mathcal{W}_r\Omega^*(\mathcal{L}[2n]) \to Gr_r^{\mathcal{W}}\Omega^*(\mathcal{L}[2n]) \to 0$$

to prove d)(resp. c)) by descending (resp. ascending)indices from $\mathcal{W}_r$ to $\mathcal{W}_{r-1}$ for $r \geq 0$ (resp. $r-1$ to r for $r < 0$) applying at each step the inductive hypothesis to $Gr_r^{\mathcal{W}}$.

Proof of d). For r big enough $\mathcal{W}_r\Omega^*(\mathcal{L}[2n])$ coincides with the whole complex $\mathbf{j}_*\mathcal{L}[2n]$, then the dual condition is true for r big enough. Now to check d) for $Gr_r^{\mathcal{W}}\Omega^*(\mathcal{L}[2n])$, we apply d) to a component with support $Y_{K'}$ with $|K'| = k'$. We choose $k > k'$ and consider $j'_{2k} : (Y^{2k} \cap Y_{K'} - Y^{2(k+1)} \cap Y_{K'}) \to (Y_{K'} - Y^{2(k+1)} \cap Y_{K'})$ (notice that $Y^{2k} \cap Y_{K'} = (Y \cap Y_{K'})^{2(k-k')}$ is of codim. $2(k-k')$ in $Y_{K'}$), then for $S.'$ equal to the intermediate extension of $\mathcal{L}_r^{K'}[2n-2k']$ on $Y_{K'}$ we have on $Y_{K'}$ the property $\mathbf{H}^m(j'^{!}_{2k}S.'^{2(k-k')+1)}) = 0$ for all $(k-k') \geq 1$ and all $m > k - k' - 2(n-k') = k + k' - 2n$ which gives for $S.'[k']$ on X: $\mathbf{H}^m(j^{!}_{2k}S.'^{2(k+1)}[k']) = 0$ for all $k > k'$ and all $m > k - 2n$, hence d) is true.
If $k = k'$, then $Y^{2k} \cap Y_{K'} = Y_{K'}$ and we have a local system in degree $k'-2n$ on $Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ hence d) is still true, and for $k < k'$, the support $Y_{K'} - Y^{2(k+1)} \cap Y_{K'}$ of $S.'$ is empty. From the decomposition theorem and the induction, this argument apply to $Gr_r^{\mathcal{W}}$ and hence apply by induction on $r \geq 0$ to $\mathcal{W}_0$ and also to $\mathcal{W}_{-1}$.

Proof of c). Dually, the vanishing condition is true for r small enough since then $\mathcal{W}_r$ coincides with the extension by zero of $\mathcal{L}[2n]$ on $X - Y$.
Now we use the filtration for $r < 0$, for $S.'$ equal to the intermediate extension of $\mathcal{L}_r^{K'}[2n-2k']$ on $Y^{K'}$ we have for $k > k'$: $\mathbf{H}^m(S.'^{2(k-k')+1}) = 0$ for all $m > k + k' - 2n$, which gives for $S.'[k'+1]$, $(r < 0)$ on X: $\mathbf{H}^m(S.^{2(k+1)}) = \mathbf{H}^m(S.^{2k+1}) = 0$ for all $m > k - 1 - 2n$. If $k = k'$, then $S.'[k'+1]$ is a local system in degree $-2n + k - 1$ on $Y_{K'} - Y^{k+1}$ and for $k < k'$, $Y_{K'} - Y^{k+1}$ is empty.

Corollary. *The weight filtration $\mathcal{W}$ of $\Omega^*\mathcal{L}$ is defined over $\mathbb{Q}$.*

The proof is based on the following lemma applied to $\mathbf{j}_*\mathcal{L}^c \simeq \Omega^*\mathcal{L}$ with its filtration $\mathcal{W}$.

Lemma. *Let K be a $\mathbb{Q}-$perverse sheaf such that $K^c = K \otimes \mathbb{C}$ is filtered by a finite filtration W^c of complex perverse sub-sheaves s.t.$Gr_r^{W^c}K^c$ is rationally defined and the rational filtration $W^r = W^c \cap K$ induces the rational structure on $Gr_r^{W^c}K^c$, then W^r is a rational filtration by perverse sub-sheaves of K such that $W^r \otimes \mathbb{C} \simeq W^c$.*

The proof is similar to the case of local systems and is by induction on the weight i since by hypothesis it applies to the lowest weight. Considering the extension $0 \to W_i^c K^c \to W_{i+1}^c K^c \xrightarrow{\pi} Gr_{i+1}^{W^c}K^c \to 0$, then $W_{i+1}^r K \otimes \mathbb{C} \simeq W_{i+1}^c K$ follows from the hypothesis $(Gr_{i+1}^{W^r}K) \otimes \mathbb{C} \simeq (Gr_{i+1}^{W^c}K)$ and the inductive isomorphism for $W_i^r K$.

Corollary. *If X is proper and if we forget the negative weights in the filtration $\mathcal{W}$ that is we consider $\mathcal{W}''$ with $\mathcal{W}''_i = \mathcal{W}_i$ for $i \geq 0$ and $\mathcal{W}''_{-1} = 0$, then the bi-filtered complex*

$$(\Omega^*(\mathcal{L}), \mathcal{W}''[m], F)$$

is a mixed Hodge complex .

5§. The complex of nearby cycles $\Psi_f(\mathcal{L})$.

Let $f\colon X \to D$ and suppose $Y = f^{-1}(0)$ a NCD, the complex of sheaves $\Psi_f\mathcal{L}$ of nearby co-cycles on Y has been introduced in [11]; its cohomology fiber $H^i((\Psi_f\mathcal{L})_y) \simeq H^i(F_y, \mathcal{L})$ at a point y in Y

is isomorphic to the cohomology of the Milnor fiber F_y at y. The monodromy induces an action $\mathcal{T}$ on the complex itself. If $dim\, X = n$, $\Psi_f\mathcal{L}[n-1]$ is perverse on Y. Since the local system $\mathcal{L}$ is defined over $\mathbb{Q}$, the monodromy decomposes in the abelian category of $\mathbb{Q}-$perverse sheaves as the product $\mathcal{T} = \mathcal{T}^s\mathcal{T}^u$ of simple and unipotent endomorphisms. Let $\mathcal{N} = Log\mathcal{T}^u$, then Deligne's filtration $\mathcal{W}(\mathcal{N})$ is defined over $\mathbb{Q}$. The aim of this section is to describe the structure of a mixed Hodge complex (MHC) on $\Psi_f\mathcal{L}$ with weight filtration $\mathcal{W}(\mathcal{N})$. This problem is closely related to the weight filtration in the open case since we have the following *relation between* $\Psi_f^u\mathcal{L}$*, the direct image* $\mathbf{j}_*\mathcal{L}$ *and* $\mathbf{j}_{!*}\mathcal{L}$ as explained in [1] and [2]

$$i_Y^*(\mathbf{j}_*\mathcal{L}[n]/\mathbf{j}_{!*}\mathcal{L}[n]) \quad \simeq \quad Coker(\mathcal{N}\colon \Psi_f^u\mathcal{L}[n-1] \to \Psi_f^u\mathcal{L}[n-1])$$

The filtration $W(\mathcal{N})$ on $\Psi_f^u(\mathcal{L})$ induces a filtration $\mathcal{W}$ on $Coker(\mathcal{N}/\Psi_f^u\mathcal{L})$, hence on $\mathbf{j}_*\mathcal{L}/\mathbf{j}_{!*}\mathcal{L}$. *The induced filtration on* $\mathbf{j}_*\mathcal{L}/\mathbf{j}_{!*}\mathcal{L}$ *is independent of the choice of* f. For a rigorous proof one should use the result of Verdier [34]. A path in the space of functions between two local equations f and f' of Y gives rise to an isomorphism between $\Psi_f\mathcal{L}$ and $\Psi_{f'}\mathcal{L}$; it is only modulo $Coker\mathcal{N}$ that this isomorphism is independent of the path. We do check here that the weight filtration $\mathcal{W}$ on $\Omega^*\mathcal{L}$ is induced locally by $W(\mathcal{N})$.

I. The weight filtration on the nearby co-cycles $\Psi_f\mathcal{L}$

When we consider the coefficients in the complex local system $\mathcal{L}\otimes\mathbb{C}$ (denoted also $\mathcal{L}$), the method to compute Ψ_f as explained in [11] uses the restriction $i_Y^*\mathbf{j}_*\mathcal{L}$ of the higher direct image of $\mathcal{L}$ to Y and the cup-product $H^i(X^*,\mathcal{L})\otimes H^1(X^*,\mathbb{Q}) \xrightarrow{\smile\eta} H^{i+1}(X^*,\mathcal{L})$ by the inverse image $\eta = f^*c \in H^1(X^*,\mathbb{Q})$ of a generator c of the cohomology $H^1(D^*,\mathbb{Q})$. We construct effectively, using Deligne's bundle extension, a bi-filtered complex on which η is defined as a morphism (of degree 1), $\eta\colon\ i_Y^*(\Omega_X^*(LogY)\otimes\mathcal{L}_X) \to i_Y^*(\Omega_X^*(LogY)\otimes\mathcal{L}_X)[1]$ satisfying $\eta^2 = 0$ so to get a double complex whose simple associated complex is quasi-isomorphic to $\Psi_f^u(\mathcal{L})$.

5.1 The global weighted complex $(\Psi_f^u(\mathcal{L}),\mathcal{W},F)$

Let t denotes a coordinate on the disc D and $\eta\colon= f^*(\frac{dt}{t})$, then $\wedge\eta$ defines a morphism of degree one on $i_Y^*\Omega_X^*(LogY)\otimes\mathcal{L}_X$. We consider the simple complex

$$\Psi_f^u\mathcal{L}_X\colon= s(i_Y^*(\Omega_X^*(LogY)\otimes\mathcal{L}_X)[p],\eta)_{p\le 0} \tag{16}$$

defined by the double logarithmic complex ($\oplus_{p\le 0} i_Y^*(\Omega_X^i(LogY)\otimes\mathcal{L}_X)$ is in degree i). To define as previously a constant combinatorial resolution of $\Psi_f^u\mathcal{L}_X$, we put $\Psi_f^u\mathcal{L}_X(s.) = \Psi_f^u\mathcal{L}_X$ for each $s.\in S(I)$ and let

$$\Psi_f^u\mathcal{L}\colon= s(\Psi_f^u\mathcal{L}_X(s.))_{s.\in S(I)} \simeq s(i_Y^*\Omega^*\mathcal{L}[p],\eta)_{p\le 0} \tag{17}$$

which can be viewed also as $s(\Omega^*\mathcal{L}[p],\eta)_{p\le 0}$; then we define the weight filtration and the Hodge filtration by

$$\mathcal{W}_r(\Psi_f^u\mathcal{L}) = s(i_Y^*\mathcal{W}_{r+2p-1}\Omega^*\mathcal{L}[p],\eta)_{p\le 0},\ F^r(\Psi_f^u\mathcal{L}) = s(i_Y^*F^{r+p}\Omega^*\mathcal{L}[p],\eta)_{p\le 0} \tag{18}$$

The logarithm of the monodromy $\mathcal{N}$ is defined on this complex and we want to show that the filtration $\mathcal{W}$ above coincides with $W(\mathcal{N})$.

5.2 Theorem. *Suppose that $\mathcal{L}$ underlies a unipotent variation of polarised Hodge structures of weight m, then $W(\mathcal{N}) = \mathcal{W}$.*

With this result we can conclude that the weight filtration in the open case is induced locally by the weight filtration defined by the monodromy on the nearby co-cycles.
The proof of this theorem is based on the results in the local case, that is for the nilpotent orbit defined by $\mathcal{L}$ at a point $y \in Y^*_M$, $M \subset I$.

5.3 Local description of the weight and Hodge filtrations Near a point $y \in Y^*_M, M \subset I$, we can find coordinates z_i for $i \in M$ defining Y^*_M locally and non zero integers n_i s.t. $f = \prod_{i=1}^{n} z_i^{n_i}$ where we do suppose $i \in [1, n]$, where $|M| = n$ (n is less or equal to the *dim* of X), then in DeRham cohomology $\eta = f^*(\frac{dt}{t}) = \sum_{i=1}^{n} n_i \frac{dz_i}{z_i}$.
Thus η defines a morphism of degree one on the DeRham complex $\Omega(L, N_i)_{i\in[1,n]}$ satisfying $\eta^2 = 0$. We define

$$\Psi^0(L) = s(\Omega(L, N_i)_{i\in[1,n]})[p], \eta)_{p\leq 0}$$

as the simple complex defined by the double complex for $p \leq 0$.
Remark: In order to take into account the action of $N = LogT^u$ we may write $L[N^p]$ for $L[p]$ and $L[N^{-1}]$ for the direct sum over $p \leq 0$, so that the action of N is just multiplication by N, then

$$\Psi^0(L) \simeq \Omega(L[N^{-1}], N_i - n_i N)_{i\in[1,n]}.$$

is the Koszul complex on $N_i - n_i N$ acting on $L[N^{-1}]$.
The complex $\Psi^0_M L$. To describe the weight in terms of the filtrations $(\Omega^* L, \mathcal{W}, F)$ associated to L, we need to use the constant complex with index $s. \in S(M)$, $\Psi^0 L(s.) = \Psi^0 L$ and introduce the complex

$$\Psi^0_M L := s(\Psi^0 L(s.))_{s.\in S(M)} \tag{19}$$

which can be viewed also as $s(\Omega^* L[p], \eta)_{p\leq 0}$, then we define on it the *weight and Hodge filtrations*

$$\mathcal{W}_r(\Psi^0_M L) = s(\mathcal{W}_{r+2p-1}\Omega^* L[p], \eta)_{p\leq 0}, \quad F^r(\Psi^0_M L) = s(F^{r+p}\Omega^* L[p], \eta)_{p\leq 0}. \tag{20}$$

Monodromy.

The logarithm $\mathcal{N}$ of the monodromy is defined by an endomorphism ν of the complex $\Psi^0_M L$, given by the formula

$$\forall a. = \Sigma_{p\leq 0} a_p \in \Psi^0_M L : \quad (\nu(a.))_p = a_{p-1}$$

such that $\nu(\mathcal{W}_r) \subset \mathcal{W}_{r-2}$ and $\nu(F^r) \subset F^{r-1}$.

Lemma 1. *The local quasi-isomorphism on the stalk at a point $y \in Y^*_M$ of the logarithmic complex with coefficients in Deligne's extension $(\Omega^*_X(LogY) \otimes \mathcal{L}_X)_y$ extends to a quasi-isomorphism from $(\Psi^u_f \mathcal{L}_X)_y$ (16) to $\Psi^0 L$ (resp. from $(\Psi^u_f \mathcal{L})_y$ (17) to $\Psi^0_M L$) respecting the weight and Hodge filtrations*

$$((\Psi^u_f \mathcal{L}_X)_y, W, F) \simeq (\Psi^0 L, W, F) \quad , \quad ((\Psi^u_f \mathcal{L})_y, \mathcal{W}, F) \simeq (\Psi^0_M L, \mathcal{W}, F)$$

This lemma is the needed link between the global and local cases.

Lemma 2. *We have a triangle in the derived category represented by the exact sequence*
$0 \to i^*_Y(\Omega^*_X(LogY) \otimes \mathcal{L}_X \to \Psi^u_f \mathcal{L}_X \xrightarrow{\nu} \Psi^u_f \mathcal{L}_X \to 0$

II. Main local results

Proof of the theorem. The proof can be reduced to the local case at a point $y \in Y_M^*$. We introduce first the following complexes.
The morphism η induces a morphism denoted also by $\eta : C_r^{KM}L \to C_{r+2}^{KM}L[1]$ so that we can define a double complex and the associated simple complex

$$\Psi_r^{KM}L = s(C_{r+2p-1}^{KM}L[p], \eta)_{p\leq 0},\ \Psi_r^K L := \Psi_r^{KK}L \tag{21}$$

i) *Decomposition of $Gr_r^{\mathcal{W}}(\Psi_M^0 L)$. There exist natural injections of $\Psi_r^{KM}L$ into $Gr_r^{\mathcal{W}}\Psi_M^0 L$ and a decomposition*

$$Gr_r^{\mathcal{W}}\Psi_M^0 L \simeq \oplus_{K\subset M}\Psi_r^{KM}L \tag{22}$$

Proof: The result follows from the decomposition of $Gr_r^{\mathcal{W}}\Omega^* L$ in the previous open case, applied to the spectral sequence with respect to p in the double complex above.
ii) We introduce now the the complex A_i^K and prove

5.4 Basic lemma. *For all $i \geq 1$, the complex $A_i^K := s[C_{i+2p-1}^K L[p], \eta]_{1-i\leq p\leq 0} \cong 0$ is acyclic.*
Proof: We view A_i^K as a double complex where η of degree 1 is a differential of the direct sum of complexes $C_{i+2p-1}^K L$ without shift in degrees :

$$A_i^K := C_{-i+1}^K \xrightarrow{\eta} \cdots \xrightarrow{\eta} C_{i+2p-1}^K L \cdots \xrightarrow{\eta} C_{i-1}^K L, \quad A_i^K(J) = \oplus_{1-i\leq p\leq 0} C_{i+2p-1}^K L(J), J \subset K$$

We may filter A_i^K by sub-complexes U_i. One way is to take $U_0 = C_{i-1}^K L$, $U_1 = s[C_{i+2p-1}^K L[p], \eta]_{1-i<p\leq 0}$ and $U_2 = A_i^K$, that is we write A_i^K as:

$$s[C_{-(i-1)}^K L[-2], s[C_{i+2p-1}^K L[p], \eta]_{0<p<i-1}[-1], C_{i-1}^K L, \eta]$$

so to use by induction $Gr_1^U = A_{i-2}^K \cong 0$.
However we will use the technique of the spectral sequence defined by the increasing filtration $U_r = s[C_{i+2p-1}^K L[p], \eta]_{-r\leq p\leq 0}$ for $0 \leq r \leq i-1$
with Deligne's notations

$$E_1^{a,b} = H^{a+b}(Gr_{-a}^U) = H^{a+b}(C_{i+2a-1}^K L)$$

Since $H^u(C_j^K L) = 0$ if $j < 0$ and $u \neq |K| - 1$ or $j > 0$ and $u \neq |K|$, we get
1) if $2a > 1 - i$, $E_1^{a,|K|-1-a} = H^{|K|-1}(C_{i+2a-1}^K L) = Gr_{|K|+i+2a-1}^{W^K}[(\cap_{i\in K} kerN_i : L \to L]$
and 0 otherwise
2) if $2a < 1 - i$, $E_1^{a,|K|-a} = H^{|K|}(C_{i+2a-1}^K L) = Gr_{-|K|+i+2a-1}^{W^K}[L/(\Sigma_{i\in K} N_i L)]$
and 0 otherwise
3) if $2a = i - 1$, $E_1^{a,b} = 0$.

Starting at the level 1, the term $E_1^{-(i-1)-d,|K|-1+(i-1+d)}$ remains unchanged until we reach the level $r = i - 1 + 2d$ where the only non trivial differential appears and we want to show it is an isomorphism. The proof is based on the following study of this differential.
The cohomology space of $C_r^K L$ for will be identified with the following polarised subspace of $Gr_{r-|K|}^{W^K}L$. For each $r > 0$, and $(m_1, \ldots, m_n) \in T(r)$ (that is $m_i > 1, \Sigma m_i = r + |K|$), let
$P(m.)L = \cap_{i\in[1,n]}(kerN_i^{m_i-1} : Gr_{m_n-2}^{W^n} \cdots Gr_{m_1-2}^{W^1}L \to Gr_{-m_n}^{W^n} \cdots Gr_{-m_1}^{W^1}L) \subset Gr_{r-|K|}^{W^K}L$,
be the primitive polarised subset, then we have the isomorphism
$P(m.)L \simeq Gr_{m_n-2}^{W^n} \cdots Gr_{m_1-2}^{W^1}(L/(\Sigma_i N_i L))$.

Sub-lemma. *Let $N_K = \Sigma_{i\in K} n_i N_i$ and for each $(m.) \in T(i+2d-1)$, let $P(m.)L$ denotes the primitive sub-HS as above. The differential at the level $r = i-1+2d$ of the spectral sequence*

$$E_r^{-(i-1)-d,|K|-2+i+d} \to E_r^{d,|K|-d}, \quad r = i-1+2d$$

is given by the inverse of the isomorphism up to a constant

$$Gr^{W^K}_{-|K|+r}[L/(\Sigma_{i\in K} N_i L)] \simeq \oplus_{m.\in T(r)} P(m.)L \xrightarrow{N_K^{r-|K|}} \oplus_{m.\in T'(r)} P(m.)L \simeq Gr^{W^K}_{|K|-r}(\cap_{i\in K} kerN_i).$$

inducing for each $(m.)$ precisely the inverse of

$$(-1/(n_1^{(m_1-1)}\dots n_n^{(m_n-1)})N_1^{(m_1-2)}\dots N_n^{(m_n-2)} : P((m_1,\dots,m_n)) \to Gr^{W^n}_{-m_n+2}\cdots Gr^{W^1}_{-m_1+2}(\cap_{i\in K} kerN_i).$$

1) We start the proof for $n=1$, that is one dimensional nilpotent orbit (L,N). Then we need to prove that the differential is the inverse of

$$-\tfrac{1}{n^r} N^{r-1} : Gr^W_{r-1} L/NL \simeq P(r-1)L \to Gr^W_{-(r-1)} kerN$$

which can be checked on the diagram

$$\begin{array}{ccccccc} Gr^W_{-i+2}L & & Gr^W_{-i+4}L & \cdots & Gr^W_{i-2}L & & Gr^W_i L \\ N\downarrow & \eta\searrow & N\downarrow & \cdots & N\downarrow & \eta\searrow & N\downarrow \\ Gr^W_{-i}L & & Gr^W_{-i+2}L & \cdots & Gr^W_{i-4}L & & Gr^W_{i-2}L \end{array}$$

where $\eta = -nId$. For $a \in Gr^W_{i-2}L$ primitive, the element $\sum_{0\le j\le i-2}(1/n)^{i-1-j}N^{i-2-j}(a) \in \oplus Gr^W_{i-2+2j}L$ is a cohomology class modulo the complex $C^{\{0\}}_{i-1}L$ inducing the cohomology class $((1/n)^{i-1}N^{i-2}(a)$ in $C^{\{0\}}_{1-i}L$ whose image by η is $-a$ the original primitive element up to sign.

2) In general we notice that $(N_K)^{r-2|K|}$ decomposes on $P(m.)L$ as the product $(n_nN_n)^{m_n-2}\circ\cdots\circ(n_1N_1)^{m_1-2}$.

We use this relation to give an inductive proof on the number of endomorphisms N_i.

The cohomology of an elementary complex $K(m_1,\cdots,m_n)$ is isomorphic to the cohomology of a complex with the unique endomorphism N_n acting on $L_n = \cap_{i\notin J(m.),i\neq n} kerN_i : L/(\Sigma_{i\in J(m.)-\{n\}} N_i L)$ with L_n depending on m_i for $i\neq n$, then the diagram is similar to the case of one variable until we reach $-m_n+2$ for which the morphism $N_n^{m_n-2}$ is needed. The morphisms $N_i^{m_i-2}$ will appear inductively with the variable i. So we can deduce in general:

$N_1^{m_1-2}\cdots N_n^{m_n-2} : Gr^{W^n}_{m_n}\cdots Gr^{W^1}_{m_1-2}P(m.)L \simeq Gr^{W^n}_{-m_n+2}\cdots Gr^{W^1}_{-m_1+2}(\cap_{i\in K} kerN_i)$

the sum over $\{(m_1\ge 2,\cdots,m_n\ge 2) : \Sigma_{i\in K} m_i = i+2d-1+|K|\ \}$ induces an isomorphism:

$$\gamma : Gr^{W^K}_{i+2d-1-|K|}[L/(\Sigma_{i\in K} N_i L)] \to Gr^{W^K}_{-(i-1)-2d+|K|}[(\cap_{i\in K}(kerN_i : L\to L].$$

Example. Consider L, N_1, N_2 in dimension 2, $K=\{1,2\}$ the origin in $\mathbb{C}^2$, A_5^K, $a \in Gr_2^{W^K}L$, $(m.) = (m_2 = 4, m_1 = 2)$ with the following conventions for differentials : the restriction from $s. = K \supset \{1\}$ to K is $-I$ (I is Identity), the restriction from $s. = K \supset \{2\}$ to K is I, the differentials on $C_r^K L$ are ($-N_1$ on dz_2,N_i otherwise and η is n_1 on dz_2, $-n_i$ otherwise.

An element a of A_5^K is written as the sum of various components of the underlying groups in $C^K_{-4}L \oplus C^K_{-2}L \oplus C^K_0 L \oplus C^K_2 L \oplus C^K_4 L$, that is $a = \sum a(dz_J, s., r)$ where dz_J stands for $\wedge_{i\in J} dz_i$, $s.$ is $\{1,2\}\supset\{1\}$, or $\{1,2\}\supset\{2\}$ or K for $\{1,2\}$ and r for $a(dz_J, s., r) \in C_r^K L$ ($r = -4,-2,0,2,4$). Still we need to specify for an element $b \in Gr_r^{W^K}L \simeq \oplus_{m_1+m_2=r} Gr^{W^1}_{m_1} Gr^{W^2}_{m_2} L$ its components $b = \Sigma_{m_1+m_2=r} b(m_1,m_2)$. A bi-primitive element $a \in P(2,4) \subset Gr_0^{W^1}Gr_2^{W^2}L \subset Gr_2^{W^K}L, a = a(dz_K, (s. = K\supset\{1\}, 4)$ in $C_4^K L$ (hence $N_1 a = 0, N_2^3 a = 0$) defines the following cohomology class $\beta(a) \in A_5^K/C_4^K L$ modulo $C_4^K L$:

$[N_2^2 a(m_2 = -2, m_1 = 0)(dz_\varnothing, s. = K, -4) \in Gr_{-2}^{W^K}L$ in $C^K_{-4}L$, $(n_2N_2 a((0,0)(\varnothing, K, -2) \in Gr_0^{W^K}L$ in $C^K_{-2}L$,

$n_1N_2^2(a)(0,-2)(dz_1, s. = K \supset \{2\}, -2) \in Gr_{-2}^{W^K}L$ in $C_{-2}^K L$, $n_1n_2N_2(a)(0,0)(dz_1, (s. = K \supset \{1\}, 0) \in Gr_2^{W^K}L$ in $C_0^K L(s. = K \supset \{1\})$, $(n_2)^2a((2,0)(dz_\varnothing, s. = K, 0) \in Gr_2^{W^K}L$ in $C_0^K L$, $(n_2)^2n_1a(2,0)(dz_1, s. = (K \supset \{2\}), 2)$ in $C_2^K L(s. = K \supset \{2\}), -(n_2)^3a(2,0)(dz_2, s. = (K \supset \{2\}), 2)$ in $C_2^K L(s. = K \supset \{2\})]$ which ends the components of $\beta(a)$,
whose image by η has two components $-(n_2)^3n_1a((2,0)(dz_K, s. = (K \supset \{2\}), 4)$ in $C_4^K L(s. = K, 2)$ and $-(n_2)^3n_1a((2,0)(dz_K, s. = (K \supset \{1\}), 4)$ in $C_4^K L(s. = (K \supset \{1\})$ which represents the cycle $-a$ in the cohomology as represented by the complex defined by $T(r)$ with $r = 4$.
Notice that the conditions for $s. = (K \supset \{1\}), r = 2$ are , $W_1(L) \leq 2, W_1(Ldz_1) \leq 0, W_1(Ldz_2) \leq 2$ are satisfied by $a(dz_2)$ while the conditions for $s. = (K \supset \{2\}), r = 2), r = 2$ are $W_2 \leq 2, W_2(dz_1) \leq 2, W_2(dz_2) \leq 0$ are not satisfied since $W_2(adz_2) = 2$ which forces the lifting in $C_2^K L(s. = (K \supset \{1\})$.

Corollary 1. *For all $i > 0$, the complex $A_i^{KM} := s[C_{i+2p-1}^{KM}L[p], \eta]_{-i<p\leq 0}$ is acyclic.*

Proof. We can easily check, as in the previous open case, that the cohomology of A_i^{KM} is quasi-isomorphic to the stalk at y of the intermediate extension of the local system on Y_K^* defined by the cohomology of A_i^K, hence it is quasi-isomorphic to zero since $A_i^K \cong 0$.

The iterated monodromy morphism defines an exact sequence

$0 \to ker\, \nu_X^i \to \Psi_f^u\mathcal{L}_X \xrightarrow{\nu_X^i} \Psi_f^u\mathcal{L}_X \to 0$ where $ker\, \nu_X^i = s(\Omega_X^*(LogY)[p], \eta)_{-i<p\leq 0}$,

$0 \to ker\, \nu^i \to \Psi_f^u\mathcal{L} \xrightarrow{\nu^i} \Psi_f^u\mathcal{L} \to 0$ where $ker\, \nu^i = s(\Omega^*\mathcal{L}[p], \eta)_{-i<p\leq 0}$.

Corollary 2. *For all $i \geq 1$, $Gr_i^{\mathcal{W}} ker\, \nu^i = s(Gr_{i+2p-1}^{\mathcal{W}}\Omega^*L[p], \eta)_{-i<p\leq 0}$ is acyclic.*

Proof: By the decomposition (19) we have: $Gr_i^{\mathcal{W}} ker\, \nu^i \cong \oplus_{K\subset M} A_i^{KM}$.

Corollary 3. *For all $i \geq 1$, $\nu^i : Gr_i^{\mathcal{W}}(\Psi_M^0 L) \simeq Gr_{-i}^{\mathcal{W}}(\Psi_M^0 L)$.*

The equivalence between Corollaries 2 and 3 follows from the exact sequence

$0 \to Gr_i^{\mathcal{W}} ker\, \nu^i \to Gr_{-i}^{\mathcal{W}}(\Psi_M^0 L) \xrightarrow{\nu^i} Gr_{-i}^{\mathcal{W}}(\Psi_M^0 L) \to 0$.

The theorem follows from the corollaries.

5.5 The global weighted complex $(\Psi_f^u(\mathcal{L}), \mathcal{W}, F)$

Returning to the global situation, we need to define the Hodge filtration on $\Psi_f^u(\mathcal{L}_X)$. We could define

$$\Psi_f^u(\mathcal{L}_X) \cong s(i_Y^*(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i+1], \eta)_{i\geq 0},\ (\Psi_f^u\mathcal{L}) := s(\Psi_f^u\mathcal{L}(s.))_{s.\in S(M)}$$

First F extends to the logarithmic complex by the formula under (15) in (4.1), then F extends to $\Psi_f^u(\mathcal{L}_X)$ via the formula

$$F^r(s(i_Y^*(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i+1])_{i\geq 0}) = s(F^{r+i+1}(i_Y^*(\Omega_X^*(LogY) \otimes \mathcal{L}_X)[i+1], \eta)_{i\geq 0}$$

The definition of the global weight filtration reduces to the local construction at a point $y \in Y_M^*$, using the quasi-isomorphism $(\Psi_f^u\, \mathcal{L})_y \simeq \Psi^0 L$.
We suppose again $\mathcal{L}$ unipotent and define as above $\Psi_f^u\mathcal{L}$ which can be viewed also as $s(i_Y^*\Omega^*\mathcal{L}[i+1], \eta)_{i\geq 0}$, then we define the weight and Hodge filtrations

$$\mathcal{W}_r(\Psi_f^u\mathcal{L}) = s(i_Y^*(\mathcal{W}_{r+2i+1}\Omega^*\mathcal{L})[i+1], \eta)_{i\geq 0},\ F^r(\Psi_f^u\mathcal{L}) = s(i_Y^*F^{r+i+1}\Omega^*\mathcal{L}[i+1], \eta)_{i\geq 0}$$

The logarithm of the monodromy $\mathcal{N}$ is defined on this complex as in the local case. The filtration $W(\mathcal{N})$ is defined on $\Psi_f^u\mathcal{L}$ in the abelian category of perverse sheaves.

5.6 Theorem. *Suppose $\mathcal{L}$ underlies a unipotent variation of polarised Hodge structures of weight m, then the graded part of the weight filtration of the complex*

$$(\Psi_f^u\mathcal{L}, \mathcal{W}[m], F)$$

decomposes into a direct sum of intermediate extension of polarised VHS. Moreover it is a MHC for X proper.
We have $W(\mathcal{N}) = \mathcal{W}$; $\mathcal{W}_{r+1}\Omega^*\mathcal{L}$ is induced by $\mathcal{W}_r\Psi_f^u\mathcal{L}$ on $Coker\mathcal{N}$ and $Gr_{r+1}^{\mathcal{W}} i_Y^*\Omega^*\mathcal{L}$ is the primitive part of $Gr_r^{\mathcal{W}}\Psi_f^u\mathcal{L}$.

The proof of this theorem reduces by definition to show that $(Gr_r^{\mathcal{W}[m]}(\Psi_f^u(\mathcal{L}_X), F)$ decomposes which result can be reduced to the local case where it follows from

Lemma. *Let i_0 be a positive integer large enough to have $Gr_j^{\mathcal{W}}\Omega^*\mathcal{L} = 0$ for all $|j| > i_0$ and $I(r) = \{p \geq 0 : |r| + 1 \leq r + 2p + 1 \leq i_0\}$, then*

$$Gr_r^{\mathcal{W}}(\Psi^0 L)_M = s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\in I(r)};\ Gr_r^{\mathcal{W}}(\Psi^0 L)_M \overset{\nu^r}{\underset{\cong}{\to}} Gr_{-r}^{\mathcal{W}}(\Psi^0 L)_M, r \geq 0$$

Proof: Suppose $r > 0$, then $Gr_r^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_{r+2p+1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\in I(r)}$ since $Gr_{r+2p+1}^{\mathcal{W}}\Omega^* L \cong 0$ for $r \notin I(r)$, while for $r < 0$, the sum
$s(Gr_{r+2p+1}^{\mathcal{W}}\Omega^* L[p], \eta)_{-1<p<-r} \cong s(Gr_i^{\mathcal{W}}\Omega^* L[p], \eta)_{i\in[r+1,-(r+1)]}$ is acyclic.
The quasi-isomorphism defined by ν^r follows since the formula is symmetric in r and $-r$.
Example. $Gr_0^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_1^{\mathcal{W}}\Omega^* L[1] \xrightarrow{\eta} Gr_3^{\mathcal{W}}\Omega^* L[2] \ldots)$
$Gr_1^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_2^{\mathcal{W}}\Omega^* L[1] \xrightarrow{\eta} Gr_4^{\mathcal{W}}\Omega^* L[2] \ldots)$
$Gr_2^{\mathcal{W}}(\Psi^0 L)_M \cong s(Gr_3^{\mathcal{W}}\Omega^* L[1] \xrightarrow{\eta} Gr_5^{\mathcal{W}}\Omega^* L[2] \ldots)$
For $r = i_0 - 1$ and $r = i_0 - 2$, $Gr_r^{\mathcal{W}}(\Psi^0 L)_M \cong Gr_{i_0}^{\mathcal{W}}\Omega^* L[1]$

Corollary. *The graded part of $(\Psi^0 L)_M$ is non zero for only a finite number of indices which decomposes into direct sum of of intermediate extension of polarised VHS.*

Proof. We can introduce the notion of primitive parts starting with $Gr_r^{\mathcal{W}}(\Psi^0 L)_M$ for $r = i_0 - 1$ and $r = i_0 - 2$ which are direct sum of of intermediate extension of polarised VHS, then using the decomposition into direct sum of primitive parts we prove by induction that such decomposition is valid for all indices r (there is no extension by η).
Then the theorem follows easily.

Remark(*dual statement*). Let us return to $(\Psi_f^u\mathcal{L} := s(\Psi_f^u\mathcal{L}(s.))_{s.\in S(I)}$ defined with indices $p \leq 0$, which can be viewed locally at a point $y \in Y$ as $s(\Omega^*\mathcal{L}[p], \eta)_{p\leq 0}$, the weight and Hodge filtrations defined by the formula (18).

Theorem.(*dual statement*) *Suppose $\mathcal{L}$ underlies a unipotent variation of polarised Hodge structures of weight m, then the graded part of the weight filtration of the complex*

$$(\Psi_f^u\mathcal{L}, \mathcal{W}[m], F)$$

decomposes into a direct sum of intermediate extension of polarised VHS. Moreover it is a MHC for X proper.
We have $W(\mathcal{N}) = \mathcal{W}$; $\mathcal{W}_{r-1} i_Y^*\Omega^*\mathcal{L}$ is induced by $\mathcal{W}_r\Psi_f^u\mathcal{L}$ on $Ker\mathcal{N}$ for $r \leq 0$ and $Gr_{r-1}^{\mathcal{W}} i_Y^*\Omega^*\mathcal{L}$ is isomorphic to the primitive part of $Gr_{-r}^{\mathcal{W}}\Psi_f^u\mathcal{L}$.

The proof of this theorem reduces by definition to show that $(Gr_r^{\mathcal{W}[m]}\Psi_f^u\mathcal{L}, F)$ decomposes; such result can be reduced to the local case where it follows from

Lemma. *Let i_0 be a positive integer large enough to have $Gr_j^{\mathcal{W}}\Omega^*\mathcal{L} = 0$ for all $|j| > i_0$ and $I(r) = \{p \leq 0, -i_0 + 1 \leq r + 2p - 1 \leq -|r| - 1\}$, then*

$$Gr_r^{\mathcal{W}}(\Psi^0 L)_M = s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\in I(r)},\ Gr_r^{\mathcal{W}}(\Psi^0 L)_M \overset{\nu^r}{\underset{\cong}{\rightarrow}} Gr_{-r}^{\mathcal{W}}(\Psi^0 L)_M, r \geq 0$$

Proof: Suppose $r > 0$, then $Gr_r^{\mathcal{W}}(\Psi^0 L)_M$ is the sum of $s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{-r<p\leq 0} \cong 0$ and $s(Gr_{r+2p-1}^{\mathcal{W}}\Omega^* L[p], \eta)_{p\leq -r}$. In particular, $Gr_r^{\mathcal{W}}(\Psi^0 L)_M \cong 0$ for all r such that $|r| \geq i_0$. Using (21) and (22) we get a decomposition into $\Psi_r^{KM} L = s(C_{r+2p-1}^{KM} L[p], \eta)_{p\in I(r)}$ where $C_{r+2p-1}^{KM} L$ is the fiber of an intermediate extension of a local system defined by $C_{r+2p-1}^{K} L$ for $r + 2p - 1 < 0$ on Y_K^*.
Then the proof is similar to the previous case in the dual definition of $\Psi_f^u \mathcal{L}$.

ACKNOWLEDGMENTS. I would like to acknowledge the valuable discussions I had with J.L. Verdier at the time I started to work on this subject and to thank J.P. Serre for accepting the note [14].

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