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Abstract

We study the homogenization of a linear kinetic equation which models the evolution of the density of charged particles submitted to a highly oscillating electric field. The electric field and the initial density are assumed to be random and stationary. We identify the asymptotic microscopic and macroscopic profiles of the density, and we derive formulas for these profiles when the space dimension is equal to one.

1 Introduction

This note is concerned with the homogenization of a linear transport equation in a stationary ergodic setting. The equation studied here describes the evolution of the density of charged particles in a rapidly oscillating random electric potential. This equation can be derived by passing to the semi-classical limit in the Schrödinger equation (see [9], [11], and the presentation in [7]). Our work generalizes a result of E. Frénod and K. Hamdache (see [7]) which was obtained in a periodic setting. The strategy of proof we have chosen here is different from the one of [7], and allows us to retrieve some of the results in [7] in a rather simple and explicit fashion.

Let us mention a few related works on the homogenization of linear transport equations; we emphasize that this list is by no means exhaustive. In [1], Y. Amirat, K. Hamdache and A. Ziani study the homogenization of a linear transport equation in a periodic setting and give an application to a model describing a multidimensional miscible flow in a porous media. In [3] (see also [8]), Laurent Dumas and François Golse focus on the homogenization of linear transport equations with absorption and scattering terms, in periodic and stationary ergodic settings. And in [4], Weinan E derives strong convergence results for the homogenization of linear and nonlinear transport equations with oscillatory incompressible velocity fields in a periodic setting.

Let us now present the context we will be working in : let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\tau_x)_{x\in\mathbb{R}^N}$ be a group transformation acting on Ω. We assume that τ_x preserves the probability measure P for all $x \in \mathbb{R}^N$, and the group transformation is ergodic, which means

$$\forall A \in \mathcal{F}, \quad \left(\tau_x A = A \ \forall x \in \mathbb{R}^N \Rightarrow P(A) = 0 \text{ or } 1\right).$$

The periodic setting can be embedded the stationary ergodic setting (see [13]). We will denote by $E[\cdot]$ the expectation with respect to the probability measure P; in the periodic case, we will write $\langle f \rangle$ rather than $E[f]$ to refer to the average of f over one period.

We consider a potential function $u = u(y,\omega) \in L^\infty(\mathbb{R}^N \times \Omega)$ which is assumed to be stationary, i.e.

$$u(y+z,\omega) = u(y,\tau_z\omega) \quad \forall (y,z,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega$$

Moreover, we assume that $0 \le u(y,\omega) \le u_{\max} = \sup u$ for all $y \in \mathbb{R}^N, \omega \in \Omega$, and $u(\cdot,\omega) \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ for almost every $\omega \in \Omega$, so that $\nabla_y u\,(y,\omega)$ is well-defined and locally Lipschitz continuous with respect to its y variable.

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Let $f^\varepsilon = f^\varepsilon(t,x,\xi,\omega)$, $(t \geq 0,\ x \in \mathbb{R}^N, \xi \in \mathbb{R}^N, \omega \in \Omega)$ be the solution of the transport equation

$$\begin{cases} \partial_t f^\varepsilon(t,x,\xi,\omega) + \xi \cdot \nabla_x f^\varepsilon(t,x,\xi,\omega) - \frac{1}{\varepsilon}\nabla_y u\left(\frac{x}{\varepsilon},\omega\right) \cdot \nabla_\xi f^\varepsilon(t,x,\xi,\omega) = 0, \\ f^\varepsilon(t=0,x,\xi,\omega) = f_0\left(x, \frac{x}{\varepsilon}, \xi, \omega\right). \end{cases} \tag{1}$$

Here, we assume that the initial data $f_0 = f_0(x,y,\xi,\omega)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$ and is stationary in y, i.e.

$$f_0(x, y+z, \xi, \omega) = f_0(x,y,\xi,\tau_z\omega) \quad \text{for all } (x,y,z,\xi,\omega) \in \mathbb{R}^{4N} \times \Omega.$$

It is well-known from the classical theory of linear transport equations that for every $\omega \in \Omega$, there exists a unique solution f^ε of (1) in $L^\infty_{\text{loc}}((0,\infty), L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi))$. The goal of this paper is to study the asymptotic behavior of f^ε as $\varepsilon \to 0$. Thus, following [7], we define the constraint space $\mathbb{K}$:

Definition 1.1. *Let*

$$\xi \cdot \nabla_y f(y,\xi,\omega) - \nabla_y u(y,\omega) \cdot \nabla_\xi f(y,\xi,\omega) = 0 \tag{2}$$

be the constraint equation, and let

$$\mathbb{K} := \{f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega));\ f \text{ satisfies } (2) \text{ in } \mathcal{D}'(\mathbb{R}^N_y \times \mathbb{R}^N_\xi) \text{ a.s. in } \omega\}.$$

We also define the projection P onto the constraint space $\mathbb{K}$, characterised by $P(f) \in \mathbb{K}$ for $f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega))$ stationary, and

$$\int_{\mathbb{R}^N \times \Omega} (P(f) - f)(y,\xi,\omega)\, g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0 \quad \text{for a.e. } y \in \mathbb{R}^N$$

for all stationary functions $g \in L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega) \cap \mathbb{K}$, with compact support in ξ.

(A more precise definition of the projection P will be given in the second section).

Finally, we define $\mathbb{K}^\perp$ as

$$\mathbb{K}^\perp := \{f \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega)); \exists g \in L^1_{loc}(\mathbb{R}^N_\xi \times \mathbb{R}^N_y, L^1(\Omega)), \quad f = P(g) - g\}.$$

Remark 1.1. *Let us indicate that the constraint equation can easily be derived thanks to a formal two-scale Ansatz : indeed, assume that*

$$f^\varepsilon(t,x,\xi,\omega) \approx f\left(t, x, \frac{x}{\varepsilon}, \xi, \omega\right) \quad \text{as } \varepsilon \to 0;$$

inserting this asymptotic expansion in equation (1), we see that f necessarily satisfies the constraint equation (2).

Remark 1.2. *Let $f, g \in L^\infty(\mathbb{R}^N_y, L^2(\mathbb{R}^N_\xi \times \Omega))$ be stationary, and assume that $f \in \mathbb{K}$ and $g \in \mathbb{K}^\perp$. Then for a.e. $y \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N \times \Omega} f(y,\xi,\omega) g(y,\xi,\omega)\, d\xi\, dP(\omega) = 0.$$

This is a characterization of $\mathbb{K}^\perp$ for the class of stationary functions in $L^\infty(\mathbb{R}^N_y, L^2(\mathbb{R}^N_\xi \times \Omega))$.

Here, we provide another proof for the result of E. Frénod and K. Hamdache in [7] in the "non-perturbed case". Our proof is based on the use of the ergodic theorem, and gives a more concrete insight of the projection P and of the microscopic behavior of the sequence f^ε. Moreover, it allows us to retrieve the explicit formulas of the integrable case.

The first result we prove in this paper is the following

Theorem 1. *Let $f_0 \in L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y; L^1(\Omega))$ stationary.*

Let $f^\varepsilon = f^\varepsilon(t,x,\xi,\omega)$ be the solution of (1). Then for all $\varepsilon > 0$, there exist $f = f(t,x,y,\xi,\omega)$ and $g = g(t,x;\tau,y,\xi,\omega)$, both stationary in y, and a sequence $\{r^\varepsilon(t,x,\xi,\omega)\}_{\varepsilon>0}$ such that

$$f^\varepsilon(t,x,\xi,\omega) = f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) + g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) + r^\varepsilon(t,x,\xi,\omega)$$

and :

- $||r^\varepsilon||_{L^1_{loc}((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))} \to 0$ *as* $\varepsilon \to 0$*;*
- $f \in L^\infty_{loc}((0,\infty); L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y; L^1(\Omega)))$*, and* $f(t,x) \in \mathbb{K}$ *for a.e.* $t \geq 0$*,* $x \in \mathbb{R}^N$*;*
- *For all* $T > 0$*, for all compact* $K \subset \mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \mathbb{R}^N_y$*,*

$$\sup_{0\leq t\leq T, 0\leq \tau\leq T} ||g||_{L^1(K\times\Omega)} < \infty.$$

  Moreover, $g(t,x;\tau,\cdot) \in \mathbb{K}^\perp$ *for a.e.* $(t,x,\tau) \in (0,\infty)\times\mathbb{R}^N\times(0,\infty)$*;*

- *Microscopic evolution equation for* g *: for a.e.* $t,x \in (0,\infty)\times\mathbb{R}^N$*,* $g(t,x;\cdot)$ *is a solution of*

$$\frac{\partial g}{\partial \tau} + \xi\cdot\nabla_y g - \nabla_y u \cdot \nabla_\xi g = 0. \tag{3}$$

  Moreover, for all $T > 0$

$$\left|\left|\int_0^T g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\, dt\right|\right|_{L^1_{loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))} \to 0 \quad \text{as } \varepsilon \to 0.$$

- *Macroscopic evolution equation :* f *and* g *satisfy*

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} + \xi^\sharp(y,\xi,\omega)\cdot\nabla_x \begin{pmatrix} f \\ g \end{pmatrix} = 0, \tag{4}$$

  where

$$\xi^\sharp(y,\xi,\omega) := P(\xi)(y,\xi,\omega);$$

- *Initial data :*

$$f(t=0,x,y,\xi,\omega) = P(f_0)(x,y,\xi,\omega),$$
$$g(t=0,x;\tau=0,y,\xi,\omega) = [f_0 - P(f_0)]\,(x,y,\xi,\omega).$$

Before going any further, we wish to make a few comments on the above results. First, let us stress that it is not obvious that the function g is well-defined : indeed, let $S(t)$ $(t \geq 0)$ denote the semi-group associated to the macroscopic evolution equation (4), and let $T(\tau)$ $(\tau \geq 0)$ be the semi-group associated to the microscopic evolution equation (3). Then g is well defined if and only if, for all stationary function $g_0 = g_0(x,y,\xi,\omega)$, for all $t,\tau \geq 0$,

$$T(\tau)\left[S(t)g_0\right] = S(t)\left[T(\tau)g_0\right].$$

This identity follows from the fact that the speed $\xi^\sharp(y,\xi,\omega)$ appearing in equation (4) is a stationary solution of (3) by definition of the projection P, and is thus invariant by the semi-group $T(\tau)$.

Next, let us explain briefly the meaning of theorem 1. The idea is the following : write f_0 as $f_0 = f_{0\parallel} + f_{0\perp}$, with $f_{0\parallel}(x,\cdot) \in \mathbb{K}$ and $f_{0\perp}(x,\cdot) \in \mathbb{K}^\perp$ a.e. Then f^ε can be written as $f^\varepsilon_\parallel + f^\varepsilon_\perp$, where $f^\varepsilon_\parallel$ (resp. $f^\varepsilon_\perp$) is the solution of equation (1) with initial data $f_{0\parallel}\left(x,\frac{x}{\varepsilon},\xi,\omega\right)$ (resp. $f_{0\perp}\left(x,\frac{x}{\varepsilon},\xi,\omega\right)$). Theorem 1 states that

$$f^\varepsilon_\parallel - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) \to 0$$

strongly in L^1_{loc} norm. In particular, there are no microscopic oscillations in time in this part of f^ε. We wish to emphasize that this result appears to us to be new.

We now focus on the other part, namely $f^\varepsilon_\perp$. An easy consequence of the theorem is

$$\int_0^T f^\varepsilon_\perp(t,x,\xi,\omega)\, dt \to 0$$

in $L^\infty_{\rm loc}(\mathbb{R}^N_x; L^1_{\rm loc}(\mathbb{R}^N_\xi; L^1(\Omega)))$ and for all $T>0$. However, it would be wrong to think that $f^\varepsilon_\perp$ vanishes in $L^1_{\rm loc}((0,\infty)\times\mathbb{R}^N_x\times\mathbb{R}^N_\xi, L^1(\Omega))$, for instance. Indeed

$$f^\varepsilon_\perp(t,x,\xi,\omega)\approx g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)$$

in $L^1_{\rm loc}$, and

$$||g(t=0,x;\tau,y)||_{L^1(\mathbb{R}^N_\xi\times\Omega)}=||f_{0\perp}(x,y)||_{L^1(\mathbb{R}^N_\xi\times\Omega)}$$

as soon as $f_{0\perp}(x,y)\in L^1(\mathbb{R}^N_\xi\times\Omega)$ for almost every x,y. Consequently, if $f_{0\perp}\neq 0$, then for all $T>0$ and for all compact $K\subset\mathbb{R}^N$, there exists a constant $C>0$ depending only on K, $||f_{0\perp}||_{L^1_{\rm loc}}$, and T such that

$$\int_0^T\left|\left|g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right|\right|_{L^1(K\times\mathbb{R}^N_\xi\times\Omega)}dt\geq C.$$

Hence $f^\varepsilon_\perp$ does not vanish strongly in general. In other words, there are fast oscillations in time, due to the ill-preparedness of the initial data (i.e. $f_0(x,\cdot)\notin\mathbb{K}$), but these oscillations do not cancel out as ε vanishes.

Let us now explain briefly here how our strategy of proof differs from the one of E. Frénod and K. Hamdache. The key of our analysis lies in the study of the behavior as $\varepsilon\to 0$ of the Hamiltonian system

$$\left\{\begin{array}{l}\dot{Y}^\varepsilon(t,x,\xi,\omega)=-\Xi^\varepsilon(t,x,\xi,\omega),\quad t>0\\ \dot{\Xi}^\varepsilon(t,y,\xi,\omega)=\frac{1}{\varepsilon}\nabla_y u(Y^\varepsilon(t,x,\xi,\omega),\omega),\quad t>0\\ Y^\varepsilon(t=0,x,\xi,\omega)=x,\ \Xi^\varepsilon(t=0,x,\xi,\omega)=\xi,\quad (x,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega.\end{array}\right.$$

Indeed,

$$f^\varepsilon(t,x,\xi,\omega)=f_0\left(Y^\varepsilon(t,x,\xi,\omega),\frac{Y^\varepsilon(t,x,\xi,\omega)}{\varepsilon},\Xi^\varepsilon(t,y,\xi,\omega),\omega\right),$$

so that we can deduce the asymptotic behavior of f^ε from the one of $(Y^\varepsilon,\Xi^\varepsilon)$. And it is easily checked that

$$\begin{array}{l}Y^\varepsilon(t,x,\xi,\omega)=\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\\ \Xi^\varepsilon(t,y,\xi,\omega)=\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\end{array}$$

where (Y,Ξ) is the solution of the system

$$\left\{\begin{array}{l}\dot{Y}(t,y,\xi,\omega)=-\Xi(t,y,\xi,\omega),\quad t>0\\ \dot{\Xi}(t,y,\xi,\omega)=\nabla_y u(Y(t,y,\xi,\omega),\omega),\quad t>0\\ Y(t=0,y,\xi,\omega)=y,\ \Xi(t=0,y,\xi,\omega)=\xi,\quad (y,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega.\end{array}\right.\tag{5}$$

Hence, in order to study the limit of f^ε as $\varepsilon\to 0$, we have to investigate the long time behavior of the system (Y,Ξ), and this will be achieved with the help of the ergodic theorem in the second section.

In the case when $N=1$, we can give explicit formulas for $\xi^\sharp(y,\xi,\omega)$; the proof of this formula in the stationary ergodic case is strongly linked to methods from the Aubry-Mather theory (see [5], [6], [12]), and thus also to the homogenization of Hamilton-Jacobi equations. Let us first recall the definition of the homogenized Hamiltonian $\bar{H}$ (see [10])

$$\bar{H}(p)=u_{\max}+\frac{1}{2}\left\{\begin{array}{ll}0 & \text{if } |p|<E\left[\sqrt{2(u_{\max}-u)}\right]\\ \lambda & \text{if } |p|\geq E\left[\sqrt{2(u_{\max}-u)}\right],\quad\text{where } |p|=E\left[\sqrt{2(u_{\max}-u)+\lambda}\right]\end{array}\right.$$

Proposition 1.1. *Assume that $N=1$.*

Let $(y,\xi,\omega)\in\mathbb{R}\times\mathbb{R}\times\Omega$ such that $H(y,\xi,\omega)>u_{max}$. Assume that for all $Q\in\mathbb{R}$, for all $\omega\in\Omega$, there exists a Lipschitz continuous function $v(\cdot,\omega)$, viscosity solution of

$$H(y,Q+\nabla_y v(y,\omega),\omega)=\bar{H}(Q)$$

such that

$$\frac{v(y,\omega)}{1+|y|} \to 0 \quad \text{as } |y| \to \infty \tag{6}$$

a.s. in ω.

Let $P = P(y,\xi,\omega) \in \mathbb{R}$ *such that* $\bar{H}(P) = H(y,\xi,\omega)$ *and* $sgn(P) = sgn(\xi)$. *Then*

$$\xi^\sharp(y,\xi,\omega) = \bar{H}'(P)$$

Moreover, if L *is the dual function of* H, *i.e.*

$$L(y,p,\omega) = \sup_{\xi\in\mathbb{R}} \left(p\xi - H(y,\xi\,\omega)\right) = \frac{1}{2}|p|^2 - u(y,\omega),$$

and $\bar{L}$ *is the homogenized Lagrangian, then*

$$P(L)(y,\xi,\omega) = \bar{L}(\xi^\sharp(y,\xi,\omega).)$$

In the periodic case, we will give another proof of the above result; the strategy chosen in that case is inspired from techniques and calculations in classical mechanics. It also allows to give a formula for $\xi^\sharp$ for low energies in the periodic setting only:

Proposition 1.2. *Assume that* $N = 1$ *and that the environment is periodic.*

Let $(y,\xi) \in \mathbb{R}^2$ *such that* $H(y,\xi) < u_{max}$. *Then* $\xi^\sharp(y,\xi) = 0$.

The organisation of this note is the following : in the second section, we derive some preliminary results on the long-time behavior of the system (Y,Ξ) thanks to the ergodic theorem. Those will be useful in the proof of theorem 1, to which is devoted the third section. Eventually, the fourth and last section is concerned with results in the integrable case, both in the periodic and the stationary ergodic settings.

2 Preliminaries

This section is largely devoted to the study of the long-time behavior of the Hamiltonian system (Y,Ξ) defined by (5). First, notice that the Hamiltonian $H(y,\xi,\omega) := \frac{1}{2}|\xi|^2 + u(y,\omega)$ is constant along the curves of the system (Y,Ξ), and if $f \in L^\infty(\Omega, \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi))$ is stationary, then

$$f \in \mathbb{K} \iff f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega),\omega\right) = f(y,\xi,\omega) \quad \forall (y,\xi,\omega) \in \mathbb{R}^N \times \mathbb{R}^N \times \Omega.$$

Indeed, for all $f \in L^\infty(\Omega, \mathcal{C}^1(\mathbb{R}^N_y \times \mathbb{R}^N_\xi))$, we have

$$\frac{\partial}{\partial t} f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega),\omega\right) = \{H,f\}\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega),\omega\right),$$

where $\{H,f\}$ denotes the Poisson bracket of f and H, i.e.

$$\{H,f\}(y,\xi,\omega) = \xi \cdot \nabla_y f(y,\xi,\omega) - \nabla_y u(y,\xi,\omega) \cdot \nabla_\xi f(y,\xi,\omega).$$

Let us mention an easily checked property of the trajectories (Y,Ξ) which will be used extensively in the rest of the article : for all $(y,z,\xi) \in \mathbb{R}^{3N}$, for all $\omega \in \Omega$, $t \geq 0$,

$$\begin{aligned} Y(t,y,\xi,\tau_z\omega) + z &= Y(t,y+z,\xi,\omega), \\ \Xi(t,y,\xi,\tau_z\omega) &= \Xi(t,y+z,\xi,\omega). \end{aligned} \tag{7}$$

In the periodic case, this invariance entails that the hamiltonian system (Y,Ξ) can be considered as a dynamical system on the N dimensional torus $[0,2\pi)^N$. In this periodic setting, it is somewhat natural to introduce the semi-group of transformations $(\mathcal{T}_t)_{t\geq 0}$ on $[0,2\pi)^N \times \mathbb{R}^N$ given by

$$\mathcal{T}_t(y,\xi) = (Y(t,y,\xi), \Xi(t,y,\xi)), \quad y \in [0,2\pi)^N,\ \xi \in \mathbb{R}^N.$$

According to Liouville's theorem, this semi-group preserves the Lebesgue measure on $[0,2\pi)^N \times \mathbb{R}^N$; moreover, we can construct a family of finite invariant measures on $[0,2\pi)^N \times \mathbb{R}^N$ by setting $m_c(y,\xi) = \mathbf{1}_{H(y,\xi)\leq c}\, dy\, d\xi$ for $c>0$ (remember that the Hamiltonian is constant along the hamiltonian curves). This construction is the root of the ergodic theorem (see corollary 2.1), and thus of the study of the long-time behavior of the system (Y,Ξ).

In the stationary ergodic setting, this construction can be generalized as follows : we define the transformation $T_t : \mathbb{R}^N_\xi \times \Omega \to \mathbb{R}^N_\xi \times \Omega$ by

$$T_t(\xi,\omega) = \left(\Xi(t,0,\xi,\omega), \tau_{Y(t,0,\xi,\omega)}\omega\right)$$

together with the family of measures

$$\mu_c := \mathbf{1}_{\mathcal{H}(\xi,\omega)\leq c}\, d\xi\, dP(\omega)$$

where $\mathcal{H}(\xi,\omega) := \frac{1}{2}|\xi|^2 + u(0,\omega)$. It is obvious that for all $c \in (0,\infty)$, μ_c is a finite measure on $\mathbb{R}^N_\xi \times \Omega$.

Notice that the "good" generalization to the stationary ergodic setting of the semi-group $(\mathcal{T}_t)$ is a semi-group which acts on $\mathbb{R}^N_\xi \times \Omega$ rather than $\mathbb{R}^N_y \times \mathbb{R}^N_\xi$. Thanks to the group of transformations $(\tau_x)_{x\in\mathbb{R}^N}$, the transformations in Ω can result in transformations in $\mathbb{R}^N_y$, but the definition chosen here allows us to define a family of finite invariant measures, whereas such a construction is rather difficult if one tries to define a semi-group acting on $\mathbb{R}^N_y \times \mathbb{R}^N_\xi$. This will be fundamental in the rest of the proof.

Lemma 2.1. *$(T_t)_{t\geq 0}$ is a semi-group on $\mathbb{R}^N_\xi \times \Omega$ and preserves the family of measures μ_c.*

Proof. Let us first prove the semi-group property : let $t,s \in [0,\infty)$, and $(\xi,\omega) \in \mathbb{R}^N \times \Omega$; then

$$\begin{aligned} T_t \circ T_s(\xi,\omega) &= T_t\left(\Xi(s,0,\xi,\omega), \tau_{Y(s,0,\xi,\omega)}\omega\right) \\ &= \left(\Xi(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega), \omega'\right) \end{aligned}$$

and using the properties (7) we deduce

$$\begin{aligned} \Xi(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega) &= \Xi(t,Y(s,0,\xi,\omega),\Xi(s,0,\xi,\omega),\omega), \\ &= \Xi(t+s,0,\xi,\omega) \end{aligned}$$

and

$$\begin{aligned} \omega' &= \tau_{Y(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega)}\tau_{Y(s,0,\xi,\omega)}\omega \\ &= \tau_{Y(t,0,\Xi(s,0,\xi,\omega),\tau_{Y(s,0,\xi,\omega)}\omega)+Y(s,0,\xi,\omega)}\omega \\ &= \tau_{Y(t,Y(s,0,\xi,\omega),\Xi(s,0,\xi,\omega),\omega)}\omega \\ &= \tau_{Y(t+s,0,\xi,\omega)}\omega \end{aligned}$$

Thus

$$T_t \circ T_s(\xi,\omega) = \left(\Xi(t+s,0,\xi,\omega), \tau_{Y(t+s,0,\xi,\omega)}\omega\right) = T_{t+s}(\xi,\omega).$$

Since it is obvious that $T_0 = \mathrm{Id}$, $(T_t)_{t\geq 0}$ is a semi-group on $\mathbb{R}^N \times \Omega$.

We now have to check the invariance property; let $F \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$ arbitrary. We set $f(y,\xi,\omega) := F(\xi,\tau_y\omega)$ for $(y,\xi,\omega) \in \mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega$, and we compute

$$\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega) = E\left[\int_{R^N} f(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\mathbf{1}_{H(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\leq c}\, d\xi\right]$$

Since the probability measure P is invariant by the group of transformation τ_y, and

$$f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) = f(Y(t,0,\xi,\tau_y\omega),\Xi(t,0,\xi,\tau_y\omega),\tau_y\omega),$$

we have, for all $y \in \mathbb{R}^N$

$$\begin{aligned} &E\left[f(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)1_{H(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega)\leq c}\right] = \\ &\qquad = E\left[f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)1_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\leq c}\right]. \end{aligned}$$

Take an arbitrary function $\phi \in L^1(\mathbb{R}^N_y)$, and write

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega)\\
=\ & E\left[\int_{\mathbb{R}^{2N}} dy\, d\xi \phi(y) f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) 1_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\leq c}\right]
\end{aligned}$$

We change variables in the integral in (y,ξ) by setting $(x,v)=(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega))$; according to Liouville's theorem, the jacobian of this change of variables is equal to 1, and

$$(x,v)=(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega)) \iff (y,\xi)=(X(t,x,v,\omega),V(t,x,v,\omega)),$$

where (X,V) is a solution of the Hamiltonian system

$$\begin{cases} \dot X = V,\\ \dot V = -\nabla u(X,\omega),\\ (X,V)(t=0,x,v)=(x,v). \end{cases}$$

Observe that in the present case, we have simply

$$X(t,x,v,\omega)=Y(t,x,-v,\omega),$$

so that (X,V) satisfies relations (7).

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} dy\, d\xi \phi(y) f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) 1_{H(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\leq c}\\
=\ & \int_{\mathbb{R}^{2N}} dx\, dv \phi(X(t,x,v,\omega)) f(x,v,\omega) 1_{H(x,v,\omega)\leq c}\\
=\ & \int_{\mathbb{R}^{2N}} dx\, dv \phi(X(t,0,v,\tau_x\omega)+x) F(v,\tau_x\omega) 1_{\mathcal{H}(v,\tau_x\omega)\leq c}
\end{aligned}$$

so that

$$\begin{aligned}
&\int_{\mathbb{R}^N\times\Omega} F(T_t(\xi,\omega))\, d\mu_c(\xi,\omega)\\
=\ & E\left[\int_{\mathbb{R}^{2N}} dx\, dv \phi(X(t,0,v,\tau_x\omega)+x) F(v,\tau_x\omega) 1_{\mathcal{H}(v,\tau_x\omega)\leq c}\right]\\
=\ & E\left[\int_{\mathbb{R}^{2N}} dx\, dv \phi(X(t,0,v,\omega)+x) F(v,\omega) 1_{\mathcal{H}(v,\omega)\leq c}\right]\\
=\ & E\left[\int_{\mathbb{R}^N} dv \left(\int_{\mathbb{R}^N} \phi(X(t,0,v,\omega)+x)\, dx\right) F(v,\omega) 1_{\mathcal{H}(v,\omega)\leq c}\right]\\
=\ & E\left[\int_{\mathbb{R}^N} dv F(v,\omega) 1_{\mathcal{H}(v,\omega)\leq c}\right] = \int_{\mathbb{R}^N\times\Omega} F\, d\mu_c
\end{aligned}$$

since the integral of ϕ is equal to 1.

□

The following corollary is an immediate consequence of Birkhoff's ergodic theorem:

Corollary 2.1. *Let $F \in L^1(\mathbb{R}^N\times\Omega;\mu_c)$. There exists a function $\bar F \in L^1(\mathbb{R}^N\times\Omega;\mu_c)$ such that as $T\to\infty$,*

$$\frac{1}{T}\int_0^T F(T_t(\xi,\omega))\, dt \to \bar F(\xi,\omega)$$

a.e. on $\mathbb{R}^N\times\Omega$ and in $L^1(\mu_c)$. Moreover, $\bar F$ is invariant by T_t for all $t>0$, and

$$\int_{\mathbb{R}^N\times\Omega} F\, d\mu_c = \int_{\mathbb{R}^N\times\Omega} \bar F\, d\mu_c. \tag{8}$$

Additionally, if $\bar f = \bar f(y,\xi,\omega)$ is the stationary function associated to $\bar F$, that is, $\bar f(y,\xi,\omega) = \bar F(\xi,\tau_y\omega)$, then $\bar f$ is invariant by the hamiltonian flow (Y,Ξ); precisely, for all $(y,\xi,\omega)\in\mathbb{R}^{2N}\times\Omega$, $t>0$

$$\bar f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) = \bar f(y,\xi,\omega).$$

Proof. We only have to prove the invariance of $\bar f$ by the Hamiltonian flow; first, for $y=0$, we have

$$\begin{aligned}\bar f(Y(t,0,\xi,\omega),\Xi(t,0,\xi,\omega),\omega) &= \bar F(\Xi(t,0,\xi,\omega),\tau_{Y(t,0,\xi,\omega)}\omega) = \bar F(T_t(\xi,\omega))\\ &= \bar F(\xi,\omega) = \bar f(0,\xi,\omega)\end{aligned}$$

and the property is proved when $y=0$.

For $y\in\mathbb{R}^N$ arbitrary,

$$\begin{aligned}\bar f(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega) &= \bar f(Y(t,0,\xi,\tau_y\omega)+y,\Xi(t,0,\xi,\tau_y\omega),\omega)\\ &= \bar f(Y(t,0,\xi,\tau_y\omega),\Xi(t,0,\xi,\tau_y\omega),\tau_y\omega)\\ &= \bar f(0,\xi,\tau_y\omega) = \bar f(y,\xi,\omega)\end{aligned}$$

according to the result in the case $y=0$.

□

Remark 2.1. *We mention here an important but easy consequence of the relations* (7) *and the invariance of the measure P w.r.t. τ_y, $y\in\mathbb{R}^N$: for any stationary function $f=f(y,\xi,\omega)=F(\xi,\tau_y\omega)$, $F\in L^\infty(\mathbb{R}^N\times\Omega)$, we have*

$$E[f(Y(t,y,\xi,\cdot),\Xi(t,y,\xi,\cdot),\cdot)] = E[F(T_t(\xi,\cdot))]$$

for all $t>0$, $y,\xi\in\mathbb{R}^N$; in particular, the left-hand side of the above equality does not depend on y.

This property was used in the proof of lemma 2.1

Remark 2.2. *Let us precise a little what happens when the function $F\in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$. In that case, $F\in L^1(\mathbb{R}^N_\xi\times\Omega;\mu_c)$ for all $c>0$. Consequently, for any $c>0$, we can define the function $\bar F_c$ associated to F by corollary 2.1.*

It is then easily proved that for any $0<c<c'$, $\bar F_c=\bar F_{c'}$, μ_c-almost everywhere. Setting $A_n = \{(\xi,\omega)\in \mathrm{Supp}\mu_n;\ \bar F_n(y,\xi)\neq \bar F_{n+1}(y,\xi)\}$, and $A=\cup_{n=0}^\infty A_n$, we see that $\mu_c(A)=0$ for all $c>0$. Moreover, for all $(\xi,\omega)\in\mathbb{R}^N\times\Omega\setminus A$, for all integers k,l such that $(\xi,\omega)\in\mathrm{Supp}\mu_k\cap\mathrm{Supp}\mu_l$, we have $\bar F_k(\xi,\omega)=\bar F_l(\xi,\omega)$. We can thus define a function $\bar F(\xi,\omega)$ on $\mathbb{R}^N\times\Omega\setminus A$ by

$$\bar F(\xi,\omega) = \bar F_n(\xi,\omega)\quad \text{for any } n\in\mathbb{N} \text{ such that } (\xi,\omega)\in\mathrm{Supp}\mu_n$$

We then now that

$$\frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt \to \bar F(\xi,\omega) \tag{9}$$

as $T\to\infty$, and the convergence holds in $L^1(\mu_c)$ for all $c>0$, and μ_n almost everywhere for $n\in\mathbb{N}$. Eventually, setting

$$B := \{(\xi,\omega)\in\mathbb{R}^N\times\Omega\setminus A;\ \frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt \text{ does not converge towards } \bar F(\xi,\omega)\}$$

it is easily proved that $\mu_c(B)=0$ for all $c>0$ (the equality is true for $c\in\mathbb{N}$, and is then deduced for $c>0$ arbitrary because the family of measures (μ_c) is increasing in c).

Eventually, we have found a function $\bar F\in L^1_{loc}(\mathbb{R}^N, L^1(\Omega))$, independent of c, such that (9) *holds in $L^1(\mu_c)$ and μ_c-almost everywhere for all $c>0$.*

Remark 2.3. *The construction above allows us to make more precise what we mean by projection P: let $f=f(y,\xi,\omega)$ be a stationary function, $f\in L^\infty(\mathbb{R}^N_y, L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega)))$, and set $F(\xi,\omega)=f(0,\xi,\omega)\in$*

$L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$. *We can then associate to* F *a function* $\bar{F} \in L^1_{loc}(\mathbb{R}^N_\xi, L^1(\Omega))$ *such that* (9) *holds in* $L^1(\mu_c)$ *for all* c *(see remark 2.2). We set*

$$P(f)(y,\xi,\omega) := \bar{F}(\xi, \tau_y\omega).$$

It follows from corollary 2.1 that $P(f)$ *is invariant by the hamiltonian flow* (5)*, and thus satisfies the constraint equation. From now on, we take this definition for the projection* P*, instead of the one given in the introduction. Notice that, for all* $y \in \mathbb{R}^N$ *and* μ_c*-almost everywhere,*

$$\begin{aligned} P(f)(y,\xi,\omega) &= \lim_{T\to\infty} \frac{1}{T}\int_0^T F(T_t(\xi,\tau_y\omega))\,dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(Y(t,0,\xi,\tau_y\omega), \Xi(t,0,\xi,\tau_y\omega), \tau_y\omega\right)\,dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega), \omega\right)\,dt \end{aligned}$$

And we also give a more precise definition of $\xi^\sharp(y,\xi,\omega)$ *: let*

$$\hat{\xi} : \begin{array}{ccc} \mathbb{R}^N \times \Omega & \to & \mathbb{R}^N \\ (\xi,\omega) & \mapsto & \xi \end{array}.$$

Then

$$\xi^\sharp(y,\xi,\omega) = P(\hat{\xi})(y,\xi,\omega) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt$$

a.e. and in $L^1(\mu_c)$ *for all* $0 < c < \infty$.

Eventually, we mention here a property that will be used in the proof of the theorem; with the same notations as above, let

$$\phi(\tau,y,\xi,\omega) = F\left(T_\tau(\xi,\tau_y\omega)\right).$$

Then ϕ is a solution of the evolution equation

$$\partial_\tau\phi + \xi\cdot\nabla_y\phi - \nabla_y u\cdot\nabla_\xi\phi = 0,$$

with initial data $\phi(\tau=0,y,\xi,\omega) = f(y,\xi,\omega) = F\left(\xi,\tau_y\omega\right)$.

3 The general N-dimensional case

This section is devoted to the proof of theorem 1. The proof is divided in three steps : first, we study the case of an initial data which does not depend on x, then the case when the initial data only depends on x (and not on y,ξ,ω), and eventually, we treat the general case.

3.1 First case : f_0 does not depend on x

Here, we assume that $f_0 = f_0(y,\xi,\omega) \in L^1_{\mathrm{loc}}(\mathbb{R}^N_\xi; L^\infty(\mathbb{R}^N_y \times \Omega)$. Recall that f_0 is stationary, i.e. $f_0(y+z,\xi,\omega) = f_0(y,\xi,\tau_z\omega)$ a.s. in ω, for all $(y,z,\xi) \in \mathbb{R}^{3N}$. In the rest of the subsection, we set

$$F_0(\xi,\omega) := f_0(0,\xi,\omega)$$

and

$$\bar{F}_0(\xi,\omega) := \lim_{T\to\infty} \frac{1}{T}\int_0^T F(T_t(\xi,\omega))\,dt, \quad \bar{f}_0(y,\xi,\omega) = \bar{F}_0(\xi,\tau_y\omega).$$

Notice that $F \in L^1_{\mathrm{loc}}(\mathbb{R}^N_\xi; L^\infty(\Omega))$, and thus $F \in L^1(\mathbb{R}^N \times \Omega; \mu_c)$ for all $c > 0$.

In that case,

$$\begin{aligned}
f^\varepsilon(t,x,\xi,\omega) &= f_0\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\\
&= f_0\left(Y\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right),\Xi\left(\frac{t}{\varepsilon},0,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right),\tau_{\frac{x}{\varepsilon}}\omega\right)\\
&= F_0\left(T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right)\\
&= \bar{f}_0\left(\frac{x}{\varepsilon},\xi,\omega\right)+\left\{F_0\left(T_{\frac{t}{\varepsilon}}\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right)-\bar{F}_0\left(\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)\right\}
\end{aligned}$$

In accordance with theorem 1, we set

$$g(\tau,y,\xi,\omega)=\left(F_0-\bar{F}_0\right)\left(T_\tau\left(\xi,\tau_y\omega\right)\right),$$

and $r^\varepsilon=0$. Then g satisfies the microscopic evolution equation (3) thanks to the remark at the end of the preceding section. Moreover, $g(\tau)\in\mathbb{K}^\perp$ by definition of $\mathbb{K}^\perp$ and because $P\left(F_0\left(T_\tau\left(\xi,\tau_y\omega\right)\right)\right)=\bar{F}_0(\xi,\tau_y\omega)$.

It only remains to check that

$$\int_0^T g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\to 0\quad\text{as }\varepsilon\to 0$$

in $L^1_{\text{loc}}(\mathbb{R}^N_x,L^1(\mathbb{R}^N\times\Omega,\mu_c))$ for all $T>0$ and $c>0$.

The invariance of the measure P with respect to the group of transformations $(\tau_x)_{x\in\mathbb{R}^N}$ (see remark 2.1) entails that

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N_\xi}\left|\frac{1}{\frac{T}{\varepsilon}}\int_0^{\frac{T}{\varepsilon}} f_0\left(Y\left(t,\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(t,\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\,dt-\bar{f}_0\left(\frac{x}{\varepsilon},\xi,\omega\right)\right|\,d\mu_c(\xi,\omega)\\
=&\int_{\Omega\times\mathbb{R}^N_\xi}\left|\frac{1}{\frac{T}{\varepsilon}}\int_0^{\frac{T}{\varepsilon}} F_0\left(T_t(\xi,\omega)\right)\,dt-\bar{F}_0\left(\xi,\omega\right)\right|\,d\mu_c(\xi,\omega)
\end{aligned}$$

and the term above goes to 0 as $\varepsilon\to 0$ according to corollary 2.1 and is independent of $x\in\mathbb{R}^N$. There remains to check that $\bar{f}_0=P(f_0)$. This follows directly from remark 2.3. Thus theorem 1 is proved in the case when f_0 does not depend on the macroscopic variable x.

The following remark will prove to be useful when treating the general case :

Remark 3.1. *If $f_0\in L^\infty$, then for any function $a\in L^\infty((0,\infty)\times\mathbb{R}^N_y\times\mathbb{R}^N_\xi\times\Omega)$, stationary in y, we have*

$$\int_0^T a\left(t,\frac{x}{\varepsilon},\xi,\omega\right)g\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\,dt\to 0\quad\text{as }\varepsilon\to 0$$

in $L^1_{loc}(\mathbb{R}^N_x,L^1(\mathbb{R}^N\times\Omega,\mu_c))$ for all $T>0$ and $c>0$.

Indeed, prove the property first for $a=a_1(t)a_2(y,\xi,\omega)$, with a_1, $a_2\in L^\infty$. For a arbitrary, take a sequence a_δ with $\delta>0$, converging to a in L^1_{loc}, and such that

$$a_\delta=\sum_{k=0}^{n_\delta}a_1^\delta(t)a_2^\delta(y,\xi,\omega).$$

with a_1^δ,a_2^δ in L^∞. The property is known for a_δ, and it is thus easily deduced for a.

3.2 Second case : $f_0=f_0(x)$

Unlike the preceding subsection, we now focus on the case when f_0 only depends on the macroscopic variable x. In order to simplify the analysis, we assume that $f_0\in W^{1,\infty}(\mathbb{R}^N_x)$ (the case when f_0 is not smooth in x will be treated in the next subsection). In that case,

$$f^\varepsilon(t,x,\xi,\omega)=f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right).$$

Hence we have to investigate the behavior as $\varepsilon \to 0$ of

$$\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right).$$

We prove the following

Lemma 3.1. *Let $T > 0$ arbitrary. As ε vanishes,*

$$\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) \to 0$$

in $L^\infty((0,T)\times\mathbb{R}^N_x; L^1(\mathbb{R}^N_\xi\times\Omega,\mu_c))$.

Proof. Let us write, for $t > 0$

$$\begin{aligned}
\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right) &= \varepsilon\int_0^{\frac{t}{\varepsilon}} \dot{Y}\left(s, \frac{x}{\varepsilon}, \xi, \omega\right)\, ds + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\\
&= -t\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \Xi\left(s, \frac{x}{\varepsilon}, \xi, \omega\right)\, ds + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\\
&= -t\left\{\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \hat{\xi}\left(T_s(\xi, \tau_{\frac{x}{\varepsilon}}\omega)\right)\, ds - \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right\}
\end{aligned}$$

Let $0 < \alpha < T$ arbitrary. For $\alpha \leq t \leq T$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N_\xi\times\Omega} \left|\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right|\, d\mu_c(\xi,\omega)\\
=\;& t\int_{\mathbb{R}^N_\xi\times\Omega} \left|\frac{\varepsilon}{t}\int_0^{\frac{t}{\varepsilon}} \hat{\xi}\left(T_s(\xi,\omega)\right)\, ds - \xi^\sharp\left(0,\xi,\omega\right)\right|\, d\mu_c(\xi,\omega)\\
\leq\;& T\sup_{\tau\geq\frac{\alpha}{\varepsilon}} \left|\left|\frac{1}{\tau}\int_0^\tau \hat{\xi}\left(T_s(\xi,\omega)\right)\, ds - \xi^\sharp\left(0,\xi,\omega\right)\right|\right|_{L^1(\mathbb{R}^N\times\Omega,\mu_c)}
\end{aligned}$$

and the upper-bound vanishes as $\varepsilon \to 0$ for any $\alpha > 0$ thanks to corollary 2.1. Notice that the upper-bound does not depend on x, hence the convergence holds in $L^\infty(\mathbb{R}^N_x; L^1(\mu_c))$.

We now have to investigate what happens when t is close to 0; notice that

$$\sup_{x\in\mathbb{R}^N} ||\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)||_{L^1(\mathbb{R}^N\times\Omega,\mu_c)} \leq C_0$$

where the constant C_0 only depends on N and c. Similarly, for all $t \geq 0$,

$$\sup_{x\in\mathbb{R}^N} \left|\left|\hat{\xi}\left(T_s(\xi, \tau_{\frac{x}{\varepsilon}})\right)\right|\right|_{L^1(\mathbb{R}^N\times\Omega,\mu_c)} \leq C_0.$$

Hence, if $0 \leq t \leq \alpha$, we have

$$\sup_{x\in\mathbb{R}^N}\int_{\mathbb{R}^N_\xi\times\Omega} \left|\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + \xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right|\, d\mu_c(\xi,\omega) \leq 2\alpha C_0.$$

Eventually,

$$\begin{aligned}
&\left|\left|\varepsilon Y\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon}, \xi, \omega\right)\right|\right|_{L^\infty((0,T)\times\mathbb{R}^N; L^1(\mu_c))} \leq\\
&\qquad\leq \inf_{0<\alpha<T}\left\{2C_0\alpha + T\sup_{\tau\geq\frac{\alpha}{\varepsilon}}\left|\left|\frac{1}{\tau}\int_0^\tau \hat{\xi}\left(T_s(\xi,\omega)\right)\, ds - \xi^\sharp\left(0,\xi,\omega\right)\right|\right|_{L^1(\mu_c)}\right\}
\end{aligned}$$

and the lemma is proved. □

We easily deduce that theorem 1 is true when $f_0 \in W^{1,\infty}(\mathbb{R}^N)$ with

$$f(t,x,y,\xi,\omega) := f_0(x - t\xi^\sharp(y,\xi,\omega)), \quad g = 0,$$
$$r^\varepsilon(t,x,\xi,\omega) := f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right)$$

and it is easily checked that f satisfies $P(f) = f$, $f(t=0) = P(f_0) = f_0$ (since f_0 is independent of y and ξ), and that f is a solution of the evolution equation (4).

3.3 Third case : f_0 arbitrary

We now tackle the case of an arbitrary stationary function $f_0 \in L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$. We begin with the case when

$$f_0(x,y,\xi,\omega) = a(x)b(y,\xi,\omega),$$

with $a \in W^{1,\infty}(\mathbb{R}^N)$ and $b \in L^1_{\text{loc}}(\mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega)) \cap L^\infty(\mathbb{R}^N_y \times \mathbb{R}^N_\xi \times \Omega)$, b stationary. This case follows directly from the two first subsections. Indeed, let

$$f(t,x,y,\xi,\omega) = a(x - t\xi^\sharp(y,\xi,\omega))\, P(b)(y,\xi,\omega),$$

and

$$g(t,x;\tau,y,\xi,\omega) = a(x - t\xi^\sharp(y,\xi,\omega))\ \ (b - P(b))\,(T_\tau(y,\xi,\omega)).$$

It is already known that f and g satisfy (4), that $f(t,x,\cdot) \in \mathbb{K}$, and that g satisfies (3) thanks to the preceding subsections and the fact that $\xi^\sharp(y,\xi,\omega)$ is invariant by the Hamiltonian flow (Y,Ξ). Notice that it is capital here that the coefficient $\xi^\sharp(y,\xi,\omega)$ in the transport equation (4) belongs to $\mathbb{K}$.

There remains to check that $g(t,x;\tau,\cdot) \in \mathbb{K}^\perp$, that the remainder r^ε goes to 0 strongly in L^1_{loc} and that $g(t,x;t/\varepsilon,x/\varepsilon,\xi,\omega)$ goes weakly to 0 in the sense of theorem 1. First, notice that $a(x - t\xi^\sharp(y,\xi,\omega)) \in \mathbb{K}$ and $(b - P(b))\,(T_\tau(y,\xi,\omega) \in \mathbb{K}^\perp$. Thus, $a(x - t\xi^\sharp)P(b) = P(a(x - t\xi^\sharp)b)$ almost everywhere (because $a(x - t\xi^\sharp(0,\xi,\omega))$ is invariant by the semi-group T_τ), and consequently

$$g(t,x;\tau,y,\xi,\omega) = \left[a(x - t\xi^\sharp)b - P\left(a(x - t\xi^\sharp)b\right)\right](T_\tau(\xi,\tau_y\omega)).$$

Hence $g(t,x;\tau) \in \mathbb{K}^\perp$ a.e.

Then, setting

$$r^\varepsilon(t,x,\xi,\omega) = f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) - g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),$$

we have to prove that r^ε goes to 0 strongly in L^1_{loc}. We compute the difference

$$\begin{aligned}
& f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) - g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) \\
= \ & a\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right) b\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right) \\
& - a\left(x - t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)\ P(b)\left(\frac{x}{\varepsilon},\xi,\omega\right) \\
& - a\left(x - t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)[b - P(b)]\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right) \\
= \ & \left[a\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right)\right) - a\left(x - t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right)\right]\ b\left(Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)
\end{aligned}$$

The right-hand side of the above equality is bounded by

$$||a||_{W^{1,\infty}}||b||_{L^\infty}\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) - x + t\xi^\sharp\left(\frac{x}{\varepsilon},\xi,\omega\right)\right|$$

and thus converges to 0 as $\varepsilon \to 0$ in $L^\infty((0,T)\times\mathbb{R}^N_x; L^1(\mathbb{R}^N_\xi \times \Omega, \mu_c)))$ according to the second subsection.

Moreover, it is easily proved that as $\varepsilon \to 0$,

$$\int_0^T g\left(t, x; \frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \xi, \omega\right)\, dt \to 0$$

strongly in $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^1(\Omega))$ thanks to remark 3.1. Hence theorem 1 is proved in that case.

Now, let $f_0 \in L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$ arbitrary, and set $F_0(x,\xi,\omega) := f_0(x,0,\xi,\omega)$. Take a sequence of functions $F_n \in L^1(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \Omega)$ such that

- $F_n \to F_0$ as $n \to \infty$ in $L^1_{\text{loc}}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi \times \Omega)$;
- For all $n \in \mathbb{N}$, there exist functions $a^n_k \in L^1 \cap W^{1,\infty}(\mathbb{R}^N)$, $b^n_k \in L^1 \cap L^\infty(\mathbb{R}^N_\xi \times \Omega)$, $1 \le k \le n$ such that
$$F_n(x,\xi,\omega) = \sum_{k=1}^n a^n_k(x)\, b^n_k(\xi,\omega) \quad \text{a.e.}$$

Let f^ε_n be the solution of (1) with initial data $F_n\left(x,\xi,\tau_{\frac{x}{\varepsilon}}\omega\right)$, and let $f_n = f_n(t,x,y,\xi,\omega)$, $g_n = g_n(t,x;\tau,y,\xi,\omega)$ be the functions associated to f^ε_n by theorem 1 for all n.

Let $f(t,x,y,\xi,\omega)$, $g(t,x;\tau,y,\xi,\omega)$ be the solutions of the system

$$\begin{gathered}
P(f) = f, \quad P(g) = 0,\\
\partial_t \begin{pmatrix} f \\ g \end{pmatrix} + \xi^\sharp(y,\xi,\omega)\cdot\nabla_x \begin{pmatrix} f \\ g \end{pmatrix} = 0,\\
\partial_\tau g + \xi\cdot\nabla_y g - \nabla_y u(y,\omega)\cdot\nabla_\xi g = 0,\\
f(t=0) = P(f_0), \quad g(t=0,x;\tau=0,y,\xi,\omega) = \left[f_0 - P(f_0)\right](x,y,\xi,\omega).
\end{gathered}$$

We have already proved that f_n, g_n satisfy the above system. We denote by $\bar{F}_0$, $\bar{F}_n$, the functions associated to F_0, F_n respectively by corollary 2.1, so that $P(f_0)(x,y,\xi,\omega) = \bar{F}_0(x,\xi,\tau_y\omega)$, and $f_n(t=0,x,y,\xi,\omega) = \bar{F}_n(x,\xi,\tau_y\omega)$, $g_n(t=0,x,\tau=0,y,\xi,\omega) = (F_n - \bar{F}_n)(x,\xi,\tau_y\omega)$.

We use the following lemma, of which we postpone the proof :

Lemma 3.2. *Let g^ε be a solution of (1) with initial data $g_0\left(x,\frac{x}{\varepsilon},\xi,\omega\right)$, and $g_0 \in L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$ stationary. Then for all $R, R', T > 0$, for all $t \in [0,T]$,*

$$\int_{x\in B_R,\ \xi\in B_{R'}} |g^\varepsilon(t,x,\xi,\omega)|\, dx\, d\xi \le ||g_0||_{L^1(K_{T,R,R'}, L^\infty(\mathbb{R}^N_y\times\Omega))}$$

where

$$K_{T,R,R'} = \left\{(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N, |x| \le R + T\sqrt{R'^2 + 2u_{max}},\ |\xi| \le \sqrt{R'^2 + 2u_{max}}\right\}.$$

Similarly, if g is a solution of (4) with initial data $g_0 \in L^1_{loc}(\mathbb{R}^N_x \times \mathbb{R}^N_\xi, L^\infty(\mathbb{R}^N_y \times \Omega))$, then

$$\int_{x\le R} |g(t,x,y,\xi,\omega)|\, dx \le \int_{x \le R + T\sqrt{\xi^2 + 2u_{max}}} |g_0(x,y,\xi,\omega)|\, dx$$

Consequently, with $C_{R,T,\xi} := \{x \in \mathbb{R}^N,\ |x| \le R + T\sqrt{\xi^2 + 2u_{\max}}\}$, we have

$$\begin{aligned}
||f(t,\cdot,y,\xi,\omega) - f_n(t,\cdot,y,\xi,\omega)||_{L^1(B_R)} &\le \left|\left|\bar{F}_0(\cdot,\xi,\tau_y\omega) - \bar{F}_n(\cdot,\xi,\tau_y\omega)\right|\right|_{L^1(C_{R,T,\xi})}\\
&\le \left|\left|\overline{|F_0 - F_n|}(\cdot,\xi,\tau_y\omega)\right|\right|_{L^1(C_{R,T,\xi})}\\
||f(t,x,y,\xi,\omega) - f_n(t,x,y,\xi,\omega)||_{L^1(B_R\times\mathbb{R}^N_\xi\times\Omega, dx\, d\mu_c(\xi,\omega))} &\le \left|\left|\overline{|F_0 - F_n|}(x,\xi,\omega)\right|\right|_{L^1(C_{R,T,\sqrt{2c}}\times\mathbb{R}^N_\xi\times\Omega, dxd\mu_c(\xi,\omega))}\\
&\le ||F_0 - F_n(x,\xi,\omega)||_{L^1(C_{R,T,\sqrt{2c}}\times\mathbb{R}^N_\xi\times\Omega, dxd\mu_c(\xi,\omega))}.
\end{aligned}$$

In the last inequality, we have used property (8).

And similarly,

$$||f^\varepsilon(t,x,\xi,\omega) - f_n^\varepsilon(t,x,\xi,\omega)||_{L^1(B_R\times B_{R'}\times\Omega)} \le ||F_0 - F_n||_{L^1(K_{T,R,R'},L^\infty(\Omega))},$$

$$||g(t,x;\tau,y,\xi,\omega) - g_n(t,x;\tau,y,\xi,\omega)||_{L^1(B_R\times\mathbb{R}^N_\xi\times\Omega,dx\,d\mu_c(\xi,\omega))} \le 2\,||F_0 - F_n(x,\xi,\omega)||_{L^1(C_{R,T,\sqrt{2c}}\times\mathbb{R}^N_\xi\times\Omega,dxd\mu_c(\xi,\omega))}.$$

The above inequalities are true for all $t\in[0,T]$ and for all $\tau\ge 0$.

Set

$$r^\varepsilon(t,x,\xi,\omega) := f^\varepsilon(t,x,\xi,\omega) - f\left(t,x,\frac{x}{\varepsilon},\xi,\omega\right) - g\left(t,x;\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right).$$

Then for all $t\in[0,T]$, for all $n\in\mathbb{N}$, setting $c=\frac{1}{2}R'^2+u_{\max}$,

$$\begin{aligned}
||r^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)} &\le ||f^\varepsilon(t) - f_n^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)}\\
&\quad + ||f(t) - f_n(t)||_{L^\infty(\mathbb{R}^N_y;L^1(B_{Rx}\times\mathbb{R}^N_\xi\times\Omega,dx\,d\mu_c(\xi,\omega)))}\\
&\quad + ||g(t) - g_n(t)||_{L^\infty((0,\infty)_\tau\times\mathbb{R}^N_y;)L^1(B_R\times\mathbb{R}^N_\xi\times\Omega,dx\,d\mu_c(\xi,\omega))}\\
&\quad + ||r_n^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)}\\
&\le 4\,||F_0 - F_n||_{L^1(C_{R,T,\sqrt{2c}}\times\mathbb{R}^N_\xi\times\Omega,dxd\mu_c(\xi,\omega))} + ||r_n^\varepsilon(t)||_{L^1(B_R\times B_{R'}\times\Omega)}
\end{aligned}$$

Thus $r^\varepsilon\to 0$ as $\varepsilon\to 0$ in $L^\infty([0,\infty);L^1_{\rm loc}(\mathbb{R}^N_x\times\mathbb{R}^N_\xi;L^1(\Omega))$.

There only remains to check that $\int_0^T g(t,x;t/\varepsilon,x/e,\xi,\omega)\,dt$ goes strongly to 0 in $L^1_{\rm loc}$ norm as ε vanishes; this result follows immediately from the same property for g_n and the above inequalities. Therefore, we skip its proof.

Proof of Lemma 3.2. First, let us recall that

$$f^\varepsilon(t,x,\xi,\omega) = f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right),$$

and the Jacobian of the change of variables

$$(x,\xi)\to\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)$$

is equal to 1.

On the other hand, since

$$\frac{1}{2}\left|\Xi(t,y,\xi,\omega)\right|^2 + u\left(Y(t,y,\xi,\omega)\right) = \frac{1}{2}|\xi|^2 + u(y,\omega)$$

we have

$$|\Xi(t,y,\xi,\omega)| \le \sqrt{|\xi|^2+2u(y,\omega)} \le \sqrt{|\xi|^2+2u_{\max}}$$

and

$$\left|\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right) - x\right| \le t\sqrt{|\xi|^2+2u_{\max}}.$$

Thus

$$\begin{aligned}
&\int_{x\in B_R,\ \xi\in B_{R'}} |f^\varepsilon(t,x,\xi,\omega)|\;dx\,d\xi\\
=\;& \int_{x\in B_R,\ \xi\in B_{R'}} \left|f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\right|\;dx\,d\xi\\
\le\;& \int_{x\in B_R,\ \xi\in B_{R'}} \sup_{y\in\mathbb{R}^N}\left|f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), y, \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\right|\;dx\,d\xi\\
\le\;& \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathbf{1}_{(\varepsilon Y(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega),\Xi(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega))\in K_{T,R,R'}} \sup_{y\in\mathbb{R}^N}\left|f_0\left(\varepsilon Y\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right), y, \Xi\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon},\xi,\omega\right),\omega\right)\right|\;dx\,d\xi\\
=\;& \int_{K_{T,R,R'}} \sup_{y\in\mathbb{R}^N}\left|f_0\left(x,y,\xi,\omega\right)\right|\;dx\,d\xi
\end{aligned}$$

The proof of the other inequality goes along the same lines.

□

4 The integrable case

In this section, we treat independently the periodic and the stationary ergodic case. Indeed, some results of the periodic case are no longer true in the stationary ergodic setting, and the results which do remain valid are not proved with the same tools.

Let us make precise what we mean about “integrable case” : in the periodic case, we take a function $u(y)$ which has the form

$$u(y) = \sum_{i=1}^{N} u_i(y_i), \tag{10}$$

where each function u_i is periodic with period 1 ($1 \le i \le N$). The Hamiltonian $H(y,\xi)$ can be written

$$H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y) = \sum_{i=1}^{N} H_i(y_i,\xi)$$

where $H_i(y_i,\xi) = \frac{1}{2}|\xi_i|^2 + u_i(y_i)$ ($1 \le i \le N$). And the Hamiltonian system (5) becomes

$$\begin{cases} \dot{Y}_i = -\Xi_i, \\ \dot{\Xi}_i = u_i'(Y_i), \\ Y_i(t=0) = y_i, \quad \Xi_i(t=0) = \xi_i. \end{cases} \tag{11}$$

Thus it is enough to investigate the behavior of each one-dimensional Hamiltonian system (11) individually, and for most calculations, we can assume without loss of generality that $N = 1$, and we drop all indices i. However, for the calculation of the projection P, a more thorough discussion will be needed, and we will come back to the case when $N > 1$ in the corresponding paragraph.

In the stationary ergodic setting, expression (10) can be transposed in the following way : assume that $\Omega = \Pi_{i=1}^{N}\Omega_i$, where each Ω_i is a probability space, and assume that for $1 \le i \le N$, an ergodic group transformation, denoted by $(\tau_{i,y})_{y\in\mathbb{R}}$, acts on each Ω_i.

Then for $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$, and $y = (y_1, \cdots, y_N) \in \mathbb{R}^N$, we set $\tau_y\omega := (\tau_{1,y_1}\omega_1, \cdots, \tau_{N,y_N}\omega_N)$. And we assume that the function u can be written

$$u(y,\omega) = \sum_{i=1}^{N} U_i\left(\tau_{i,y_i}\omega_i\right),$$

where $U_i \in L^\infty(\Omega_i)$ for all $1 \le i \le N$. The same remarks as in the periodic case can be made, and thus we will only consider the case $N = 1$; note that in the stationary ergodic case, we are unable to compute the projection P when $N > 1$.

4.1 Periodic setting

The goal of this subsection is to give another proof of the results of K. Hamdache and E. Frénod in [7], based on the study of the system

$$\begin{cases} \dot{Y} = -\Xi, \\ \dot{\Xi} = u'(Y), \\ Y(t=0) = y, \quad \Xi = \xi, \quad y \in \mathbb{R}, \ \xi \in \mathbb{R}. \end{cases} \tag{12}$$

The Hamiltonian $H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y)$ is constant along the trajectories of the system (12), so that

$$\frac{1}{2}|\Xi(t,y\xi)|^2 + u(Y(t,y,\xi)) = H(y,\xi).$$

We now fix $y, \xi \in \mathbb{R}^N$. Without any loss of generality, we assume $y \in [-\frac{1}{2}, \frac{1}{2})$, and we set $\mathcal{E} := H(y, \xi)$. The above equation describes the movement of a single particle in a periodic potential u, with $0 \leq u \leq u_{\max}$. It is well-known that there are two kinds of behavior, depending on the value of the energy $\mathcal{E}$: if $\mathcal{E} < u_{\max}$, the particle is "trapped" in a well of potential around y, and $Y(t)$ remains bounded as $t \to \infty$. In that case, the trajectories in the phase space are closed curves. If $\mathcal{E} > u_{\max}$, the trajectory of the particle is unconstrained and $|Y(t)| \to \infty$ as $t \to \infty$. We study more precisely these two cases and their consequences on the expression of the projection P in the following; we refer for instance to [2] for further calculations and results about Hamiltonian dynamics and ordinary differential equations in general.

4.1.1 Expression of $\xi^\sharp(y, \xi)$

We begin with the case when $H(y, \xi) < u_{\max}$. In that case, $u(y) \leq H(y, \xi) < u_{\max}$. By continuity of the potential u, there exists $y_- < y$ and $y_+ > y$ such that $H(y, \xi) < u(y_\pm) < u_{\max}$, and the periodicity of u allows us to choose $y_\pm$ such that $|y_+ - y_-| < 1$. Then $y_- < Y(t, y, \xi) < y_+$ for all $t \geq 0$. Indeed, assume that there exists $t > 0$ such that $Y(t, y, \xi) \geq y_+ > y = Y(t = 0, y, \xi)$. Since the trajectory Y is continuous in time, there exists $0 < t_0 \leq t$ such that $Y(t = t_0, y, \xi) = y_+$, which is absurd since

$$H(Y(t = t_0, y, \xi), \Xi(t = t_0, y, \xi)) = H(y, \xi) \geq u(Y(t = t_0, y, \xi)) > H(y, \xi).$$

Thus $Y(t, y, \xi)$ is bounded. Since

$$\xi^\sharp(y, \xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Xi(t, y, \xi)\, dt = -\lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{Y}(t, y, \xi)\, dt = \lim_{T \to \infty} \frac{y - Y(T, y, \xi)}{T}$$

we deduce that $\xi^\sharp(y, \xi) = 0$ for all y, ξ such that $H(y, \xi) < u_{\max}$.

We now study the case $H(y, \xi) > u_{\max}$. Since

$$|\dot{Y}(t, y, \xi)|^2 = 2\left(H(y, \xi) - u(Y(t, y, \xi))\right) \geq 2(H(y, \xi) - u_{\max}) > 0$$

$\dot{Y}$ does not vanish for $t \geq 0$. Consequently,

$$\Xi(t, y, \xi) = -\dot{Y}(t, y, \xi) = \operatorname{sgn}(\xi)\sqrt{2\left(H(y, \xi) - u(Y(t, y, \xi))\right)},$$

and since $|Y(t, y, \xi) - y| \geq \sqrt{2\left(H(y, \xi) - u_{\max}\right)}t$, $|Y(t)| \to \infty$ as $t \to \infty$. We immediately deduce that $\Xi(t, y, \xi)$ is periodic in time: indeed, there exists $t_0 > 0$ such that

$$Y(t_0, y, \xi) = y - \operatorname{sgn}(\xi).$$

And

$$\Xi(t = t_0, y, \xi) = \operatorname{sgn}(\xi)\sqrt{2\left(H(y, \xi) - u(Y(t_0, y, \xi))\right)} = \operatorname{sgn}(\xi)\sqrt{2\left(H(y, \xi) - u(y)\right)} = \xi = \Xi(t = 0, y, \xi),$$

so that for $s \geq 0$,

$$\begin{gathered} Y(t_0 + s, y, \xi) = Y(s, y, \xi) - \operatorname{sgn}(\xi), \\ \Xi(t_0 + s, y, \xi) = \Xi(s, y\xi), \end{gathered}$$

and Ξ is periodic with period t_0.

Consequently,

$$\xi^\sharp(y, \xi) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \Xi(t, y, \xi)\, dt = \frac{1}{t_0} \int_0^{t_0} \Xi(t, y, \xi)\, dt.$$

But

$$\begin{aligned} \int_0^{t_0} \Xi(t, y, \xi)\, dt &= -\int_0^{t_0} \dot{Y}(t, y, \xi)\, dt \\ &= -\left(Y(t_0, y, \xi) - y\right) \\ &= \operatorname{sgn}(\xi). \end{aligned}$$

Thus we only have to compute t_0. With this aim in view, we use the change of variables $s = Y(t)$, with Jacobian $ds = \dot{Y} dt$ (recall that $\dot{Y}(t, y, \xi) = -\mathrm{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(Y(t,y,\xi))\right)}$), in the formula

$$\begin{aligned} t_0 &= \int_0^{t_0} dt \\ &= \int_{Y(t=0)}^{Y(t_0)} \frac{1}{-\mathrm{sgn}(\xi)\sqrt{2\left(H(y,\xi) - u(s)\right)}} ds \\ &= -\mathrm{sgn}(\xi) \int_y^{y-\mathrm{sgn}(\xi)} \frac{1}{\sqrt{2\left(H(y,\xi) - u(s)\right)}} ds \\ &= \int_0^1 \frac{1}{\sqrt{2\left(H(y,\xi) - u(s)\right)}} ds \end{aligned}$$

Eventually, we deduce

$$\xi^\sharp(y,\xi) = \mathrm{sgn}(\xi)\varphi(H(y,\xi)),$$

where

$$\varphi(\mathcal{E}) = \sqrt{2}\mathbf{1}_{\mathcal{E}>u_{\max}} \frac{1}{\left\langle \frac{1}{\sqrt{(\mathcal{E}-u(s))}} \right\rangle}$$

We close this paragraph with a calculation which allows us to express $\xi^\sharp$ in terms of the homogenized Hamiltonian $\bar{H}$. The result we will obtain will be justified in more abstract and theoretical terms in the last subsection, using arguments similar to those of the theory of Aubry-Mather.

First, let us recall the expression of the homogenized Hamiltonian $\bar{H}$ (see [10]) : we have

$$H(y,\xi) = \frac{1}{2}|\xi|^2 + u(y), \quad \text{with} \inf u = 0,\ \sup u = u_{\max},$$

and thus

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\begin{cases} 0 & \text{if } p < \left\langle \sqrt{2(u_{\max} - u)} \right\rangle \\ \lambda & \text{if } |p| \geq \left\langle \sqrt{2(u_{\max} - u)} \right\rangle, \quad \text{where } |p| = \left\langle \sqrt{2(u_{\max} - u) + \lambda} \right\rangle \end{cases}$$

In other words, setting

$$\theta : \begin{array}{ccc} [0,\infty) & \to & [0,\infty) \\ \lambda & \mapsto & \left\langle \sqrt{2(u_{\max} - u) + \lambda} \right\rangle \end{array}$$

we have

$$\bar{H}(p) = u_{\max} + \frac{1}{2}\mathbf{1}_{|p|\geq\theta(0)}\theta^{-1}(|p|).$$

Hence,

$$\bar{H}'(p) = \mathrm{sgn}(p)\frac{1}{2}\mathbf{1}_{|p|\geq\theta(0)}\frac{1}{\theta'\left(\theta^{-1}(|p|)\right)};$$

and

$$\theta'(\lambda) = \frac{1}{2}\left\langle \frac{1}{\sqrt{2(u_{\max} - u) + \lambda}} \right\rangle,$$

$$\theta^{-1}(|p|) = 2\left(\bar{H}(p) - u_{\max}\right) \quad \forall |p| \geq \theta(0),$$

$$|p| > \theta(0) \iff \bar{H}(p) > u_{\max} \quad \forall p.$$

Gathering all the terms, we are led to

$$\begin{aligned} \bar{H}'(p) &= \mathrm{sgn}(p)\sqrt{2}\mathbf{1}_{\bar{H}(p)>u_{\max}} \frac{1}{\left\langle \frac{1}{\sqrt{\bar{H}(p)-u}} \right\rangle} \\ &= \mathrm{sgn}(p)\varphi\left(\bar{H}(p)\right) \end{aligned}$$

Thus, the final expression is

$$\xi^\sharp(y,\xi) = \bar{H}'(p),$$

where p is such that

$$\bar{H}(p) = H(y,\xi) \vee u_{max}, \quad \mathrm{sgn}(p) = \mathrm{sgn}(\xi).$$

4.1.2 Expression of the projection P

We also mention here how to find a general expression of the projection P in the special case $N = 1$, and we explain how to generalize this expression in some particular cases when $N > 1$. Recall that if $f = f(y,\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^N_y \times \mathbb{R}^N_\xi)$ is periodic in y, then

$$P(f)(y,\xi) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt$$

and the limit holds almost everywhere and in $L^1([0,1)\times\mathbb{R}^N, m_c)$, with $dm_c(y,\xi) = \mathbf{1}_{H(y,\xi)\le c}\,dy\,d\xi$.

We begin with the case $H(y,\xi) > u_{\max}$. We have seen in the previous paragraph that there exists $t_0 > 0$, which depends only on $H(y,\xi)$ such that for all $t > 0$, for all $k \in \mathbb{N}$

$$Y(t + kt_0, y, \xi) = Y(t,y,\xi) - k\mathrm{sgn}(\xi), \quad \Xi(t + kt_0, y, \xi) = \Xi(t,y,\xi).$$

Thus $f(Y(t),\Xi(t))$ is periodic in time with period t_0, and

$$P(f)(y,\xi) = \frac{1}{t_0}\int_0^{t_0} f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt.$$

We use once again the change of variables $s = Y(t)$, so that

$$\begin{aligned}
&\int_0^{t_0} f(Y(t,y,\xi),\Xi(t,y,\xi))\,dt\\
=\ &\int_y^{y-\mathrm{sgn}(\xi)} f(s,\mathrm{sgn}(\xi)\sqrt{2\,(H(y,\xi)-u(s))})\frac{1}{-\mathrm{sgn}(\xi)\sqrt{2\,(H(y,\xi)-u(s))}}\,ds\\
=\ &\left\langle f\left(\cdot,\mathrm{sgn}(\xi)\sqrt{2\,(H(y,\xi)-u(\cdot))}\right)\frac{1}{\sqrt{2\,(H(y,\xi)-u(\cdot))}}\right\rangle.
\end{aligned}$$

And eventually,

$$P(f)(y,\xi) = \bar{f}(\mathrm{sgn}(\xi), H(y,\xi)) \tag{13}$$

with

$$\bar{f}(\eta,\mathcal{E}) := \frac{\left\langle f\left(\cdot,\eta\sqrt{2\,(\mathcal{E}-u(\cdot))}\right)\frac{1}{\sqrt{(\mathcal{E}-u(\cdot))}}\right\rangle}{\left\langle\frac{1}{\sqrt{(\mathcal{E}-u)}}\right\rangle} \quad \eta = \pm 1,\ \mathcal{E} > u_{\max}$$

We now focus on the case $0 < \mathcal{E} < u_{\max}$. In order to simplify the analysis we assume that $\mathcal{E} \notin \{u(y)\,; u \text{ has a local extremum at } y\}$ (this set is finite or countable), and that

$$\forall y \in \mathbb{R}, \quad u'(y) = 0 \Rightarrow u \text{ has a local extremum at } y.$$

In that case, it can be easily proved that $Y(t,y,\xi)$ is periodic in t; this follows directly from the fact that the trajectory in the phase space is closed (see [2]). Indeed, pushing a little further the analysis of the previous paragraph, we construct $z_\pm$ such that

$$\begin{gathered}
|z_+ - z_-| < 2\pi, \quad z_- < z_+,\\
u(z_\pm) = \mathcal{E},\\
z_- \le y \le z_+,\\
u(z) < \mathcal{E} \quad \forall z \in (z_-, z_+).
\end{gathered}$$

Then the particle starting from y with initial speed $-\xi$ reaches either z_+ or z_- in finite time; the speed of the particle is 0 at that moment since

$$|\dot{Y}|^2 = 2(\mathcal{E} - u(Y)),$$

but its acceleration is $-u'(z_\pm) \neq 0$, so the particle turns around and goes back in the reverse direction. It then reaches the other extremity of the interval (z_-, z_+) in finite time, and the same phenomena occurs. Hence after a finite time t_0, the particle is back at its starting point y with the same speed $-\xi$. Consequently, the movement of the particle is periodic in time with period t_0. Thus, we have

$$P(f)(y,\xi) = \frac{1}{t_0}\int_{t_1}^{t_1+t_0} f(Y(t,y,\xi), \Xi(t,y,\xi))\, dt,$$

where $t_1 \geq 0$ is arbitrary. It is convenient to choose for t_1 the first time when the particle hits z_-. In that case, it is easily seen that t_0 is twice the time it takes to the particle to go from z_- to z_+, so that

$$\frac{t_0}{2} = \int_{t_1}^{t_1+t_0/2} dt = \int_{z_-}^{z_+} \frac{1}{\sqrt{2\left(\mathcal{E} - u(s)\right)}} ds = \left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{2\left(\mathcal{E} - u\right)}} \right\rangle$$

and

$$\int_{t_1}^{t_1+\frac{t_0}{2}} f(Y(t,y,\xi), \Xi(t,y,\xi))\, dt = \left\langle \mathbf{1}_{u<\mathcal{E}} f(s, -\frac{1}{\sqrt{2\left(\mathcal{E} - u\right)}}) \frac{1}{\sqrt{2\left(\mathcal{E} - u\right)}} \right\rangle,$$

$$\int_{t_1+\frac{t_0}{2}}^{t_1+t_0} f(Y(t,y,\xi), \Xi(t,y,\xi))\, dt = \left\langle \mathbf{1}_{u<\mathcal{E}} f(s, \frac{1}{\sqrt{2\left(\mathcal{E} - u\right)}}) \frac{1}{\sqrt{2\left(\mathcal{E} - u\right)}} \right\rangle.$$

Gathering all the terms, we are led to

$$P(f)(y,\xi) = \frac{\left\langle \mathbf{1}_{u<\mathcal{E}} \left[f\left(\cdot, \frac{1}{\sqrt{2(\mathcal{E}-u)}}\right) + f\left(\cdot, -\frac{1}{\sqrt{2(\mathcal{E}-u)}}\right)\right] \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle}{2\left\langle \mathbf{1}_{u<\mathcal{E}} \frac{1}{\sqrt{(\mathcal{E}-u)}} \right\rangle} \tag{14}$$

Expressions (13) and (14) are compatible with the ones in [7].

Let us now come back to the case when $N > 1$, and take a function $\varphi(y,\xi) = \varphi_1(y_1,\xi_1)\cdots\varphi_N(y_N,\xi_N)$, where each φ_i is periodic with period 1. We want to compute the limit

$$\frac{1}{T}\int_0^T \varphi_1(Y_1(t,y_1,\xi_1), \Xi_1(t,y_1,\xi_1))\cdots\varphi_N(Y_N(t,y_N,\xi_N), \Xi_N(t,y_N,\xi_N))\, dt.$$

In general, knowing the behavior of each trajectory (Y_i, Ξ_i) independently is not enough to compute such a product. However, here, we recall that each function $\varphi_i(Y_i(t,y_i,\xi_i), \Xi_i(t,y_i,\xi_i))$ $(1 \leq i \leq N)$ is periodic in time. The period depends only on $H_i(y_i,\xi_i)$ and on the function u_i. More precisely, setting

$$T_i(\mathcal{E}) := \sqrt{2}\int_0^1 \mathbf{1}_{u_i(z)<\mathcal{E}} \frac{1}{\sqrt{\mathcal{E} - u_i(z)}}\, dz \quad \forall \mathcal{E} > 0, \mathcal{E} \neq u_{\max},$$

$\varphi_i(Y_i(t,y_i,\xi_i), \Xi_i(t,y_i,\xi_i))$ is periodic in time with period $T_i(H_i(y_i,\xi_i))$.

We can thus use the following result :

Lemma 4.1. *Let $f_1, \cdots, f_N \in L^\infty(\mathbb{R})$ such that f_i is periodic with period θ_i, and set $\langle f_i \rangle = \frac{1}{\theta_i}\int_0^{\theta_i} f_i$.*

Assume that

$$\frac{k_1}{\theta_1} + \cdots + \frac{k_N}{\theta_N} \neq 0 \quad \forall (k_1, \cdots, k_N) \in \mathbb{Z}^N \setminus \{0\}. \tag{15}$$

Then as $T \to \infty$,

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\, dt \to \langle f_1 \rangle \cdots \langle f_N \rangle.$$

Sketch of proof. By density, it is enough to prove the lemma for $f_1, \cdot, f_N \in \mathcal{C}^\infty(\mathbb{R})$. Write f_i as a Fourier series (the series converges thanks to the regularity assumption), and use the fact that for all $\alpha \neq 0$,

$$\frac{1}{T}\int_0^T e^{i\alpha t}\, dt \to 0 \quad \text{as } T \to \infty.$$

$\square$

In the present setting, we deduce the following result :

Proposition 4.1. *Let $\varphi : (y,\xi) \mapsto \varphi_1(y_1,\xi_1)\cdots\varphi_N(y_N,\xi_N)$, where $\varphi_i \in L^\infty_{per}(\mathbb{R}_y \times \mathbb{R}^\xi)$.*

Let $(y,\xi) \in [0,1)^N \times \mathbb{R}^N$, and let $\theta_i = \theta_i(y_i,\xi_i) = T_i(H_i(y_i,\xi_i))$ for $1 \le i \le N$. Assume that $(\theta_1,\cdots,\theta_N)$ satisfy condition (15). *Then*

$$P(\varphi)(y,\xi) = P_1(\varphi_1)(y_1,\xi_1)\cdots P_N(\varphi_N)(y_N,\xi_N) \tag{16}$$

where each P_i is the projection in dimension 1 with potential u_i, given by expressions (13) *and* (14).

In particular, when the set

$$\{(y,\xi) \in [0,1]^N \times \mathbb{R}^N;\ (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N)) \text{ satisfy condition (15)}\}$$

has zero Lebesgue measure, equality (16) holds almost everywhere. It can then be generalized to arbitrary functions $\varphi \in L^\infty_{\text{per}}(\mathbb{R}^N \times \mathbb{R}^N)$ (always by linearity and density). The correct expression of the projection P is then

$$P = P_1 \circ P_2 \circ \cdots \circ P_N, \tag{17}$$

where each projection P_i acts on the variables (y_i,ξ_i) only. Notice that all projections P_i thus commute with one another; hence the order in which they are taken is unimportant.

We wish to emphasize that on the open set $\{(y,\xi) \in \mathbb{R}^{2N}, \forall i \in \{1,\cdots,N\}\ H_i(y_i,\xi_i) > \max u_i\}$, the expression (17) is true. Indeed, the function T_i is strictly decreasing on $(\max u_i, +\infty)$, and thus the set

$$\{(\mathcal{E}_1,\cdots,\mathcal{E}_N) \in \mathbb{R}^N;\ \mathcal{E}_i > \max u_i \text{ and } \exists k \in \mathbb{Z}^N \setminus \{0\},\ k_1/T_1(\mathcal{E}_1) + \cdots + k_N/T_N(\mathcal{E}_N) = 0\}$$

is countable. As a consequence, the set

$$\{(y,\xi) \in \mathbb{R}^{2N}, H_i(y_i,\xi_i) > \max u_i\ \forall i \text{ and } (\theta_1(y_1,\xi_1),\cdots,\theta_N(y_N,\xi_N)) \text{ satisfy condition (15)}\}$$

has zero Lebesgue measure.

However, let us mention here that in general, condition (15) cannot be relaxed : indeed, assume for instance that $u_i = u_j := u$ for $i \neq j$ and assume that the function u is such that

$$\exists y_0 > 0, \quad u(y) = y^2 \text{ for } |y| < y_0,$$

and $u(y) > y_0^2$ if $y \in [-\frac{1}{2},\frac{1}{2}] \setminus [-y_0,y_0]$.

Then if $|\mathcal{E}| \le \sqrt{y_0}$, we have

$$T(\mathcal{E}) = \int_{-\sqrt{\mathcal{E}}}^{\sqrt{\mathcal{E}}} \frac{1}{\sqrt{\mathcal{E}-y^2}}\, dy = 2\int_0^1 \frac{1}{\sqrt{1-z^2}}\, dz =: T_0$$

Thus, if $H_i(y_i,\xi_i) \le \sqrt{y_0}$, then $(Y_i,\Xi_i)(t,y_i,\xi_i)$ is periodic with period T_0. Notice that T_0 does not depend on the energy $H_i(y_i,\xi_i)$

In that case, the function $\varphi(Y(t),\Xi(t))$ is also periodic with period T_0. Thus we have to compute the limit of

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\, dt$$

as $T \to \infty$, where the f_i are arbitrary functions with period T_0. It is then easily proved that

$$\frac{1}{T}\int_0^T f_1(t)\cdots f_N(t)\,dt \to \sum_{\substack{k \in \mathbb{Z}^N,\\ k_1+\cdots+k_N=0}} a_{1,k_1}\cdots a_{N,k_N} \tag{18}$$

where

$$a_{j,l} = \frac{1}{T_0}\int_0^{T_0} f_j(t)e^{-\frac{2il\pi t}{T_0}}\,dt, \quad 1 \le j \le N,\ l \in \mathbb{Z}.$$

In general, the right-hand side of (18) differs from $a_{1,0}\cdots a_{N,0}$, and thus

$$P \neq P_1 \circ \cdots \circ P_N$$

for (y,ξ) in a neighbourhood of the origin.

4.2 Stationary ergodic setting

In the stationary ergodic setting, some of the expressions or properties above are no longer true. The most significant difference occurs when the energy $H(y,\xi) < u_{\max}$; indeed, in that case the particle is not necessarily trapped, depending on the profile of the potential u. Hence, in the rest of the subsection, we focus on the case $H(y,\xi) > u_{\max}$. In that case, the movement of the particle is unbounded and has many similarities with the periodic case. In particular, the particle sees "all the potential" during its evolution, and this will be fundamental in the use of the ergodic theorem.

4.2.1 Expression of $\xi^\sharp(y,\xi,\omega)$

This paragraph is devoted to the proof of proposition 1.1 in the stationary ergodic setting. We refer for instance to [12] for conditions on the existence of correctors for all $P \in \mathbb{R}$ in the case of a general coercive hamiltonian. In the present case, there exist correctors if

$$\{y \in \mathbb{R};\ u(y,\omega) = u_{\max}\} \neq \emptyset$$

a.s. in $\omega \in \Omega$.

Remark 4.1. *We wish to point out that the expressions in the periodic and in the stationary ergodic case when $H(y,\xi,\omega) > u_{max}$ are exactly the same (compare proposition 1.1 and the end of paragraph 4.1.1). This expression, and more precisely, the equality $\xi^\sharp = \bar{H}'(P)$ for some P, is in fact strongly linked to Aubry-Mather theory. Indeed,*

$$\xi^\sharp(y,\xi,\omega) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt = -\lim_{T\to\infty}\frac{Y(T,y,\xi,\omega)-y}{T},$$

and $\xi^\sharp(y,\xi,\omega)$ is thus (up to a multiplication by -1) the rotation number associated to the Hamiltonian flow starting at (y,ξ). The interested reader should compare our proposition 1.1 to lemma 2.8 in [5] or theorem 4.1 in [6], and our proof to the ones in these articles. We refer to [5, 6] for further references to Aubry-Mather theory and its applications to partial differential equations.

Proof of proposition 1.1. In all the proof, we fix y,ξ,ω such that $H(y,\xi,\omega) > u_{\max}$, and we set $P = P(y,\xi,\omega)$. Let $Q \in \mathbb{R}$ be arbitrary, and let v be a corrector, i.e.

$$H(y, Q + \nabla_y v(y,\omega), \omega) = \bar{H}(Q).$$

Then according to the theory of viscosity solutions, for all $(y,\xi,\omega) \in \mathbb{R}\times\mathbb{R}\times\Omega$, for all $T>0$,

$$v(y,\omega) \le v(Y(T,y,\xi,\omega)) + \int_0^T L(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\,dt + Q\,[Y(T,y,\xi,\omega)-y] + \bar{H}(Q)T. \tag{19}$$

Hence

$$\frac{1}{T}\int_0^T L(Y(t,y,\xi,\omega),\Xi(t,y,\xi,\omega),\omega)\,dt \geq \frac{v(y)-v(Y(T,y,\xi,\omega))}{T} - Q\frac{Y(T,y,\xi,\omega)-y}{T} - \bar{H}(Q)T \quad (20)$$

and

$$\frac{1}{2}|\dot{Y}(t,y,\xi,\omega)| = H(y,\xi,\omega) - u \geq H(y,\xi,\omega) - u_{\max} > 0 \quad \forall t > 0.$$

Thus there exist constants $\alpha, \beta > 0$ depending only on $H(y,\xi,\omega)$ and $u_{\max}$, such that

$$0 < \alpha \leq \left|\frac{Y(T,y,\xi,\omega)-y}{T}\right| \leq \beta \quad \forall T > 0.$$

Consequently, $Y(T) \to \infty$ as $T \to \infty$ and

$$\frac{v(y)-v(Y(T,y,\xi,\omega))}{T} = \frac{v(y)-v(Y(T,y,\xi,\omega))}{Y(T,y,\xi,\omega)-y}\,\frac{Y(T,y,\xi,\omega)-y}{T} \to 0 \quad \text{as } T \to \infty.$$

(remember (6)).

On the other hand, as $T \to \infty$,

$$\frac{Y(T,y,\xi,\omega)-y}{T} = -\frac{1}{T}\int_0^T \Xi(t,y,\xi,\omega)\,dt \to -\xi^\sharp(y,\xi,\omega).$$

Hence, passing to the limit in (20), we derive

$$P(L)(y,\xi,\omega) \geq Q\xi^\sharp(y,\xi,\omega) - \bar{H}(Q) \quad \forall Q \in \mathbb{R}. \quad (21)$$

Thus

$$P(L)(y,\xi,\omega) \geq \bar{L}\left(\xi^\sharp(y,\xi,\omega)\right). \quad (22)$$

In order to prove the proposition, we have to find a special $Q_0 \in \mathbb{R}$ such that equality holds in (19). This will entail that

$$P(L)(y,\xi,\omega) = \sup_{Q\in\mathbb{R}}\left(Q\xi^\sharp(y,\xi,\omega) - \bar{H}(Q)\right) = \bar{L}\left(\xi^\sharp(y,\xi,\omega)\right),$$

and the sup is obtained for $\xi^\sharp(y,\xi,\omega) = \bar{H}'(Q_0)$.

Let us thus prove that with $Q = P = P(y,\xi,\omega)$, equality holds in (19).

First, notice that

$$v(y,\omega) := \operatorname{sgn}(P)\int_0^y \sqrt{2(\bar{H}(P)-u(z,\omega))}\,dz - Py$$

is a viscosity solution of

$$H(y, P + \nabla_y v, \omega) = \bar{H}(P),$$

and as $y \to \infty$

$$\frac{1}{y}\int_0^y \sqrt{2(\bar{H}(P)-u(z,\omega))}\,dz \to E\left[\sqrt{2(\bar{H}(P)-U)}\right].$$

By definition of $\bar{H}$,

$$E\left[\sqrt{2(\bar{H}(P)-U)}\right] = |P|;$$

consequently,

$$\frac{1}{1+|y|}\left(\operatorname{sgn}(P)\int_0^y \sqrt{2(\bar{H}(P)-u(z,\omega))}\,dz - Py\right) \to 0$$

as $y \to \infty$, a.s. in ω. Thus v satisfies (6), and $v \in L^\infty(\Omega;\mathcal{C}^1(\mathbb{R}^N))$. Thus the method of characteristics, for instance, can be used to prove that equality holds in (19), with (y,ξ) replaced by any couple $(y',\xi') = (y', P + \nabla_y v(y',\omega))$. We have to prove that we can take $(y',\xi') = (y,\xi)$. First, notice that

$$\nabla_y v(y',\omega) = \operatorname{sgn}(P)\sqrt{2(\bar{H}(P)-u(y',\omega))} - P,$$

and thus $\operatorname{sgn}(P + \nabla_y v(y',\omega)) = \operatorname{sgn}(\xi') = \operatorname{sgn}(P) = \operatorname{sgn}(\xi)$. Hence, take $y' = y$. Then $|\xi|^2 = |\xi'|^2$ because $H(y,\xi) = \bar{H}(P) = H(y,\xi')$ by definition of ξ'. Thus $\xi = \xi'$, and equality holds in (19).

□

4.3 Expression of the projection P

The same method as in the periodic case can be used in order to find the expression of the projection P when $H(y,\xi,\omega) =: \mathcal{E} > u_{\max}$; indeed, in that case, remember that

$$P(f)(y,\xi,\omega) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega), \omega\right)\, dt$$

and we can use the change of variables

$$dt = \frac{1}{\dot{Y}}\, dY = \frac{1}{-\mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(Y,\omega))}}\, dY$$

in order to obtain

$$\begin{aligned} &\int_0^T f\left(Y(t,y,\xi,\omega), \Xi(t,y,\xi,\omega), \omega\right)\, dt \\ = &\int_y^{Y(T,y,\xi,\omega)} f\left(z, \mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(z,\omega))}, \omega\right) \frac{1}{-\mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(z,\omega))}}\, dz. \end{aligned}$$

Since the group transformation (τ_x) is ergodic, and $Y(T)\to\infty$ as $T\to\infty$, for all $\mathcal{E} > u_{\max}$,

$$\begin{aligned} \frac{1}{Y(T)-y}\int_y^{Y(T,y,\xi,\omega)} f\left(z, \mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(z,\omega))}, \omega\right) \frac{1}{-\mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(z,\omega))}}\, dz \to \\ \to E\left[F\left(\mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(0,\omega))}, \omega\right) \frac{1}{-\mathrm{sgn}(\xi)\sqrt{2(\mathcal{E}-u(0,\omega))}}\right] \end{aligned}$$

Thus, we obtain

$$P(f)(y,\xi,\omega) = \xi^\sharp(y,\xi,\omega)\bar{f}(sgn(\xi), H(y,\xi,\omega)),$$

where

$$\bar{f}(\eta,\mathcal{E}) = E\left[F\left(\eta\sqrt{2(\mathcal{E}-u(0,\omega))}, \omega\right) \frac{1}{\eta\sqrt{2(\mathcal{E}-u(0,\omega))}}\right].$$

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