Hölder regularity for operator scaling stable random fields

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Abstract. We investigate the sample paths regularity of operator scaling α-stable random fields. Such fields were introduced in [6] as anisotropic generalizations of self-similar fields and satisfy the scaling property $\{X(c^E x); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}$ where E is a $d \times d$ real matrix and $H > 0$. In the case of harmonizable operator scaling random fields, the sample paths are locally Hölderian and their Hölder regularity is characterized by the eigen decomposition of $\mathbb{R}^d$ with respect to E. In particular, the directional Hölder regularity may vary and is given by the eigenvalues of E. In the case of moving average operator scaling random α-stable random fields, with $\alpha \in (0, 2)$ and $d \geq 2$, the sample paths are almost surely discontinous.

1. Introduction

In this paper we consider operator scaling stable random fields as introduced in [6]. More precisely, for E a real $d \times d$ matrix with positive real parts of the eigenvalues, a scalar valued random field $(X(x))_{x\in\mathbb{R}^d}$ is called *operator scaling* for E and $H > 0$ if for every $c > 0$

$$\{X(c^E x); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}, \tag{1}$$

where $\overset{(fdd)}{=}$ means equality of finite dimensional distributions and as usual $c^E = \exp(E \log c)$ with $\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$ is the matrix exponential. Let us remark that up to consider the matrix E/H, we may assume, without loss of generality, that $H = 1$.

These fields can be seen as anisotropic generalizations of self-similar random fields. Let us recall that a scalar valued random field $(X(x))_{x\in\mathbb{R}^d}$ is said to be H*-self-similar* with $H \in \mathbb{R}$ if

$$\{X(cx); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{c^H X(x); x \in \mathbb{R}^d\}$$

for every $c > 0$. Then a H-self-similar field is also an operator scaling field for the matrix $E = I_d/H$, where I_d is the identity matrix of size $d\times d$. Self-similar random fields are used in various applications such as internet traffic modelling [29], ground water modelling and mathematical finance, just to mention a few. Various examples can be found for instance in the books [18, 1, 27]. A very important class of such fields are given by Gaussian random fields and especially by the fractional Brownian field B_H, where $H \in (0, 1)$ is the so-called Hurst parameter. The random field B_H is H-self-similar and has stationary increments, e.g. $\{B_H(x + h) - B_H(h); x \in \mathbb{R}^d\} \overset{(fdd)}{=} \{B_H(x); x \in \mathbb{R}^d\}$ for any $h \in \mathbb{R}^d$. It is an isotropic generalization of the famous fractional Brownian motion, implicitely introduced in [12] and defined in [19].

Nevertheless, Gaussian random fields are not convenient for some heavy tails phenomena modelling. For this purpose, α-stable random fields have been introduced. A scalar valued random field $\{X(x); x \in \mathbb{R}^d\}$ is said to be symmetric α-stable (SαS), for $\alpha \in (0, 2)$, if any linear combination $\sum_{k=1}^{n} a_k X(x_k)$ is SαS. We address to the book [27] for a well understanding of such fields. The self-similar α-stable fields with stationary increments have been extensively used to propose alternative to Gaussian modelling (see [23, 29] for instance) and are isotropic.

However, certain applications (see, e.g., [5, 7] and references therein) require that the random field is anisotropic and satisfies a scaling relation. This scaling relation should have different Hurst

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indices in different directions and these directions should not necessarily be orthogonal. To reach this goal the authors of [6] propose a new class of random fields with an anisotropic behavior driven by a $d \times d$ matrix E. More precisely, they introduce α-stable random fields which have stationary increments and satisfy the operator scaling property (1). Two different classes of such fields are defined and analyzed, using a moving average representation as well as an harmonizable one. In the Gaussian case $\alpha = 2$, according to [6] there exist modifications of these fields which are almost surely Hölder-continuous of certain indices. We give similar results here in the stable case $\alpha \in (0, 2)$ for harmonizable operator scaling stable random fields. Actually, we obtain their critical global and directional Hölder exponents, which are given by the eigenvalues of E. In general, such fields are anisotropic and their sample paths properties varies with the direction. In particular, in the case where E is diagonalizable, for any eigenvector θ_j associated with the real eigenvalue λ_j, harmonizable operator scaling stable random field admits $H_j = 1/\lambda_j$ as critical Hölder exponent in direction θ_j. Let us point out that we establish an accurrate upper bound for the modulus of continuity. Such upper bound has already been given in the case of real harmonizable fractional stable motions, which are self-similar, in [14] and in the case of some Gaussian random processes in [13]. Then, in this paper, we generalize these results to any dimension d and any harmonizable operator scaling stable fields. From this upper bound, we can deduce the Hölder sample paths regularity. Let us point out that we also obtain this upper bound in the case of Gaussian operator scaling random field, which improves the sample paths properties establishes in [6].

Furthermore, whereas in the Gaussian case $\alpha = 2$, moving average and harmonizable fields have the same kind of regularity properties, this is no more true in the case $\alpha \in (0, 2)$. In particular, we show that for $d \geq 2$, a moving average operator scaling stable random field does not admit any continuous modification. Remark that if $d = 1$, the sample paths regularity properties are already known since the processes studied are self-similar moving average stable processes, see for example [27, 14, 28].

One of the main tool for the study of sample paths of operator scaling random fields is the change of polar coordinates with respect to the matrix E introduced in [22] and used in [6]. For X a Gaussian operator scaling random field with stationary increments, using (1), we can write the variogramme of X as

$$v^2(x) = \mathbb{E}\big(X^2(x)\big) = \tau_E(x)^{2H}\mathbb{E}\big(X^2(\ell_E(x))\big),$$

where $\tau_E(x)$ is the radial part of x with respect to E and $\ell_E(x)$ is its polar part. Therefore, in the Gaussian case, the sample paths regularity depends on the behavior of the polar coordinates $(\tau_E(x), \ell_E(x))$ around $x = 0$. Such property also holds in the stable case $\alpha \in (0, 2)$. The Hölder regularity of the sample paths follows from estimates of $\tau_E(x)$ compared to $\|x\|$. These estimates are given in Section 3 and their proof are postponed to the Appendix.

Furthermore, to study the sample paths in the stable case, the other main tool we use is a series representation of harmonizable operator scaling α-stable random fields. Such representations in series of infinitely divisible laws have been studied in [17, 16, 25, 24]. As in [14], our study is based on a LePage series representation. Actually, the main idea is to choose a representation which is a conditionnally Gaussian series.

In Section 2, we recall the definition of harmonizable operator scaling random fields. Then, Sections 3 and 4 are devoted to the main tools we need for the study of the sample paths of these fields. More precisely, Section 3 deals with the polar coordinates with respect to a matrix E and Section 4 gives the LePage series representation. In Section 5, the sample paths properties and the Hausdorff dimension of the graph are given. Section 6 is concerned with moving average operator scaling random fields.

2. Harmonizable representation

Let us recall the definition of harmonizable operator scaling stable random fields, given by [6].

Let E be a real $d \times d$ matrix. Let $\lambda_1, \dots, \lambda_d$ be the complex eigenvalues of E and $a_j = \Re(\lambda_j)$ for each $j = 1, \dots, d$. We assume that

$$\min_{1 \le j \le d} a_j > 1. \tag{2}$$

Let $\psi : \mathbb{R}^d \to [0, \infty)$ be a continuous, E^t-homogeneous function, which means according to Definition 2.6 of [6] that

$$\psi(c^{E^t} x) = c\psi(x) \text{ for all } c > 0.$$

We assume moreover that $\psi(x) \neq 0$ for $x \neq 0$. Such functions were studied in detail in [22], Chapter 5 and various examples are given in Theorem 2.11 and Corollary 2.12 of [6].

Let $0 < \alpha \le 2$ and $W_\alpha(d\xi)$ be a complex isotropic α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure (see [27] p.281). If $\alpha = 2$, $W_\alpha(d\xi)$ is a complex isotropic Gaussian random measure. Let $q = \operatorname{trace}(E)$.

Definition 2.1. *Since* (2) *is fulfilled, the random field*

$$X_\alpha(x) = \Re \int_{\mathbb{R}^d} \left(e^{i\langle x, \xi \rangle} - 1\right) \psi(\xi)^{-1-q/\alpha}\, W_\alpha(d\xi) \ , x \in \mathbb{R}^d, \tag{3}$$

is well defined and called harmonizable operator scaling stable random field.

From Theorem 4.1 and Corollary 4.2 of [6], X_α is stochastically continuous, has stationary increments and satisfies the following operator scaling property

$$\forall c > 0, \left\{X_\alpha(c^E x); x \in \mathbb{R}^d\right\} \stackrel{(fdd)}{=} \left\{cX_\alpha(x); x \in \mathbb{R}^d\right\}. \tag{4}$$

For notational sake of simplicity we denote the kernel function by

$$f(x, \xi) = \left(\mathrm{e}^{i\langle x, \xi \rangle} - 1\right) \psi(\xi)^{-1-q/\alpha}. \tag{5}$$

Let us recall that $f(x, \cdot) \in L^\alpha(\mathbb{R}^d)$ for any $x \in \mathbb{R}^d$, which is a necessary and sufficient condition for X_α to be defined.

Now, let us give some examples of such random fields.

Example 2.1. Let I_d be the identity matrix of size $d \times d$, $E = I_d/H$ (with $0 < H < 1$) and $\psi(x) = \|x\|^H$ with $\|\cdot\|$ the Euclidean norm. Then the random field defined by (3) is a real harmonizable stable random field (see [27] for details on such fields). In this case, X_α is self-similar with exponent H, i.e.

$$\forall c > 0, \{X_\alpha(cx); x \in \mathbb{R}^d\} \stackrel{(fdd)}{=} \{c^H X_\alpha(x); x \in \mathbb{R}^d\}.$$

Let us quote that, if $\alpha = 2$, X_α is a fractional Brownian field and its critical Hölder exponent is given by its Hurst index H (see Theorem 8.3.2 of [2] for instance).

Example 2.2. Assume that E is diagonalizable. Then, all the eigenvalues of E are real given by $(a_j)_{1 \le j \le d}$. Let $(\theta_j)_{1 \le j \le d}$ be a basis of some corresponding eigenvectors and consider the function ψ defined by

$$\psi(x) = \left(\sum_{j=1}^d |\langle x, \theta_j \rangle|^{2/a_j}\right)^{1/2}, \ x \in \mathbb{R}^d.$$

The function ψ is clearly continuous and non negative on $\mathbb{R}^d$. Moreover, since $\langle c^{E^t} x, \theta_j \rangle = \langle x, c^E \theta_j \rangle = c^{a_j} \langle x, \theta_j \rangle$, it is also E^t-homogeneous. Finally, since $(\theta_j)_{1 \le j \le d}$ is a basis of $\mathbb{R}^d$ we have that $\psi(x) = 0$ if and only if $x = 0$. Then we can define X_α by (3) and in this case the operator scaling property (4) implies that

$$\forall j = 1, \dots, d, \ \forall c > 0, \ \{X_\alpha(ct\theta_j); t \in \mathbb{R}\} \stackrel{(fdd)}{=} \left\{c^{1/a_j} X_\alpha(t\theta_j); t \in \mathbb{R}\right\}.$$

The random field X_α is self-similar with Hurst index $H_j = 1/a_j$ in the direction θ_j. In particular, in the Gaussian case ($\alpha = 2$), $(X_2(t\theta_j))_{t \in \mathbb{R}}$ is a fractional Brownian motion with Hurst index H_j. Then, in this case, its critical Hölder exponent is equal to H_j.

The main tool in the study of operator scaling random fields is the change of coordinates in a kind of polar coordinates with respect to the matrix E. Then, before we study the sample paths of X_α, we recall in the next section the definition of these coordinates and give some estimates of the radial part.

3. Polar coordinates

According to Chapter 6 of [22], since E is a real $d\times d$ matrix with positive real parts of the eigenvalues there exists a norm $\|\cdot\|_E$ on $\mathbb{R}^d$ such that for the unit sphere $S_E=\{x\in\mathbb{R}^d:\|x\|_E=1\}$ the map

$$\begin{array}{rccc}\Psi_E: & (0,\infty)\times S_E & \longrightarrow & \mathbb{R}^d\setminus\{0\}\\ & (r,\theta) & \longmapsto & r^E\theta\end{array}$$

is a homeomorphism. Hence we can write any $x\in\mathbb{R}^d\backslash\{0\}$ uniquely as $x=\tau_E(x)^E\ell_E(x)$ for some *radial part* $\tau_E(x)>0$ and some direction $\ell_E(x)\in S_E$ such that $x\mapsto\tau_E(x)$ and $x\mapsto\ell_E(x)$ are continuous. Observe that $S_E=\{x\in\mathbb{R}^d:\tau_E(x)=1\}$ is compact and set

$$m_E=\min_{S_E}\|x\| \quad\text{and}\quad M_E=\max_{S_E}\|x\|. \tag{6}$$

We know that $\tau_E(x)\to\infty$ as $x\to\infty$ and $\tau_E(x)\to 0$ as $x\to 0$. Hence we can extend $\tau_E(\cdot)$ continuously by setting $\tau_E(0)=0$.
Let us recall the formula of integration in *polar coordinates* established in [6].

Proposition 3.1. *There exists a unique finite Radon measure σ_E on S_E such that for all $f\in L^1(\mathbb{R}^d,dx)$ we have*

$$\int_{\mathbb{R}^d} f(x)\,dx=\int_0^\infty\int_{S_E} f(r^E\theta)\,\sigma_E(d\theta)\,r^{q-1}\,dr.$$

In the Gaussian case ($\alpha=2$), the variogramme of X_2 can be rewritten as follows

$$v^2(x)=\tau_E(x)^2\mathbb{E}\left(X_2\left(\ell_E(x)\right)^2\right),$$

using the operator scaling property (4). Then, the Hölder regularity of X_2 follows from estimates of $\tau_E(x)$ compared to $\|x\|$ around $x=0$, e.g. the Hölder regularity of τ_E around 0, see [6]. Then, in order to get an upper bound for the modulus of continuity (for any α), we need precise estimates of $\tau_E(x)$.

As done in [21] for the study of operator-self-similar Gaussian random fields we use the Jordan decomposition of the matrix E to get estimates of τ_E. From the Jordan decomposition's theorem (see [10] p. 129 for instance), there exists a real invertible $d\times d$ matrix P such that $D=P^{-1}EP$ is of the real canonical form, which means that D is composed of diagonal blocks which are either Jordan cell matrix of the form

$$\begin{pmatrix}\lambda & 1 & \dots & 0\\ 0 & \lambda & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 0 & \dots & 0 & \lambda\end{pmatrix}$$

with λ a real eigenvalue of E or blocks of the form

$$\begin{pmatrix}\Lambda & I_2 & 0 & \dots & 0\\ 0 & \Lambda & I_2 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & \ddots & I_2\\ 0 & \dots & \dots & 0 & \Lambda\end{pmatrix}\ \text{with } \Lambda=\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\ \text{and } I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{7}$$

where the complex numbers $a\pm ib$ ($b\neq 0$) are complex conjugated eigenvalues of E.

Let us denote by $\|\cdot\|$ the subordinated norm of the Euclidean norm on the matrix space. Precise estimates of τ_E follow from the next lemma.

Lemma 3.2. *Let J be either a Jordan cell matrix of size l or a block of the form (7) of size $2l$ associated with the eigenvalue $\lambda = a + ib$. Then, for any $t \in (0, e^{-1}] \cup [e, +\infty)$*

$$t^a \le \|t^J\| \le \sqrt{2l}\, e\, t^a \, |\log t|^{l-1} .$$

Proof. see the Appendix. □

Let us be more precise on the Jordan decomposition of E. Let us recall that the eigenvalues of E are denoted by λ_j, $j = 1 \dots d$ and that $a_j = \Re(\lambda_j) > 1$ for $j = 1, \dots, d$. There exist $J_1, \dots, J_m$, where each J_j is either a Jordan cell matrix or a block of the form (7), and P a real $d \times d$ invertible matrix such that

$$E = P \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & J_m \end{pmatrix} P^{-1}.$$

We can assume that each J_j is associated with the eigenvalue λ_j of E and that

$$1 < a_1 \le \dots \le a_m.$$

We also set $H_j = a_j^{-1}$ and have

$$0 < H_m \le \dots \le H_1 < 1. \tag{8}$$

If $\lambda_j \in \mathbb{R}$, J_j is a Jordan cell matrix of size $\tilde{l_j} = l_j \in \mathbb{N}\backslash\{0\}$. If $\lambda_j \in \mathbb{C}\backslash\mathbb{R}$, J_j is a block of the form (7) of size $\tilde{l_j} = 2l_j \in 2\mathbb{N}\backslash\{0\}$. Then for any $t > 0$,

$$t^E = P \begin{pmatrix} t^{J_1} & 0 & \dots & 0 \\ 0 & t^{J_2} & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & t^{J_m} \end{pmatrix} P^{-1}$$

We denote by $(e_1, \dots, e_d)$ the canonical basis of $\mathbb{R}^d$ and set $f_j = Pe_j$ for every $j = 1, \dots, d$. Hence, $(f_1, \dots, f_d)$ is a basis of $\mathbb{R}^d$. For all $j = 1, \dots, m$, let

$$W_j = \mathrm{Vect}\left(f_k \; ; \; \sum_{i=1}^{j-1} \tilde{l_i} + 1 \le k \le \sum_{i=1}^{j} \tilde{l_j} \right). \tag{9}$$

Then, each W_j is a E-invariant set and $E = \bigoplus_{j=1}^m W_j$.

The following result gives bounds on the growth rate of $\tau_E(x)$ in terms of the real parts of the eigenvalues of E.

Proposition 3.3. *For any $r \in (0,1)$ there exist $c_1, c_2 > 0$ such that for every $1 \le j_0 \le j \le m$,*

$$c_1 \|x\|^{H_{j_0}} \, |\log \|x\||^{-(p_{j_0,j}-1)H_{j_0}} \le \tau_E(x) \le c_2 \|x\|^{H_j} \, |\log \|x\||^{(p_{j_0,j}-1)H_j}$$

holds for any $x \in \oplus_{k=j_0}^{j} W_k$ with $\|x\| \le r$ and $p_{j_0,j} = \max\limits_{j_0 \le k \le j} l_k$.

Proof. See the Appendix. □

Then, we easily deduce the following corollary.

Corollary 3.4. *For any $r \in (0,1)$ there exist $c_1, c_2 > 0$ such that for any $x \in W_j$ with $\|x\| \le r$*

$$c_1 \|x\|^{H_j} \, |\log \|x\||^{-(l_j-1)H_j} \le \tau_E(x) \le c_2 \|x\|^{H_j} \, |\log \|x\||^{(l_j-1)H_j} \tag{10}$$

and for any $x \in \mathbb{R}^d$ with $\|x\| \le r$

$$c_1 \|x\|^{H_1} \, |\log \|x\||^{-(l-1)H_1} \le \tau_E(x) \le c_2 \|x\|^{H_m} \, |\log \|x\||^{(l-1)H_m} \tag{11}$$

where $l = \max_{1\leq j\leq m} l_j$.

Therefore we have precise estimates for the Hölder regularity of the radial part. Let us remark that we improve the first statement of Lemma 2.1 of [6] and that the second one can also be improved in a similar way. From these estimates we deduce the Hölder regularity of X_α in Section 5. As already mentionned, the study of the sample paths is based on a series representation. Then, before we state regularity properties, it remains to give the LePage series representation of harmonizable operator scaling random fields X_α defined by (3).

4. Representation as a LePage series

An overview on series representations of infinitely divisible random variable without Gaussian part can be found for example in [24, 26] and references therein. In particular, LePage series representation ([17, 16]) have been used in [15, 14] to study the sample paths regularity of some self-similar α-stable random motions with $\alpha \in (0,2)$. Here, this representation is also the main representation we use in the case $\alpha \in (0,2)$. Actually, in the Gaussian case $\alpha = 2$, such representation does not hold.

Let us now introduce some notations that will be used throughout the paper. Let μ be an arbitrary probability measure equivalent to the Lebesgue measure on $\mathbb{R}^d$ and let m be its Radon-Nikodym derivative that is $\mu(d\xi) = m(\xi)d\xi$.

Notation Let $(T_n)_{n\geq1}$, $(g_n)_{n\geq1}$ and $(\xi_n)_{n\geq1}$ be independent sequences of random variables.

- T_n is the nth arrival time of a Poisson process with intensity 1.
- $(g_n)_{n\geq1}$ is a sequence of i.i.d. isotropic complex random variables so that $g_n \stackrel{(d)}{=} \mathrm{e}^{i\theta} g_n$ for any $\theta \in \mathbb{R}$. We also assume that $0 < \mathbb{E}(|g_n|^\alpha) < +\infty$.
- $(\xi_n)_{n\geq1}$ is a sequence of i.i.d. random variables with common law $\mu(d\xi) = m(\xi)d\xi$.

According to Chapter 3 and Chapter 4 of [27], stochastic integrals with respect to an α-stable random measure Λ can be represented as a LePage series as soon as the control measure of Λ is a finite measure. The next proposition generalizes this to W_α whose control measure is the Lebesgue measure. It is a consequence of Lemma 4.1 of [15], which is a correction of Lemma 1.4 of [20]. This proposition can also be deduced from [24, 25], concerned with series representations of stochastic integrals with respect to infinitely divisible random measures.

Proposition 4.1. *Assume that $\alpha \in (0,2)$. Then, for every complex valued function $h \in L^\alpha(\mathbb{R}^d)$, the series*

$$Y^h = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} h(\xi_n) g_n \tag{12}$$

converges almost surely. Furthermore,

$$C_\alpha Y^h \stackrel{(d)}{=} \int_{\mathbb{R}^d} h(\xi) W_\alpha(d\xi)$$

with

$$C_\alpha = \mathbb{E}(|\Re(g_1)|^\alpha)^{-1/\alpha} \left(\frac{1}{2\pi}\int_0^\pi |\cos{(x)}|^\alpha dx\right)^{1/\alpha} \left(\int_0^{+\infty} \frac{\sin{(x)}}{x^\alpha} dx\right)^{-1/\alpha}. \tag{13}$$

Remark 4.1. *According to Proposition 4.1, taking $\alpha \in (0,2)$, the random measure*

$$\Lambda_\alpha(d\xi) = C_\alpha \sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} g_n \delta_{\xi_n}(d\xi)$$

is a complex isotropic α-stable random measure.

Proof. Let $V_n = m(\xi_n)^{-1/\alpha} h(\xi_n) g_n$. Then, V_n, $n \geq 1$, are i.i.d. isotropic complex random variables. By Lemma 4.1 in [15], Y^h converges almost surely and

$$\forall z \in \mathbb{C},\, \mathbb{E}\Big(\exp\Big(i\Re\Big(\bar{z}Y^h\Big)\Big)\Big) = \exp\left(-\sigma^\alpha |z|^\alpha\right)$$

with

$$\sigma^\alpha = \mathbb{E}(|\Re(V_1)|^\alpha) \int_0^{+\infty} \frac{\sin(x)}{x^\alpha} dx.$$

Since g_1 is invariant by rotation and independent with ξ_1,

$$\mathbb{E}(|\Re(V_1)|^\alpha) = \mathbb{E}\Big(m(\xi_1)^{-1}|h(\xi_1)|^\alpha\Big)\mathbb{E}(|\Re(g_1)|^\alpha) = \mathbb{E}(|\Re(g_1)|^\alpha) \int_{\mathbb{R}^d} |h(\xi)|^\alpha d\xi.$$

Moreover, by definition of an isotropic α-stable random stable measure (see [27]),

$$\forall z \in \mathbb{C}, \mathbb{E}\left(\exp\left(i\Re\left(\bar{z}\int_{\mathbb{R}^d} h(\xi)M(d\xi)\right)\right)\right) = \exp\left(-c_\alpha^\alpha(h)|z|^\alpha\right)$$

with $c_\alpha^\alpha(h) = \left(\frac{1}{2\pi}\int_0^\pi |\cos(x)|^\alpha dx\right) \int_{\mathbb{R}^d} |h(\xi)|^\alpha d\xi$. Therefore, we have

$$C_\alpha Y^h \overset{(d)}{=} \int_{\mathbb{R}^d} h(\xi) W_\alpha(d\xi)$$

where C_α is defined by (13), which concludes the proof. □

From the previous proposition, we deduce the following statement which is the main series representation we use in our investigation.

Proposition 4.2. *Let $\alpha \in (0,2)$. For every $x \in \mathbb{R}^d$, the series*

$$Y_\alpha(x) = C_\alpha \Re\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} f(x,\xi_n) g_n\right), \tag{14}$$

where C_α is defined by (13)*, converges almost surely. Furthermore,*

$$\left\{Y_\alpha(x) : x \in \mathbb{R}^d\right\} \overset{(fdd)}{=} \left\{X_\alpha(x) : x \in \mathbb{R}^d\right\}.$$

Proof. From Proposition 4.1, for any $x \in \mathbb{R}^d$, the convergence of the series follows from the fact that $f(x,\cdot) \in L^\alpha(\mathbb{R}^d)$. The equality in distribution between X_α and Y_α is obtained by linearity of both fields. □

Using LePage representation (14) of X_α and the estimates given in Section 3, we give an upper bound for the modulus of continuity of X_α and obtain the critical Hölder regularity of its sample paths in the next section.

5. Hölder regularity and Hausdorff dimension

Throughout this section we fix K a compact set of $\mathbb{R}^d$ and investigate the Hölder regularity of the sample paths of X_α on K, with X_α the harmonizable operator scaling stable random field defined by (3).

Let us recall that for the Gaussian case $\alpha = 2$, according to Theorem 5.4 of [6], the Hölder regularity of X_2 depends on the subspaces $(W_j)_{1\leq j\leq m}$ defined by (9) and associated to the eigenvalues of E. More precisely, Theorem 5.4 of [6] implies that, when restricted to the subspace W_j, the Gaussian random field $\{X_2(x); x \in W_j\}$ admits H_j as critical Hölder exponent. This follows from the fact that the regularity of X_2 is determined by the regularity of τ_E around 0, which is given by H_j according to (10). Here, we give an upper bound for the modulus of continuity of X_α in the general case $\alpha \in (0,2]$. Then we prove that the critical Hölder exponents are the same than in the Gaussian case $\alpha = 2$. Let us state our main result.

Theorem 5.1. *Let $\alpha \in (0,2)$. There exists a modification X_α^* of X_α on K such that*

$$\lim_{\delta \downarrow 0} \sup_{\substack{x,y \in K \\ \|x-y\| \leq \delta}} \frac{|X_\alpha^*(x) - X_\alpha^*(y)|}{\tau_E(x-y)|\log \tau_E(x-y)|^{1/\alpha+1/2+\varepsilon}} = 0 \ a.s. \tag{15}$$

for any $\varepsilon > 0$.

This result was proved in the case of harmonizable self-similar processes in [14], e.g. in the case of Example 2.1 with $d = 1$. The main idea is to use the LePage series representation (14) where g_n, $n \geq 1$, are Gaussian complex isotropic random variables. Furthermore, it remains to choose the density distribution m of the ξ_n. In [14], the authors choose

$$m(\xi) = \frac{c_\eta}{|\xi||\log|\xi||^{1+\eta}},\ \xi \in \mathbb{R}\backslash\{0\}$$

where $c_\eta > 0$. A straightforward generalization in higher dimension d leads to choose

$$m(\xi) = \frac{c_\eta}{\|\xi\|^d|\log\|\xi\||^{1+\eta}},\ \xi \in \mathbb{R}^d\backslash\{0\}.$$

Remark that in this case (e.g. Example 2.1) the matrix $E = I_d/H = E^t$ and that we can choose $\|\cdot\|_{E^t} = \|\cdot\|$. Then, using classical polar coordinates, we obtain that for $x \neq 0$,

$$\tau_{E^t}(x) = \|x\|^H \text{ and } \ell_{E^t}(x) = \frac{x}{\|x\|}$$

and therefore

$$m(\xi) = \frac{c_\eta}{\tau_{E^t}(\xi)^q|\log \tau_{E^t}(\xi)|^{1+\eta}}$$

since $q = \text{trace}(E) = d/H$. Then we choose this density in the general case. The advantage is that m only depends on the radial part τ_{E^t}.

Proof of Theorem 5.1. We can assume without loss of generality that $K = [0,1]^d$. According to Proposition 4.2 for every $x \in \mathbb{R}^d$

$$Y_\alpha(x) = C_\alpha \Re\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} f(x,\xi_n) g_n\right)$$

converges almost surely and $Y_\alpha \overset{(fdd)}{=} X_\alpha$. As already mentionned, we assume that g_n, $n \geq 1$ are Gaussian complex isotropic random variables and that the density distribution of ξ_n is defined by

$$m(\xi) = \frac{c_\eta}{\tau_{E^t}(\xi)^q|\log \tau_{E^t}(\xi)|^{1+\eta}},\ \xi \in \mathbb{R}^d\backslash\{0\},$$

where $\eta > 0$ and c_η is chosen such that $\int_{\mathbb{R}^d} m(\xi)d\xi = 1$. In particular, conditionally to $(T_n, \xi_n)_n$, $Y_\alpha(x)$ is a real-valued Gaussian random variable.

As in the proof of Kolmogorov-Centsov Theorem (see [11]), we first prove that almost surely for $\tau_E(x_k - x_{k'})$ small enough,

$$|Y_\alpha(x_k) - Y_\alpha(x_{k'})| \leq C\tau_E(x_k - x_{k'})|\log \tau_E(x_k - x_{k'})|^{1/\alpha+1/2+\varepsilon}.$$

where $(x_k)_k$ is some countable dense sequence of K. Then, X_α satisfies the same property. Finally, we give a modification X_α^* of X_α for which (15) holds. In the first step, we construct the sequence $(x_k)_k$ we use.

Step 1. For $k \in \mathbb{N}\backslash\{0\}$ let us choose $\nu_k \in \mathbb{N}\backslash\{0\}$ the smallest integer such that

$$c_2 d^{H_m/2} 2^{-\nu_k H_m} |\nu_k \log 2|^{(l-1)H_m} \leq 2^{-k},$$

where c_2 and l are given by Corollary 3.4 for $r = 1/2$. Remark that by definition, $\nu_k \leq \nu_{k+1}$. Up to change c_2 in Proposition 3.4, we can assume that

$$c_2 d^{H_m/2} 2^{-H_m} |\log 2|^{(l-1)H_m} > 1,$$

which implies that $\nu_k > 1$ for every k. Then, since $1 \leq \nu_k - 1 \leq \nu_k$,

$$c_2 d^{H_m/2} 2^{-(\nu_k - 1)H_m} |(\nu_k - 1)\log 2|^{(l-1)H_m} > 2^{-k}$$

and we have

$$2^{-k}\left(2\sqrt{d}\right)^{-H_m} c_2^{-1} < \left(2^{-\nu_k}\left(\nu_k \log 2\right)^{l-1}\right)^{H_m} \leq 2^{-k}\left(\sqrt{d}\right)^{-H_m} c_2^{-1}. \tag{16}$$

Let us remark that $(\nu_k)_{k\geq 1}$ is an increasing sequence and then that $\nu_k \geq k$ for every k. Furthermore, taking the logarithm of (16), one easily proves that

$$\lim_{k\to+\infty} \frac{k}{\nu_k} = H_m,$$

which implies $\nu_k = O(k)$.

For every $k \in \mathbb{N}\backslash\{0\}$ and $j = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ we set

$$x_{k,j} = \frac{j}{2^{\nu_k}},\ \mathcal{D}_k = \left\{x_{k,j} : j \in \mathbb{Z}^d \cap [0, 2^{\nu_k}]^d\right\} \text{ and } N_k = \operatorname{card} \mathcal{D}_k = (2^{\nu_k} + 1)^d.$$

Then $\mathcal{D}_k$ is a 2^{-k} net of K for τ_E in the sense that for any $x \in K$ one can find $x_{k,j} \in \mathcal{D}_k$ such that $\tau_E(x - x_{k,j}) \leq 2^{-k}$. Actually, by Proposition 3.4, it is sufficient to choose j such that $j_i \leq 2^{\nu_k} x_i < j_i + 1$ for $1 \leq i \leq d$.

Let us remark that the sequence $(\mathcal{D}_k)_k$ is increasing and set $\mathcal{D} = \bigcup_{k=1}^{+\infty} \mathcal{D}_k$.

Step 2. Almost surely, for any $x, y \in \mathcal{D}$

$$Y_\alpha(x) - Y_\alpha(y) = C_\alpha \Re\left(\sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} (f(x, \xi_n) - f(y, \xi_n)) g_n\right),$$

where C_α is defined by (13). Since the sequences $(T_n)_n$, $(\xi_n)_n$ and $(g_n)_n$ are independent and $(g_n)_n$ is a sequence of i.i.d. Gaussian complex isotropic random variables

$$R(x, y) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} m(\xi_n)^{-1/\alpha} (f(x, \xi_n) - f(y, \xi_n)) g_n$$

is a Gaussian isotropic complex random variable conditionally to $(T_n, \xi_n)_n$. Remark that $Y_\alpha(x) - Y_\alpha(y) = C_\alpha \Re(R(x, y))$ almost surely. Therefore, conditionally to $(T_n, \xi_n)_n$, $Y_\alpha(x) - Y_\alpha(y)$ is a real centered Gaussian random variable with variance

$$\begin{aligned} v^2((x, y) \mid (T_n, \xi_n)_n) &= \frac{C_\alpha^2}{2} \mathbb{E}\left(|R(x, y)|^2 \mid (T_n, \xi_n)_n\right) \\ &= \frac{C_\alpha^2}{2} \mathbb{E}\left(|g_1|^2\right) \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m(\xi_n)^{-2/\alpha} |f(x - y, \xi_n)|^2, \end{aligned}$$

since $|f(x, \xi_n) - f(y, \xi_n)| = |f(x - y, \xi_n)|$.

We consider the set

$$E_{i,j}^k = \{\omega : |Y_\alpha(x_{k,i}) - Y_\alpha(x_{k,j})| > v((x_{k,i}, x_{k,j}) \mid (T_n, \xi_n)_n)\, \varphi(\tau_E(x_{k,i} - x_{k,j}))\},$$

where, as in [13],

$$\varphi(t) = \sqrt{2C_\varphi d \log \frac{1}{t}},\ t > 0$$

and $C_\varphi > 0$ will be chosen later. Then, for every (k, i, j),

$$\mathbb{P}\left(E_{i,j}^k\right) = \mathbb{E}\left(\mathbb{E}\left(\mathbf{1}_{E_{i,j}^k} \mid (T_n, \xi_n)_n\right)\right).$$

We give in the following an upper bound of this probability for well chosen (k, i, j). Note that if Z is a real centered Gaussian random variable with variance 1, we have

$$\mathbb{E}\left(\mathbf{1}_{E_{i,j}^k} \mid (T_n, \xi_n)_n\right) = \mathbb{P}(|Z| > \varphi(\tau_E(x_{k,i} - x_{k,j}))) \text{ almost surely.}$$

Let us choose $\delta \in (0,1)$ and set for $k \in \mathbb{N}\backslash\{0\}$, $\delta_k = 2^{-(1-\delta)k}$ and

$$I_k = \left\{(i,j) \in \mathbb{Z}^d \cap [0, 2^{\nu_k}]^d : \tau_E(x_{k,i} - x_{k,j}) \le \delta_k\right\}.$$

For every $(i,j) \in I_k$, since φ is a decreasing function

$$\mathbb{P}(|Z| > \varphi(\tau_E(x_{k,i} - x_{k,j}))) \le \mathbb{P}(|Z| > \varphi(\delta_k)).$$

Then, we recall that

$$\forall u \ge 0, \mathbb{P}(Z > u) \le \frac{\mathrm{e}^{-u^2/2}}{\sqrt{2\pi}u}.$$

Therefore, for every $k \in \mathbb{N}\backslash\{0\}$ and $(i,j) \in I_k$,

$$\mathbb{P}\left(E_{i,j}^k\right) \le \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-\varphi^2(\delta_k)/2}}{\varphi(\delta_k)} = \frac{2^{-(1-\delta)kC_\varphi d}}{\sqrt{2C_\varphi d(1-\delta)k\log 2}}.$$

Hence,

$$\sum_{k=1}^{\infty} \sum_{(i,j)\in I_k} \mathbb{P}\left(E_{i,j}^k\right) \le \frac{1}{\sqrt{2C_\varphi d(1-\delta)\log 2}} \sum_{k=1}^{+\infty} 2^{-(1-\delta)kC_\varphi d} \,\mathrm{card}\,\mathrm{I_k}.$$

Since $\nu_k = O(k)$, the lower bounds of (16) and Corollary 3.4 leads to

$$\mathrm{card}\left\{j \in \mathbb{Z}^d \cap [0, 2^{\nu_k}]^d : (i,j) \in I_k\right\} = O\left(\delta_k^{d/H_1} 2^{dk/H_m} k^{2d(l-1)}\right),$$

for any $i \in \mathbb{Z}^d \cap [0, 2^{\nu_k}]^d$. Then one can find a finite constant $C > 0$ such that

$$\sum_{k=1}^{\infty} \sum_{(i,j)\in I_k} \mathbb{P}\left(E_{i,j}^k\right) \le \frac{C}{\sqrt{C_\varphi(1-\delta)}} \sum_{k=1}^{\infty} k^{3d(l-1)} 2^{-kd\left(-\frac{2}{H_m} + \frac{1-\delta}{H_1} + (1-\delta)C_\varphi\right)},$$

which is finite for $C_\varphi > \frac{2}{H_m} - \frac{1}{H_1}$ and δ small enough. By the Borel-Cantelli Lemma, almost surely there exists an integer $k^*(\omega)$ such that for every $k \ge k^*(\omega)$,

$$|Y_\alpha(x) - Y_\alpha(y)| \le v((x,y) \,|\, (T_n, \xi_n)_n)\, \varphi(\tau_E(x-y))$$

as soon as $x, y \in \mathcal{D}_k$ with $\tau_E(x-y) \le \delta_k$.

Step 3. As in [14] let us give an upper bound of

$$v^2((x,y) \,|\, (T_n,\xi_n)_n) = \frac{C_\alpha^2}{2}\mathbb{E}\Big(|g_1|^2\Big) \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m(\xi_n)^{-2/\alpha} |f(x-y,\xi_n)|^2$$

with respect to $\tau_E(x-y)$. By definition of f

$$v^2((x,y) \,|\, (T_n,\xi_n)_n) \le \frac{C_\alpha^2}{2}\mathbb{E}\Big(|g_1|^2\Big)\sigma^2\left(\tau_E(x-y)\right),$$

where for all $h > 0$

$$\sigma^2(h) = \sum_{n=1}^{+\infty} T_n^{-2/\alpha} m(\xi_n)^{-2/\alpha} \min\left(M_E \left\|h^{E^t}\xi_n\right\|, 2\right)^2 \psi(\xi)^{-2-2q/\alpha},$$

with M_E given by (6). For the sake of clearness we postpone the proof of the control of $\sigma^2(h)$ in Appendix and state it in the following lemma.

Lemma 5.2. *For any $\gamma \in (0,1)$ there exists a finite constant $c > 0$ such that*

$$\mathbb{E}\big(\sigma^2(h) \,|\, (T_n)_n\big) \le c \sum_{n=1}^{+\infty} T_n^{-2/\alpha} h^2 |\log h|^{(1+\eta)(2/\alpha-1)} \text{ almost surely}$$

as soon as $h \le 1 - \gamma$.

Following [14] let us denote

$$b(h) = h|\log h|^{(1+\eta)/\alpha}.$$

Then by Lemma 5.2,

$$\mathbb{E}\left(\sum_{k=1}^{+\infty} \frac{\sigma^2(2^{-k})}{b^2(2^{-k})}\middle| (T_n)_n\right) < +\infty.$$

Therefore by independence of $(T_n)_n$ and $(\xi_n)_n$, almost surely

$$\lim_{k\to+\infty} \frac{\sigma^2(2^{-k})}{b^2(2^{-k})} = 0.$$

Up to change the Euclidean norm $\|\cdot\|$ by the equivalent norm $\|\cdot\|_{E^t}$ defined in Lemma 6.1.5 of [22] the map $h \mapsto \left\|h^{E^t}\xi\right\|$ is increasing and so is σ^2. Also, one can conclude, as in [14], that almost surely

$$\lim_{h\to 0} \frac{\sigma^2(h)}{b^2(h)} = 0.$$

Therefore, up to change k^* one can assume that for every $k \geq k^*(\omega)$, for every $x, y \in \mathcal{D}_k$,

$$|Y_\alpha(x) - Y_\alpha(y)| \leq \sqrt{2dC_\varphi}\tau_E(x-y)|\log \tau_E(x-y)|^{(1+\eta)/\alpha+1/2}. \tag{17}$$

as soon as $\tau_E(x-y) \leq \delta_k$. Let

$$\Omega^* = \bigcup_{n=1}^{+\infty}\bigcap_{k\geq n}\bigcap_{\substack{x,y\in\mathcal{D}_k\\ \tau_E(x-y)\leq\delta_k}} \left\{|X_\alpha(x) - X_\alpha(y)| \leq \sqrt{2dC_\varphi}\tau_E(x-y)|\log \tau_E(x-y)|^{(1+\eta)/\alpha+1/2}\right\}$$

Since X_α and Y_α have the same finite dimensional margins $\mathbb{P}(\Omega^*) = 1$.

Step 4. Let $\omega \in \Omega^*$. By Step 3 there exists $k^*(\omega) \geq 1$ such that X_α satisfies (17) for $k \geq k^*(\omega)$, $x, y \in \mathcal{D}_k$ and $\tau_E(x-y) \leq \delta_k$.

Let us recall that by Lemma 2.2 of [6], there exists $K_E \geq 1$ such that for all $x, y \in \mathbb{R}^d$

$$\tau_E(x+y) \leq K_E\left(\tau_E(x) + \tau_E(y)\right).$$

Let us denote $F(h) = \sqrt{2dC_\varphi}\, h|\log h|^{(1+\eta)/\alpha+1/2}$ and choose $k_0 \in \mathbb{N}$ such that $2^{k_0}\delta_{k_0+1} > 3K_E^2$ and F is increasing on $(0, \delta_{k_0}]$. Up to change $k^*(\omega)$, we can assume that $k^*(\omega) \geq k_0$.

Let $x, y \in \mathcal{D}$ with $x \neq y$ such that $3K_E^2\tau_E(x-y) \leq \delta_{k^*(\omega)}$. Then there exists a unique $k \geq k^*(\omega)$ such that $\delta_{k+1} < 3K_E^2\tau_E(x-y) \leq \delta_k$. Since $x, y \in \mathcal{D}$, there exists $n \geq k+1$ such that $x, y \in \mathcal{D}_n$. Moreover, by Step 1, for $j = k, \ldots, n-1$, we can choose $x^{(j)}, y^{(j)} \in \mathcal{D}_j$ such that

$$\tau_E\left(x - x^{(j)}\right) \leq 2^{-j} \text{ and } \tau_E\left(y - y^{(j)}\right) \leq 2^{-j}.$$

By construction $\tau_E\left(x^{(k)} - y^{(k)}\right) \leq K_E^2\left(\tau_E(x-y) + 22^{-k}\right)$. Let us point out that since $k \geq k_0$, $2^k\delta_{k+1} \geq 2^{k_0}\delta_{k_0+1} > 3K_E^2$. Therefore, one easily sees that

$$\tau_E\left(x^{(k)} - y^{(k)}\right) \leq 3K_E^2\tau_E(x-y).$$

Since $3K_E^2\tau_E(x-y) \leq \delta_k$ we obtain by Step 3 on the one hand that

$$\left|X_\alpha\left(x^{(k)}\right) - X_\alpha\left(y^{(k)}\right)\right| \leq F\left(\tau_E\left(x^{(k)} - y^{(k)}\right)\right).$$

On the other hand we can write

$$X_\alpha(x) - X_\alpha\left(x^{(k)}\right) = \sum_{j=k}^{n-1}\left(X_\alpha\left(x^{(j+1)}\right) - X_\alpha\left(x^{(j)}\right)\right)$$

with $\tau_E\left(x^{(j+1)}-x^{(j)}\right)\le 3K_E^2 2^{-(j+1)}\le \delta_{j+1}$ since $j\ge k_0$. Moreover, note that $x^{(j)}\in\mathcal{D}_j\subset\mathcal{D}_{j+1}$ and then by Step 3

$$\left|X_\alpha(x)-X_\alpha\left(x^{(k)}\right)\right|\le\sum_{j=k}^{n-1}F\left(\tau_E\left(x^{(j+1)}-x^{(j)}\right)\right)\le CF(\delta_{k+1}),$$

where $C=\sum_{j=0}^{+\infty}(j+1)^{(1+\eta)/\alpha+1/2}\,\delta_j<+\infty$. Using similar computations for $X_\alpha(y)-X_\alpha\big(y^{(k)}\big)$, we obtain that

$$\begin{aligned}|X_\alpha(x)-X_\alpha(y)| &\le F\left(\tau_E\left(x^{(k)}-y^{(k)}\right)\right)+2CF(\delta_{k+1})\\ &\le (1+2C)F\left(3K_E^2\tau_E(x-y)\right).\end{aligned}$$

Then one can find a constant $C>0$ such that for $3K_E^2\tau_E(x-y)\le\delta_{k^*(\omega)}$

$$|X_\alpha(x)-X_\alpha(y)|\le C\tau_E(x-y)|\log\tau_E(x-y)|^{(1+\eta)/\alpha+1/2}. \tag{18}$$

We now give a modification of X_α. For $x\in\mathcal{D}$, we set

$$X_\alpha^*(x)(\omega)=X_\alpha(x)(\omega).$$

For $x\in K$ let $x^{(n)}\in\mathcal{D}$ such that $\lim_{n\to+\infty}x^{(n)}=x$. In view of (18), $\left(X_\alpha^*\big(x^{(n)}\big)(\omega)\right)_n$ is a Cauchy sequence and then converges. We set

$$X_\alpha^*(x)(\omega)=\lim_{n\to+\infty}X_\alpha^*\left(x^{(n)}\right)(\omega).$$

Remark that this limit does not depend on the choice of $\big(x^{(n)}\big)$. Moreover, since X_α is stochastically continuous, X_α^* is a modification of X_α.

By continuity of τ_E, we easily see that

$$\left|X_\alpha^*(x)(\omega)-X_\alpha^*(x')(\omega)\right|\le C\tau_E(x-y)|\log\tau_E(x-y)|^{(1+\eta)/\alpha+1/2}$$

as soon as $3K_E^2\tau_E(x-y)<\delta_{k^*(\omega)}$, which concludes the proof. □

Following the same lines as the proof of Theorem 5.1 we obtain a similar result in the Gaussian case ($\alpha=2$) for more general fields. Let us remark that Y_α is not defined for $\alpha=2$. However, in Step 2 of the proof, let us replace Y_α by X a centered Gaussian random field and $v^2((x,y)\,|\,(T_n,\xi_n)_n)$ by the variance of $X(x)-X(y)$

$$v^2((x,y))=\mathbb{E}\Big(\left(X(x)-X(y)\right)^2\Big).$$

Furthermore let us replace Step 3 by the assumption that for some $\beta\in\mathbb{R}$ and $\delta>0$ there exists a finite constant $C>0$ such that for $x,y\in K$ with $\tau_E(x-y)\le\delta$

$$\mathbb{E}\Big(\left(X(x)-X(y)\right)^2\Big)\le C\tau_E(x-y)^2|\log\tau_E(x-y)|^\beta. \tag{19}$$

Then Step 1, Step 2 and Step 4 yields the following proposition.

Proposition 5.3. *Let X be a centered Gaussian random field satisfying* (19)*. There exists a modification X^* of X on K such that*

$$\lim_{\delta\downarrow 0}\sup_{\substack{x,y\in K\\ \|x-y\|\le\delta}}\frac{|X^*(x)-X^*(y)|}{\tau_E(x-y)|\log\tau_E(x-y)|^{1/2+\beta+\varepsilon}}=0\ a.s. \tag{20}$$

for any $\varepsilon>0$.

Let us point out that if X_2 is an operator scaling Gaussian random field as defined in [6], then

$$\mathbb{E}\Big((X_2(x)-X_2(y))^2\Big)=\tau_E(x-y)^2\mathbb{E}\Big(X_2(\ell_E(x-y))^2\Big),$$

and X_2 satisfies (19) with $\beta=0$ by (5.2) of [6]. Therefore this result is more precise than one could expect from the Theorem 5.1, replacing α by 2.

Let us also mention that Marianne Clausel gives a different proof of a similar result for some Gaussian operator scaling random fields with stationary increments in [8].

Now, as in [6], we are looking for global and directional Hölder critical exponents of the harmonizable stable random field X_α. These exponents have been introduced in [7] in the Gaussian realm but can be defined for any random field. Following Definition 5.1 of [6], $H\in(0,1)$ is said to be the Hölder critical exponent of a random field $(X(x))_{x\in\mathbb{R}^d}$ if there exists a modification X^* of X such that for any $s\in(0,H)$, the sample paths of X^* satisfy almost surely a uniform Hölder condition of order s on K, that is there exists a finite positive random variable A such that almost surely

$$|X^*(x)-X^*(y)|\le A\|x-y\|^s \quad \text{for all } x,y\in K \tag{21}$$

while for any $s\in(H,1)$, almost surely (21) fails. Note that the Hölder critical exponent, if exists, is well defined since any continuous modification of X and X^* are indistinguishable. Moreover, according to Definition 5.3 of [6] we say that X admits $H(u)$ as directional regularity in direction $u\in S^{d-1}$, with S^{d-1} the Euclidean unit sphere, if the process $(X(tu))_{t\in\mathbb{R}}$ admits $H(u)$ as Hölder critical exponent on $K\cap\mathbb{R}u$.

For all $j=1,\dots,m$ we set $K_j=K\cap\bigoplus_{k=1}^j W_k$. Let us remark that $K_m=K$.

Corollary 5.4. *Let $\alpha\in(0,2]$. There exists a modification X_α^* of X_α on K such that for all $j=1,\dots,m$*

$$\lim_{\delta\downarrow 0}\sup_{\substack{x,y\in K_j\\ \|x-y\|\le\delta}}\frac{|X_\alpha^*(x)-X_\alpha^*(y)|}{\|x-y\|^{H_j}|\log\|x-y\||^{H_j(p_j-1)+\beta+1/2+\varepsilon}}=0 \ a.s.$$

for any $\varepsilon>0$, where $p_j=\max_{1\le k\le j} l_k$, $\beta=1/\alpha$ if $\alpha\neq 2$ and $\beta=0$ if $\alpha=2$.

Proof. It follows from Theorem 5.1 and Corollary 3.3, since $a_j\le a_d$ for any $j=1\dots d$. □

Corollary 5.5. *Let $\alpha\in(0,2]$. The random field X_α^* has locally H-Hölder sample paths on $\mathbb{R}^d$ for every $H\in(0,H_m)$.*

Now let us give the directional and global Hölder critical exponents of X_α.

Proposition 5.6. *The random field X_α admits H_m as Hölder critical exponent.*
Moreover, for any $j=1,\dots,m$, for any direction $u\in W_j$, the field X_α admits H_j as directional regularity in the direction u.

Proof. For Z a real SαS random variable we let

$$\|Z\|_\alpha=(-\log(\mathbb{E}(\exp(iZ))))^{1/\alpha}.$$

Then, for any $x,y\in\mathbb{R}^d$,

$$\|X_\alpha^*(x)-X_\alpha^*(y)\|_\alpha=C_\alpha(\ell_E(x-y))\tau_E(x-y)$$

where for all $\theta\in S_E$

$$C_\alpha(\theta)=\left(c_\alpha\int_{\mathbb{R}^d}\left|\mathrm{e}^{i\langle\theta,\xi\rangle}-1\right|^\alpha\psi(\xi)^{-\alpha-q}d\xi\right)^{1/\alpha} \text{ and } c_\alpha=\frac{1}{2\pi}\int_0^\pi|\cos(t)|^\alpha dt.$$

From Lebesgue's Theorem, the function C_α is continuous on the compact set S_E with positive values. Let us denote $m_\alpha=\min_{\theta\in S_E}C_\alpha(\theta)>0$. According to (10) in Corollary 3.4, for any $j=1,\dots,m$ and u a direction in W_j,

$$\|X_\alpha^*(tu)-X_\alpha^*(su)\|_\alpha\ge m_\alpha c_1\,|t-s|^{H_j}\,|\log|t-s||^{-(l_j-1)H_j}\,.$$

Therefore, for any $s > H_j$, it implies that $\frac{X^*_\alpha(tu)-X^*_\alpha(su)}{|t-s|^s}$ is almost surely unbounded as $|t-s| \downarrow 0$ so (21) fails almost surely on $K \cap \mathbb{R}u$.
Moreover, Corollary 5.4 implies that $(X^*_\alpha(tu))_{t\in\mathbb{R}}$ satisfies (21) on $K\cap\mathbb{R}u$ for any $s < H_j$ and thus H_j is the directional regularity of X_α in the direction u.
Moreover, one can find a direction $u \in S^{d-1}$ in which almost surely $(X^*_\alpha(tu))_{t\in\mathbb{R}}$ does not satisfy (21) on $K \cap \mathbb{R}u$ for any $s > H_m$. Therefore, almost surely $(X^*_\alpha(x))_{x\in\mathbb{R}^d}$ can not satisfy (21) on K for any $s > H_m$. Then, by Corollary 5.5 X_α admits H_m as Hölder critical exponent.

□

Remark 5.1. In the diagonalizable case (see Example 2.2), the W_j are the eigenspaces associated with the eigenvalues of E. In particular, for θ_j an eigenvector associated with the eigenvalue $\lambda_j = a_j$, the critical Hölder exponent in direction θ_j is $H_j = 1/a_j$.

Proposition 5.6 compared to Theorem 5.4 of [6] shows that the operator scaling stable field, defined through an harmonizable representation share the same sample paths properties as the Gaussian ones. Therefore it is natural to have also the same result of Theorem 5.6 [6] for the box- and the Hausdorff-dimensions of their graphs on a compact set. We also refer to Falconer [9] for the definitions and properties of box- and the Hausdorff-dimension and keep the notations of [6]. More precisely, we fix a compact set $K \subset \mathbb{R}^d$ and consider $\mathcal{G}(X^*_\alpha)(\omega) = \{(x, X^*_\alpha(x)(\omega)); x \in K\}$ the graph of a realization of the field X^*_α over the compact K. We denote $\dim_{\mathcal{H}}\mathcal{G}(X^*_\alpha)$, resp $\dim_{\mathcal{B}}\mathcal{G}(X^*_\alpha)$, the Hausdorff-dimension and the box-dimension of $\mathcal{G}(X^*_\alpha)$, respectively.

Proposition 5.7. *Almost surely*

$$\dim_{\mathcal{H}} \mathcal{G}(X^*_\alpha) = \dim_{\mathcal{B}} \mathcal{G}(X^*_\alpha) = d + 1 - H_m.$$

Proof. The proof is very similar to those of Theorem 5.6 [6]. It also uses same kinds of arguments as in [4]. For sake of completeness we recall the main ideas. Corollary 5.5 allows as usual to state the upper bound

$$\dim_{\mathcal{H}}\mathcal{G}(X^*_\alpha) \leq \overline{\dim_{\mathcal{B}}}\mathcal{G}(X^*_\alpha) \leq d + 1 - H_m, \text{ a.s.}$$

where $\overline{\dim_{\mathcal{B}}}$ denotes the upper box-dimension. The lower bound will also follows from Frostman criterion (Theorem 4.13 (a) in [9]). One has to prove that the integral I_s

$$I_s = \int_{K\times K} \mathbb{E}\left[\left((X^*_\alpha(x) - X^*_\alpha(y))^2 + \|x-y\|^2\right)^{-s/2}\right] dx\, dy,$$

is finite to get that almost surely $\dim_{\mathcal{H}}\mathcal{G}(X^*_\alpha) \geq s$. In our case, the fundamental lemma of [3] allows us to write this integral using the characteristic function of the SαS field X^*_α. Actually, when one remarks that, using Fourier-inversion, $(\xi^2+1)^{-s/2} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t} f_s(t)dt$, where $f_s \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, one gets

$$I_s = \int_{K\times K} \left(\frac{1}{2\pi}\|x-y\|^{-s} \int_{\mathbb{R}} e^{-|t|^\alpha \frac{\|X^*_\alpha(x)-X^*_\alpha(y)\|^\alpha_\alpha}{\|x-y\|^\alpha}} f_s(t)dt\right) dx\, dy.$$

By a change of variables, as $f_s \in L^\infty(\mathbb{R})$, one can find $C > 0$ such that

$$\begin{aligned} I_s &\leq C \int_{K\times K} \|x-y\|^{1-s} \|X^*_\alpha(x) - X^*_\alpha(y)\|^{-1}_\alpha \, dx\, dy \\ &\leq C m_\alpha^{-1} \int_{K\times K} \|x-y\|^{1-s} \tau_E(x-y)^{-1} dx\, dy, \end{aligned}$$

where $m_\alpha = \min\limits_{\theta\in S_E}\left(c_\alpha \int_{\mathbb{R}^d} \left|\mathrm{e}^{i\langle\theta,\xi\rangle} - 1\right|^\alpha \psi(\xi)^{-\alpha-q} d\xi\right)^{1/\alpha}$ as introduced in the proof of Proposition 5.6. Since $\int_{K\times K} \|x-y\|^{1-s}\tau_E(x-y)^{-1}dx\, dy$ is proved to be finite as soon as $s < d+1-H_m$ in [6],

$$\underline{\dim_{\mathcal{B}}}\mathcal{G}(X^*_\alpha) \geq \dim_{\mathcal{H}}\mathcal{G}(X^*_\alpha) \geq d + 1 - H_m \text{ a.s.}$$

and the proof is complete.

□

Harmonizable operator scaling stable random fields share many properties with Gaussian operator random fields. In particular, they have locally Hölder sample paths and critical directional Hölder exponent depending on the directions. In the Gaussian case ($\alpha = 2$), [6] establishes such properties in the framework of harmonizable and moving average Gaussian operator scaling random field. However, for stable laws, harmonizable and moving average representations do not have the same behavior as we see in the next section.

6. Moving average representation

Let us recall the definition of moving average operator scaling stable random fields introduced in [6]. Let $0 < \alpha \leq 2$. We consider $M_\alpha(dy)$ an independently scattered $S\alpha S$ random measure on $\mathbb{R}^d$ with Lebesgue control measure, see [27] for details on such random measures. Let us recall that, as before, $q = \text{trace}(E)$. Let $\varphi : \mathbb{R}^d \to [0, \infty)$ be a continuous E-homogeneous function. We assume moreover that there exists $\beta > 1$ such that φ is (β, E)-*admissible*. According to Definition 2.7 of [6] it means that $\varphi(x) \neq 0$ for $x \neq 0$ and that for any $0 < A < B$ there exists a constant $C > 0$ such that, for $A \leq \|y\| \leq B$,

$$|\varphi(x + y) - \varphi(y)| \leq C\tau_E(x)^\beta \tag{22}$$

holds for any $\tau_E(x) \leq 1$.

Definition 6.1. *Since $\beta > 1$, the α-stable random field*

$$Z_\alpha(x) = \int_{\mathbb{R}^d} \left(\varphi(x - y)^{1-q/\alpha} - \varphi(-y)^{1-q/\alpha}\right) M_\alpha(dy) \ , x \in \mathbb{R}^d. \tag{23}$$

is well defined and called moving average operator scaling stable random field.

From Theorem 3.1 and Corollary 3.2 of [6], it is stochastically continuous, has stationary increments and satisfies the following operator scaling property

$$\forall c > 0, \ \left\{Z_\alpha(c^E x); x \in \mathbb{R}^d\right\} \stackrel{(fdd)}{=} \left\{cZ_\alpha(x); x \in \mathbb{R}^d\right\},$$

as the harmonizable field X_α.

In the Gaussian case ($\alpha = 2$), the variogramme of Z_2 is similar to the one of the harmonizable field X_2. Then, [6] proves that Z_2 and X_2 admit the same critical Hölder sample paths properties. However, when $\alpha \in (0, 2)$, let us recall that moving average self-similar α-stable random motions does not have in general continuous sample paths (see [27]). The next proposition states the same property for Z_α.

Proposition 6.1. *Assume $\alpha \in (0, 2)$ and $d \geq 2$. Then, the random field Z_α is almost surely unbounded on every open ball.*

Proof. Let us remark that $\varphi(0) = 0$ by continuity and E-homogeneity and $q = \sum_{j=1}^{d} a_j > d > \alpha$, as soon as $d \geq 2$. Then, for any U open set, since $U^* = U \cap \mathbb{Q}^d$ is dense in U, for any $y \in U$

$$f^*(U^*, y) = \sup_{x \in U^*} \left|\varphi(x - y)^{1-q/\alpha} - \varphi(-y)^{1-q/\alpha}\right| = +\infty.$$

Then $\int_{\mathbb{R}^d} f^*(U^*, y)^\alpha dy = +\infty$ and the necessary condition for sample boundedness (10.2.14) of Theorem 10.2.3 p.450 of [27] fails. We conclude the proof by Corollary 10.2.4 of [27]. □

7. Appendix

Proof of lemma 3.2. The lower bound is straightforward. Actually, for any $t>0$, t^λ is an eigenvalue of the matrix t^J and therefore $t^a = \left|t^\lambda\right| \leq \|t^J\|$.

Let us prove the upper bound. First, let us assume that J is a Jordan cell matrix of size l. In this case $\lambda = a \in \mathbb{R}$ and

$$t^J = t^a \begin{pmatrix} 1 & 0 & \dots & 0 \\ \log t & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \frac{(\log t)^{l-1}}{(l-1)!} & \dots & \log t & 1 \end{pmatrix}.$$

Let us recall that the norm defined for a matrix $A = (a_{ij})_{1\leq i,j,\leq d}$ by $\|A\|_\infty = \max\limits_{1\leq i\leq d} \sum_{j=1}^d |a_{ij}|$ is the subordinated norm of the infinite norm $\|x\|_\infty = \max\limits_{1\leq i\leq d}|x_i|$ for $x\in\mathbb{R}^d$. By definition, we can deduce that $\|t^J\|_\infty = t^a \sum\limits_{j=0}^{l-1} \frac{|\log t|^j}{j!}$. Then,

$$\left\|t^J\right\| \leq \sqrt{l}\left\|t^J\right\|_\infty \leq \sqrt{l} t^a |\log(t)|^{l-1} \sum_{j=0}^{l-1} \frac{1}{j!}$$

for any $t\in(0,e^{-1}]\cup[e,+\infty)$. Therefore, for any $t\in(0,e^{-1}]\cup[e,+\infty)$, we have

$$\|t^J\| \leq \sqrt{l}\, e\, t^a \,|\log t|^{l-1}.$$

In the second case, let us assume that J is a block of the form (7) of size $2l$ associated with the eigenvalue $\lambda = a+ib$ for $b\neq 0$. Then $t^J = t^a R(t) N(t)$ where

$$R(t) = \begin{pmatrix} R_b(t) & 0 & \dots & 0 \\ 0 & R_b(t) & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & R_b(t) \end{pmatrix} \text{ with } R_b(t) = \begin{pmatrix} \cos(b\log t) & -\sin(b\log t) \\ \sin(b\log t) & \cos(b\log t) \end{pmatrix},$$

and

$$N(t) = \begin{pmatrix} I_2 & 0 & \dots & 0 \\ N_1(t) & I_2 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ N_{l-1}(t) & \dots & N_1(t) & I_2 \end{pmatrix} \text{ with } N_j(t) = \begin{pmatrix} \frac{|\log t|^j}{j!} & 0 \\ 0 & \frac{|\log t|^j}{j!} \end{pmatrix}.$$

Hence,

$$\left\|t^J\right\| \leq t^\alpha \|R(t)\| \|N(t)\|.$$

Since $R(t)$ is an orthogonal matrix, $\|R(t)\| = 1$. Furthermore, $N(t)$ is a $(2l)\times(2l)$ matrix and

$$\|N(t)\| \leq \sqrt{2l}\|N(t)\|_\infty = \sqrt{2l} \sum_{j=0}^{l-1} \frac{|\log t|^j}{j!}$$

Therefore, we also obtain that

$$\|t^J\| \leq \sqrt{2l}\, e t^a \,|\log t|^{l-1}$$

for any $t\in(0,e^{-1}]\cup[e,+\infty)$.

□

Proof of Proposition 3.4. Let $r\in(0,1)$. One can find $r_E\in(0,r)$ such that for any $\|x\|\leq r_E$ we have $\tau_E(x)\leq e^{-1}$. Let $x\in\bigoplus_{k=j_0}^j W_k$ with $\|x\|\leq r_E$. Then $x = \tau_E(x)^E \ell_E(x)$ and $l_E(x)\in\bigoplus_{k=j_0}^j W_k$.

We first establish the lower bound of Proposition 3.4. Let us write $l_E(x) = \sum_{k=1}^m l_k(x)$ where each $l_k(x) \in W_k$. Let L_k be the coordinates of $l_k(x)$ in the basis $\left(f_{\sum_{i=1}^{k-1}\tilde{l}_i+1}, \dots, f_{\sum_{i=1}^{k}\tilde{l}_i}\right)$ of W_k. Hence, by definition of P,

$$P^{-1}l_E(x) = \begin{pmatrix} L_1 \\ \vdots \\ L_m \end{pmatrix} \text{ and } x = \tau_E(x)^E l_E(x) = P \begin{pmatrix} \tau_E(x)^{J_1} L_1 \\ \vdots \\ \tau_E(x)^{J_m} L_m \end{pmatrix}.$$

Since $l_E(x) \in \bigoplus_{k=j_0}^{j} W_k$, $L_k = 0$ for $k \notin \{j_0, \dots, j\}$,

$$\|x\| \leq \|P\| \left(\sum_{k=j_0}^{j} \left\| \tau_E(x)^{J_k} L_k \right\|^2 \right)^{1/2} \leq \|P\| \left(\sum_{k=j_0}^{j} \left\| \tau_E(x)^{J_k} \right\|^2 \|L_k\|^2 \right)^{1/2}$$

By Lemma 3.2,

$$\|x\| \leq \sqrt{2}e\|P\| \left(\sum_{k=j_0}^{j} l_k \tau_E(x)^{2a_k} |\log \tau_E(x)|^{2(l_k-1)} \|L_k\|^2 \right)^{1/2}$$

since $\tau_E(x) \leq 1/e$. Hence, since $\tau_E(x) \leq 1$, $|\log \tau_E(x)| \geq 1$, $a_k \geq a_{j_0}$ and $l_k \leq p_{j_0,j} = \max_{j_0 \leq i \leq j} l_i \leq d$,

$$\begin{aligned} \|x\| &\leq \sqrt{2d}e\|P\|\tau_E(x)^{a_{j_0}} |\log \tau_E(x)|^{(p_{j_0,j}-1)} \left(\textstyle\sum_{k=j_0}^{j} \|L_k\|^2 \right)^{1/2} \\ &\leq \sqrt{2d}e\|P\|\tau_E(x)^{a_{j_0}} |\log \tau_E(x)|^{(p_{j_0,j}-1)} \left\| P^{-1} l_E(x) \right\| \end{aligned}$$

Then,

$$\|x\| \leq \sqrt{2d}eM_E\|P\|\left\|P^{-1}\right\|\tau_E(x)^{a_{j_0}} |\log \tau_E(x)|^{(p_{j_0,j}-1)} \tag{24}$$

where M_E is defined by (6).

Let us take the logarithm of this equation. Choosing $r_0 < \min(1, r_E)$ small enough, one can find $C > 0$ such that

$$|\log \tau_E(x)| \leq C|\log \|x\|| \text{ for } \|x\| < r_0. \tag{25}$$

Using in (24), we obtain that there exists $C > 0$ such that for $\|x\| \leq r_0$

$$\|x\|^{H_{j_0}} |\log \|x\||^{-H_{j_0}(p_{j_0,j}-1)} \leq C\tau_E(x).$$

By continuity of the map,

$$x \mapsto \|x\|^{H_{j_0}} |\log \|x\||^{-H_{j_0}(p_{j_0,j}-1)} \tau_E(x)^{-1}$$

on $\left\{x \in \mathbb{R}^d \,/\, 0 < \|x\| < 1\right\}$, up to change C, the previous inequality holds for $\|x\| \leq r$, which gives the lower bound in (11).

Let us now establish the upper bound in (11). We write $x = \sum_{k=1}^m x_k$ with each $x_k \in W_k$. We then denote by X_k the coordinates of x_k in the basis $\left(f_{\sum_{i=1}^{k-1}\tilde{l}_i+1}, \dots, f_{\sum_{i=1}^{k}\tilde{l}_i}\right)$ of W_k. Since $x \in \bigoplus_{k=j_0}^{j} W_k$, $X_k = 0$ for every $k \notin \{j_0, \dots, j\}$. Hence, by definition of P,

$$P^{-1}x = \begin{pmatrix} X_1 \\ \vdots \\ X_m \end{pmatrix} \text{ and } l_E(x) = \tau_E(x)^{-E} x = P \begin{pmatrix} \tau_E(x)^{-J_1} X_1 \\ \vdots \\ \tau_E(x)^{-J_m} X_m \end{pmatrix}.$$

Then, $\|l_E(x)\| \leq \|P\| \left(\sum_{k=j_0}^{j} \left\| \tau_E(x)^{-J_k} X_k \right\|^2 \right)^{1/2}$.

The Lemma 3.2 yields

$$\|l_E(x)\| \le \sqrt{2}e\|P\| \left(\sum_{j=j_0}^{j} l_k \tau_E(x)^{-2a_k} |\log \tau_E(x)|^{2(l_k-1)} \|X_k\|^2 \right)^{1/2}$$

since $\tau_E(x)^{-1} \ge e$. Hence, since $\tau_E(x)^{-1} \ge e > 1$, $a_k \le a_j$ and $l_k \le p_{j_0,j}$,

$$\begin{aligned} 0 < m_E &\le \sqrt{2d}e\|P\| \left\|P^{-1}\right\| \tau_E(x)^{-a_j} |\log \tau_E(x)|^{p_{j_0,j}-1} \sqrt{\textstyle\sum_{k=j_0}^{j} \|X_k\|^2} \\ &\le \sqrt{2d}e\|P\| \left\|P^{-1}\right\| \tau_E(x)^{-a_j} |\log \tau_E(x)|^{p_{j_0,j}-1} \left\|P^{-1}x\right\|. \end{aligned}$$

Then, using (25) and $\left\|P^{-1}x\right\| \le \left\|P^{-1}\right\| \|x\|$, there exists a constant $C > 0$ such that

$$\tau_E(x) < C\|x\|^{H_j} |\log \|x\||^{H_j(p_{j_0,j}-1)}$$

for $\|x\| \le r_0$. By continuity of the map

$$x \mapsto \frac{\tau_E(x)}{\|x\|^{H_j} |\log \|x\||^{H_j(p_{j_0,j}-1)}}$$

on $\left\{x \in \mathbb{R}^d \,/\, 0 < \|x\| < 1\right\}$, up to change C, the previous inequality holds for $\|x\| \le r$, which gives the upper bound in (11) and concludes the proof.

$\square$

Proof of Lemma 5.2. It is sufficient to consider

$$I(h) = \mathbb{E}\left(m(\xi_n)^{-2/\alpha} \min\left(M_E \left\| h^{E^t} \xi_n \right\|, 2\right)^2 \psi(\xi_n)^{-2-2q/\alpha} \right)$$

By definition,

$$I(h) = \int_{\mathbb{R}^d} m(\xi)^{1-2/\alpha} \psi(\xi)^{-2-2q/\alpha} \min\left(M_E \left\| h^{E^t} \xi \right\|, 2\right)^2 d\xi.$$

Using the formula of integration in *polar coordinates* with respect to E^t, see Proposition 3.1,

$$I(h) = \int_{S_{E^t}} \int_0^{+\infty} m\left(r^{E^t}\theta\right)^{1-2/\alpha} \psi\left(r^{E^t}\theta\right)^{-2-2q/\alpha} \min\left(M_E \left\| (hr)^{E^t} \theta \right\|, 2\right)^2 r^{q-1} dr \sigma_{E^t}(d\theta).$$

Since ψ is E^t-homogeneous,

$$\begin{aligned} I(h) &= \int_{S_{E^t}} \int_0^{+\infty} m\left(r^{E^t}\theta\right)^{1-2/\alpha} \psi(\theta)^{-2-2q/\alpha} \min\left(M_E \left\| (hr)^{E^t} \theta \right\|, 2\right)^2 r^{-2+q-1-2q/\alpha} dr \sigma_{E^t}(d\theta) \\ &= c_\eta^{1-2/\alpha} \int_{S_{E^t}} \int_0^{+\infty} \psi(\theta)^{-2-2q/\alpha} \min\left(M_E \left\| (hr)^{E^t} \theta \right\|, 2\right)^2 r^{-3} |\log(r)|^{(1+\eta)(2/\alpha-1)} dr \sigma_{E^t}(d\theta). \end{aligned}$$

By the change of variable $\rho = hr$, $I(h)$ is equal to

$$c_\eta^{1-2/\alpha} h^2 \int_{S_{E^t}} \int_0^{+\infty} \psi(\theta)^{-2-2q/\alpha} \min\left(M_E \left\| \rho^{E^t} \theta \right\|, 2\right)^2 \rho^{-3} \left| \log\left(\frac{\rho}{h}\right) \right|^{(1+\eta)(2/\alpha-1)} dr \sigma_{E^t}(d\theta).$$

For any $\gamma \in (0,1)$, there exists A_γ such that for every $\rho > 0$ and every $h \le 1-\gamma$,

$$\left| \log\left(\frac{\rho}{h}\right) \right| = |\log(\rho) - \log(h)| \le A_\gamma ||\log(\rho)| + 1| |\log(h)|.$$

Since $2/\alpha > 1$,

$$I(h) \le A_\gamma^{2/\alpha-1} c_\eta^{1-2/\alpha} h^2 |\log(h)|^{(1+\eta)(2/\alpha-1)} (I_1 + I_2)$$

with

$$I_1 = 4 \int_{S_{E^t}} \psi(\theta)^{-2-2q/\alpha} \sigma_{E^t}(d\theta) \int_1^{+\infty} \rho^{-3} ||\log(\rho)| + 1|^{(1+\eta)(2/\alpha-1)} d\rho$$

and

$$I_2 = M_E^2 M_{E^t}^2 \int_{S_{E^t}} \psi(\theta)^{-2-2q/\alpha} \sigma_{E^t}(d\theta) \int_0^1 \left\| \rho^{E^t} \right\|^2 \rho^{-3} ||\log(\rho)| + 1|^{(1+\eta)(2/\alpha-1)} d\rho,$$

where M_E and M_{E^t} are defined by (6). Since ψ is continuous with positive value on the compact set S_{E^t},

$$\int_{S_{E^t}} \psi(\theta)^{-2-2q/\alpha} \sigma(d\theta) < +\infty.$$

Hence $I_1 < +\infty$.
It follows from Proposition 3.4 that for any $\delta' \in (0,1)$, there exists a constant $c'_\delta > 0$ such that

$$\|\rho^{E^t}\| \leq C\rho^{a_1} |\log|\rho||^{l-1}$$

for all $\rho \leq \delta'$. Hence, since $a_1 > 1$,

$$\int_0^1 \left\| \rho^{E^t} \right\|^2 \rho^{-3} ||\log(\rho)| + 1|^{(1+\eta)(2/\alpha-1)} d\rho < +\infty$$

and $I_2 < +\infty$, which concludes the proof. □

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