

# The Scattering Amplitude for the Schrödinger Operator in a Layer

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## Abstract

For a layer with compactly supported inhomogeneity we give an appropriate definition of the scattering amplitude. Then we recover the relation between the far field pattern and the scattering amplitude.

## 1 Introduction

The scattering amplitude is a wellknown way to solve some inverse scattering problem. In the litterature we actually find the definition of the scattering amplitude in the whole homogeneous perturbed space [5],[3] or half space [1]. For a stratified space, this definition is more complicated [2]. The situation in the layer, not yet studied, introduces also difficulties. In this paper we are looking for a definition of the scattering amplitude in the layer. Such wave guides can modelize problems of wave propagation in geophysic, underwater acoustic and so on.

One consider the Schrödinger operator  $A = -\Delta + V(X)$  in  $L^2(\Omega)$  which derives from  $A_0 = -\Delta$  in  $L^2(\Omega)$ , where  $\Omega = \mathbb{R}^n \times (0, \pi)$ ,  $n \geq 1$  and  $D(A) = D(A_0) = H_0^1(\Omega) \cap H^2(\Omega)$ . We study the boundary value problem:

$$\begin{cases} (-\Delta + V(X))u(X) = \lambda u(X) & \text{in } L^2(\Omega), \\ u(X)|_{\partial\Omega} = 0. \end{cases} \quad (1.1)$$

We define the scattering amplitude and the outgoing solutions of (1.1). A work in progress uses the scattering amplitude to establish uniqueness theorem for the potential  $V(X)$ .

The paper is organized as follow. In section 2, we carry out the generalized eigenfunctions of  $A$  and we define the wave operators. We state results on the existence and completeness of these operators. This allows us to prove that the system of generalized eigenfunctions of  $A$  is dense. In section 3, we define the scattering operator, the scattering matrix and finally the scattering amplitude. In section 4, we give an asymptotic development of the generalized eigenfunctions of  $A$  and we link it with the scattering amplitude.

## 2 Spectral Study of $A$

### 2.1 Generalized Eigenfunctions of $A_0$

By a standard Fourier technique, we obtain the generalized eigenfunctions  $\psi_j^0$  of  $A_0$  associated to the eigenvalues  $\lambda_j(|p|)$  in the following form

$$\psi_j^0(x', y, p) = (2\pi)^n \sqrt{\frac{2}{\pi}} e^{ix' \cdot p} \sin(jy), \quad \text{and} \quad \lambda_j(|p|) = j^2 + |p|^2, \quad j \geq 1,$$

with  $X = (x', y)$ ,  $x' = (x_1, \dots, x_n) \in \mathbb{R}^n$ ,  $y \in (0, \pi)$  and  $p \in \mathbb{R}^n$ .

This family of generalized eigenfunctions  $\{\psi_j^0\}_{j \geq 1}$  is dense in  $L^2(\Omega)$ . Then to obtain a spectral representation

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of  $A_0$ , if  $f \in L^2(\Omega)$ , we define the following limit in  $L^2(\Omega)$  and the spaces  $\mathcal{H}_j$ :

$$\begin{aligned} \widetilde{f}_j(p) &= \lim_{M \rightarrow +\infty} \int_{|x'| < M} \overline{\psi_j^0(x', y, p)} f(x', y) dx' dy, \\ \mathcal{H}_j &= \{u \in L^2(\Omega); \exists \widetilde{f}_j \in L^2(\mathbb{R}^n), u(x', y) = \int_{\mathbb{R}^n} \psi_j^0(x', y, p) \widetilde{f}_j(p) dp\}. \end{aligned} \quad (2.1)$$

The unitary map  $\Phi : f \mapsto (\widetilde{f}_1(p), \widetilde{f}_2(p), \dots)$  from  $L^2(\Omega)$  into  $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \dots$  is a spectral representation of  $A_0$  in the following sense

$$\Phi(A_0 f) = (\widetilde{A_0 f_1}, \widetilde{A_0 f_2}, \dots) = (\lambda_1(|p|) \widetilde{f}_1(p), \lambda_2(|p|) \widetilde{f}_2(p), \dots).$$

## 2.2 Generalized Eigenfunctions of $A$

We look for the generalized eigenfunctions of  $A$  in the form:

$$\psi_j^\pm(x', y, p) = \psi_j^0(x', y, p) + w_j^\pm(x', y, p), \quad (2.2)$$

We denote

$$N(\lambda) = \#\{j > 1; j^2 \leq \lambda\}. \quad (2.3)$$

The function  $\psi_j^\pm$  satisfies the eigenvalue equation

$$(-\Delta + V(x', y) - \lambda_j(|p|)) \psi_j^\pm(x', y, p) = 0. \quad (2.4)$$

Then, taking into account the equations (2.2), (2.4) and the limiting absorption principle, we obtain

$$w_j^\pm(x', y, p) = -R^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p), \quad (2.5)$$

where  $R^\pm(z)$  is the resolvent of  $A$ . If we introduce the resolvent equation in (2.5), we obtain

$$w_j^\pm(x', y, p) = -R_0^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p) + R_0^\pm(\lambda_j(|p|))VR^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p). \quad (2.6)$$

## 2.3 Wave Operators

We first consider the wave operators

$$W_\pm = s - \lim_{t \rightarrow \pm\infty} e^{itA} e^{-itA_0}, \quad (2.7)$$

and the scattering operator  $S = W_+^* W_-$ . The wave operators actually exist. The proof is an adaptation of the theorem XIII-31 of [6]. Then the limits (2.7) exist, moreover we obtain the completeness of the wave operators  $W_\pm$  from  $\mathcal{H}_{ac}^0(\Omega) = L^2(\Omega)$  (subspace of absolute continuity of  $A_0$ ) into  $\mathcal{H}_{ac}(\Omega)$  (subspace of absolute continuity of  $A$ ), see [4]. To obtain a spectral representation of  $A$ , if  $f \in L^2(\Omega)$ , we define the following limit in  $L^2(\Omega)$ :

$$\widetilde{f}_j^\pm(p) = \lim_{M \rightarrow +\infty} \int_{|x'| < M} \overline{\psi_j^\pm(x', y, p)} f(x', y) dx' dy,$$

and  $\mathcal{H}_j^\pm = W_\pm \mathcal{H}_j$ , where  $\mathcal{H}_j$  is defined by (2.1). The unitary map  $\Phi^\pm : f \mapsto (\widetilde{f}_1^\pm(p), \widetilde{f}_2^\pm(p), \dots)$  from  $L^2(\Omega)$  to  $\mathcal{H}_1^\pm \oplus \mathcal{H}_2^\pm \oplus \dots$  is a spectral representation of  $A|_{\mathcal{H}_{ac}}$ .

### 3 The Scattering Amplitude in a Layer

We have defined in the previous section the scattering operator  $S = W_+^* W_-$ . Using the relations  $\Phi^+ = \Phi W_-^*$  and  $\Phi^- = \Phi W_+^*$ , the scattering operator can be written

$$S = \Phi^* \Phi^- \Phi^{+*} \Phi. \quad (3.1)$$

In a first step we define the scattering matrix as follows

$$\widetilde{S}u_j(p) = (\widetilde{S}(\lambda_j(|p|))u_j)(\omega) \quad \text{with } \omega \in S^{n-1}. \quad (3.2)$$

Note that  $\widetilde{S}(\lambda) \in \mathcal{L}((L^2(S^{n-1}))^{N(\lambda)})$ . We obtain an  $N(\lambda) \times N(\lambda)$ -matrix, where  $N(\lambda)$  was defined by (2.3). So we have to determine  $\widetilde{S}u_j(p)$ . Using (3.1) we obtain  $\Phi S u = \Phi u + (\Phi^- - \Phi^+) \theta^+ u$  where  $\theta^+ = \Phi^{+*} \Phi$ . Then we have

$$\widetilde{S}u_j(p) = \widetilde{u}_j(p) + \lim_{M \rightarrow +\infty} \int_{|x'| < M, y \in (0, \pi)} \overline{[\psi_j^-(x', y, p) - \psi_j^+(x', y, p)]} \theta^+ u(x', y) dx' dy.$$

After several technical transformations, we obtain:

$$\begin{aligned} \widetilde{S}u_j(p) &= \widetilde{u}_j(p) + \sum_{k^2 < j^2 + |p|^2} i\pi \alpha^{n-2} \\ &\left( \int_{\mathbb{R}^n \times (0, \pi)} V(x', y) \psi_j^0(x', y, p) \int_{S^{n-1}} \overline{\psi_k^0(x', y, \alpha\omega')} \widetilde{u}_k(\alpha\omega') d\omega' dx' dy \right. \\ &\left. - \int_{\mathbb{R}^n \times (0, \pi)} V(x', y) \psi_j^0(x', y, p) \int_{S^{n-1}} \overline{[R^+(\lambda_k(\alpha\omega')) V \psi_k^0(\cdot, \cdot, \alpha\omega')]}(x', y, \alpha\omega') \widetilde{u}_k(\alpha\omega') d\omega' dx' dy, \right) \end{aligned} \quad (3.3)$$

with  $\alpha = (\lambda_j(|p|) - k^2)^{1/2}$ . So taking into account (3.2) and (3.3) we obtain explicetely the scattering matrix and the scattering amplitude  $\mathcal{A}(\omega, \omega', \lambda_j(|p|)) = [\mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|))]_{1 \leq j, k \leq N(\lambda)}$  is defined as the kernel of this operator

$$((S(\lambda_j(|p|)) - I)\widetilde{u}_j)(\omega) = \int_{S^{n-1}} \sum_{k^2 < j^2 + |p|^2} \mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|)) \widetilde{u}_k(\alpha\omega') d\omega'. \quad (3.4)$$

**Proposition 3.1** *The scattering amplitude is a  $N(\lambda) \times N(\lambda)$ -matrix*

$$\begin{aligned} \mathcal{A}(\omega, \omega', \lambda) &= [\mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|))]_{1 \leq j, k \leq N(\lambda)} \\ \mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|)) &= \mathcal{B}_{j,k}(\omega, \omega', \lambda_j(|p|)) + \mathcal{C}_{j,k}(\omega, \omega', \lambda_j(|p|)), \\ \mathcal{B}_{j,k} &= i\pi \alpha^{n-2} \int_{\mathbb{R}^n \times (0, \pi)} V(x', y) \psi_j^0(x', y, p) \int_{S^{n-1}} \overline{\psi_k^0(x', y, \alpha\omega')} \widetilde{u}_k(\alpha\omega') d\omega' dx' dy \\ \mathcal{C}_{j,k} &= -i\pi \alpha^{n-2} \int_{\mathbb{R}^n \times (0, \pi)} V(x', y) \psi_j^0(x', y, p) \int_{S^{n-1}} \overline{[R^+(\lambda_k(\alpha\omega')) V \psi_k^0(\cdot, \cdot, \alpha\omega')]}(x', y, \alpha\omega') \widetilde{u}_k(\alpha\omega') d\omega' dx' dy. \end{aligned} \quad (3.5)$$

Remark: It should be mentioned the matrix-form of the scattering amplitude which differs radically from the classical one.

### 4 Asymptotic Behaviour of the Generalized Eigenfunctions

The functions  $w_j^\pm$  see (2.2) are defined by

$$\begin{aligned} w_j^\pm(x', y, p) &= -R_0^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p) + R_0^\pm(\lambda_j(|p|))V R^\pm(\lambda_j(|p|))(V\psi_j^0)(x', y, p) \\ &= I_{1,j}(x', y, p) + I_{2,j}(x', y, p). \end{aligned}$$

We are looking for the asymptotic behaviour of each term of this sum. Let us consider the  $k^{th}$  component of the first term

$$[I_{1,j}(x', y, p)]_k = \int_{\mathbb{R}^n} \psi_k^0(x', y, p) \frac{1}{|p'|^2 + k^2 - \lambda_j - i0} [V \widetilde{\psi_j^0}(\cdot, p)](p') dp'.$$

Then

$$[I_{1,j}(x', y, p)]_k = (2\pi)^{-3n} \left(\frac{2}{\pi}\right)^{3/2} \sin(ky) \int_{\mathbb{R}^n} \frac{e^{ix' \cdot p'}}{|p'|^2 + k^2 - \lambda_j - i0} f_1(p') dp'$$

with  $f_1(p') = \int_{\mathbb{R}^n \times (0, \pi)} e^{i\tilde{x}' \cdot (p-p')} \sin(k\tilde{y}) \sin(j\tilde{y}) V(\tilde{x}', \tilde{y}) d\tilde{x}' d\tilde{y}.$

Remark: To obtain an asymptotic development of  $[I_{1,j}(x', y, p)]_k$ , we use the lemma A1 of [2]. For this we have to replace  $f_1 \in C^\infty(\mathbb{R}^n)$  by a  $C_0^\infty(\mathbb{R}^n)$  function.

Let  $p'_0 = \alpha\omega'$ , ( $p'_0$  verify  $|p'_0|^2 + k^2 = \lambda_j(|p|)$ ), we define the  $C_0^\infty(\mathbb{R}^n)$  function  $\chi$  equal to 1 in a neighborhood of  $p'_0$  and 0 outside. Then

$$[I_{1,j}(x', y, p)]_k = (2\pi)^{-2n} \left(\frac{2}{\pi}\right)^{3/2} \sin(ky) (J_1(x') + K_1(x'))$$

$$J_1(x') = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix' \cdot p'} \chi(p')}{|p'|^2 + k^2 - \lambda_j - i0} f_1(p') dp',$$

$$K_1(x') = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{ix' \cdot p'} (1 - \chi(p'))}{|p'|^2 + k^2 - \lambda_j - i0} f_1(p') dp'.$$

Then, we apply the lemma A1 of [2] and we obtain

$$J_1(x') = C_{p'_0} \frac{e^{i|x'| \cdot |p'_0|}}{|x'|^{\frac{n-1}{2}}} (f_1(p'_0) + \mathcal{O}(|x'|^{-1}))$$

where  $C_{p'_0} = (4\pi)^{-1} \alpha^{\frac{n-3}{4}} (2\pi)^{\frac{3-n}{2}} e^{-i\pi \frac{n-3}{4}}$ . Moreover, we prove that  $K_1(x') = \mathcal{O}(|x'|^{-\frac{n+1}{2}})$ . So the asymptotic behaviour of the first term of  $w_j^+$ , denoted  $\overline{\overline{[I_{1,j}(x', y, p)]_k}}$ , is given by

$$\begin{aligned} \overline{\overline{[I_{1,j}(x', y, p)]_k}} &= \left(\frac{2}{\pi}\right)^{\frac{1+5n}{2}} \sin(ky) \alpha^{\frac{n-3}{2}} e^{-i\pi \frac{n-3}{4}} \frac{e^{i|x'| \cdot |p'_0|}}{|x'|^{\frac{n-1}{2}}} \\ &\quad \times \int_{\mathbb{R}^n \times (0, \pi)} e^{i\tilde{x}' \cdot (p-p'_0)} \sin(k\tilde{y}) \sin(j\tilde{y}) V(\tilde{x}', \tilde{y}) d\tilde{x}' d\tilde{y}. \end{aligned}$$

In the same way, we prove that the asymptotic behaviour of the second term of  $w_j^+$ , denoted  $\overline{\overline{[I_{2,j}(x', y, p)]_k}}$ , is given by

$$\begin{aligned} \overline{\overline{[I_{2,j}(x', y, p)]_k}} &= \frac{(2\pi)^{\frac{1-3n}{2}}}{\pi} \sin(ky) \alpha^{n-3} V(x', y) e^{-i\pi \frac{n-3}{4}} \frac{e^{i|x'| \cdot |p'_0|}}{|x'|^{\frac{n-1}{2}}} \\ &\quad \times \int_{\mathbb{R}^n \times (0, \pi)} e^{-i\tilde{x}' \cdot p'_0} \sin(k\tilde{y}) R(\lambda_j(|p|)) (V(\cdot, \cdot) \psi_j^0(\cdot, \cdot, p))(\tilde{x}', \tilde{y}, p) d\tilde{x}' d\tilde{y}. \end{aligned}$$

In conclusion, we obtain the asymptotic behaviour of the  $k^{th}$  component of  $w_j^+$ :

$$\overline{\overline{[w_j^+(x', y, p)]_k}} = \overline{\overline{[I_{1,j}(x', y, p)]_k}} + \overline{\overline{[I_{2,j}(x', y, p)]_k}}. \quad (4.1)$$

So we have to link the far field pattern with the scattering amplitude. For this, we compare (3.5) with (4.1).

**Proposition 4.1** *If we set*

$$[G(x', y, p)]_k = \sqrt{\frac{2}{\pi}} (2\pi)^{-\frac{n-1}{2}} \sin(ky) \alpha^{-\frac{n+1}{2}} e^{-i\pi \frac{n-1}{2}} \frac{e^{i|x'|\cdot\alpha}}{|x'|^{\frac{n-1}{2}}},$$

with  $\alpha = (\lambda_j(|p|) - k^2)^{1/2}$ , we can write

$$\psi_j^\pm(x', y, p) = \psi_j^0(x', y, p) + \sum_{k=1}^{N(\lambda)} \mathcal{A}_{j,k}(\omega, \omega', \lambda_j(|p|)) [G(x', y, p)]_k + o(|x'|^{-\frac{n+1}{2}}).$$

This relation allows us to give a physical sense to the scattering amplitude.

Our goal is to prove that exists an injection between the scattering amplitude and the potential  $V(X)$ . For the layer  $(\mathbb{R}^2 \times (0, \pi))$  [3] prove an uniqueness result but using the Dirichlet to Neumann map. In a strip  $(\mathbb{R} \times (0, \pi))$  does not exist any result. We think that the knowledge of the scattering amplitude at fixed energy is not sufficient to determine the potential. We want to prove that a countable infinity of energies will be sufficient.

## References

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