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The Annals of Applied Probability
2008, Vol. 18, No. 5, 1706–1736
DOI: 10.1214/07-AAP513

PROPAGATION OF CHAOS AND POINCARÉ INEQUALITIES FOR A SYSTEM OF PARTICLES INTERACTING THROUGH THEIR CDF

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In this paper, in the particular case of a concave flux function, we are interested in the long time behavior of the nonlinear process associated in [*Methodol. Comput. Appl. Probab.* **2** (2000) 69–91] to the one-dimensional viscous scalar conservation law. We also consider the particle system obtained by replacing the cumulative distribution function in the drift coefficient of this nonlinear process by the empirical cumulative distribution function. We first obtain a trajectorial propagation of chaos estimate which strengthens the weak convergence result obtained in [8] without any convexity assumption on the flux function. Then Poincaré inequalities are used to get explicit estimates concerning the long time behavior of both the nonlinear process and the particle system.

Introduction. In this paper, we are interested in the viscous scalar conservation law with C^1 flux function $-A$

$$\partial_t F_t(x) = \frac{\sigma^2}{2}\partial_{xx}F_t(x) + \partial_x(A(F_t(x))), \qquad F_0(x) = H * m(x), \tag{1}$$

where m is a probability measure on the real line and $H(x) = 1_{\{x\geq 0\}}$ denotes the Heaviside function. As a consequence, $H * m$ is the cumulative distribution function of the probability measure m. Since A appears in this equation through its derivative, we suppose without restriction that $A(0) = 0$. According to [8], one may associate the following nonlinear process with the conservation law:

$$\begin{cases} X_t = X_0 + \sigma B_t - \displaystyle\int_0^t A'(H * P_s(X_s))\,ds, \\ \forall t \geq 0, \text{the law of } X_t \text{ is } P_t, \end{cases} \tag{2}$$

Received January 2007; revised December 2007.

AMS 2000 subject classifications. 65C35, 60K35, 60E15, 35K15, 46N30.

Key words and phrases. Viscous scalar conservation law, nonlinear process, particle system, propagation of chaos, Poincaré inequality, long time behavior.

where $(B_t)_{t\geq 0}$ is a real Brownian motion independent from the initial random variable X_0 with law m and σ a positive constant. The process X is said to be nonlinear in the sense that the drift term of the SDE depends on the entire law P_t of X_t. More precisely, according to [8], this nonlinear stochastic differential equation admits a unique weak solution. Moreover, $H * P_t(x)$ is the unique bounded weak solution of (1). For $t > 0$, by the Girsanov theorem, P_t admits a density p_t with respect to the Lebesgue measure on the real line.

We want to address the long time behavior of the nonlinear process solving (2) by studying convergence of the density p_t (see [2] and [3] for a similar study in a different setting). Since the cumulative distribution function $x \to H * P_s(x)$ which appears in the drift coefficient is nondecreasing, convexity of A is a natural assumption in order to ensure ergodicity. Then the flux function $-A$ in the conservation law (1) is concave.

In the first section of the paper, after recalling results obtained in [8], we show that trajectorial uniqueness holds for (2) under convexity of A. Then we introduce a simulable system of n particles obtained by replacing in the drift coefficient the cumulative distribution function by its empirical version and the derivative A' by a suitable finite difference approximation. When A is convex, existence and trajectorial uniqueness hold for this system. Moreover, we prove a trajectorial estimation of propagation of chaos which strengthens the weak convergence result obtained in [8]. Unfortunately, because the empirical cumulative distribution function is a step function and therefore not an increasing one, this estimation is not uniform in time.

The second and main section deals with the long time behavior of both the nonlinear process and the particle system. We address the convergence of the density p_t of X_t by first studying the convergence of the associated solution $H * p_t$ of (1) to the solution F_∞ with the same expectation of the stationary equation $\frac{\sigma^2}{2}\partial_{xx}F_\infty(x) + \partial_x(A(F_\infty(x)) = 0$ obtained by removing the time derivative in (1). For this result, no convexity hypothesis is made on A. Instead, one assumes $A(u) < 0$ for $u \in (0,1)$, $A'(0) < 0$, $A(1) = 0$ and $A'(1) > 0$. In contrast, to prove exponential convergence of the density of the particle system uniform in the number n of particles, we suppose that the function A is uniformly convex. This hypothesis ensures the existence of an invariant distribution for the particle system. In [14], a necessary and sufficient condition on the drift sequence is established for existence of the invariant measure and convergence in total variation norm for the law of the particle system at time t to this measure. In the present paper, the key step to derive quantitative convergence to equilibrium consists in obtaining a Poincaré inequality for the stationary density of the particle system uniform in n. This density has exponential-like tails and therefore does not satisfy a logarithmic Sobolev inequality. So the derivation of the Poincaré inequality cannot rely on the curvature criterion, used, for instance, in [5, 6, 12] or [13] for the granular media equation. Instead we make a direct estimation of the Poincaré constant using the specific analytic form of the invariant density. To our knowledge, our study provides the first example of a particle system, for which a Poincaré inequality but no logarithmic Sobolev inequality holds uniformly in the number n of particles.

ASSUMPTION. Throughout the paper, we assume that A is a C^1 function on $[0,1]$ s.t. $A(0)=0$.

1. Propagation of chaos.

1.1. *The nonlinear process.* Let us first state existence and uniqueness for the nonlinear stochastic differential equation (2).

THEOREM 1.1. *The nonlinear stochastic differential equation (2) admits a unique weak solution $((X_t, P_t))_{t\geq 0}$. For $t>0$, P_t admits a density p_t with respect to the Lebesgue measure on $\mathbb{R}$. The function $(t,x)\mapsto H * P_t(x)$ is the unique bounded weak solution of the viscous scalar conservation law (1). Moreover,*

$$\forall t\geq 0 \qquad X_t - X_0 \text{ is integrable and } \mathbb{E}(X_t - X_0) = -A(1)t. \tag{3}$$

Last, if the function A is convex on $[0,1]$, (2) admits a unique strong solution.

PROOF. The first and third statements are consequences of Proposition 1.2 and Theorem 2.1 of [8] [uniqueness follows from uniqueness for (1) and existence is obtained by a propagation of chaos result].

According to the Yamada–Watanabe theorem, to deduce the last statement, it is enough to check that when A is convex, then trajectorial uniqueness holds for the standard stochastic differential equation

$$dX_t = \sigma\, dB_t - A'(H * Q_t(X_t))\, dt$$

where $(Q_t)_{t\geq 0}$ is the flow of time-marginals of a probability measure Q on $C([0,+\infty),\mathbb{R})$. Since for each $t\geq 0$ the function $x\mapsto A'(H*Q_t(x))$ is nondecreasing, if $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ both solve this standard SDE, then $|X_t - Y_t|$ is bounded by

$$|X_0 - Y_0| + \int_0^t \operatorname{sign}(X_s - Y_s)(A'(H * Q_s(Y_s)) - A'(H * Q_s(X_s)))\, ds,$$

and then by $|X_0 - Y_0|$ which concludes the proof of trajectorial uniqueness.

Existence of the density p_t for $t>0$ follows from the boundedness of the drift coefficient and the Girsanov theorem. To prove (3), one first remarks that by boundedness of the drift coefficient, for each $t \geq 0$, the random variable $X_t - X_0$ is integrable and

$$\begin{aligned}\mathbb{E}(X_t - X_0) &= -\int_0^t \mathbb{E}(A'(H * P_s(X_s)))\,ds \\ &= -\int_0^t \int_{\mathbb{R}} A'\left(\int_{-\infty}^x P_s(dy)\right) P_s(dx)\,ds.\end{aligned}$$

For $s > 0$, since by the Girsanov theorem P_s does not weight points,

$$\int_{\mathbb{R}} A'\left(\int_{-\infty}^x P_s(dy)\right) P_s(dx) = [A(H * P_s(x))]_{-\infty}^{+\infty} = A(1). \qquad \square$$

COROLLARY 1.2. *Assume that A is C^2 on $[0,1]$. Then the function $H * P_t(x)$ is $C^{1,2}$ on $(0,+\infty) \times \mathbb{R}$ and solves (1) in the classical sense on this domain.*

PROOF. By the Girsanov theorem, for $t_0 > 0$, the law P_{t_0} of X_{t_0} admits a density with respect to the Lebesgue measure on $\mathbb{R}$. Hence $(t,x) \mapsto H * P_t(x)$ is a continuous function on $(0,+\infty) \times \mathbb{R}$ with values in $[0,1]$. According to [11], Theorem 8.1, page 495, Remark 8.1, page 495 and Theorem 2.5, page 18, there exists a function u with values in $[0,1]$, continuous on $[0,+\infty) \times \mathbb{R}$ and $C^{1,2}$ on $(0,+\infty) \times \mathbb{R}$ such that

$$\begin{cases} \forall x \in \mathbb{R}, & u(0,x) = H * P_{t_0}(x), \\ \forall (t,x) \in (0,+\infty) \times \mathbb{R}, & \partial_t u(t,x) = \dfrac{\sigma^2}{2} \partial_{xx} u(t,x) + \partial_x (A(u(t,x))). \end{cases}$$

By the uniqueness result for bounded weak solutions of this viscous scalar conservation law recalled in Theorem 1.1, $\forall t \geq t_0,\ H * P_t(x) = u(t - t_0, x)$. The conclusion follows since t_0 is arbitrary. $\square$

1.2. *Study of the particle system.* For $n \in \mathbb{N}^*$, let $(a_n(i))_{1 \leq i \leq n}$ be a sequence of real numbers. In this section, we are interested in the n-dimensional stochastic differential equation

$$(4) \quad dX_t^{i,n} = \sigma\, dB_t^i - a_n\left(\sum_{j=1}^n 1_{\{X_t^{j,n} \leq X_t^{i,n}\}}\right) dt, \qquad X_0^{i,n} = X_0^i,\ 1 \leq i \leq n,$$

where $(B^i)_{i \geq 1}$ are independent standard Brownian motions independent from the sequence $(X_0^i)_{i \geq 1}$ of initial random variables.

In the next section devoted to the approximation of the nonlinear stochastic differential equation (2), we will choose $a_n(i)$ equal to the finite difference approximation $n(A(i/n) - A((i-1)/n))$ of $A'(\frac{i}{n})$. For this particular choice, the nondecreasing assumption made in the following proposition is implied by convexity of A.

PROPOSITION 1.3. *Assume that the sequence $(a_n(i))_{1\le i\le n}$ is nondecreasing. Then the stochastic differential equation (4) has a unique strong solution. Let $(Y_t^{1,n},\dots,Y_t^{n,n})$ denote another solution starting from $(Y_0^1,\dots,Y_0^n)$ and driven by the same Brownian motion $(B^1,\dots,B^n)$. Then*

$$\text{(5)} \qquad a.s., \forall t\ge 0 \qquad \sum_{i=1}^n (X_t^{i,n}-Y_t^{i,n})^2 \le \sum_{i=1}^n (X_0^i - Y_0^i)^2.$$

In addition, if the initial conditions $(X_0^1,\dots,X_0^n)$ and $(Y_0^1,\dots,Y_0^n)$ are s.t. a.s., $\forall i\in\{1,\dots,n\}$, $X_0^i < Y_0^i$ (resp. $X_0^i\le Y_0^i$), then

$$\text{(6)} \qquad a.s., \forall t\ge 0, \forall i\in\{1,\dots,n\} \qquad X_t^{i,n} < Y_t^{i,n} \ (resp.\ X_t^{i,n}\le Y_t^{i,n}).$$

Existence of a weak solution to (4) is a consequence of the Girsanov theorem. Therefore, according to the Yamada–Watanabe theorem, it is enough to prove (5) which implies trajectorial uniqueness to obtain existence of a unique strong solution. To do so, we will need the following lemma.

LEMMA 1.4. *Let $(a(i))_{1\le i\le n}$ and $(b(i))_{1\le i\le n}$ denote two nondecreasing sequences of real numbers. Then for any permutation $\tau\in\mathcal{S}_n$,*

$$\text{(7)} \qquad \sum_{i=1}^n a(i)b(\tau(i)) \le \sum_{i=1}^n a(i)b(i).$$

PROOF. For $n=2$, the result is an easy consequence of the inequality

$$(a(2)-a(1))(b(2)-b(1))\ge 0.$$

For $n>2$, we define τ_1 as τ if $\tau(1)=1$ and as τ composed with the transposition between 1 and $\tau^{-1}(1)$ otherwise. This way, $\tau_1(1)=1$. In addition, using the result for $n=2$, we get $\sum_{i=1}^n a(i)b(\tau(i)) \le \sum_{i=1}^n a(i)b(\tau_1(i))$.

For $2\le j\le n-1$, we define inductively τ_j as τ_{j-1} if $\tau_{j-1}(j)=j$ and as τ_{j-1} composed with the transposition between j and $\tau_{j-1}^{-1}(j)$ otherwise. This way, for $1\le i\le j$, $\tau_j(i)=i$. Again by the result for $n=2$, one has

$$\sum_{i=1}^n a(i)b(\tau(i)) \le \sum_{i=1}^n a(i)b(\tau_1(i)) \le \sum_{i=1}^n a(i)b(\tau_2(i)) \le \cdots \le \sum_{i=1}^n a(i)b(\tau_{n-1}(i)).$$

We conclude by remarking that τ_{n-1} is the identity. □

We are now ready to complete the proof of Proposition 1.3.

PROOF OF PROPOSITION 1.3. Let $(X^{1,n},\dots,X^{n,n})$ and $(Y^{1,n},\dots,Y^{n,n})$ denote two solutions. The difference

$$\sum_{i=1}^n (X_t^{i,n}-Y_t^{i,n})^2 - \sum_{i=1}^n (X_0^i - Y_0^i)^2$$

is equal to

$$(8)\ 2\int_0^t \sum_{i=1}^n (X_s^{i,n}-Y_s^{i,n})\left(a_n\left(\sum_{j=1}^n 1_{\{Y_s^{j,n}\le Y_s^{i,n}\}}\right)-a_n\left(\sum_{j=1}^n 1_{\{X_s^{j,n}\le X_s^{i,n}\}}\right)\right)ds.$$

By the Girsanov theorem, for any $s>0$ the distributions of $(X_s^{1,n},\dots,X_s^{n,n})$ and $(Y_s^{1,n},\dots,Y_s^{n,n})$ admit densities w.r.t. the Lebesgue measure on $\mathbb{R}^n$ and therefore $d\mathbb{P}\otimes ds$ a.e. the positions $X_s^{1,n},\dots,X_s^{n,n}$ (resp. $Y_s^{1,n},\dots,Y_s^{n,n}$) are distinct and there is a unique permutation $\tau_s^X\in\mathcal{S}_n$ (resp. $\tau_s^Y\in\mathcal{S}_n$) such that $X_s^{\tau_s^X(1),n}<X_s^{\tau_s^X(2),n}<\dots<X_s^{\tau_s^X(n),n}$ (resp. $Y_s^{\tau_s^Y(1),n}<Y_s^{\tau_s^Y(2),n}<\dots<Y_s^{\tau_s^Y(n),n}$). Therefore $d\mathbb{P}\otimes ds$ a.e.,

$$\sum_{i=1}^n (X_s^{i,n}-Y_s^{i,n})\left(a_n\left(\sum_{j=1}^n 1_{\{Y_s^{j,n}\le Y_s^{i,n}\}}\right)-a_n\left(\sum_{j=1}^n 1_{\{X_s^{j,n}\le X_s^{i,n}\}}\right)\right)$$

is equal to

$$\sum_{i=1}^n a_n(i)((X_s^{\tau_s^Y(i),n}-Y_s^{\tau_s^Y(i),n})-(X_s^{\tau_s^X(i),n}-Y_s^{\tau_s^X(i),n})).$$

The sequence $(a_n(i))_{1\le i\le n}$ is nondecreasing. Applying Lemma 1.4 with $b(i)=X_s^{\tau_s^X(i),n}$ and $\tau=(\tau_s^X)^{-1}\circ\tau_s^Y$ then with $b(i)=Y_s^{\tau_s^Y(i),n}$ and $\tau=(\tau_s^Y)^{-1}\circ\tau_s^X$, one obtains that the integrand in (8) is nonpositive $d\mathbb{P}\otimes ds$ a.e. Hence (5) holds.

Let us now suppose that a.s. $\forall i\in\{1,\dots,n\},\ X_0^i<Y_0^i$ and define $\nu=\inf\{t>0:\exists i\in\{1,\dots,n\}, X_t^{i,n}\ge Y_t^{i,n}\}$ with the convention $\inf\varnothing=+\infty$. From now on, we restrict ourselves to the event $\{\nu<+\infty\}$. Let $i\in\{1,\dots,n\}$ be such that $Y_\nu^{i,n}=X_\nu^{i,n}$. There is an increasing sequence $(s_k)_{k\ge1}$ of positive times with limit ν such that $\forall k\ge1, a_n(\sum_{j=1}^n 1_{\{X_{s_k}^{j,n}\le X_{s_k}^{i,n}\}})<a_n(\sum_{j=1}^n 1_{\{Y_{s_k}^{j,n}\le Y_{s_k}^{i,n}\}})$. Since $(a_n(i))_{1\le i\le n}$ is nondecreasing, by extracting a subsequence still denoted by $(s_k)_k$ for simplicity, one deduces the existence of $j\in\{1,\dots,n\}$ with $j\ne i$ such that $\forall k\ge1, X_{s_k}^{i,n}<X_{s_k}^{j,n}$ and $Y_{s_k}^{j,n}\le Y_{s_k}^{i,n}$. Since $s_k<\nu$, $X_{s_k}^{i,n}<X_{s_k}^{j,n}<Y_{s_k}^{j,n}\le Y_{s_k}^{i,n}$. By continuity of the paths, one obtains $X_\nu^{i,n}=X_\nu^{j,n}=Y_\nu^{j,n}=Y_\nu^{i,n}$. Now since the probability of the event

$$\exists i_1,i_2,i_3 \text{ dist. in } \{1,\dots,n\},\exists t>0 \qquad X_0^{i_1}+\sigma B_t^{i_1}=X_0^{i_2}+\sigma B_t^{i_2}=X_0^{i_3}+\sigma B_t^{i_3}$$

is equal to 0, the Girsanov theorem implies that a.s. $\forall l\in\{1,\dots,n\}\setminus\{i,j\}$, $X_\nu^{l,n}\ne X_\nu^{i,n}=X_\nu^{j,n}$. In the same way, $Y_\nu^{l,n}\ne Y_\nu^{i,n}=Y_\nu^{j,n}$. By continuity of the paths and definition of ν one deduces that for k large enough, and for every $t\in[s_k,\nu]$,

$$\sum_{\substack{l=1\\l\ne i,j}}^n 1_{\{Y_t^{l,n}\le Y_t^{i,n}\}}\le\sum_{\substack{l=1\\l\ne i,j}}^n 1_{\{X_t^{l,n}\le X_t^{i,n}\}};\ \sum_{\substack{l=1\\l\ne i,j}}^n 1_{\{Y_t^{l,n}\le Y_t^{j,n}\}}\le\sum_{\substack{l=1\\l\ne i,j}}^n 1_{\{X_t^{l,n}\le X_t^{j,n}\}}.$$

Since a.s. dt a.e., $Y_t^{i,n} \neq Y_t^{j,n}$ and $(a_n(i))_{1\le i\le n}$ is nondecreasing, one obtains that a.s. dt a.e. on $[s_k, \nu]$,

$$a_n\left(\sum_{l=1}^n 1_{\{Y_t^{l,n}\le Y_t^{i,n}\}}\right) + a_n\left(\sum_{l=1}^n 1_{\{Y_t^{l,n}\le Y_t^{j,n}\}}\right) \\ \le a_n\left(\sum_{l=1}^n 1_{\{X_t^{l,n}\le X_t^{j,n}\}}\right) + a_n\left(\sum_{l=1}^n 1_{\{X_t^{l,n}\le X_t^{i,n}\}}\right).$$

By integration with respect to t on $[s_k, \nu]$, this implies that a.s. $Y_\nu^{i,n} - X_\nu^{i,n} + Y_\nu^{j,n} - X_\nu^{j,n} \ge Y_{s_k}^{i,n} - X_{s_k}^{i,n} + Y_{s_k}^{j,n} - X_{s_k}^{j,n} > 0$. Therefore $\mathbb{P}(\nu < +\infty) = 0$.

When a.s. for $i \in \{1, \ldots, n\}$, $X_0^i \le Y_0^i$, one obtains that for $\varepsilon > 0$ the solution $(Y_t^{1,n,\varepsilon}, \ldots, Y_t^{n,n,\varepsilon})$ to (4) starting from $(Y_0^1 + \varepsilon, \ldots, Y_0^n + \varepsilon)$ is such that

$$\text{a.s., } \forall t \ge 0\ \forall i \in \{1, \ldots, n\} \qquad X_t^{i,n} < Y_t^{i,n,\varepsilon}.$$

Since by (5), $Y_t^{i,n,\varepsilon} \le Y_t^{i,n} + \sqrt{n}\varepsilon$, one easily concludes by letting $\varepsilon \to 0$. □

1.3. *Trajectorial propagation of chaos.* From now on, we set

$$\forall n \in \mathbb{N}^*, \forall i \in \{1, \ldots, n\} \qquad a_n(i) = n\left(A\left(\frac{i}{n}\right) - A\left(\frac{i-1}{n}\right)\right) \tag{9}$$

and assume that the initial positions $(X_0^i)_{i\ge 1}$ of the particles are independent and identically distributed according to m. We prefer to define $a_n(i)$ with the above finite difference approximation of the choice $A'(i/n)$ made in [8] because the sum $\sum_{i=1}^n a_n(i)$ which plays a role in the long time behavior of the particle system is then simply equal to $nA(1)$. One could also obtain trajectorial propagation of chaos estimates similar to Theorem 1.5 below for the choice $a_n(i) = A'(i/n)$.

In the present section, we also suppose that A is a convex function on $[0, 1]$. By Theorem 1.1, for each $i \ge 1$, the nonlinear stochastic differential equation

$$\begin{cases} X_t^i = X_0^i + \sigma B_t^i - \displaystyle\int_0^t A'(H * P_s(X_s^i))\, ds, \\ \forall t \ge 0, \text{the law of } X_t^i \text{ is } P_t, \end{cases} \tag{10}$$

has a unique solution and for all $t \ge 0$, the law P_t of X_t^i does not depend on i. Under a Lipschitz regularity assumption on A', we obtain the following trajectorial propagation of chaos estimation.

THEOREM 1.5. *If $A : [0, 1] \to \mathbb{R}$ is convex and A' is Lipschitz continuous with constant K, then*

$$\forall n \ge 1, \forall 1 \le i \le n, \forall t \ge 0 \qquad \mathbb{E}\left(\sup_{s\in[0,t]} (X_s^{i,n} - X_s^i)^2\right) \le \frac{K^2 t^2}{6n}.$$

PROOF. Let us write $\sum_{i=1}^n (X_t^{i,n} - X_t^i)^2$ as

$$2\int_0^t \sum_{i=1}^n (X_s^{i,n} - X_s^i)\left(a_n\left(\sum_{j=1}^n 1_{\{X_s^j \le X_s^i\}}\right) - a_n\left(\sum_{j=1}^n 1_{\{X_s^{j,n} \le X_s^{i,n}\}}\right)\right) ds$$

$$+ 2\int_0^t \sum_{i=1}^n (X_s^{i,n} - X_s^i) C(s, X_s^1, \ldots, X_s^n)\, ds$$

where $C(s, X_s^1, \ldots, X_s^n)$ is equal to

$$A'(H * P_s(X_s^i)) - n\left(A\left(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j \le X_s^i\}}\right) - A\left(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j \le X_s^i\}} - \frac{1}{n}\right)\right).$$

Like in the proof of trajectorial uniqueness for (4), because of the convexity of A, the first term of the r.h.s. is nonpositive. Moreover, by Lipschitz continuity of A',

$$\left(A'(H * P_s(X_s^i)) - n\left(A\left(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j \le X_s^i\}}\right) - A\left(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j \le X_s^i\}} - \frac{1}{n}\right)\right)\right)^2$$

$$= \left(\int_0^1 A'(H * P_s(X_s^i)) - A'\left(\frac{1}{n}\sum_{j=1}^n 1_{\{X_s^j \le X_s^i\}} + \frac{\theta - 1}{n}\right) d\theta\right)^2$$

$$\le \frac{K^2}{n^2}\int_0^1 \left(\sum_{j \ne i}\left(H * P_s(X_s^i) - 1_{\{X_s^j \le X_s^i\}}\right) + (H * P_s(X_s^i) - \theta)\right)^2 d\theta.$$

For $s > 0$, as the variables X_s^i are i.i.d. with common law P_s which does not weight points and $H * P_s(X_s^i)$ is uniformly distributed on $[0,1]$,

$$\int_0^1 \mathbb{E}\left(\left(\sum_{j \ne i}(H * P_s(X_s^i) - 1_{\{X_s^j \le X_s^i\}}) + (H * P_s(X_s^i) - \theta)\right)^2\right) d\theta$$

$$= \sum_{j \ne i}\mathbb{E}((H * P_s(X_s^i) - 1_{\{X_s^j \le X_s^i\}})^2) + \int_0^1 \mathbb{E}((H * P_s(X_s^i) - \theta)^2)\, d\theta$$

$$= (n-1)\mathbb{E}((H * P_s(X_s^i))(1 - H * P_s(X_s^i))) + 1/6$$

$$= n/6.$$

Using the Cauchy–Schwarz inequality, one obtains

$$\mathbb{E}\left(\sup_{s \in [0,t]}\sum_{i=1}^n (X_s^{i,n} - X_s^i)^2\right) \le 2\int_0^t \sqrt{\frac{K^2}{6n}\mathbb{E}\left(\left(\sum_{i=1}^n (X_s^{i,n} - X_s^i)\right)^2\right)}\, ds$$

$$\le \frac{2K}{\sqrt{6}}\int_0^t \sqrt{\mathbb{E}\left(\sup_{u \in [0,s]}\sum_{i=1}^n (X_u^{i,n} - X_u^i)^2\right)}\, ds.$$

By comparison with the ordinary differential equation $\alpha'(t) = 2K\sqrt{\frac{\alpha(t)}{6}}$, one concludes that

$$\forall t \geq 0 \qquad \mathbb{E}\left(\sup_{s\in[0,t]} \sum_{i=1}^{n} (X_s^{i,n} - X_s^i)^2\right) \leq \frac{K^2t^2}{6}.$$

Exchangeability of the couples $((X^{i,n}, X^i))_{i\in\{1,\ldots,n\}}$ completes the proof. $\square$

REMARK 1.6. One could think that assuming that A is uniformly convex:

$$\exists \alpha > 0, \forall 0 \leq x \leq y \leq 1 \qquad A'(y) - A'(x) \geq \alpha(y - x) \tag{11}$$

would lead to a better estimation. Indeed, then for every $i \in \{1,\ldots,n-1\}$,

$$a_n(i+1) - a_n(i) = n\int_{i/n}^{(i+1)/n} \left[A'(x) - A'\left(x - \frac{1}{n}\right)\right] dx \geq \frac{\alpha}{n}.$$

But since even in this situation, the nonpositive term

$$\sum_{i=1}^{n} (X_s^{i,n} - X_s^i)\left(a_n\left(\sum_{j=1}^{n} 1_{\{X_s^j \leq X_s^i\}}\right) - a_n\left(\sum_{j=1}^{n} 1_{\{X_s^{j,n} \leq X_s^{i,n}\}}\right)\right)$$

vanishes as soon as the order between the coordinates of $(X_s^{1,n}, \ldots, X_s^{n,n})$ is the same as the order between the coordinates of $(X_s^1, \ldots, X_s^n)$, we were not able so far to improve the estimation.

COROLLARY 1.7. *Under the hypotheses of Theorem 1.5, let $\tilde{m}$ be a probability measure on $\mathbb{R}$ such that $\forall x \in \mathbb{R}$, $H * \tilde{m}(x) \leq H * m(x)$. If for some random variable U_1 uniform on $[0,1]$ independent from $(B^i)_{i\geq 1}$, $X_0^1 = \inf\{x : H * m(x) \geq U_1\}$ and $(Y_t^1)_{t\geq 0}$ denotes the solution of the nonlinear stochastic differential equation*

$$\begin{cases} Y_t^1 = Y_0^1 + \sigma B_t^1 - \int_0^t A'(H * \tilde{P}_s(Y_s^1))\, ds, \\ \forall t \geq 0, \textit{the law of } Y_t^1 \textit{ is } \tilde{P}_t, \end{cases} \tag{12}$$

with $Y_0^1 = \inf\{x : H * \tilde{m}(x) \geq U_1\}$, then

$$\mathbb{P}(\forall t \geq 0, X_t^1 \leq Y_t^1) = 1.$$

Moreover $\forall t \geq 0$, $\forall x \in \mathbb{R}$, $H * \tilde{P}_t(x) \leq H * P_t(x)$. Last, the function $t \mapsto \mathbb{E}|Y_t^1 - X_t^1|$ is constant.

REMARK 1.8. At least when m and $\tilde{m}$ do not weight points, one has a.s. $A'(H * P_0(X_0^1)) = A'(H * \tilde{P}_0(Y_0^1))$ since $H * m(X_0^1) = H * \tilde{m}(Y_0^1) = U_1$. Therefore a.s. $d(Y^1 - X^1)_0 = 0$ and one may wonder whether a.s. $Y_t^1 -$

X_t^1 does not depend on t. If this property holds, necessarily, a.s. dt a.e. $A'(H*P_t(X_t^1)) = A'(H*\tilde{P}_t(Y_t^1))$. If A' is increasing, a.s. for all $t>0$, $H*p_t(X_t^1) = H*\tilde{p}_t(Y_t^1)$ with p_t and $\tilde{p}_t$ denoting the respective densities of P_t and $\tilde{P}_t$. If A is C^2, the Brownian contribution in $d(H*p_t(X_t^1) - H*\tilde{p}_t(Y_t^1))$ given by Itô's formula vanishes, that is, $p_t(X_t^1) = \tilde{p}_t(Y_t^1)$ and $\forall u \in]0,1[$, $p_t((H*p_t)^{-1}(u)) = \tilde{p}_t((H*\tilde{p}_t)^{-1}(u))$ or equivalently $((H*p_t)^{-1})'(u) = ((H*\tilde{p}_t)^{-1})'(u)$. Hence $Y_t^1 = X_t^1 + c$ for a deterministic constant c which does not depend on t according to (3). Letting $t \to 0$, one obtains $Y_0^1 = X_0^1 + c$. This necessary condition turns out to be sufficient as $(X_t^1+c)_{t\geq 0}$ obviously solves the nonlinear stochastic differential equation (2) starting from $X_0^1 + c$.

Proof of Corollary 1.7. For $(U_i)_{i\geq 2}$ a sequence of independent uniform random variables independent from $(U_1,(B^i)_{i\geq 1})$, we set

$$\forall i \geq 2 \qquad X_0^i = \inf\{x : H*m(x) \geq U_i\} \quad \text{and} \quad Y_0^i = \inf\{x : H*\tilde{m}(x) \geq U_i\}.$$

Since $H*\tilde{m} \leq H*m$, a.s. $\forall i \geq 1$, $Y_0^i \geq X_0^i$. From Proposition 1.3, one deduces that the solutions $(X_t^{1,n},\dots,X_t^{n,n})$ and $(Y_t^{1,n},\dots,Y_t^{n,n})$ to (4) respectively starting from $(X_0^1,\dots,X_0^n)$ and $(Y_0^1,\dots,Y_0^n)$ are such that

$$\text{a.s., } \forall n \geq 1, \forall i \in \{1,\dots,n\}, \forall t \geq 0 \qquad Y_t^{i,n} \geq X_t^{i,n}.$$

Since, by Theorem 1.5, for fixed $t \geq 0$, one may extract from $(X_t^{1,n}, Y_t^{1,n})_{n\geq 1}$ a subsequence almost surely converging to (X_t^1, Y_t^1), one easily deduces that $\mathbb{P}(\forall t \geq 0, X_t^1 \leq Y_t^1) = 1$. Hence

$$\forall t \geq 0, \forall x \in \mathbb{R} \qquad H*\tilde{P}_t(x) = \mathbb{P}(Y_t^1 \leq x) \leq \mathbb{P}(X_t^1 \leq x) = H*P_t(x).$$

Since $|Y_t^1 - X_t^1| - |Y_0^1 - X_0^1| = Y_t^1 - Y_0^1 - (X_t^1 - X_0^1)$, (3) ensures that $\mathbb{E}|Y_t^1 - X_t^1| \in [0,+\infty]$ does not depend on t. $\square$

2. Long time behavior. In this section we are interested in the long time behavior of both the nonlinear process and the particle system. According to (3) and the equality $\sum_{i=1}^n a_n(i) = nA(1)$ which follows from (9), we have to suppose $A(1) = 0$ in order to obtain convergence of the densities as t tends to infinity. We address the convergence of the density p_t of X_t by first studying the convergence of the associated cumulative distribution function F_t under the following hypothesis denoted by (H) in the sequel:

$$\text{(H)} \qquad \begin{gathered} A(0) = A(1) = 0, \qquad A'(0) < 0, \\ A'(1) > 0 \quad \text{and} \quad \forall u \in (0,1) \qquad A(u) < 0. \end{gathered}$$

These assumptions determine the spatial behavior at infinity of the drift coefficient in (2).

To prove exponential convergence of the density of the particle system uniform in the number n of particles, we make the stronger assumption of uniform convexity on A. The key step in the proof is to obtain a Poincaré inequality uniform in n for the stationary density of the particle system. This density has exponential-like tails and therefore does not satisfy a logarithmic Sobolev inequality. So the derivation of the Poincaré inequality cannot rely on the curvature criterion, used, for instance, by Malrieu [12, 13] when dealing with the granular media equation. Instead, we take advantage of the following nice feature: up to reordering of the coordinates, the stationary density is the density of the image by a linear transformation of a vector of independent exponential variables. And it turns out that the control of the constant in the n-dimensional Poincaré inequality relies on the Hardy inequality stated in Lemma 2.18 which is a one-dimensional Poincaré-like inequality. To our knowledge, our study provides the first example of a particle system, for which a Poincaré inequality but no logarithmic Sobolev inequality holds uniformly in the number n of particles.

2.1. *The nonlinear process.* In this section, we are first going to obtain necessary and sufficient conditions on the function A ensuring existence for the stationary Fokker–Planck equation obtained by removing the time-derivative in the nonlinear Fokker–Planck equation

$$\partial_t p_t = \frac{\sigma^2}{2}\partial_{xx} p_t + \partial_x(A'(H * p_t)p_t) \tag{13}$$

satisfied by the density of the solution of (2). Under a slightly stronger condition, the solutions satisfy a Poincaré inequality.

LEMMA 2.1. *A necessary and sufficient condition for the existence of a probability measure μ solving the stationary Fokker–Planck equation*

$$\frac{\sigma^2}{2}\partial_{xx}\mu + \partial_x(A'(H * \mu(x))\mu) = 0$$

in the distribution sense is $A(1) = 0$ and $A(u) < 0$ for all $u \in (0,1)$. Under that condition, all the solutions are the translations of a probability measure with a C^1 density f which satisfies

$$\begin{aligned} \forall x \in \mathbb{R} \qquad f(x) &= -\frac{2}{\sigma^2}A(H * f(x)) \quad \text{and} \\ f'(x) &= -\frac{2}{\sigma^2}A'(H * f(x))f(x). \end{aligned} \tag{14}$$

If $A'(0) < 0$ and $A'(1) > 0$, then

$$f(x) \sim \begin{cases} -\dfrac{2A'(0)}{\sigma^2}\displaystyle\int_{-\infty}^{x} f(y)\,dy, & \text{when } x \to -\infty, \\ \dfrac{2A'(1)}{\sigma^2}\displaystyle\int_{x}^{+\infty} f(y)\,dy, & \text{when } x \to +\infty, \end{cases}$$

(15)
$$\int_0^x \frac{dy}{f(y)} \sim \begin{cases} \dfrac{-\sigma^2}{2A'(0)f(x)}, & \text{when } x \to -\infty, \\ \dfrac{\sigma^2}{2A'(1)f(x)}, & \text{when } x \to +\infty, \end{cases}$$

and all the solutions satisfy a Poincaré inequality and have a finite expectation. Last, if the function A is C^2 on $[0,1]$, then f is C^2 and satisfies

$$f''(x) = -\frac{2}{\sigma^2} A''(H * f(x)) f^2(x) + \frac{f'^2(x)}{f(x)}. \tag{16}$$

Proof. Let μ be a probability measure on $\mathbb{R}$ solving the stationary Fokker–Planck equation. The equality $\frac{\sigma^2}{2}\partial_{xx}\mu = -\partial_x(A'(H*\mu(x))\mu)$ ensures that μ does not weight points. Hence the stationary equation is equivalent to $\partial_{xx}(\frac{\sigma^2}{2}\mu + A(H*\mu(x))) = 0$. One deduces that μ possesses a C^1 density f such that

$$\forall x \in \mathbb{R} \qquad f(x) = -\frac{2}{\sigma^2} A(H * f(x)) + \alpha x + \beta, \tag{17}$$

for some constants α and β. Since $A(0) = 0$, letting $x \to -\infty$ then $x \to +\infty$ in the last equality, one obtains $\alpha = \beta = A(1) = 0$. For $u \in (0,1)$, since $u = H * f(x)$ for some $x \in \mathbb{R}$ and $H * f$ is not constant and equal to u, the Cauchy–Lipschitz theorem and (17) imply that $A(u) \neq 0$. Since f is nonnegative, $A(u) < 0$. Hence $A(1) = 0$ and $A(u) < 0$ for all $u \in (0,1)$ is a necessary condition.

Under that condition, a probability measure μ solves the stationary Fokker–Planck equation if and only if its cumulative distribution function $H * \mu(x)$ is a C^2 solution to the differential equation

$$\varphi'(x) = -\frac{2}{\sigma^2} A(\varphi(x)), \qquad x \in \mathbb{R}. \tag{18}$$

By the Cauchy–Lipschitz theorem, for each $v \in [0,1]$ this equation admits a unique solution φ_v defined on $\mathbb{R}$ with values in $[0,1]$ such that $\varphi_v(0) = v$. Moreover, as $A(0) = A(1) = 0$, $\varphi_0 \equiv 0$ and $\varphi_1 \equiv 1$ and

$$\forall v \in (0,1), \forall x \in \mathbb{R} \qquad 0 < \varphi_v(x) < 1. \tag{19}$$

For $v \in (0,1)$, since φ_v is nondecreasing and $\varphi_v(x) = v - \frac{2}{\sigma^2}\int_0^x A(\varphi_v(y))\,dy$, necessarily $\lim_{y\to+\infty} \varphi_v(y) = 1$. In the same way, $\lim_{y\to-\infty}\varphi_v(y) = 0$ and φ_v is an increasing function from $\mathbb{R}$ to $(0,1)$ with inverse denoted by φ_v^{-1}. The uniqueness result for (18) implies that $\forall v \in (0,1), \forall x \in \mathbb{R}, \varphi_v(x) = \varphi_{1/2}(x + \varphi_{1/2}^{-1}(v))$. Therefore the solutions to the stationary Fokker–Planck equation are the probability measures obtained by spatial translation of the probability measure with density $f(x) = \varphi'_{1/2}(x)$ which satisfies (14) according to (18).

Let us now suppose that $A'(0) < 0$ and $A'(1) > 0$. When $x \to +\infty$,

$$f(x) = -\frac{2}{\sigma^2} A\left(1 - \int_x^{+\infty} f(y)\, dy\right) \sim \frac{2A'(1)}{\sigma^2} \int_x^{+\infty} f(y)\, dy.$$

By (14), $\frac{f'(x)}{f(x)} = (\log f(x))' = -\frac{2}{\sigma^2} A'(\varphi_{1/2}(x))$ converges to $-\frac{2A'(1)}{\sigma^2}$ as $x \to +\infty$. This implies that $\frac{\log(f(x))}{x}$ converges to $-\frac{2A'(1)}{\sigma^2}$ and that $xf(x)1_{\{x \geq 0\}}$ is integrable. Moreover, since $\int_0^{+\infty} \frac{dy}{f(y)} = +\infty$, $\int_0^x \frac{dy}{f(y)} \sim \frac{\sigma^2}{2A'(1)} \int_0^x -\frac{f'(y)}{f^2(y)}\, dy \sim \frac{\sigma^2}{2A'(1)f(x)}$, as $x \to +\infty$. In the same way, one obtains the equivalents given in (15) when $x \to -\infty$ and checks the integrability of the function $xf(x)1_{\{x \leq 0\}}$. From (15), one has

$$\lim_{x \to -\infty} \int_{-\infty}^x f(y)\, dy \int_x^0 \frac{dy}{f(y)} = \frac{\sigma^4}{4(A'(0))^2}$$

and

$$\lim_{x \to +\infty} \int_x^{+\infty} f(y)\, dy \int_0^x \frac{dy}{f(y)} = \frac{\sigma^4}{4(A'(1))^2}.$$

By Theorem 6.2.2, page 99 of [1], one concludes that the measure with density f satisfies a Poincaré inequality.

By (14), the function f is C^2 as soon as the function A is C^2 on $[0,1]$. Moreover, $f''(x) = -\frac{2}{\sigma^2} A''(H * f(x)) f^2(x) - \frac{2}{\sigma^2} A'(H * f(x)) f'(x)$ which combined with (14) implies (16). $\square$

REMARK 2.2. When A is a C^1 convex function on $[0,1]$ such that $A(0) = A(1) = 0$ and $A'(u) < 0$ for some $u \in (0,1)$, then the necessary and sufficient condition in Lemma 2.1 is obviously satisfied. Since (14) implies

$$\begin{aligned}(\log f(x))'' &= \left(\frac{f'(x)}{f(x)}\right)' = \left(\frac{-2/\sigma^2 A'(H * f(x)) f(x)}{f(x)}\right)' \\ &= -\frac{2}{\sigma^2} A''(H * f(x)) f(x) \leq 0,\end{aligned}$$

the probability measures solving the stationary Fokker–Planck equation admit log-concave densities with respect to the Lebesgue measure. Log-concavity is a property stronger than the existence of a Poincaré inequality (see [7]).

EXAMPLE 2.3. Using (18) and (19), the following two choices for A lead to exact computations and different tails for the stationary densities:

- if $A(x) = \frac{1}{2}x(x-1)$, one gets $\log(\frac{\varphi_{1/2}(x)}{1-\varphi_{1/2}(x)}) = x/\sigma^2$, that is,

$$\varphi_{1/2}(x) = \frac{e^{x/\sigma^2}}{1+e^{x/\sigma^2}} \quad \text{and} \quad \varphi'_{1/2}(x) = \frac{1}{4\sigma^2\cosh^2(x/2\sigma^2)};$$

- if $A(x) = x^3 - x = x(x-1)(x+1)$,

$$\varphi_{\sqrt{1/2}}(x) = \frac{1}{\sqrt{1+e^{-4x/\sigma^2}}} \quad \text{and} \quad \varphi'_{\sqrt{1/2}}(x) = \frac{2e^{-4x/\sigma^2}}{\sigma^2(1+e^{-4x/\sigma^2})^{3/2}}.$$

When $A(1) = 0$ and $A(u) < 0$ for all $u \in (0,1)$, a natural question is how to link the translation parameter of the candidate long time limit of the marginal P_t solving the stationary Fokker–Planck equation to the initial marginal m. When $\int_{\mathbb{R}} |x| m(dx) < +\infty$, by (3), for all $t \geq 0$, $\mathbb{E}(X_t^1) = \mathbb{E}(X_0^1)$. Therefore the translation parameter is chosen in order to ensure that the invariant measure has the same mean as the initial measure m.

Let us denote by p_t the density of P_t and by $F_t = H * P_t$ its cumulative distribution function.

THEOREM 2.4. *Let A be C^2 on $[0,1]$ satisfying (H). Assume that m admits a density p_0 such that $\int_{\mathbb{R}} |x| p_0(x)\,dx < +\infty$ and $\int_{\mathbb{R}} \frac{(p_0(x)-p_\infty(x))^2}{p_\infty(x)}\,dx$ is small enough where p_∞ denotes the stationary distribution with same expectation as p_0. Last, we suppose that A and p_0 are such that p is a smooth solution of (13). Then $\int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx$ converges to 0 exponentially fast as $t \to +\infty$.*

By a smooth solution of (13), we mean that p possesses enough regularity and integrability so that the formal computations made in the proof below are justified.

EXAMPLE 2.5. When $A(x) = \frac{1}{2}(x^2 - x)$, one easily checks that the function $\phi(t,x) = -F_t(x + \frac{t}{2})$ solves Burgers' equation

$$\partial_t \phi = \frac{\sigma^2}{2}\partial_{xx}\phi - \frac{1}{2}\partial_x \phi^2, \qquad \phi(0,x) = -F_0(x).$$

By the Cole–Hopf transformation, $\psi(t,x) = \exp(-\frac{1}{\sigma^2}\int_{-\infty}^x \phi(t,y)\,dy)$ solves the heat equation

$$\partial_t \psi = \frac{\sigma^2}{2}\partial_{xx}\psi, \qquad \psi(0,x) = \exp\biggl(\frac{1}{\sigma^2}\int_{-\infty}^x F_0(y)\,dy\biggr).$$

Since $F_t(x) = \sigma^2 \frac{\partial_x \psi}{\psi}(t, x - \frac{t}{2})$, one deduces that

$$F_t(x) = \frac{\int_{\mathbb{R}} e^{-(x-t/2-y)^2/2\sigma^2 t} F_0(y)\psi(0,y)\,dy/(\sigma\sqrt{2\pi t})}{\int_{\mathbb{R}} e^{-(x-t/2-y)^2/2\sigma^2 t}\psi(0,y)\,dy/(\sigma\sqrt{2\pi t})}. \tag{20}$$

If $\bar{x}$ denotes the expectation associated with the cumulative distribution function F_0, one has $\int_{-\infty}^{\bar{x}} F_0(z)\,dz = \int_{\bar{x}}^{+\infty}(1-F_0(z))\,dz$. Since

$$\int_{-\infty}^{x} F_0(z)\,dz = \int_{-\infty}^{\bar{x}} F_0(z)\,dz - \int_{\bar{x}}^{x}(1-F_0(z))\,dz + (x-\bar{x}),$$

one deduces that the function $\tilde{\psi}(0,x) = e^{-(x-\bar{x})/\sigma^2}\psi(0,x)$ [resp. $\psi(0,x)$] is bounded on $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) and converges to 1 as x tends to $+\infty$ (resp. $-\infty$).

Let us deduce the limit of $F_t(x)$ as $t \to +\infty$. Writing the integral for $y \in \mathbb{R}$ as the sum of the integrals for $y \in \mathbb{R}_-$ and for $y \in \mathbb{R}_+$, and making the change of variables $z = \frac{y-x+t/2}{\sigma\sqrt{t}}$ (resp. $z = \frac{y-x-t/2}{\sigma\sqrt{t}}$) in the first (resp. second) integral, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}} e^{-(y-x+t/2)^2/(2\sigma^2 t)} F_0(y)\psi(0,y)\frac{dy}{\sigma\sqrt{2\pi t}} \\
&\quad = \int_{\mathbb{R}} e^{-z^2/2} 1_{\{z \le \sqrt{t}/(2\sigma) - x/(\sigma\sqrt{t})\}} \\
&\qquad\quad \times F_0\Big(\sigma\sqrt{t}z + x - \frac{t}{2}\Big)\psi\Big(0, \sigma\sqrt{t}z + x - \frac{t}{2}\Big)\frac{dz}{\sqrt{2\pi}} \\
&\qquad + e^{(x-\bar{x})/\sigma^2}\int_{\mathbb{R}} e^{-z^2/2} 1_{\{z \ge -\sqrt{t}/(2\sigma) - x/(\sigma\sqrt{t})\}} \\
&\qquad\qquad\quad \times F_0\Big(\sigma\sqrt{t}z + x + \frac{t}{2}\Big)\tilde{\psi}\Big(0, \sigma\sqrt{t}z + x + \frac{t}{2}\Big)\frac{dz}{\sqrt{2\pi}}.
\end{aligned}$$

By the Lebesgue theorem, the first term of the right-hand side converges to 0 whereas the second term converges to $e^{(x-\bar{x})/\sigma^2}$. Replacing F_0 by 1 in the above computation, one obtains that the denominator in (20) converges to $1 + e^{(x-\bar{x})/\sigma^2}$. Therefore

$$\forall x \in \mathbb{R} \qquad \lim_{t\to+\infty} F_t(x) = \frac{e^{(x-\bar{x})/\sigma^2}}{1+e^{(x-\bar{x})/\sigma^2}}.$$

Notice that in the same way, one may also obtain the limit of the density

$$\begin{aligned}
p_t(x) = {}& \frac{\int_{\mathbb{R}}((y+t/2-x)/(\sigma^2 t))e^{-(x-t/2-y)^2/(2\sigma^2 t)}F_0(y)\psi(0,y)\,dy/(\sigma\sqrt{2\pi t})}{\int_{\mathbb{R}} e^{-(x-\frac{t}{2}-y)^2/(2\sigma^2 t)}\psi(0,y)\,dy/(\sigma\sqrt{2\pi t})} \\
& - \frac{1}{\sigma^2}\left(\frac{\int_{\mathbb{R}} e^{-(x-t/2-y)^2/(2\sigma^2 t)}F_0(y)\psi(0,y)\,dy/(\sigma\sqrt{2\pi t})}{\int_{\mathbb{R}} e^{-(x-t/2-y)^2/(2\sigma^2 t)}\psi(0,y)\,dy(\sigma\sqrt{2\pi t})}\right)^2.
\end{aligned}$$

One easily checks

$$\begin{aligned}
\forall x \in \mathbb{R} \qquad \lim_{t\to+\infty} p_t(x) &= \frac{1}{\sigma^2}\left(\frac{e^{(x-\bar{x})/\sigma^2}}{1+e^{(x-\bar{x})/\sigma^2}} - \frac{e^{2(x-\bar{x})/\sigma^2}}{(1+e^{(x-\bar{x})/\sigma^2})^2}\right) \\
&= \frac{1}{4\sigma^2\cosh^2((x-\bar{x})/2\sigma^2)}.
\end{aligned}$$

In order to prove Theorem 2.4, we are first going to check exponential convergence of F_t to the cumulative distribution function F_∞ of p_∞. Let $G_t = F_t - F_\infty$. Since for a random variable X with cumulative distribution function F, $\mathbb{E}(X) = \int_0^{+\infty}(1-F(x))\,dx - \int_{-\infty}^0 F(x)\,dx$, the equality of the expectations associated to F_t and F_∞ writes $\int_{\mathbb{R}} G_t(x)\,dx = 0$. This very convenient expression of the link between p_t and p_∞ is one main reason for first considering the convergence of G_t to 0. In order to prove this convergence, we need the following result.

Lemma 2.6. *Under the assumptions of Theorem 2.4, one has*

$$\int_{\mathbb{R}} \frac{G_t^2(x)}{p_\infty(x)}\,dx \le c \int_{\mathbb{R}} \left(\frac{G_t(x)}{p_\infty(x)}\right)'^2 p_\infty(x)\,dx \tag{21}$$

where c denotes the constant in the Poincaré inequality satisfied by p_∞. Moreover

$$\begin{aligned}\int_{\mathbb{R}} &\frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx \\ &= \int_{\mathbb{R}} \left(\frac{G_t(x)}{p_\infty(x)}\right)'^2 p_\infty(x)\,dx + \frac{2}{\sigma^2}\int_{\mathbb{R}} G_t(x)^2 A''(F_\infty)(x)\,dx\end{aligned} \tag{22}$$

and

$$\int_{\mathbb{R}} \frac{G_t(x)^2}{p_\infty(x)}\,dx \le \tilde{c} \int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx. \tag{23}$$

Remark 2.7. When A is convex, (23) is a consequence of (22) and (21).

Proof of Lemma 2.6. As $\int_{\mathbb{R}} G_t(x)\,dx = 0$, (21) is the Poincaré inequality satisfied by p_∞ written for the function G_t/p_∞.

Since $(\frac{G_t(x)}{p_\infty(x)})' = \frac{G_t'(x)}{p_\infty(x)} - \frac{G_t(x)p_\infty'(x)}{p_\infty(x)^2}$, one has

$$\begin{aligned}\int_{\mathbb{R}} \left(\frac{G_t(x)}{p_\infty(x)}\right)'^2 p_\infty(x)\,dx &= \int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx - \int_{\mathbb{R}} \frac{G_t^2(x)' p_\infty'(x)}{p_\infty^2(x)}\,dx \\ &\quad + \int_{\mathbb{R}} \frac{G_t^2(x) p_\infty'(x)^2}{p_\infty^3(x)}\,dx \\ &= \int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx + \int_{\mathbb{R}} \frac{G_t^2(x) p_\infty''(x)}{p_\infty^2(x)}\,dx \\ &\quad - \int_{\mathbb{R}} \frac{G_t^2(x) p_\infty'(x)^2}{p_\infty^3(x)}\,dx.\end{aligned}$$

Since p_∞ solves (16), one easily deduces (22).

Writing $G_t^2(y)$ as

$$2\Big(1_{\{y\le 0\}}\int_{-\infty}^{y} G_t(p_t-p_\infty)(x)\,dx - 1_{\{y>0\}}\int_{y}^{+\infty} G_t(p_t-p_\infty)(x)\,dx\Big),$$

one obtains

$$\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx = -2\int_{\mathbb{R}} G_t(p_t-p_\infty)(x)\int_0^x \frac{1}{p_\infty(y)}\,dy\,dx. \tag{24}$$

By (15), and since $\frac{1}{p_\infty}$ is bounded from below and above on each compact subset of the real line,

$$\exists C>0, \forall x\in\mathbb{R} \qquad \left|\int_0^x \frac{1}{p_\infty(y)}\,dy\right| \le \frac{C}{p_\infty(x)}.$$

Using the Cauchy–Schwarz inequality in (24), and inserting the latter bound, one obtains

$$\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx \le 2C\Big(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx\Big)^{1/2}.$$

One easily deduces (23). $\square$

According to (23), the exponential convergence of $\int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx$ to zero is a stronger result than the exponential convergence stated in the next lemma.

LEMMA 2.8. *Under the assumptions of Theorem 2.4, there is a positive constant C such that if $\int_{\mathbb{R}} \frac{G_0^2}{p_\infty}(x)\,dx$ is small enough, then*

$$\forall t\ge 0 \qquad \int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx \le \frac{e^{-Ct}}{C}\int_{\mathbb{R}} \frac{G_0^2}{p_\infty}(x)\,dx.$$

PROOF. According to (14), one has $\frac{\sigma^2}{2}F_\infty'' + (A(F_\infty))' = 0$ which also writes $\frac{p_\infty'}{p_\infty} = -\frac{2}{\sigma^2}A'(F_\infty)$. Combining these equations with (1), then using Young's inequality, one easily obtains for $\varepsilon > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx \\
&\quad = -\frac{\sigma^2}{2}\int_{\mathbb{R}} \Big(\frac{G_t(x)}{p_\infty(x)}\Big)'^2 p_\infty(x)\,dx \\
&\qquad - \int_{\mathbb{R}} (A(F_t)-A(F_\infty)-A'(F_\infty)G_t)(x)\Big(\frac{G_t(x)}{p_\infty(x)}\Big)'\,dx \\
&\quad \le \Big(\varepsilon - \frac{\sigma^2}{2}\Big)\int_{\mathbb{R}} \Big(\frac{G_t}{p_\infty}(x)\Big)'^2 p_\infty(x)\,dx + \frac{\|A''\|_\infty^2}{16\varepsilon}\int_{\mathbb{R}} \frac{G_t^4(x)}{p_\infty(x)}\,dx.
\end{aligned} \tag{25}$$

Since

$$\begin{aligned}\|G_t\|_\infty^2 &\le \left(\int_{\mathbb{R}} \frac{|p_t(x)-p_\infty(x)|}{\sqrt{p_\infty(x)}}\sqrt{p_\infty(x)}\,dx\right)^2\\ &\le \int_{\mathbb{R}} \frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx,\end{aligned}\tag{26}$$

$|G_t|$ is bounded by 1 and $p_\infty A''(F_\infty) = -\frac{2}{\sigma^2}A \times A''(F_\infty)$ is bounded, one deduces from (22) that

$$\|G_t\|_\infty^2 \le \frac{4}{\sigma^4}\|AA''\|_\infty \int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx + \left(1 \wedge \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx\right).$$

Inserting this bound in (25) and using Young's inequality, one deduces that for $\eta > 0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx\\ &\quad\le \left(\varepsilon - \frac{\sigma^2}{2}\right)\int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx\\ &\qquad + \frac{\|AA''\|_\infty\|A''\|_\infty^2}{4\varepsilon\sigma^4}\left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx\right)^2\\ &\qquad + \eta\left(1 \wedge \int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx\right)^2 + \frac{\|A''\|_\infty^4}{1024\varepsilon^2\eta}\left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx\right)^2\\ &\quad\le \left(\varepsilon + \eta - \frac{\sigma^2}{2}\right)\int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx\\ &\qquad + \left(\frac{\|AA''\|_\infty\|A''\|_\infty^2}{4\varepsilon\sigma^4} + \frac{\|A''\|_\infty^4}{1024\varepsilon^2\eta}\right)\left(\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx\right)^2.\end{aligned}$$

One easily concludes with (21) and Lemma 2.10 below. □

REMARK 2.9. (i) After reading this proof, one may wonder whether one could replace the upper bound in (25) by

$$\left(\varepsilon - \frac{\sigma^2}{2}\right)\int_{\mathbb{R}} \left(\frac{G_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx + \frac{\|A''\|_\infty^2}{16\varepsilon}\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx$$

using $\|G_t\|_\infty \le 1$. If the constant c in the Poincaré inequality (21) was smaller than $\frac{\sigma^4}{\|A''\|_\infty^2}$, one could deduce exponential convergence of $\int_{\mathbb{R}} \frac{G_t^2}{p_\infty}(x)\,dx$ to 0 even for large values of $\int_{\mathbb{R}} \frac{G_0^2}{p_\infty}(x)\,dx$. In case $A(x) = \frac{1}{2}(x^2 - x)$ (see Example

2.5), one has $\|A''\|_\infty = 1$ and

$$c \geq \int_{\mathbb{R}} x^2 p_\infty(x)\,dx - \left(\int_{\mathbb{R}} x p_\infty(x)\,dx\right)^2 = \int_0^{+\infty} \frac{x^2}{2\sigma^2\cosh^2(x/(2\sigma^2))}\,dx$$
$$> 4\sigma^4 \int_0^{+\infty} y^2 e^{-2y}\,dy = \sigma^4 = \frac{\sigma^4}{\|A''\|_\infty^2},$$

and this approach does not work.

(ii) Convexity of A implies nonnegativity of the term $A(F_t) - A(F_\infty) - A'(F_\infty)G_t$ which appears in the right-hand side of the first displayed equality in the proof. One may wonder if one could exploit this property to obtain exponential convergence of p_t to p_∞ even if p_0 is not close to p_∞. We have not been able to do so.

Proof of Theorem 2.4. By (14), $p'_\infty = -\frac{2}{\sigma^2}A'(F_\infty)p_\infty$ and $\|p_\infty\|_\infty \leq \frac{2\|A\|_\infty}{\sigma^2}$. The Fokker–Planck equation (13) for p_t ensures that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{(p_t(x) - p_\infty(x))^2}{p_\infty(x)}\,dx \\
&\quad = -\frac{\sigma^2}{2}\int_{\mathbb{R}} \left(\frac{p_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx \\
&\qquad - \int_{\mathbb{R}} (A'(F_t) - A'(F_\infty))(x)(p_t - p_\infty)(x)\left(\frac{p_t}{p_\infty}(x)\right)'\,dx \\
&\qquad - \int_{\mathbb{R}} (A'(F_t) - A'(F_\infty))(x)p_\infty(x)\left(\frac{p_t}{p_\infty}(x)\right)'\,dx.
\end{aligned}$$

Then, using Young's inequality and (26), one easily checks that for $\varepsilon, \eta > 0$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} \frac{(p_t(x) - p_\infty(x))^2}{p_\infty(x)}\,dx \\
&\quad \leq \left(\eta + \varepsilon - \frac{\sigma^2}{2}\right)\int_{\mathbb{R}} \left(\frac{p_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx \\
&\qquad + \frac{1}{4\varepsilon}\int_{\mathbb{R}} (A'(F_t)(x) - A'(F_\infty)(x))^2 \frac{(p_t(x) - p_\infty(x))^2}{p_\infty(x)}\,dx \\
&\qquad + \frac{1}{4\eta}\int_{\mathbb{R}} (A'(F_t)(x) - A'(F_\infty)(x))^2 p_\infty(x)\,dx \\
&\quad \leq \left(\eta + \varepsilon - \frac{\sigma^2}{2}\right)\int_{\mathbb{R}} \left(\frac{p_t}{p_\infty}(x)\right)'^2 p_\infty(x)\,dx \\
&\qquad + \frac{\|A''\|_\infty^2}{4\varepsilon}\left(\int_{\mathbb{R}} \frac{(p_t(x) - p_\infty(x))^2}{p_\infty(x)}\,dx\right)^2
\end{aligned}$$

$$+\frac{\|A''\|_\infty^2}{4\eta}\times\frac{4\|A\|_\infty^2}{\sigma^4}\int_{\mathbb{R}}\frac{G_t^2}{p_\infty}(x)\,dx.$$

By (23) and Lemma 2.8, for $\int_{\mathbb{R}}\frac{(p_0(x)-p_\infty(x))^2}{p_\infty(x)}\,dx$ small enough, the last term of the r.h.s. is smaller than $\frac{\tilde{c}e^{-Ct}}{C}\int_{\mathbb{R}}\frac{(p_0(x)-p_\infty(x))^2}{p_\infty(x)}\,dx$. Since $\int_{\mathbb{R}}(\frac{p_t}{p_\infty}(x))'^2 p_\infty(x)\,dx$ is greater than $\frac{1}{c}\int_{\mathbb{R}}\frac{(p_t(x)-p_\infty(x))^2}{p_\infty(x)}\,dx$, one easily concludes by Lemma 2.10 below. $\square$

LEMMA 2.10. *Assume that* $u:\mathbb{R}_+\to\mathbb{R}_+$ *satisfies*

$$\forall t\geq 0\qquad \frac{du}{dt}(t)\leq \beta u(t)(u(t)-\alpha)+\gamma e^{-\delta t}$$

for some constants $\alpha,\beta,\delta>0$ *and* $\gamma\geq 0$.

If $\gamma=0$ *and* $u(0)<\alpha$, *then*

$$\forall t\geq 0\qquad u(t)\leq\frac{\alpha u(0)e^{-\alpha\beta t}}{\alpha+u(0)(e^{-\alpha\beta t}-1)}.$$

If $u(0)<\frac{\alpha}{2}$ *and* $\gamma<\frac{\beta\alpha^2}{4}$, *then* $u(t)$ *converges to* 0 *exponentially fast as* $t\to+\infty$.

PROOF. When $\gamma=0$, as long as $u(t)\in(0,\alpha)$, one has

$$\frac{du}{dt}(t)\left(\frac{1}{u(t)}+\frac{1}{\alpha-u(t)}\right)\leq-\alpha\beta$$

and after integration one obtains the desired estimation. Since the upper bound is not greater than $u(0)$ and $u(t)=0\Rightarrow\forall s\geq t, u(s)=0$ one easily concludes.

Now when $\gamma\in(0,\frac{\beta\alpha^2}{4})$, one has $\beta a(\alpha-a)=\gamma$ for some $a\in(0,\frac{\alpha}{2})$ and

$$\frac{d}{dt}\left(u(t)\wedge\frac{\alpha}{2}-a\right)^+=1_{\{a<u(t)<\alpha/2\}}\frac{du}{dt}(t)\leq 0.$$

Hence when $u(0)<\frac{\alpha}{2}$, $\forall t\geq 0$, $u(t)\leq u(0)\vee a$ and

$$\frac{du}{dt}(t)\leq-\beta(\alpha-u(0)\vee a)u(t)+\gamma e^{-\delta t}.$$

For $v(t)=e^{\beta(\alpha-u(0)\vee a)t}u(t)$ one deduces

$$\frac{dv}{dt}(t)\leq\gamma e^{(\beta(\alpha-u(0)\vee a)-\delta)t}$$

and one concludes by integration of this inequality that $u(t)$ is bounded by $C(1+t)e^{-[(\beta(\alpha-u(0)\vee a))\wedge\delta]t}$. $\square$

2.2. *The particle system (4).* Let us suppose that $A(1)=0$ and that the first-order moment associated with the initial probability measure m is defined and equal to $\bar{x}$. As in the case of the granular media equation considered by Malrieu [12, 13], the direction $(1,1,\dots,1)$ is quite singular for the particle system. Indeed,

$$d(X_t^{1,n}+\cdots+X_t^{n,n})=\sigma\sum_{i=1}^n dB_t^i,$$

which prevents the law of $(X_t^{1,n},\dots,X_t^{n,n})$ from converging as $t\to+\infty$. Following [12, 13], one introduces the hyperplane $\mathcal{M}_n=\{y=(y_1,\dots,y_n)\in\mathbb{R}^n : y_1+\cdots+y_n=n\bar{x}\}$ orthogonal to this singular direction and denotes by $\bar{P}$ the orthogonal projection on $\mathcal{M}_n$ and by P the orthogonal projection on $\{y=(y_1,\dots,y_n)\in\mathbb{R}^n : y_1+\cdots+y_n=0\}$. Since $\sum_{i=1}^n a_n(i)=n(A(1)-A(0))=0$, the orthogonal projection $(Y_t^{i,n}=\bar{x}+X_t^{i,n}-\frac{1}{n}\sum_{j=1}^n X_t^{j,n})_{1\le i\le n}$ of the original particle system on $\mathcal{M}_n$ is a diffusion on this hyperplane solving

$$dY_t^{i,n}=\sigma\frac{n-1}{n}\,dB_t^i-\frac{\sigma}{n}\sum_{j\neq i}dB_t^j-a_n\left(\sum_{j=1}^n 1_{\{Y_t^{j,n}\le Y_t^{i,n}\}}\right)dt. \tag{27}$$

Propagation of chaos for the projected system is a consequence of the following estimate.

PROPOSITION 2.11. *Assume that A is convex, such that A' is Lipschitz continuous with constant K and $A(1)=0$ and that the initial measure m has a finite second order moment. Then, $\forall i\in\{1,\dots,n\},\forall t\ge 0$,*

$$\mathbb{E}[(X_t^i-Y_t^{i,n})^2]\le\frac{1}{n}\left[\frac{K^2t^2}{6}+\mathbb{E}[(X_0-\bar{x})^2]+\sigma^2t+2\int_0^t\int_{\mathbb{R}}A(F_s(x))\,dx\,ds\right],$$

where X^i is solution of (10).

PROOF. Denoting $X_1^n(t)=(X_t^1,\dots,X_t^n)$, $X_1^{n,n}(t)=(X_t^{1,n},\dots,X_t^{n,n})$ and $Y_1^{n,n}(t)=(Y_t^{1,n},\dots,Y_t^{n,n})$, one has

$$\begin{aligned}|X_1^n(t)-Y_1^{n,n}(t)|^2&=|X_1^n(t)-\bar{P}X_1^{n,n}(t)|^2 \quad (28)\\&=|X_1^n(t)-\bar{P}X_1^n(t)|^2+|\bar{P}X_1^n(t)-\bar{P}X_1^{n,n}(t)|^2\\&\le\frac{1}{n}\left(\sum_{i=1}^n(X_t^i-\bar{x})\right)^2+\sum_{i=1}^n(X_t^i-X_t^{i,n})^2. \quad (29)\end{aligned}$$

Since $(X_t-\bar{x})^2\le 3((X_0-\bar{x})^2+\sigma^2B_t^2+\|A'\|_\infty^2t^2)$, the variable X_t is square integrable. As

$$\begin{aligned}\forall x>0\qquad |(x-\bar{x})A(F_t(x))|&\le\|A'\|_\infty(1-F_t(x))(x+|\bar{x}|)\\&\le\|A'\|_\infty\left(\frac{\mathbb{E}(X_t^2)}{x}+|\bar{x}|(1-F_t(x))\right),\end{aligned}$$

one has $\lim_{x\to+\infty}(x-\bar{x})A(F_t(x))=0$. Similarly $(x-\bar{x})A(F_t(x))$ also vanishes as $x\to-\infty$ and $\int_{\mathbb{R}}(x-\bar{x})A'(F_t(x))p_t(x)\,dx=-\int_{\mathbb{R}}A(F_t(x))\,dx$. Computing $(X_t-\bar{x})^2$ by Itô's formula and taking expectations, one deduces that

$$\mathbb{E}((X_t-\bar{x})^2)=\mathbb{E}((X_0-\bar{x})^2)+\sigma^2 t+2\int_0^t\int_{\mathbb{R}}A(F_s(x))\,dx\,ds.$$

Moreover, by (3), $\mathbb{E}(X_t-\bar{x})=-A(1)t=0$. One concludes by taking expectations in (29) then using Theorem 1.5 and exchangeability of the particles. □

Let us now study the long time behavior of the projected particle system.

THEOREM 2.12. *Assume that the function A is uniformly convex on $[0,1]$ with constant α [see (11)] and such that $A(1)=0$. Then, the probability measure with density*

$$p_\infty^n(y)=\frac{1}{Z_n}e^{-2/\sigma^2\sum_{i=1}^n a_n(i)y_{(i)}}$$

with respect to the Lebesgue measure dy on $\mathcal{M}_n$ is invariant for the projected dynamics (27). Here $y_{(1)}\le y_{(2)}\le\cdots\le y_{(n)}$ denotes the increasing reordering of the coordinates of $y=(y_1,\dots,y_n)$ and $Z_n=\int_{\mathcal{M}_n}e^{-\frac{2}{\sigma^2}\sum_{i=1}^n a_n(i)y_{(i)}}\,dy$. Moreover, if $(Y_0^{1,n},\dots,Y_0^{n,n})$ admits a symmetric density $p_0^n(y)$ with respect to the Lebesgue measure on $\mathcal{M}_n$, then for all $t\ge 0$, $(Y_t^{1,n},\dots,Y_t^{n,n})$ admits a symmetric density $p_t^n(y)$ which is such that

$$\forall t\ge 0\qquad \int_{\mathcal{M}_n}\left(\frac{p_t^n}{p_\infty^n}(x)-1\right)^2 p_\infty^n(x)\,dx \le e^{-\lambda_n t}\int_{\mathcal{M}_n}\left(\frac{p_0^n}{p_\infty^n}(x)-1\right)^2 p_\infty^n(x)\,dx \tag{30}$$

where the sequence $(\lambda_n)_n$ is bounded from below by $\frac{\alpha^2}{12^3\sigma^2}$.

In order to deduce long time properties of the nonlinear process from long time properties of the projected system, it is not restrictive to assume that p_0^n is symmetric (see Remark 2.15 to get some intuition about this hypothesis). But the lack of uniformity in time of the estimation given in Proposition 2.11 is a real problem.

REMARK 2.13. In case $n=2$, the process $Y_t=Y_t^{2,2}-Y_t^{1,2}$ solves the stochastic differential equation

$$dY_t=\sigma(dB_t^2-dB_t^1)-\operatorname{sgn}(Y_t)(a_2(2)-a_2(1))\,dt$$

and the density of Y_t converges exponentially to $\frac{a_2(2)-a_2(1)}{2\sigma^2}e^{(-(a_2(2)-a_2(1))/\sigma^2)|y|}$ when the density of Y_0 is close enough to this limit. As $(Y_t^{1,2}, Y_t^{2,2}) = \overline{x} + \frac{1}{2}(-Z_t, Z_t)$, one easily deduces exponential convergence of the density of $(Y_t^{1,2}, Y_t^{2,2})$ on the straight line $\mathcal{M}_2$ to $\frac{a_2(2)-a_2(1)}{\sqrt{2}\sigma^2}e^{-(a_2(2)/\sigma^2)2y_{(2)}}e^{(a_2(1)/\sigma^2)(-2y_{(1)})}$.

The proof of Theorem 2.12 relies on the following Poincaré inequality.

Proposition 2.14. *Under the assumptions of Theorem 2.12, the density*

$$\tilde{p}^n_\infty(y) = \frac{n!1_{\{y_1 \le y_2 \le \cdots \le y_n\}}}{Z_n} e^{-(2/\sigma^2)\sum_{i=1}^n a_n(i) y_i}$$

on $\mathcal{M}_n$ is such that for $f:\mathbb{R}^n \to \mathbb{R}$ regular enough,

$$\begin{aligned}\int_{\mathcal{M}_n}\left(f(y) - \int_{\mathcal{M}_n} f(y)\tilde{p}^n_\infty(y)\,dy\right)^2 \tilde{p}^n_\infty(y)\,dy \\ \le \frac{\sigma^2}{\lambda_n}\int_{\mathcal{M}_n}|P\nabla f(y)|^2 \tilde{p}^n_\infty(y)\,dy\end{aligned} \tag{31}$$

where the sequence $(\lambda_n)_n$ is bounded from below by $\frac{\alpha^2}{12^3\sigma^2}$.

Proof of Theorem 2.12. Let us first check the following Green formula: for $f:\mathbb{R}^n \to \mathbb{R}$ and $u:\mathbb{R}^n \to \mathbb{R}^n$ regular enough,

$$\int_{\mathcal{M}_n} f\nabla\cdot(Pu)(y)\,dy = -\int_{\mathcal{M}_n} P\nabla f\cdot(Pu)(y)\,dy. \tag{32}$$

Let $\mathbf{1} \in \mathbb{R}^n$ denote the vector with all coordinates equal to 1. For $\varphi:\mathbb{R}\to\mathbb{R}$ and $v:\mathbb{R}^n \to \mathbb{R}^n$, one has

$$\begin{aligned}&\int_{\mathbb{R}}\varphi(\sqrt{n}z)\int_{\mathcal{M}_n}\nabla\cdot(Pv)\left(y + \frac{z\mathbf{1}}{\sqrt{n}}\right)dy\,dz \\ &\quad = \int_{\mathbb{R}^n}\varphi(x_1 + \cdots + x_n - n\bar{x})\nabla\cdot(Pv)(x)\,dx \\ &\quad = -\int_{\mathbb{R}^n}\varphi'(x_1 + \cdots + x_n - n\bar{x})\mathbf{1}\cdot(Pv)(x)\,dx = 0.\end{aligned}$$

The function φ being arbitrary, one deduces that $\int_{\mathcal{M}_n}\nabla\cdot(Pv)(y)\,dy = 0$. Since $\nabla\cdot P(fu) = \nabla f\cdot(Pu) + f\nabla\cdot(Pu) = P\nabla f\cdot(Pu) + f\nabla\cdot(Pu)$, (32) follows for the choice $v = fu$.

By weak uniqueness for (27), when $(Y_0^{1,n}, \ldots, Y_0^{n,n})$ has a symmetric density p_0^n with respect to the Lebesgue measure on $\mathcal{M}_n$, the particles $Y^{i,n}$, $i \in \{1, \ldots, n\}$ are exchangeable and for each $t \ge 0$, $(Y_t^{1,n}, \ldots, Y_t^{n,n})$ has a symmetric density p_t^n. By composition with the projection $\bar{P}$, one obtains an extension of p_t^n on $\mathbb{R}^n$ that we still denote by p_t^n. Since $\sum_{i=1}^n a_n(i) = n(A(1)-A(0)) = 0$, setting

$$b(y) = \sum_{\tau\in\mathcal{S}_n} 1_{\{y_{\tau(1)}\le y_{\tau(2)}\le\cdots\le y_{\tau(n)}\}} \begin{pmatrix} ca_n(\tau^{-1}(1)) \\ a_n(\tau^{-1}(2)) \\ \vdots \\ a_n(\tau^{-1}(n)) \end{pmatrix},$$

one has $Pb = b$ and the infinitesimal generator associated with (27) is $L\psi = \frac{\sigma^2}{2}\nabla\cdot(P\nabla\psi) - Pb\cdot\nabla\psi$. Computing $d\psi(Y_t^{1,n},\dots,Y_t^{n,n})$ by Itô's formula and taking expectations then using (32), one obtains

$$\begin{aligned}\int_{\mathcal{M}_n}\psi(y)\partial_t p_t^n(y)\,dy &= \int_{\mathcal{M}_n} L\psi(y)p_t^n(y)\,dy \\ &= \int_{\mathcal{M}_n}\psi(y)\nabla\cdot P\biggl(\frac{\sigma^2}{2}\nabla p_t^n + bp_t^n\biggr)(y)\,dy.\end{aligned}$$

Hence the densities solve the Fokker–Planck equation

$$\partial_t p_t^n = \nabla\cdot P\biggl(\frac{\sigma^2}{2}\nabla p_t^n + bp_t^n\biggr).$$

Now using (32) and $b = -\frac{\sigma^2\nabla p_\infty^n}{2p_\infty^n}$, one deduces

$$\begin{aligned}&\partial_t\int_{\mathcal{M}_n}\biggl(\frac{p_t^n}{p_\infty^n}(y)-1\biggr)^2 p_\infty^n(y)\,dy \\ &\qquad = 2\int_{\mathcal{M}_n}\frac{p_t^n}{p_\infty^n}(y)\nabla\cdot P\biggl(\frac{\sigma^2}{2}\nabla p_t^n + bp_t^n\biggr)(y)\,dy \\ &\qquad = -\sigma^2\int_{\mathcal{M}_n} P\nabla\frac{p_t^n}{p_\infty^n}(y)\cdot P\frac{\nabla p_t^n + (2bp_t^n/\sigma^2)}{p_\infty^n}(y)p_\infty^n(y)\,dy \\ &\qquad = -\sigma^2\int_{\mathcal{M}_n}\biggl|P\nabla\frac{p_t^n}{p_\infty^n}(y)\biggr|^2 p_\infty^n(y)\,dy.\end{aligned} \tag{33}$$

By symmetry of the function $\frac{p_t^n}{p_\infty^n}$ and (31),

$$\begin{aligned}\sigma^2\int_{\mathcal{M}_n}\biggl|P\nabla\frac{p_t^n}{p_\infty^n}(y)\biggr|^2 p_\infty^n(y)\,dy &= \sigma^2\int_{\mathcal{M}_n}\biggl|P\nabla\frac{p_t^n}{p_\infty^n}(y)\biggr|^2\tilde{p}_\infty^n(y)\,dy \\ &\ge \lambda_n\int_{\mathcal{M}_n}\biggl(\frac{p_t^n}{p_\infty^n}(y)-1\biggr)^2\tilde{p}_\infty^n(y)\,dy \\ &\ge \lambda_n\int_{\mathcal{M}_n}\biggl(\frac{p_t^n}{p_\infty^n}(y)-1\biggr)^2 p_\infty^n(y)\,dy\end{aligned}$$

and the conclusion follows. $\square$

Notice that the computation in (33) is formal and can only be justified when p_t^n is a smooth solution of the Fokker–Planck equation.

REMARK 2.15. Let us denote by $Y_t^{(1),n} \leq \cdots \leq Y_t^{(n),n}$ the increasing reordering of $(Y_t^{1,n}, \ldots, Y_t^{n,n})$. According to [9], the reordered system is a diffusion process normally reflected at the boundary of the closed convex set $\{y \in \mathcal{M}_n : y_1 \leq y_2 \leq \cdots \leq y_n\}$. More precisely,

$$
\left\{
\begin{array}{l}
dY_t^{(i),n} = \sigma\, d\beta_t^i - a_n(i)\, dt + (\gamma_t^i - \gamma_t^{i+1})\, d|K|_t, \\
\left(\displaystyle\int_0^t (\gamma_s^i - \gamma_s^{i+1})\, d|K|_s, 1 \leq i \leq n\right)_{t \geq 0} \text{ is a continuous process} \\
\qquad \text{with finite variation equal to } |K|_t, \\
\gamma^1 \equiv \gamma^{n+1} \equiv 0, \\
d|K|_t \text{ a.e. } \forall 2 \leq i \leq n, \gamma_t^i \geq 0 \text{ and } \gamma_t^i (Y_t^{(i),n} - Y_t^{(i-1),n}) = 0,
\end{array}
\right.
\tag{34}
$$

where $(\beta^1, \ldots, \beta^n)$ is a Brownian motion such that $\frac{\langle \beta^i, \beta^j \rangle_t}{t} = 1_{\{i=j\}} - 1/n$.

If the initial condition $(Y_0^{(1),n} \leq \cdots \leq Y_0^{(n),n})$ admits a density $\tilde{p}_0^n$ with respect to the Lebesgue measure on $\mathcal{M}_n$, then the law of $(Y_t^{(1),n}, \ldots, Y_t^{(n),n})$ is the image by increasing reordering of the symmetric law of the solution $(Y_t^{1,n}, \ldots, Y_t^{n,n})$ to (27) starting from $(Y_0^{1,n}, \ldots, Y_0^{n,n})$ with density p_0^n obtained by symmetrization of $\tilde{p}_0^n$. Therefore $(Y_t^{(1),n}, \ldots, Y_t^{(n),n})$ has the density $\tilde{p}_t^n(y) = n! p_t^n(y) 1_{\{y_1 \leq \cdots \leq y_n\}}$ and (30) holds with p^n replaced by $\tilde{p}^n$.

In order to prove Proposition 2.14, we take advantage of the specific form of the density $\tilde{p}_\infty^n$. Remarking that $\tilde{p}_\infty^n$ is the density of the image of a vector of independent exponential random variables by a linear transformation, one first obtains the following result.

LEMMA 2.16. *The Poincaré inequality (31) holds with the constant* λ_n *greater than* $\frac{\alpha^2}{4\sigma^2}$ *multiplied by the smallest eigenvalue* $\tilde{\lambda}_n$ *of the* $(n-1) \times (n-1)$ *matrix* Q^n *defined by* $\forall 1 \leq i, j \leq n-1$, $Q_{ij}^n = b_n(i) L_{ij}^n b_n(j)$ *where*

$$
b_n(i) = \frac{i(n-i)}{n} \quad \textit{and} \quad L^n = \begin{pmatrix}
2 & -1 & 0 & \ldots & \ldots & \ldots & 0 \\
-1 & 2 & -1 & 0 & \ldots & \ldots & 0 \\
0 & -1 & 2 & -1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \ldots & 0 & -1 & 2 & -1 & 0 \\
0 & \ldots & \ldots & 0 & -1 & 2 & -1 \\
0 & \ldots & \ldots & \ldots & 0 & -1 & 2
\end{pmatrix}.
$$

The last statement in Proposition 2.14 then follows from the next lemma which is obtained by interpreting Q^n as a finite element rigidity matrix associated with the operator $-x(1-x)\partial_{xx}(x(1-x).)$ acting on functions on $(0,1)$. The Hardy inequality stated in Lemma 2.18 ensures that it is enough to bound the smallest eigenvalue of the corresponding mass matrix from below. The resort to this one-dimensional Poincaré-like inequality in order to estimate the constant in the n-dimensional Poincaré inequality (31) is striking.

Lemma 2.17. *The sequence $(\tilde{\lambda}_n)_n$ is bounded from below by $1/(16\times 27)$.*

Proof of Lemma 2.16. Let f be such that $\int_{\mathcal{M}_n} f(y)\tilde{p}^n_\infty(y)\,dy = 0$. Since the left-hand side in the Poincaré inequality (31) only depends on the restriction of f to $\mathcal{M}_n$, one may assume that $\forall x\in\mathbb{R}^n, f(x) = f(\bar{P}x)$, which ensures that for $(x_1,\dots,x_n)\in\mathbb{R}^n$ such that $x_1+\dots+x_n = 0$, $f(\bar{x}+x_1,\dots,\bar{x}+x_n) = f(x_1,\dots,x_n)$ and $P\nabla f(\bar{x}+x_1,\dots,\bar{x}+x_n) = \nabla f(x_1,\dots,x_n)$. Therefore the Poincaré inequality (31) is equivalent to $I(f)\le \frac{\sigma^2}{\lambda_n} I(|\nabla f|)$ where

$$I(g) = \int_{\mathbb{R}^{n-1}} (g^2\tilde{p}^n_\infty)(-(x_2+\dots+x_n), x_2^n)\,dx_2^n \qquad \text{with } x_2^n = (x_2,\dots,x_n).$$

To integrate the coordinates over independent domains, we make the change of variables $z_2^n = Mx_2^n$ where

$$M = \begin{pmatrix} 2 & 1 & 1 & \dots & \dots & 1 \\ -1 & 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 1 \end{pmatrix}.$$

One easily checks that for $2\le i\le n$, $z_2+\dots+z_i = x_2+\dots+x_n+x_i$ and deduce that $(n-1)z_2+(n-2)z_3+\dots+2z_{n-1}+z_n = n(x_2+\dots+x_n)$. Therefore

$$M^{-1} = \frac{1}{n}\begin{pmatrix} 1 & 2-n & 3-n & 4-n & \dots & -1 \\ 1 & 2 & 3-n & 4-n & \dots & -1 \\ 1 & 2 & 3 & 4-n & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & n-2 & -1 \\ 1 & 2 & 3 & \dots & \dots & n-1 \end{pmatrix}$$

and denoting

$$N = \begin{pmatrix} \frac{1-n}{n} & \frac{2-n}{n} & \dots -\frac{2}{n} & -\frac{1}{n} \\ & & M^{-1} & \end{pmatrix},$$

one has

$$I(f) = \frac{n!}{Z_n} \int_{(\mathbb{R}_+)^{n-1}} f^2(Nz_2^n) e^{(-2/\sigma^2)\sum_{i=2}^n \beta_n(i) z_i} \frac{dz_2^n}{|M|}$$

where

$$\begin{aligned} \beta_n(i) &= \frac{1}{n}[(i-1)(a_n(i)+\cdots+a_n(n)) \\ &\qquad - (n+1-i)(a_n(1)+\cdots+a_n(i-1))] \\ &= -nA((i-1)/n) > 0. \end{aligned}$$

Here $|M|$ denotes the determinant of the matrix M; it is equal to n by an easy computation. The one-dimensional exponential density with parameter c satisfies the Poincaré inequality with optimal constant $4/c^2$. Tensorizing this inequality (see Chapters 3 and 6 in [1] for further details), one obtains

$$\begin{aligned} I(f) &\le \frac{n!}{Z_n} \int_{(\mathbb{R}_+)^{n-1}} \sum_{j=2}^n \frac{\sigma^4}{\beta_n^2(j)} \left(\sum_{k=1}^n N_{kj-1} \partial_k f(Nz_2^n) \right)^2 e^{(-2/\sigma^2)\sum_{i=2}^n \beta_n(i) z_i} \frac{dz_2^n}{|M|} \\ &= \int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^n \sum_{j=2}^n \frac{\sigma^4}{\beta_n^2(j)} N_{kj-1} N_{lj-1} \partial_k f \partial_l f \tilde{p}_\infty^n(-(x_2+\cdots+x_n), x_2^n)\, dx_2^n. \end{aligned}$$

Since A is uniformly convex with constant α and $A(0) = A(1) = 0$,

$$\beta_n(i) = -nA((i-1)/n) \ge -\frac{n\alpha}{2} \times \frac{i-1}{n} \left(\frac{i-1}{n} - 1 \right) = \frac{\alpha}{2} b_n(i-1).$$

Therefore

$$\begin{aligned} I(f) &\le \frac{4\sigma^4}{\alpha^2} \int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^n \sum_{j=1}^{n-1} \frac{N_{kj} N_{lj}}{b_n^2(j)} \partial_k f \partial_l f \tilde{p}_\infty^n(-(x_2+\cdots+x_n), x_2^n)\, dx_2^n \\ &\le \frac{4\sigma^2}{\alpha^2 \tilde{\lambda}_n} I(|\nabla f|) \end{aligned}$$

where $\tilde{\lambda}_n$ denotes the inverse of the largest eigenvalue of the symmetric positive semidefinite matrix $\bar{N}\bar{N}^*$ defined by $\bar{N}_{ij} = \frac{N_{ij}}{b_n(j)}$. To prove Proposition 2.14 with a possibly modified lower bound, it is enough to check that the largest eigenvalue is bounded from above uniformly in n. Unfortunately, the trace of the matrix can be bounded from below by a positive constant multiplied by $\log(n)$. Therefore one has to be more precise.

Let w be an eigenvector associated with the largest eigenvalue: $\bar{N}\bar{N}^* w = \frac{1}{\tilde{\lambda}_n} w$. Of course $\bar{N}^* w$ is nonzero and multiplying the previous equality by $\bar{N}^*$, one obtains that $\bar{N}^* w$ is an eigenvector of $\bar{N}^*\bar{N}$ associated with the eigenvalue $\frac{1}{\bar{\lambda}_n}$. By symmetry, $\frac{1}{\bar{\lambda}_n}$ is also the largest eigenvalue of $\bar{N}^*\bar{N}$. We are going to check that the latter matrix is invertible with inverse equal to Q^n in order to conclude the proof. Because of the definition of $\bar{N}$, it is enough to check that N^*N is invertible with inverse equal to L_n.

By construction of the matrix N, for the equation $Nz_2^n = x$ where $x \in \mathbb{R}^n$ to have a solution z_2^n, it is necessary and sufficient that $x_1 = -(x_2 + \cdots + x_n)$ and then $z_2^n = Mx_2^n$.

Now for fixed $y \in \mathbb{R}^{n-1}$, let us find $x_2^n \in \mathbb{R}^{n-1}$ such that $N^*x = y$ where $x = -(x_2 + \cdots + x_n, x_2^n)$. This equation writes

$$\left((M^{-1})^* - \begin{pmatrix} N_{11} & N_{11} & \dots & N_{11} \\ N_{12} & N_{12} & \dots & N_{12} \\ \vdots & \vdots & \vdots & \vdots \\ N_{1n-1} & N_{1n-1} & \dots & N_{1n-1} \end{pmatrix} \right) x_2^n = y.$$

One easily checks that the $(n-1) \times (n-1)$ matrix in the left-hand side is equal to

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & 1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \qquad \text{with inverse } R = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -1 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Combining $x_2^n = Ry$ with the solution of the previous problem, one obtains that the unique solution of the equation $N^*Nz_2^n = y$ is $z_2^n = MRy$. One concludes by checking that the matrix MR is equal to L_n. □

Proof of Lemma 2.17. For $i \in \{1, \dots, n-1\}$, the functions

$$u_i(x) = \begin{cases} 0, & \text{if } x \in (0,1) \Big\backslash \left[\dfrac{i-1}{n}, \dfrac{i+1}{n}\right], \\ \dfrac{i(n-i)(x-(i-1)/n)}{\sqrt{n}x(1-x)}, & \text{if } x \in \left[\dfrac{i-1}{n}, \dfrac{i}{n}\right], \\ \dfrac{i(n-i)((i+1)/n - x)}{\sqrt{n}x(1-x)}, & \text{if } x \in \left[\dfrac{i}{n}, \dfrac{i+1}{n}\right], \end{cases}$$

are such that

$$\forall i, j \in \{1, \dots, n-1\} \qquad Q_{ij}^n = \int_0^1 (x(1-x)u_i(x))'(x(1-x)u_j(x))'\, dx.$$

By the Hardy inequality stated in Lemma 2.18 below, the smallest eigenvalue of the matrix Q^n is greater than the smallest eigenvalue of the $(n-1) \times (n-1)$ tridiagonal matrix $R_{ij}^n = \int_0^1 u_i(x)u_j(x)\, dx$ divided by 16.

For $i \in \{1, \dots, n-2\}$, let $r_i^n = \int_{i/n}^{(i+1)/n} u_i(u_i - u_{i+1})(x)\,dx$ and

$$r_{n-1}^n = \int_{(n-1)/n}^1 u_{n-1}^2(x)\,dx = \frac{(n-1)^2}{n}\int_{(n-1)/n}^1 \frac{1}{x^2}\,dx = \frac{n-1}{n}.$$

Using the change of variables $y = 1-x$, one easily checks that

$$\forall i \in \{1, \dots, n-1\} \qquad R_{ii}^n - R_{ii-1}^n - R_{ii+1}^n = r_i^n + r_{n-i}^n,$$

where by convention $R_{10}^n = R_{n-1n}^n = 0$. We are going to prove that

$$\forall n \geq 3\ \forall i \in \{2, \dots, n-3\} \qquad r_i^n \geq \tfrac{1}{27},$$

and that r_1^n and r_{n-2}^n are nonnegative. For $y \in \mathbb{R}^{n-1}$, one deduces that

$$\begin{aligned} y^* R^n y &= \sum_{i=1}^{n-1} R_{ii}^n y_i^2 + 2\sum_{i=1}^{n-2} R_{ii+1}^n y_i y_{i+1} \\ &= \sum_{i=1}^{n-1}(R_{ii}^n - R_{ii-1}^n - R_{ii+1}^n) y_i^2 + \sum_{i=1}^{n-2} R_{ii+1}^n (y_i + y_{i+1})^2 \geq \frac{|y|^2}{27} \end{aligned}$$

and the conclusion follows.

Let us first suppose that $i \leq \lfloor \frac{n}{2} \rfloor - 1$, which ensures that the function $f(x) = x^2(1-x)^2$ is increasing on $[i/n, (i+1)/n]$. Let $g(x) = u_i(u_i - u_{i+1})(x)$. One easily checks that

$$\begin{aligned} \int_{i/n}^{(i+1)/n} g(x)\,dx &= \frac{i^2(n-i)^2}{n^4}\left(\frac{1}{3} - \frac{(i+1)(n-i-1)}{6i(n-i)}\right) \\ &\geq \begin{cases} 0, & \text{if } i = 1, \\ \dfrac{i^2(n-i)^2}{12n^4}, & \text{if } i \geq 2. \end{cases} \end{aligned}$$

Since there is some $x_i \in [i/n, (i+1)/n]$ such that the function $g(x)$ is nonnegative on $[i/n, x_i]$ then nonpositive on $[x_i, (i+1)/n]$, and f is positive and increasing, one deduces that for all $x \in [i/n, (i+1)/n]$, $\int_{i/n}^x \frac{g(y)}{f(y)}\,dy \geq 0$. This ensures that $\forall x \in [i/n, (i+1)/n]$

$$\frac{d}{dx}\left(f(x)\int_{i/n}^x \frac{g(y)}{f(y)}\,dy\right) = f'(x)\int_{i/n}^x \frac{g(y)}{f(y)}\,dy + g(x) \geq g(x).$$

Therefore

$$\begin{aligned} r_i^n &= \int_{i/n}^{(i+1)/n} \frac{g(y)}{f(y)}\,dy \geq \frac{1}{f((i+1)/n)}\int_{i/n}^{(i+1)/n} g(y)\,dy \\ &\geq \begin{cases} 0, & \text{if } i = 1, \\ \dfrac{i^2(n-i)^2}{12(i+1)^2(n-i-1)^2} \geq \dfrac{1}{27}, & \text{if } i \geq 2. \end{cases} \end{aligned}$$

Let us now suppose that $i \geq \lfloor \frac{n+1}{2} \rfloor$ so that the function f is decreasing on $[i/n, (i+1)/n]$. We deduce that

$$\begin{aligned} r_i^n &\geq \frac{1}{f(i/n)} \int_{i/n}^{(i+1)/n} (fu_i^2)(x)\,dx \\ &\quad - \frac{1}{f((i+1)/n)} \int_{i/n}^{(i+1)/n} (fu_i u_{i+1})(x)\,dx \\ &= \frac{1}{3} - \frac{i(n-i)}{6(i+1)(n-i-1)} \end{aligned}$$

and the left-hand side is greater than $1/12$ for $i \leq n-3$ and nonnegative for $i = n-2$.

We still have to deal with the case n odd and $i = (n-1)/2$. Then, f is not monotonic on $I_n = [i/n, (i+1)/n] = [1/2 - 1/2n, 1/2 + 1/2n]$. But by symmetry,

$$\begin{aligned} r_{(n-1)/2}^n &= \frac{(n-1)^2(n+1)^2}{16n} \int_{1/2-1/2n}^{1/2+1/2n} \frac{(1/2+1/2n-x)(1-2x)}{x^2(1-x)^2}\,dx \\ &= \frac{(n-1)^2(n+1)^2}{32n} \int_{1/2-1/2n}^{1/2+1/2n} \frac{(1-2x)^2}{x^2(1-x)^2}\,dx \\ &\geq \frac{(n-1)^2(n+1)^2}{2n} \int_{1/2-1/2n}^{1/2+1/2n} (1-2x)^2\,dx = \frac{(n^2-1)^2}{6n^4}, \end{aligned}$$

which completes the proof. $\square$

LEMMA 2.18. *For all $u \in L^2(0,1)$ such that the distribution derivative $(x(1-x)u(x))'$ belongs to $L^2(0,1)$,*

$$\int_0^1 u^2(x)\,dx \leq 16 \int_0^1 ((x(1-x)u(x))')^2\,dx.$$

PROOF. For v a C^∞ function with compact support on $(0,1)$, by the integration by parts formula,

$$\begin{aligned} \int_0^{1/2} \frac{v^2(x)}{x^2(1-x)^2}\,dx &\leq 4\int_0^{1/2} \frac{v^2(x)}{x^2}\,dx = 8\biggl(\int_0^{1/2} \frac{vv'(x)}{x}\,dx - v^2(1/2)\biggr) \\ &\leq 8\biggl(\int_0^{1/2} \frac{v^2(x)}{x^2}\,dx\biggr)^{1/2} \biggl(\int_0^{1/2} (v'(x))^2\,dx\biggr)^{1/2}. \end{aligned}$$

Dealing with the integral on $(1/2,1)$ in a symmetric way, one deduces

$$\int_0^1 \frac{v^2(x)}{x^2(1-x)^2}\,dx \leq 16 \int_0^1 (v'(x))^2\,dx. \tag{35}$$

Now approximating $v \in H_0^1(0,1)$ by a sequence of C^∞ functions with compact support converging in the H^1 norm and almost everywhere, one deduces with the Fatou lemma that the inequality still holds for $v \in H_0^1$.

For u satisfying the hypotheses in the lemma, $v(x) = x(1-x)u(x)$ belongs to $H^1(0,1)$. According to Theorem VIII.2, page 122 of [4], v admits a representative continuous on $[0,1]$ still denoted by v. Moreover, since $u(x) = \frac{v(x)}{x(1-x)}$ belongs to $L^2(0,1)$, necessarily, $v(0) = v(1) = 0$. By Theorem VIII.11, page 133 of [4], v belongs to $H_0^1(0,1)$ and the conclusion follows from (35). $\square$

Acknowledgment. We warmly thank Tony Lelièvre (CERMICS) for fruitful discussions concerning the analysis of the long time behavior of the nonlinear process.

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