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## To cite this version:

Rémi Carles. On instability for the cubic nonlinear Schrodinger equation. Comptes rendus de l'Académie des sciences. Série I, Mathématique, 2007, 344 (8), pp.483-486. 10.1016/j.crma.2007.03.006 . hal-00127816

HAL Id: hal-00127816
https://hal.science/hal-00127816
Submitted on 29 Jan 2007

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# ON THE INSTABILITY FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION 

RÉMI CARLES


#### Abstract

We study the flow map associated to the cubic Schrödinger equation in space dimension at least three. We consider initial data of arbitrary size in $H^{s}$, where $0<s<s_{c}, s_{c}$ the critical index, and perturbations in $H^{\sigma}$, where $\sigma<s_{c}$ is independent of $s$. We show an instability mechanism in some Sobolev spaces of order smaller than $s$. The analysis relies on two features of super-critical geometric optics: creation of oscillation, and ghost effect.


## 1. Introduction

We consider the Cauchy problem for the cubic, defocusing Schrödinger equation:

$$
\begin{equation*}
i \partial_{t} \psi+\frac{1}{2} \Delta \psi=|\psi|^{2} \psi, x \in \mathbb{R}^{n} \quad ; \quad \psi_{\mid t=0}=\varphi \tag{1.1}
\end{equation*}
$$

Formally, the mass and energy associated to this equation are independent of time:

$$
\begin{aligned}
& \text { Mass: } M[\psi](t)=\int_{\mathbb{R}^{n}}|\psi(t, x)|^{2} d x \equiv M[\psi](0)=M[\varphi] \\
& \text { Energy: } E[\psi](t)=\int_{\mathbb{R}^{n}}|\nabla \psi(t, x)|^{2} d x+\int_{\mathbb{R}^{n}}|\psi(t, x)|^{4} d x \equiv E[\psi](0)=E[\varphi] .
\end{aligned}
$$

Scaling arguments yield the critical value for the Cauchy problem in $H^{s}\left(\mathbb{R}^{n}\right)$ :

$$
s_{c}=\frac{n}{2}-1
$$

Assume $n \geqslant 3$, so that $s_{c}>0$. It was established in (3) that (1.1) is locally wellposed in $H^{s}\left(\mathbb{R}^{n}\right)$ if $s \geqslant s_{c}$. On the other hand, (1.1) is ill-posed in $H^{s}$ if $s<s_{c}$ ([4]). Moreover, the following norm inflation phenomenon was proved in [4] (see also [1, 2]): if $0<s<s_{c}$, we can find $\left(\varphi_{j}\right)_{j \in \mathbb{N}}$ in the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n}\right)$ with

$$
\begin{equation*}
\left\|\varphi_{j}\right\|_{H^{s}} \underset{j \rightarrow+\infty}{\longrightarrow} 0 \tag{1.2}
\end{equation*}
$$

and a sequence of positive times $\tau_{j} \rightarrow 0$, such that the solution $\psi_{j}$ to (1.1) with initial data $\varphi_{j}$ satisfy:

$$
\left\|\psi_{j}\left(\tau_{j}\right)\right\|_{H^{s}} \underset{j \rightarrow+\infty}{ }+\infty
$$

In [2], this was improved to: we can find $t_{j} \rightarrow 0$ such that

$$
\left.\left.\left\|\psi_{j}\left(t_{j}\right)\right\|_{H^{k}} \underset{j \rightarrow+\infty}{\longrightarrow}+\infty, \quad \forall k \in\right] \frac{s}{1+s_{c}-s}, s\right]
$$

[^0]Note that (1.2) means that we consider the flow map near the origin. We show that inside rings of $H^{s}$, the situation is yet more involved: for data bounded in $H^{s}$, with $0<s<s_{c}$, we consider perturbations which are small in $H^{\sigma}$ for any $\sigma<s_{c}$, and infer a similar conclusion.

Theorem 1.1. Let $n \geqslant 3$ and $0 \leqslant s<s_{c}=\frac{n}{2}-1$. Fix $C_{0}, \delta>0$. We can find two sequences of initial data $\left(\varphi_{j}\right)_{j \in \mathbb{N}}$ and $\left(\widetilde{\varphi}_{j}\right)_{j \in \mathbb{N}}$ in the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n}\right)$, with:

$$
C_{0}-\delta \leqslant\left\|\varphi_{j}\right\|_{H^{s}},\left\|\widetilde{\varphi}_{j}\right\|_{H^{s}} \leqslant C_{0}+\delta \quad ; \quad\left\|\varphi_{j}-\widetilde{\varphi}_{j}\right\|_{H^{\sigma}}^{j \rightarrow+\infty} \underset{\longrightarrow}{\longrightarrow} \quad \forall \sigma<s_{c}
$$

and a sequence of positive times $t_{j} \rightarrow 0$, such that the solutions $\psi_{j}$ and $\widetilde{\psi}_{j}$ to (1.1), with initial data $\varphi_{j}$ and $\widetilde{\varphi}_{j}$ respectively, satisfy:

$$
\begin{gathered}
\left.\left.\left\|\psi_{j}\left(t_{j}\right)-\widetilde{\psi}_{j}\left(t_{j}\right)\right\|_{H^{k}} \underset{j \rightarrow+\infty}{\longrightarrow}+\infty, \quad \forall k \in\right] \frac{s}{1+s_{c}-s}, s\right] \quad(\text { if } s>0), \\
\liminf _{j \rightarrow+\infty}\left\|\psi_{j}\left(t_{j}\right)-\widetilde{\psi}_{j}\left(t_{j}\right)\right\|_{H^{\frac{s}{1+s_{c}-s}}>0}
\end{gathered}
$$

The main novelty in this result is the fact that the initial data are close to each other in $H^{\sigma}$, for any $\sigma<s_{c}$. In particular, this range for $\sigma$ is independent of $s$.

Remark 1.2. Like in [1 2], we consider initial data of the form

$$
\varphi_{j}(x)=j^{\frac{n}{2}-s} a_{0}(j x)
$$

for some $a_{0} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ independent of $j$. The above result holds for all $a_{0} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ with, say $^{1},\left\|a_{0}\right\|_{H^{s}}=C_{0}$, and $\widetilde{\varphi}_{j}(x)=\left(j^{\frac{n}{2}-s}+j\right) a_{0}(j x)$ (see Section 2).

Considering the case $s=\frac{n}{4}$, we infer from the proof of Theorem 1.1
Corollary 1.3. Let $n \geqslant 5$ and $C_{0}, \delta>0$. We can find two sequences of initial data $\left(\varphi_{j}\right)_{j \in \mathbb{N}}$ and $\left(\widetilde{\varphi}_{j}\right)_{j \in \mathbb{N}}$ in the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n}\right)$, with:

$$
C_{0}-\delta \leqslant E\left[\varphi_{j}\right], E\left[\widetilde{\varphi}_{j}\right] \leqslant C_{0}+\delta \quad ; \quad M\left[\varphi_{j}\right]+M\left[\widetilde{\varphi}_{j}\right]+E\left[\varphi_{j}-\widetilde{\varphi}_{j}\right] \underset{j \rightarrow+\infty}{\longrightarrow} 0
$$

and a sequence of positive times $t_{j} \rightarrow 0$, such that the solutions $\psi_{j}$ and $\widetilde{\psi}_{j}$ to (1.1) with initial data $\varphi_{j}$ and $\widetilde{\varphi}_{j}$ respectively, satisfy:

$$
\liminf _{j \rightarrow+\infty} E\left[\psi_{j}-\widetilde{\psi}_{j}\right]\left(t_{j}\right)>0
$$

## 2. Reduction of the problem: Super-Critical geometric optics

We now proceed as in [2]. We set $\varepsilon=j^{s-s_{c}}: \varepsilon \rightarrow 0$ as $j \rightarrow+\infty$. We change the unknown function as follows:

$$
u^{\varepsilon}(t, x)=j^{s-\frac{n}{2}} \psi_{j}\left(\frac{t}{j^{s_{c}+2-s}}, \frac{x}{j}\right) .
$$

Note that we have the relation:

$$
\left\|\psi_{j}(t)\right\|_{\dot{H}^{m}}=j^{m-s}\left\|u^{\varepsilon}\left(j^{s_{c}+2-s} t\right)\right\|_{\dot{H}^{m}}
$$

With initial data of the form $\varphi_{j}(x)=j^{\frac{n}{2}-s} a_{0}(j x)+j a_{1}(j x)$, (1.1) becomes:

$$
\begin{equation*}
i \varepsilon \partial_{t} u^{\varepsilon}+\frac{\varepsilon^{2}}{2} \Delta u^{\varepsilon}=\left|u^{\varepsilon}\right|^{2} u^{\varepsilon} \quad ; \quad u^{\varepsilon}(0, x)=a_{0}(x)+\varepsilon a_{1}(x) . \tag{2.1}
\end{equation*}
$$

[^1]We emphasize two features for the WKB analysis associated to (2.1). First, even if the initial datum is independent of $\varepsilon$, the solution instantly becomes $\varepsilon$-oscillatory. This is the argument of the proof of [2] Cor. 1.7]. Second, the aspect which was not used in the proof of [2] Cor. 1.7] is what was called ghost effect in gas dynamics ([6]): a perturbation of order $\varepsilon$ of the initial datum may instantly become relevant at leading order. These two features are direct consequences of the fact that (2.1) is super-critical as far as WKB analysis is concerned (see e.g. [2]).

Consider the two solutions $u^{\varepsilon}$ and $\widetilde{u}^{\varepsilon}$ of (2.1) with $a_{1}=0$ and $a_{1}=a_{0}$ respectively. Then Theorem 1.1 stems from the following proposition, which in turn is essentially a reformulation of [2, Prop. 1.9 and 5.1].

Proposition 2.1. Let $n \geqslant 1$ and $a_{0} \in \mathcal{S}\left(\mathbb{R}^{n} ; \mathbb{R}\right) \backslash\{0\}$. There exist $T>0$ independent of $\varepsilon \in] 0,1]$, and $a, \phi, \phi_{1} \in C\left([0, T] ; H^{s}\right)$ for all $s \geqslant 0$, such that:

$$
\left\|u^{\varepsilon}-a e^{i \phi / \varepsilon}\right\|_{L^{\infty}\left([0, T] ; H_{\varepsilon}^{s}\right)}+\left\|\widetilde{u}^{\varepsilon}-a e^{i \phi_{1}} e^{i \phi / \varepsilon}\right\|_{L^{\infty}\left([0, T] ; H_{\varepsilon}^{s}\right)}=\mathcal{O}(\varepsilon), \quad \forall s \geqslant 0
$$

where

$$
\|f\|_{H_{\varepsilon}^{s}}^{2}=\int_{\mathbb{R}^{n}}\left(1+|\varepsilon \xi|^{2}\right)^{s}|\widehat{f}(\xi)|^{2} d \xi
$$

and $\widehat{f}$ stands for the Fourier transform of $f$. In addition, we have, in $H^{s}$ :

$$
\phi(t, x)=-t\left|a_{0}(x)\right|^{2}+\mathcal{O}\left(t^{3}\right) \quad ; \quad \phi_{1}(t, x)=-2 t\left|a_{0}(x)\right|^{2}+\mathcal{O}\left(t^{3}\right) \quad \text { as } t \rightarrow 0
$$

Therefore, there exists $\tau>0$ independent of $\varepsilon$, such that:

$$
\liminf _{\varepsilon \rightarrow 0} \varepsilon^{s}\left\|u^{\varepsilon}(\tau)-\widetilde{u}^{\varepsilon}(\tau)\right\|_{\dot{H}^{s}}>0, \quad \forall s \geqslant 0
$$

## 3. Outline of the proof of Proposition 2.1

The idea, due to E. Grenier [5], consists in writing the solution to (2.1) as $u^{\varepsilon}(t, x)=a^{\varepsilon}(t, x) e^{i \phi^{\varepsilon}(t, x) / \varepsilon}$, where $a^{\varepsilon}$ is complex-valued, and $\phi^{\varepsilon}$ is real-valued. We assume that $a_{0}, a_{1} \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ are independent of $\varepsilon$. For simplicity, we also assume that they are real-valued. Impose:

$$
\left\{\begin{array}{l}
\partial_{t} \phi^{\varepsilon}+\frac{1}{2}\left|\nabla \phi^{\varepsilon}\right|^{2}+\left|a^{\varepsilon}\right|^{2}=0 \quad ; \quad \phi^{\varepsilon}(0, x)=0  \tag{3.1}\\
\partial_{t} a^{\varepsilon}+\nabla \phi^{\varepsilon} \cdot \nabla a^{\varepsilon}+\frac{1}{2} a^{\varepsilon} \Delta \phi^{\varepsilon}=i \frac{\varepsilon}{2} \Delta a^{\varepsilon} \quad ; \quad a^{\varepsilon}(0, x)=a_{0}(x)+\varepsilon a_{1}(x)
\end{array}\right.
$$

Working with the unknown function $\mathbf{u}^{\varepsilon}={ }^{t}\left(\operatorname{Re} a^{\varepsilon}, \operatorname{Im} a^{\varepsilon}, \partial_{1} \phi^{\varepsilon}, \ldots, \partial_{n} \phi^{\varepsilon}\right)$, (3.1) yields a symmetric quasi-linear hyperbolic system: for $s>n / 2+2$, there exists $T>0$ independent of $\varepsilon \in] 0,1]$ (and of $s$, from tame estimates), such that (3.1) has a unique solution $\left(\phi^{\varepsilon}, a^{\varepsilon}\right) \in C\left([0, T] ; H^{s}\right)^{2}$. Moreover, the bounds in $H^{s}\left(\mathbb{R}^{n}\right)$ are independent of $\varepsilon$, and we see that $\left(\phi^{\varepsilon}, a^{\varepsilon}\right)$ converges to $(\phi, a)$, solution of:

$$
\begin{cases}\partial_{t} \phi+\frac{1}{2}|\nabla \phi|^{2}+|a|^{2}=0 \quad ; \quad \phi(0, x)=0  \tag{3.2}\\ \partial_{t} a+\nabla \phi \cdot \nabla a+\frac{1}{2} a \Delta \phi=0 \quad ; \quad a(0, x)=a_{0}(x)\end{cases}
$$

More precisely, energy estimates for symmetric systems yield:

$$
\left\|\phi^{\varepsilon}-\phi\right\|_{L^{\infty}\left([0, T] ; H^{s}\right)}+\left\|a^{\varepsilon}-a\right\|_{L^{\infty}\left([0, T] ; H^{s}\right)}=\mathcal{O}(\varepsilon), \quad \forall s \geqslant 0
$$

One can prove that $\phi^{\varepsilon}$ and $a^{\varepsilon}$ have an asymptotic expansion in powers of $\varepsilon$. Consider the next term, given by:

$$
\left\{\begin{array}{l}
\partial_{t} \phi^{(1)}+\nabla \phi \cdot \nabla \phi^{(1)}+2 \operatorname{Re}\left(\bar{a} a^{(1)}\right)=0 ;\left.\phi^{(1)}\right|_{t=0}=0 . \\
\partial_{t} a^{(1)}+\nabla \phi \cdot \nabla a^{(1)}+\nabla \phi^{(1)} \cdot \nabla a+\frac{1}{2} a^{(1)} \Delta \phi+\frac{1}{2} a \Delta \phi^{(1)}=\frac{i}{2} \Delta a ;\left.a^{(1)}\right|_{t=0}=a_{1}
\end{array}\right.
$$

Then $a^{(1)}, \phi^{(1)} \in L^{\infty}\left([0, T] ; H^{s}\right)$ for every $s \geqslant 0$, and

$$
\left\|a^{\varepsilon}-a-\varepsilon a^{(1)}\right\|_{L^{\infty}\left(\left[0, T_{*}\right] ; H^{s}\right)}+\left\|\Phi^{\varepsilon}-\phi-\varepsilon \phi^{(1)}\right\|_{L^{\infty}\left(\left[0, T_{*}\right] ; H^{s}\right)} \leqslant C_{s} \varepsilon^{2}, \quad \forall s \geqslant 0
$$

Observe that since $a$ is real-valued, $\left(\phi^{(1)}, \operatorname{Re}\left(\bar{a} a^{(1)}\right)\right)$ solves an homogeneous linear system. Therefore, if $\operatorname{Re}\left(\bar{a} a^{(1)}\right)=0$ at time $t=0$, then $\phi^{(1)} \equiv 0$.

Considering the cases $a_{1}=0$ and $a_{1}=a_{0}$ for $u^{\varepsilon}$ and $\widetilde{u}^{\varepsilon}$ respectively, we obtain the first assertion of Prop. 2.1. Note that the above $\mathcal{O}\left(\varepsilon^{2}\right)$ becomes an $\mathcal{O}(\varepsilon)$ only, since we divide $\phi^{\varepsilon}$ and $\phi$ by $\varepsilon$. This also explains why the first estimate of Prop. 2.1 is stated in $H_{\varepsilon}^{s}$ and not in $H^{s}$. The rest of the proposition follows easily.
Remark 3.1. We could use the ghost effect at higher order. For $N \in \mathbb{N}$, assume $\widetilde{u}_{\mid t=0}^{\varepsilon}=\left(1+\varepsilon^{N}\right) a_{0}$ for instance. Then for some $\tau>0$ independent of $\varepsilon$, we have

$$
\liminf _{\varepsilon \rightarrow 0}\left(\varepsilon^{s}\left\|u^{\varepsilon}(\tau)-\widetilde{u}^{\varepsilon}(\tau)\right\|_{\dot{H}^{s}} \times \varepsilon^{1-N}\right)>0, \quad \forall s \geqslant 0
$$

Back to the functions $\psi$, the range for $k$ becomes:

$$
k \geqslant \frac{s+\left(s_{c}-s\right)(N-1)}{1+s_{c}-s}
$$

For this lower bound to be strictly smaller than $s$, we have to assume $s>N-1$.

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[^2]
[^0]:    2000 Mathematics Subject Classification. 35B33; 35B65; 35Q55; 81Q05; 81Q20.
    Support by the ANR project SCASEN is acknowledged.

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