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## ON THE INSTABILITY FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION

RÉMI CARLES

ABSTRACT. We study the flow map associated to the cubic Schrödinger equation in space dimension at least three. We consider initial data of arbitrary size in  $H^s$ , where  $0 < s < s_c$ ,  $s_c$  the critical index, and perturbations in  $H^\sigma$ , where  $\sigma < s_c$  is independent of  $s$ . We show an instability mechanism in some Sobolev spaces of order smaller than  $s$ . The analysis relies on two features of super-critical geometric optics: creation of oscillation, and ghost effect.

### 1. INTRODUCTION

We consider the Cauchy problem for the cubic, defocusing Schrödinger equation:

$$(1.1) \quad i\partial_t \psi + \frac{1}{2} \Delta \psi = |\psi|^2 \psi, \quad x \in \mathbb{R}^n \quad ; \quad \psi|_{t=0} = \varphi.$$

Formally, the mass and energy associated to this equation are independent of time:

$$\text{Mass: } M[\psi](t) = \int_{\mathbb{R}^n} |\psi(t, x)|^2 dx \equiv M[\psi](0) = M[\varphi],$$

$$\text{Energy: } E[\psi](t) = \int_{\mathbb{R}^n} |\nabla \psi(t, x)|^2 dx + \int_{\mathbb{R}^n} |\psi(t, x)|^4 dx \equiv E[\psi](0) = E[\varphi].$$

Scaling arguments yield the critical value for the Cauchy problem in  $H^s(\mathbb{R}^n)$ :

$$s_c = \frac{n}{2} - 1.$$

Assume  $n \geq 3$ , so that  $s_c > 0$ . It was established in [3] that (1.1) is locally well-posed in  $H^s(\mathbb{R}^n)$  if  $s \geq s_c$ . On the other hand, (1.1) is ill-posed in  $H^s$  if  $s < s_c$  ([4]). Moreover, the following norm inflation phenomenon was proved in [4] (see also [1, 2]): if  $0 < s < s_c$ , we can find  $(\varphi_j)_{j \in \mathbb{N}}$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$  with

$$(1.2) \quad \|\varphi_j\|_{H^s} \xrightarrow{j \rightarrow +\infty} 0,$$

and a sequence of positive times  $\tau_j \rightarrow 0$ , such that the solution  $\psi_j$  to (1.1) with initial data  $\varphi_j$  satisfy:

$$\|\psi_j(\tau_j)\|_{H^s} \xrightarrow{j \rightarrow +\infty} +\infty.$$

In [2], this was improved to: we can find  $t_j \rightarrow 0$  such that

$$\|\psi_j(t_j)\|_{H^k} \xrightarrow{j \rightarrow +\infty} +\infty, \quad \forall k \in \left] \frac{s}{1 + s_c - s}, s \right].$$

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Note that (1.2) means that we consider the flow map near the origin. We show that inside rings of  $H^s$ , the situation is yet more involved: for data bounded in  $H^s$ , with  $0 < s < s_c$ , we consider perturbations which are small in  $H^\sigma$  for *any*  $\sigma < s_c$ , and infer a similar conclusion.

**Theorem 1.1.** *Let  $n \geq 3$  and  $0 \leq s < s_c = \frac{n}{2} - 1$ . Fix  $C_0, \delta > 0$ . We can find two sequences of initial data  $(\varphi_j)_{j \in \mathbb{N}}$  and  $(\tilde{\varphi}_j)_{j \in \mathbb{N}}$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$ , with:*

$$C_0 - \delta \leq \|\varphi_j\|_{H^s}, \|\tilde{\varphi}_j\|_{H^s} \leq C_0 + \delta \quad ; \quad \|\varphi_j - \tilde{\varphi}_j\|_{H^\sigma} \xrightarrow{j \rightarrow +\infty} 0, \quad \forall \sigma < s_c,$$

and a sequence of positive times  $t_j \rightarrow 0$ , such that the solutions  $\psi_j$  and  $\tilde{\psi}_j$  to (1.1), with initial data  $\varphi_j$  and  $\tilde{\varphi}_j$  respectively, satisfy:

$$\|\psi_j(t_j) - \tilde{\psi}_j(t_j)\|_{H^k} \xrightarrow{j \rightarrow +\infty} +\infty, \quad \forall k \in \left] \frac{s}{1+s_c-s}, s \right] \quad (\text{if } s > 0),$$

$$\liminf_{j \rightarrow +\infty} \|\psi_j(t_j) - \tilde{\psi}_j(t_j)\|_{H^{\frac{s}{1+s_c-s}}} > 0.$$

The main novelty in this result is the fact that the initial data are close to each other in  $H^\sigma$ , for any  $\sigma < s_c$ . In particular, this range for  $\sigma$  is independent of  $s$ .

*Remark 1.2.* Like in [1, 2], we consider initial data of the form

$$\varphi_j(x) = j^{\frac{n}{2}-s} a_0(jx),$$

for some  $a_0 \in \mathcal{S}(\mathbb{R}^n)$  independent of  $j$ . The above result holds for *all*  $a_0 \in \mathcal{S}(\mathbb{R}^n)$  with, say<sup>1</sup>,  $\|a_0\|_{H^s} = C_0$ , and  $\tilde{\varphi}_j(x) = (j^{\frac{n}{2}-s} + j)a_0(jx)$  (see Section 2).

Considering the case  $s = \frac{n}{4}$ , we infer from the proof of Theorem 1.1:

**Corollary 1.3.** *Let  $n \geq 5$  and  $C_0, \delta > 0$ . We can find two sequences of initial data  $(\varphi_j)_{j \in \mathbb{N}}$  and  $(\tilde{\varphi}_j)_{j \in \mathbb{N}}$  in the Schwartz class  $\mathcal{S}(\mathbb{R}^n)$ , with:*

$$C_0 - \delta \leq E[\varphi_j], E[\tilde{\varphi}_j] \leq C_0 + \delta \quad ; \quad M[\varphi_j] + M[\tilde{\varphi}_j] + E[\varphi_j - \tilde{\varphi}_j] \xrightarrow{j \rightarrow +\infty} 0,$$

and a sequence of positive times  $t_j \rightarrow 0$ , such that the solutions  $\psi_j$  and  $\tilde{\psi}_j$  to (1.1) with initial data  $\varphi_j$  and  $\tilde{\varphi}_j$  respectively, satisfy:

$$\liminf_{j \rightarrow +\infty} E[\psi_j - \tilde{\psi}_j](t_j) > 0.$$

## 2. REDUCTION OF THE PROBLEM: SUPER-CRITICAL GEOMETRIC OPTICS

We now proceed as in [2]. We set  $\varepsilon = j^{s-s_c}$ :  $\varepsilon \rightarrow 0$  as  $j \rightarrow +\infty$ . We change the unknown function as follows:

$$u^\varepsilon(t, x) = j^{s-\frac{n}{2}} \psi_j \left( \frac{t}{j^{s_c+2-s}}, \frac{x}{j} \right).$$

Note that we have the relation:

$$\|\psi_j(t)\|_{\dot{H}^m} = j^{m-s} \|u^\varepsilon(j^{s_c+2-s}t)\|_{\dot{H}^m}.$$

With initial data of the form  $\varphi_j(x) = j^{\frac{n}{2}-s} a_0(jx) + j a_1(jx)$ , (1.1) becomes:

$$(2.1) \quad i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = |u^\varepsilon|^2 u^\varepsilon \quad ; \quad u^\varepsilon(0, x) = a_0(x) + \varepsilon a_1(x).$$

<sup>1</sup>Provided that we choose  $j$  sufficiently large.

We emphasize two features for the WKB analysis associated to (2.1). First, even if the initial datum is independent of  $\varepsilon$ , the solution instantly becomes  $\varepsilon$ -oscillatory. This is the argument of the proof of [2, Cor. 1.7]. Second, the aspect which was not used in the proof of [2, Cor. 1.7] is what was called *ghost effect* in gas dynamics ([6]): a perturbation of order  $\varepsilon$  of the initial datum may instantly become relevant at leading order. These two features are direct consequences of the fact that (2.1) is super-critical as far as WKB analysis is concerned (see e.g. [2]).

Consider the two solutions  $u^\varepsilon$  and  $\tilde{u}^\varepsilon$  of (2.1) with  $a_1 = 0$  and  $a_1 = a_0$  respectively. Then Theorem 1.1 stems from the following proposition, which in turn is essentially a reformulation of [2, Prop. 1.9 and 5.1].

**Proposition 2.1.** *Let  $n \geq 1$  and  $a_0 \in \mathcal{S}(\mathbb{R}^n; \mathbb{R}) \setminus \{0\}$ . There exist  $T > 0$  independent of  $\varepsilon \in ]0, 1]$ , and  $a, \phi, \phi_1 \in C([0, T]; H^s)$  for all  $s \geq 0$ , such that:*

$$\|u^\varepsilon - ae^{i\phi/\varepsilon}\|_{L^\infty([0, T]; H_\varepsilon^s)} + \|\tilde{u}^\varepsilon - ae^{i\phi_1}e^{i\phi/\varepsilon}\|_{L^\infty([0, T]; H_\varepsilon^s)} = \mathcal{O}(\varepsilon), \quad \forall s \geq 0,$$

where

$$\|f\|_{H_\varepsilon^s}^2 = \int_{\mathbb{R}^n} (1 + |\varepsilon\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi,$$

and  $\widehat{f}$  stands for the Fourier transform of  $f$ . In addition, we have, in  $H^s$ :

$$\phi(t, x) = -t|a_0(x)|^2 + \mathcal{O}(t^3) \quad ; \quad \phi_1(t, x) = -2t|a_0(x)|^2 + \mathcal{O}(t^3) \quad \text{as } t \rightarrow 0.$$

Therefore, there exists  $\tau > 0$  independent of  $\varepsilon$ , such that:

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon^s \|u^\varepsilon(\tau) - \tilde{u}^\varepsilon(\tau)\|_{\dot{H}^s} > 0, \quad \forall s \geq 0.$$

### 3. OUTLINE OF THE PROOF OF PROPOSITION 2.1

The idea, due to E. Grenier [5], consists in writing the solution to (2.1) as  $u^\varepsilon(t, x) = a^\varepsilon(t, x)e^{i\phi^\varepsilon(t, x)/\varepsilon}$ , where  $a^\varepsilon$  is complex-valued, and  $\phi^\varepsilon$  is real-valued. We assume that  $a_0, a_1 \in \mathcal{S}(\mathbb{R}^n)$  are independent of  $\varepsilon$ . For simplicity, we also assume that they are real-valued. Impose:

$$(3.1) \quad \begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 = 0 & ; \quad \phi^\varepsilon(0, x) = 0. \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i \frac{\varepsilon}{2} \Delta a^\varepsilon & ; \quad a^\varepsilon(0, x) = a_0(x) + \varepsilon a_1(x). \end{cases}$$

Working with the unknown function  $\mathbf{u}^\varepsilon = {}^t(\operatorname{Re} a^\varepsilon, \operatorname{Im} a^\varepsilon, \partial_1 \phi^\varepsilon, \dots, \partial_n \phi^\varepsilon)$ , (3.1) yields a symmetric quasi-linear hyperbolic system: for  $s > n/2 + 2$ , there exists  $T > 0$  independent of  $\varepsilon \in ]0, 1]$  (and of  $s$ , from tame estimates), such that (3.1) has a unique solution  $(\phi^\varepsilon, a^\varepsilon) \in C([0, T]; H^s)^2$ . Moreover, the bounds in  $H^s(\mathbb{R}^n)$  are independent of  $\varepsilon$ , and we see that  $(\phi^\varepsilon, a^\varepsilon)$  converges to  $(\phi, a)$ , solution of:

$$(3.2) \quad \begin{cases} \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + |a|^2 = 0 & ; \quad \phi(0, x) = 0. \\ \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = 0 & ; \quad a(0, x) = a_0(x). \end{cases}$$

More precisely, energy estimates for symmetric systems yield:

$$\|\phi^\varepsilon - \phi\|_{L^\infty([0, T]; H^s)} + \|a^\varepsilon - a\|_{L^\infty([0, T]; H^s)} = \mathcal{O}(\varepsilon), \quad \forall s \geq 0.$$

One can prove that  $\phi^\varepsilon$  and  $a^\varepsilon$  have an asymptotic expansion in powers of  $\varepsilon$ . Consider the next term, given by:

$$\begin{cases} \partial_t \phi^{(1)} + \nabla \phi \cdot \nabla \phi^{(1)} + 2 \operatorname{Re}(\bar{a} a^{(1)}) = 0; & \phi^{(1)}|_{t=0} = 0. \\ \partial_t a^{(1)} + \nabla \phi \cdot \nabla a^{(1)} + \nabla \phi^{(1)} \cdot \nabla a + \frac{1}{2} a^{(1)} \Delta \phi + \frac{1}{2} a \Delta \phi^{(1)} = \frac{i}{2} \Delta a; & a^{(1)}|_{t=0} = a_1. \end{cases}$$

Then  $a^{(1)}, \phi^{(1)} \in L^\infty([0, T]; H^s)$  for every  $s \geq 0$ , and

$$\|a^\varepsilon - a - \varepsilon a^{(1)}\|_{L^\infty([0, T_*]; H^s)} + \|\Phi^\varepsilon - \phi - \varepsilon \phi^{(1)}\|_{L^\infty([0, T_*]; H^s)} \leq C_s \varepsilon^2, \quad \forall s \geq 0.$$

Observe that since  $a$  is real-valued,  $(\phi^{(1)}, \operatorname{Re}(\bar{a} a^{(1)}))$  solves an homogeneous linear system. Therefore, if  $\operatorname{Re}(\bar{a} a^{(1)}) = 0$  at time  $t = 0$ , then  $\phi^{(1)} \equiv 0$ .

Considering the cases  $a_1 = 0$  and  $a_1 = a_0$  for  $u^\varepsilon$  and  $\tilde{u}^\varepsilon$  respectively, we obtain the first assertion of Prop. 2.1. Note that the above  $\mathcal{O}(\varepsilon^2)$  becomes an  $\mathcal{O}(\varepsilon)$  only, since we divide  $\phi^\varepsilon$  and  $\phi$  by  $\varepsilon$ . This also explains why the first estimate of Prop. 2.1 is stated in  $H_\varepsilon^s$  and not in  $H^s$ . The rest of the proposition follows easily.

*Remark 3.1.* We could use the *ghost effect* at higher order. For  $N \in \mathbb{N}$ , assume  $\tilde{u}_{|t=0}^\varepsilon = (1 + \varepsilon^N) a_0$  for instance. Then for some  $\tau > 0$  independent of  $\varepsilon$ , we have

$$\liminf_{\varepsilon \rightarrow 0} (\varepsilon^s \|u^\varepsilon(\tau) - \tilde{u}^\varepsilon(\tau)\|_{\dot{H}^s} \times \varepsilon^{1-N}) > 0, \quad \forall s \geq 0.$$

Back to the functions  $\psi$ , the range for  $k$  becomes:

$$k \geq \frac{s + (s_c - s)(N - 1)}{1 + s_c - s}.$$

For this lower bound to be strictly smaller than  $s$ , we have to assume  $s > N - 1$ .

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