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Dispersion relations and Omnès representations for $K \to \pi\pi$ decay amplitudes

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Abstract

We derive dispersion relations for $K \to \pi\pi$ decay, using the Lehmann-Symanzik-Zimmermann formalism, which allows the analytic continuation of the amplitudes with respect to the momenta of the external particles. No off-shell extrapolation of the field operators is assumed. We obtain generalized Omnès representations, which incorporate the $\pi\pi$ and πK S-wave phase shifts in the elastic region of the direct and crossed channels, according to Watson theorem. The contribution of the inelastic final-state and initial-state interactions is parametrized by the technique of conformal mappings. We compare our results with previous dispersive treatments and indicate how the formalism can be combined with lattice calculations to yield physical predictions.

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1 Introduction

The weak decay $K \to \pi\pi$ has been a continuous challenge for the theoretical investigations. Chiral perturbation theory (ChPT) was extensively used [1], but the large number of unknown counterterms and renormalization constants render the numerical predictions difficult beyond the leading order. Most lattice calculations (see [2] and references therein) simulate matrix elements of the type $\langle \pi | O_i | K \rangle$, related to the physical matrix elements $\langle \pi\pi | O_i | K \rangle$ by lowest order ChPT [3]. In this procedure the higher order final state interactions are completely missing, while it is expected that they play an important role for the $\Delta I = 1/2$ rule and the CP-violating ratio ϵ'/ϵ. The finite-volume techniques developed in [4] can take into account FSI, but they are numerically very demanding [2]. In a combined approach proposed recently, the results obtained by lattice simulations at unphysical points are extrapolated to the physical configuration by using calculations to NLO in ChPT [5, 6].

An alternative way to connect the on-shell amplitude to lattice results at unphysical points and to spectral functions measured experimentally is based on dispersion relations. This formalism was used some time ago for the CP-conserving amplitudes in order to explain the $\Delta I = 1/2$ rule [7], and more recently in [8, 9] for evaluating the effects of final state interactions upon ϵ'/ϵ. The last works use an Omnès representation [10] for the decay amplitude, written by analogy with the case of the scalar form factor. This approach was investigated further in Refs. [11]-[15], where some critical remarks about the method were advanced. An alternative dispersive framework for $K \to \pi\pi$ decay was proposed in [14], by assuming that the weak hamiltonian carries a non-zero momentum. Then the matrix element of the decay becomes equivalent with the πK elastic scattering amplitude, for which Mandelstam representation is assumed.

In the references mentioned above, the dispersion relations for the weak decay were written by using the analogy with the familiar cases of the form factors or the scattering amplitudes. However, in the weak decay a continuation in the external momenta is necessary in order to obtain a dispersion relation. As a proof of the dispersion relations in this case is missing, their meaning was not always clear and led to some confusion. The dispersive variable was interpreted either as the mass or the momentum of an off-shell particle. The clarification of this point is possible only by a systematic derivation in the frame of a field theoretic formalism. In the present work we ad-

dress this problem, by performing the continuation in the external momenta with the Lehmann-Symanzik-Zimmermann (LSZ) formalism [16]. Our main result is a general Omnès representation for the $K \to \pi\pi$ amplitudes, including final and initial state interactions in both the direct ($K \to \pi\pi$) and the crossed ($\pi \to \pi K$) channels. The derivation clarifies the significance of the dispersive variables, allowing to make contact with lattice calculations done at unphysical points.

In the next section we present the derivation of the dispersion relations, using LSZ reduction and hadronic unitarity. We follow to some extent the dispersive treatment of $B \to \pi\pi$ decay considered in [17, 18]. However, the different masses of the decaying particles in the two processes require specific treatments. In section 3, we derive a generalized Omnès representation for $K \to \pi\pi$ decay by solving the inhomogeneous Hilbert problem [19, 20] in the direct and the crossed channels. In Section 4, we compare our results with the dispersion relations considered previously in the literature, and show how to combine them with lattice calculations in order to predict the physical amplitude.

2 LSZ reduction and dispersion relations

We consider decay amplitudes A_I of definite isospin, $I = 0, 2$, defined as

$$A_I = \langle (\pi(k_1)\,\pi(k_2))_I\,;\mathrm{out}\,|\mathcal{H}_w(0)|K(p)\,;\mathrm{in}\rangle\,, \tag{1}$$

where the "in" and "out" states are defined with respect to the strong interactions and $\mathcal{H}_w$ is the weak effective hamiltonian density [21]

$$\mathcal{H}_w(x) = \frac{G_F}{\sqrt{2}} \sum_{k=u,c} V_{kd} V_{ks}^* \tag{2}$$
$$\times \left[C_1(\mu) O_1^k(x,\mu) + C_2(\mu) O_2^k(x,\mu) + \sum_{j=3,\ldots,8} C_j(\mu) O_j(x,\mu) \right] .$$

Here O_j are local $\Delta S = 1$, $\Delta B = 0$ operators and C_j the corresponding Wilson coefficients, which take into account perturbatively the strong dynamics at distances shorter than $1/\mu$. We assume that a factor i was included in the definition of the operators, so that the amplitudes (1) satisfy time reversal invariance, up to the complex coefficients in (2).

For our purpose it is more convenient to start from the S-matrix element

$$S_I = \langle \pi(k_1)\, \pi(k_2); \mathrm{out}\, | K(p); \mathrm{in} \rangle \,, \tag{3}$$

where the transition from the "in" to the "out" states is achieved by both the strong and weak interactions. By expanding the $S-$matrix to first order in the weak interactions one obtains the expression (1) of the decay amplitude. Alternatively, by applying the LSZ reduction [16] to the K meson in Eq. (3), the decay amplitude (1) is expressed as

$$A_I = \frac{1}{\sqrt{2p_0}} \langle \pi(k_1)\, \pi(k_2); \mathrm{out}\, |\, \eta_K(0)\, |0\rangle \,, \tag{4}$$

where $\eta_K(x) = \mathcal{K}_x \phi_K(x)$ denotes the source operator ($\mathcal{K}_x$ is the Klein-Gordon operator and ϕ_K the interpolating field of the kaon). In a Lagrangian theory the source operator has the formal expression

$$\eta_K(x) = \frac{\delta \mathcal{L}_{int}}{\delta \phi_K} - \partial_\mu \frac{\delta \mathcal{L}_{int}}{\delta \partial_\mu \phi_K} \,, \tag{5}$$

i.e. it has contributions from both the strong and weak parts of the interaction Lagrangian. In what follows we do not need the explicit expressions of the sources, but only the significance of the matrix elements involving them. We stress also that throughout the derivation the sources are on-shell operators, defined in terms of the physical interpolation fields.

The matrix element (4) depends on the momenta k_1 and k_2 of the two pions. We shall consider it as a function of the invariant variables $s = (k_1 + k_2)^2$, $t = k_1^2$, $u = k_2^2$. The physical amplitude corresponds to the values $s = m_K^2$, $t = m_\pi^2$ and $u = m_\pi^2$. The extrapolation to arbitrary external momenta can be achieved by the LSZ reduction formalism [16]. We remark that Eq. (4) is similar to the definition of the electromagnetic form factor of the pion, where η_K is replaced by the electromagnetic current J_μ. We can apply therefore the standard methods used in deriving the dispersion relations for the pion form factor [22]. Making the LSZ reduction of one final pion in Eq. (4), we obtain

$$A_I(s,t) = \frac{i}{\sqrt{4k_{10}p_0}} \int \mathrm{d}x e^{ik_1 x} \theta(x_0) \langle \pi(k_2) | [\eta_\pi(x), \eta_K(0)] | 0 \rangle \,, \tag{6}$$

where $\eta_\pi(x)$ is the source of the reduced pion. We left aside the so-called "degenerate terms" which are polynomial of the Lorentz invariant variables

[22]. Then Eq. (6) defines a function holomorphic for those values of the external squared momenta s and t for which the integral is convergent (for the unreduced pion we take the physical value $u = m_\pi^2$). Due to the presence of $\theta(x_0)$, the integral upon x_0 converges in the upper half of the k_{10} complex plane, $\mathrm{Im}\, k_{10} > 0$. The causality property of the commutator restricts the integral upon the spatial variables to $|\mathbf{x}| < |x_0|$ [22]. We choose the particular Lorentz frame where the unreduced pion is at rest ($\mathbf{k_2} = 0$), when $k_{10} = (s - t - m_\pi^2)/2m_\pi$ and $\mathbf{k}_1^2 = [s - (\sqrt{t} + m_\pi)^2][s - (\sqrt{t} - m_\pi)^2]/(2m_\pi)^2$. Then the integral in (6) represents a function of s and t, analytic for complex values of these variables, with possible discontinuities along the real axes. The rigorous proof of the analyticity in the external masses is actually a difficult problem and requires a more detailed analysis [23]. Here we do not attempt to give a proof, but only use the LSZ representation to understand the meaning of the dispersive variables and to read off the contributions to the spectral functions appearing in the dispersion relation.

The discontinuity across the real axis is obtained formally from the expression (6) by replacing $i\theta(x_0)$ by $1/2$, inserting a complete set of intermediate states in the commutator $[\eta_\pi(x), \eta_K(0)]$ and using translational invariance [22]. The two terms in the commutator allow us to decompose the spectral function as

$$\sigma = \sigma_s + \sigma_t \,, \tag{7}$$

where

$$\sigma_s = \frac{1}{2\sqrt{4k_{01}p_0}} \sum_n \delta(k_1 + k_2 - p_n) \langle \pi(k_2)|\eta_\pi(0)|n\rangle\langle n|\eta_K(0)|0\rangle \,, \tag{8}$$

and

$$\sigma_t = \frac{1}{2\sqrt{4k_{01}p_0}} \sum_n \delta(k_1 + p_n) \langle \pi(k_2)|\eta_K(0)|n\rangle\langle n|\eta_\pi(0)|0\rangle \,. \tag{9}$$

In these relations the summation is over intermediate states consisting of physical particles, with an implicit integration upon their momenta. By the subscripts s (t), we anticipate the fact that σ_s receives contributions from the $s-$channel ($K \to \pi\pi$) and σ_t from the $t-$channel ($\pi \to K\pi$). Accordingly, we can write the amplitude as a sum of two terms, A_s^I and A_t^I, obtained by a dispersion representation involving the spectral function σ_s and σ_t, respectively. In order to evaluate the spectral functions, we recall that the sources contain contributions from both the strong and the weak interactions, the last ones being treated to first order.

Let us consider first the spectral function σ_s defined in (8). As discussed in Ref. [18], the intermediate states n which contribute to the unitarity sum are generated by either the weak or the strong part of the source η_K, undergoing a rescattering to the final $\pi\pi$ state by a strong (weak) process, respectively. The first contributions represent the so-called "final state interactions" (FSI), while the second are usually interpreted as "initial state interactions" (ISI). The lowest intermediate state contributing to FSI consists of two-pions, which produces the branch-point $s = 4m_\pi^2$, while for the ISI the lowest intermediate state is the pair $K^*\pi$, responsible for the threshold $s = (m_{K^*} + m_\pi)^2$. In order to write the specific contributions, we recall that in the LSZ formalism the matrix elements of the sources represent, up to kinematical factors, the physical decay or scattering amplitudes [22]. Thus, according to Eq. (4), $\langle\pi\pi|\eta_K(0)|0\rangle$ is the amplitude of the weak decay of K into two pions (here only the weak part of the source contributes), while $\langle\pi(k_2)|\eta_\pi(0)|n\rangle$ is the amplitude of either the strong or the weak $n \to \pi\pi$ transition, depending on which part of the source is considered.

A remarkable property of σ_s is that it depends only on s, being independent of the variable t [18], [22]. This can be easily seen by recalling that the intermediate states $|n\rangle$ consist of physical particles. By choosing the c.m. system, where $p_n^2 = s$ is the total energy squared, we see that the matrix elements in (8) depend only on s and the physical masses. In the two-particle approximation of the unitarity sum, the integral in (8) can be performed exactly [17]. According to the discussion above we can write

$$\sigma_s(s) = \sigma_{FSI}(s) + \sigma_{ISI}(s)\,, \tag{10}$$

where the first contributions to each term are

$$\begin{aligned}
\sigma_{FSI}(s) &= \theta(s-4m_\pi^2)M^*_{\pi\pi\to\pi\pi}A_{K\to\pi\pi} + \theta(s-4m_K^2)M^*_{\bar{K}K\to\pi\pi}A_{K\to\bar{K}K} + \dots\,, \\
\sigma_{ISI}(s) &= \theta(s-(m_{K^*}+m_\pi)^2)A^*_{K^*\pi\to\pi\pi}M_{K\to K^*\pi}(s) + \dots\,.
\end{aligned} \tag{11}$$

In the above relations $M_{\pi\pi\to\pi\pi}$ and $M_{\bar{K}K\to\pi\pi}$ denote on-shell S-wave strong scattering amplitudes at c.m. energy squared s, and $A_{K\to\pi\pi}$ ($M_{K\to K^*\pi}$), etc., are weak (strong) decay amplitudes. The amplitude A_s^I can be recovered from the discontinuity by means of a dispersion integral. Neglecting for the moment possible subtractions and polynomials in the Mandelstam variables, we have

$$A_s^I(s) = \frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\sigma_{FSI}(s')}{s'-s}\mathrm{d}s' + \frac{1}{\pi}\int_{(m_{K^*}+m_\pi)^2}^{\infty}\frac{\sigma_{ISI}(s')}{s'-s}\mathrm{d}s'\,. \tag{12}$$

We consider now the spectral function σ_t defined in (9). The weak part of the source η_π is responsible for the FSI in the t-channel, with the lowest branch-point at $t = (m_K + m_\pi)^2$. The strong part of the source η_π generates the ISI in the t-channel, with the lowest branch-point $t = 9m_\pi^2$. As above, it is easy to show that σ_t does not depend on s and can be written as

$$\sigma_t(t) = \tilde{\sigma}_{FSI}(t) + \tilde{\sigma}_{ISI}(t)\,, \tag{13}$$

where, according to the above discussion, the lowest terms are

$$\begin{aligned} \tilde{\sigma}_{FSI}(t) &= \theta(t - (m_K + m_\pi)^2) N^*_{\pi K \to \pi K} A_{\pi \to \pi K} + \dots\,, \\ \tilde{\sigma}_{ISI}(t) &= \theta(t - 9m_\pi^2)\, N^*_{\pi \to 3\pi} A_{3\pi \to \pi K} + \dots\,. \end{aligned} \tag{14}$$

Here $N_{\pi K \to \pi K}$ is the πK S-wave scattering amplitude at c.m. energy squared equal to t, and $A_{\pi \to \pi K}$ ($N_{\pi \to 3\pi}$) etc., are weak (strong) decay amplitudes. Neglecting again possible subtractions, we write the amplitude A_t^I in terms of its discontinuity in the t-channel by a dispersion integral

$$A_t^I(t) = \frac{1}{\pi} \int\limits_{(m_K+m_\pi)^2}^{\infty} \frac{\tilde{\sigma}_{FSI}(t')}{t' - t} \mathrm{d}t' + \frac{1}{\pi} \int\limits_{9m_\pi^2}^{\infty} \frac{\tilde{\sigma}_{ISI}(t')}{t' - t} \mathrm{d}t'\,. \tag{15}$$

The total amplitude $A_I(s,t)$ is then expressed as the sum

$$A_I(s,t) = A_s^I(s) + A_t^I(t)\,, \tag{16}$$

the physical amplitude being obtained for $s = m_K^2$ and $t = m_\pi^2$. We note in particular that $A_t^I(m_\pi^2)$ is a real number, since the point $t = m_\pi^2$ is situated below the cuts in the dispersion relation (15).

The significance of the variables s and t is clear from the above discussion: s is defined in terms of the pion momenta as $s = (k_1 + k_2)^2$ and t is equal to the external momentum squared of one pion, $t = k_1^2$. Therefore, it represents the mass squared of one external pion. We recall that in the above formalism no off-shell extrapolation was assumed, the sources entering the matrix elements being on-shell operators.

3 Omnès representations

It is convenient to write the above dispersion relations in terms of the phases of the rescattering amplitudes in the elastic region, according to Watson theorem [24]. To illustrate the method, we consider first the amplitude $A_I(s, m_\pi^2)$

as a function of s at physical $t = m_\pi^2$. The general case of $A_I(s,t)$ will be treated in subsection 3.2. This generalization is useful in order to incorporate information available on the decay amplitude at nonphysical pion masses.

3.1 Amplitude $A_I(s, m_\pi^2)$

In this case, as mentioned above, the last term in Eq. (16) is a real constant. Denoting $A_\pm = A_I(s \pm i\epsilon, m_\pi^2)$, we write the unitarity relation in the s-channel as

$$\frac{A_+ - A_-}{2i} = \theta(s - 4m_\pi^2) M_I^*(s) A_+ + \theta(s - s_{in})\, \sigma_{in}(s)\,, \tag{17}$$

where

$$M_I(s) = \frac{\eta_0^I e^{2i\delta_0^I} - 1}{2i}\,, \tag{18}$$

is the S-wave $\pi\pi$ scattering amplitude of isospin I. In Eq. (17) σ_{in} denotes the sum of the inelastic FSI and the ISI spectral functions (we take s_{in} equal to the FSI inelastic branch-point $4m_K^2$, which is lower than the ISI branch-point $(m_{K^*} + m_\pi)^2$). In the r.h.s. of Eq. (17) we note the presence of the amplitude $A_+ = A_I(s + i\epsilon, m_\pi^2)$, due to the fact that the intermediate two pions in the unitarity sum are physical particles, as we mentioned above.

The relation (17) can be written as an inhomogeneous Hilbert equation [19]

$$A_+(1 - 2iM_I^*) - A_- = 2i\,\theta(s - s_{in})\,\sigma_{in}(s)\,, \quad s \geq 4m_\pi^2\,. \tag{19}$$

We shall construct the solution by imposing time reversal invariance, which implies that the amplitudes satisfy the reality condition $A_I(s^*) = A_I^*(s)$ and the discontinuity across the cut is equal to the imaginary part [20]. Using the expression (18) we obtain from (19), for $s \geq 4m_\pi^2$,

$$\mathrm{Im}A_I \cos\delta_0^I - \mathrm{Re}A_I \sin\delta_0^I = \theta(s - s_{in}) \frac{2\,\mathrm{Re}\,[\sigma_{in} e^{i\delta_0^I}]}{1 + \eta_0^I}\,. \tag{20}$$

We define now the Omnès function

$$\Omega_I(s) = \exp\left[\frac{s - s_0}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\delta_0^I(s')\,\mathrm{d}s'}{(s' - s_0)(s' - s)}\right]\,, \tag{21}$$

assuming that one subtraction is sufficient. The boundary values of $\Omega_I(s)$ satisfy the relations $\Omega_I(s \pm i\epsilon) = \exp(\pm i\delta_0^I)|\Omega_I(s)|$ (we recall that the modulus

$|\Omega_I(s)|$ is obtained from (21) by taking the principal value of the integral). Then Eq. (20) can be written as

$$\mathrm{Im}\left[\frac{A_I(s,m_\pi^2)}{\Omega_I(s)}\right] = \theta(s-s_{in})\frac{2}{1+\eta_0^I}\,\frac{\mathrm{Re}\,[\sigma_{in}\,e^{i\delta_0^I}]}{|\Omega_I(s)|}\,. \tag{22}$$

We define now the function $G_I(s)$ through the relation

$$A_I(s,m_\pi^2) = \Omega_I(s)\,G_I(s)\,, \tag{23}$$

and express it by a dispersion relation in terms of its imaginary part given in (22):

$$G_I(s) = P_I(s) + \frac{s-s_0}{\pi}\int\limits_{s_{in}}^{\infty}\mathrm{d}s'\frac{2}{1+\eta_0^I}\,\frac{\mathrm{Re}\,[\sigma_{in}(s')e^{i\delta_0^I(s')}]}{|\Omega_I(s')|\,(s'-s_0)(s'-s)}\,. \tag{24}$$

Here $P_I(s)$ is a polynomial and, for convenience, we wrote the integral with one subtraction. The subtractions are actually not relevant in our method, since we shall parametrize the function $G_I(s)$ in a different way, using the technique of conformal mappings[2]. Namely, since by construction $G_I(s)$ is analytic in the $s-$plane cut for $s > s_{in}$, we consider the variable

$$z(s) = \frac{\sqrt{s_{in}-m_\pi^2}-\sqrt{s_{in}-s}}{\sqrt{s_{in}-m_\pi^2}+\sqrt{s_{in}-s}}\,, \tag{25}$$

which maps the s-plane cut along the real axis for $s > s_{in}$ onto the disk $z < 1$ of the plane $z = z(s)$. Actually, the mapping of the s-plane onto the unit disk is not unique [26]. For further convenience we choosed the mapping such that $z(m_\pi^2) = 0$. Now we expand $G_I(s)$ in powers the variable z

$$G_I(s) = \sum_n a_n^{(I)}\,[z(s)]^n\,, \tag{26}$$

where $a_n^{(I)}$ are real numbers. This series converges in the whole disk $|z| < 1$, *i.e,* in the whole s-plane cut along $s > s_{in}$, in particular at the physical point $s = m_K^2$.

[2]The method of conformal mappings was proposed in particle physics a long time ago [25]. In a context similar to the present one, the method was applied to the pion electromagnetic form factor in Ref. [26].

Inserting (26) into (23) we obtain a representation of the amplitude

$$A_I(s, m_\pi^2) = \Omega_I(s) \sum_n a_n^{(I)} [z(s)]^n \,, \tag{27}$$

in terms of the known $S-$wave $\pi\pi$ phase shifts entering the Omnès function $\Omega_I(s)$, and the real Taylor coefficients $a_n^{(I)}$. In Section 5 we shall discuss how these coefficients can be determined by using lattice results at unphysical values of s. The physical amplitude is obtained from (27) by setting $s = m_K^2$.

3.2 Amplitude $A_I(s,t)$

We consider now the amplitude $A_I(s,t)$ for arbitrary arguments. It turns out that the elastic unitarity for $A_I(s,t)$ can not be immediately solved by means of an Omnès representation, as in the above treatment of $A_I(s, m_\pi^2)$. Indeed, from the relations (16) and (10-12) the discontinuity of $A_I(s,t)$ across the cut along $s > 4m_\pi^2$ at fixed t is

$$\frac{A_I(s+i\epsilon,t) - A_I(s-i\epsilon,t)}{2i} = M_I^*(s) A_I(s+i\epsilon, m_\pi^2) + \theta(s-s_{in})\sigma_{in}(s) \,. \tag{28}$$

We note the presence, in the r.h.s., of the amplitude $A_I(s+i\epsilon, m_\pi^2)$, due to the fact that the intermediate states in the unitarity sum consist of physical particles. Therefore the functions which appear on the two sides of (28) are different, and this relation can not be written as a Hilbert boundary value equation.

This difficulty can be circumvented if we treat separately the functions $A_s^I(s)$ and $A_t^I(t)$ defined in Eqs. (12) and (15), respectively. Denoting $A_{s,\pm} = A_s^I(s \pm i\epsilon)$ we obtain from Eqs. (10-12)

$$\frac{A_{s,+} - A_{s,-}}{2i} = \theta(s-4m_\pi^2) M_I^*(s) A_I(s+i\epsilon, m_\pi^2) + \theta(s-s_{in})\,\sigma_{in}(s) \,. \tag{29}$$

In order to bring this relation to a form convenient for the Muskhelishvili-Omnès technique, we express, according to Eq. (16),

$$A_I(s+i\epsilon, m_\pi^2) = A_{s,+} + A_t^I(m_\pi^2) \,. \tag{30}$$

Then Eq. (29) becomes:

$$A_{s,+}(1-2iM_I^*) - A_{s,-} = 2i\,[\theta(s-4m_\pi^2) M_I^* A_t^I(m_\pi^2) + \theta(s-s_{in})\,\sigma_{in}(s)] \,. \tag{31}$$

This equation is similar to Eq. (19), excepted for an additional term in the r.h.s., which contributes to the imaginary part above the elastic threshold $4m_\pi^2$. Therefore, $A_s(s)$ will be of the form (23), with the function $G_I(s)$ given by a dispersion relation similar to (24), containing in addition the term

$$A_t^I(m_\pi^2)\,\frac{s-s_0}{\pi}\int\limits_{4m_\pi^2}^{\infty}\mathrm{d}s'\frac{2}{1+\eta_0^I}\frac{\mathrm{Re}\,[M_I^*(s')e^{i\delta_0^I(s')}]}{|\Omega_I(s')|(s'-s_0)(s'-s)}\,, \tag{32}$$

where we took into account that $A_t^I(m_\pi^2)$ is a real constant, as mentioned below Eq. (16).

It is convenient to separate in this integral the contribution of the inelastic region $s > s_{in}$, combining it with the contribution of the inelastic term σ_{in} and expanding them in powers of the variable z defined in (25). Therefore, we express $A_s^I(s)$ as

$$A_s^I(s) = \Omega_I(s)\left[A_t^I(m_\pi^2)\,f_I(s) + \sum_n c_n^{(I)}\,[z(s)]^n\right]\,, \tag{33}$$

with

$$f_I(s) = \frac{s-s_0}{\pi}\int\limits_{4m_\pi^2}^{s_{in}}\frac{\sin\delta_0^I(s')\,\mathrm{d}s'}{|\Omega_I(s')|(s'-s_0)(s'-s)}\,, \tag{34}$$

where we took along the elastic region $\eta_0^I(s) = 1$ and $M_I = e^{i\delta_0^I}\,\sin\delta_0^I$.

It is convenient to choose the subtraction point $s_0 = m_\pi^2$ in both the expression (21) of the Omnès function and the definition (34) of the function $f_I(s)$. This implies $\Omega_I(m_\pi^2) = 1$ and $f_I(m_\pi^2) = 0$. By recalling also that the conformal mapping (25) was defined such as $z(m_\pi^2) = 0$, it follows that the amplitude $A_s^I(s)$ given by Eq. (33) is normalized as

$$A_s^I(m_\pi^2) = c_0^{(I)}\,. \tag{35}$$

We consider now the second term in the decomposition (16), namely the amplitude $A_t^I(t)$. In order to obtain an Omnès representation, we must work with amplitudes $\tilde{A}_J(s,t)$ of definite isospin in the t channel, $\pi \to \pi K$. By crossing symmetry we can write

$$A_I(s,t) = \sum_{J=\frac{1}{2},\frac{3}{2}} C_{IJ}\tilde{A}_J(s,t)\,,\quad I = 0,2\,, \tag{36}$$

where the matrix C_{IJ} is known from the elastic πK scattering [27]. Each amplitude $\tilde{A}_J(s,t)$ admits a decomposition similar to (16)

$$\tilde{A}_J(s,t) = \tilde{A}_s^J(s) + \tilde{A}_t^J(t)\,. \tag{37}$$

We consider the amplitude $\tilde{A}_t^J(t)$ and define $\tilde{A}_{t,\pm} = \tilde{A}_t^J(t \pm i\epsilon)$. Then the unitarity relation in the t-channel, given by Eqs. (13-15), can be written as

$$\frac{\tilde{A}_{t,+} - \tilde{A}_{t,-}}{2i} = \theta(t-(m_K+m_\pi)^2)N_J^*(t)\tilde{A}_J(m_K^2, t+i\epsilon) + \theta(t-t_{in})\,\tilde{\sigma}_{in}(t)\,, \tag{38}$$

where

$$N_J(t) = \frac{\tilde{\eta}_0^J e^{2i\tilde{\delta}_0^J} - 1}{2i}\,, \tag{39}$$

denotes the S-wave πK scattering amplitudes at c.m. energy squared equal to t, and $\tilde{\sigma}_{in}(t)$ the contribution of the inelastic FSI and ISI t-channels. We take t_{in} equal to the ISI branch-point $9m_\pi^2$, which is lower than the inelastic FSI branch-point $(m_K + m_\eta)^2$.

Since the intermediate πK state in the unitarity sum (9) consists of physical particles, in the r.h.s. of (38) contributes the amplitude $\tilde{A}_J(m_K^2, t+i\epsilon)$. By expressing $\tilde{A}_J(m_K^2, t+i\epsilon)$ according to (37) we obtain

$$\begin{aligned} \tilde{A}_{t,+}(1-2iN_J^*) - \tilde{A}_{t,-} &= 2i\,[\theta(t-(m_\pi+m_K)^2)N_J^*\tilde{A}_s^J(m_K^2) \\ &+ \theta(t-t_{in})\,\tilde{\sigma}_{in}(t)]\,. \end{aligned} \tag{40}$$

The solution of this equation can be obtained following the procedure applied to the function $A_s^I(s)$. We introduce the Omnès function

$$\tilde{\Omega}_J(t) = \exp\left[\frac{t-t_0}{\pi}\int\limits_{(m_\pi+m_K)^2}^{\infty} \frac{\tilde{\delta}_0^J(t')}{(t'-t_0)(t'-t)}\mathrm{d}t'\right]\,, \tag{41}$$

and express the ratio $\tilde{A}_t^J(t)/\tilde{\Omega}_J(t)$ through a dispersion relation in terms of its imaginary part calculated from (40). Then $\tilde{A}_t^J(t)$ can be written as

$$\begin{aligned} \tilde{A}_t^J(t) &= \tilde{\Omega}_J(t)\left[\tilde{P}_J(t) + \frac{t-t_0}{\pi}\int\limits_{(m_\pi+m_K)^2}^{\infty} \mathrm{d}t' \frac{2}{1+\tilde{\eta}_0^J}\frac{\mathrm{Re}\,[\tilde{A}_s^J(m_K^2)N_J^*(t')e^{i\tilde{\delta}_0^J}]}{|\tilde{\Omega}_J(t')|(t'-t_0)(t'-t)}\right. \\ &+ \left.\frac{t-t_0}{\pi}\int\limits_{t_{in}}^{\infty} \mathrm{d}t' \frac{2}{1+\tilde{\eta}_0^J}\frac{\mathrm{Re}\,[\sigma_{in}(t')e^{i\tilde{\delta}_0^J}]}{|\tilde{\Omega}_J(t')|(t'-t_0)(t'-t)}\right]\,. \end{aligned} \tag{42}$$

We recall that the quantities $\tilde{A}_s^J(m_K^2)$ entering this relation are complex numbers.

We further separate in Eq. (42) the contribution of the elastic part of the cut, and take into account the higher singularities by means of a conformal mapping. Namely, we define the variable

$$w(t) = \frac{\sqrt{t_{in} - m_\pi^2} - \sqrt{t_{in} - t}}{\sqrt{t_{in} - m_\pi^2} - \sqrt{t_{in} - t}}, \tag{43}$$

which maps the t-plane cut for $t > t_{in}$ onto the disk $|w| < 1$, such that $w(m_\pi^2) = 0$, and expand the polynomial and the inelastic part of Eq. (42) in powers of this variable:

$$\tilde{P}_J + \frac{t - t_0}{\pi} \int_{t_{in}}^{\infty} \mathrm{d}t' \frac{2}{1 + \tilde{\eta}_0^J} \frac{\mathrm{Re}\{[\tilde{A}_s^J(m_K^2)\, N_J^*(t') + \sigma_{in}(t')] e^{i\tilde{\delta}_0^J}\}}{|\tilde{\Omega}_J(t')|(t' - t_0)(t' - t)} = \sum \tilde{c}_n^{(J)} \, [w(t)]^n \,, \tag{44}$$

the coefficients $\tilde{c}_n^{(J)}$ being real. Then Eq. (42) becomes

$$\tilde{A}_t^J(t) = \tilde{\Omega}_J(t) \left[\mathrm{Re}\tilde{A}_s^J(m_K^2)\, g_J(t) + \sum_n \tilde{c}_n^{(J)} \, [w(t)]^n \right] , \tag{45}$$

where we defined

$$g_J(t) = \frac{t - t_0}{\pi} \int_{(m_\pi + m_K)^2}^{t_{in}} \frac{\sin \delta_0^J \, \mathrm{d}t'}{|\tilde{\Omega}_J(t')|(t' - t_0)(t' - t)} \,. \tag{46}$$

We took into account the fact that in the elastic region $\tilde{\eta}_0^J = 1$ and $\tilde{N}_J^* e^{i\tilde{\delta}} = \sin \tilde{\delta}_0^J$.

It is convenient to choose the subtraction point $t_0 = m_\pi^2$ in both the Omnès function (41) and the definition (46) of $g_J(t)$. This means that $\tilde{\Omega}_J(m_\pi^2) = 1$ and $g_J(m_\pi^2) = 0$. Recalling also that the conformal variable $w(t)$ was defined in (43) such that $w(m_\pi^2) = 0$, we obtain from (45)

$$\tilde{A}_t^J(m_\pi^2) = \tilde{c}_0^{(J)} \,. \tag{47}$$

Collecting Eqs. (16, 33, 36) and (45), we express the amplitude $A_I(s,t)$ as

$$\begin{aligned} A_I(s,t) &= \Omega_I(s) \left[A_t^I(m_\pi^2)\, f_I(s) + \sum_n c_n^{(I)} [z(s)]^n \right] \\ &+ \sum_J C_{IJ}\, \tilde{\Omega}_J(t) \left[\mathrm{Re}\tilde{A}_s^J(m_K^2)\, g_J(t) + \sum_n \tilde{c}_n^{(J)} [w(t)]^n \right] , \end{aligned} \tag{48}$$

where the functions $f_I(s)$ and $g_J(t)$ are defined in Eqs. (34) and (46), respectively (we recall that $s_{in} = 4m_K^2$ and $t_{in} = 9m_\pi^2$). In Eq. (48) we must insert, according to (16,36,37) and (47),

$$A_t^I(m_\pi^2) = \sum_J C_{IJ}\, \tilde{c}_0^{(J)} \,. \tag{49}$$

Also, using the crossing relation (36), we express the quantity $\mathrm{Re}\,\tilde{A}_s^J(m_K^2)$ entering (48) as

$$\mathrm{Re}\,\tilde{A}_s^J(m_K^2) = \sum_{L=0,2} C_{JL}^{-1}\,\mathrm{Re}\,A_s^L(m_K^2)\,, \quad J = \frac{1}{2}\,,\frac{3}{2}\,, \tag{50}$$

where $\mathrm{Re}\,A_s^L(m_K^2)$ is obtained from (33). The relations (48)-(50) provide a system of coupled equations, which express each amplitude $A_I(s,t)\,(I=0,2)$ in terms of $\pi\pi$ and πK S-wave phase shifts and the real coefficients $c_n^{(I)}$ and $\tilde{c}_n^{(J)}\,(J = 1/2\,,\,3/2)$.

At fixed $t = m_\pi^2$ the amplitude (48) takes the simple form

$$A_I(s, m_\pi^2) = \Omega_I(s)\left[\sum_J C_{IJ}\tilde{c}_0^{(J)}\,\{f_I(s) + \Omega_I^{-1}(s)\} + \sum_n c_n^{(I)}[z(s)]^n\right]\,, \tag{51}$$

where we introduced the last term of Eq. (48), *i.e.* $A_t^I(t)$ evaluated for $t = m_\pi^2$, inside the brackets, and expressed $A_t^I(m_\pi^2)$ according to Eq. (49).

It is easy to verify that the function multiplying the Omnès factor in the relation (51) is real for $s < s_{in}$. Indeed, the coefficients $c_n^{(I)}$ and $\tilde{c}_0^{(J)}$ are real, and for values of s below the inelastic threshold the variable $z(s)$ is real. The only terms having an imaginary part for $s < s_{in}$ are the function $f_I(s)$ and the Omnès function $\Omega_I(s)$. But it is easy to check, using (21) and (34), that their imaginary parts compensate in Eq. (51):

$$\mathrm{Im}\,[f_I(s) + \Omega_I^{-1}(s)] = \frac{\sin\delta_0^I(s)}{|\Omega_I(s)|} - \frac{\sin\delta_0^I(s)}{|\Omega_I(s)|} = 0\,. \tag{52}$$

Therefore, the first term in the r.h.s. of (51) is real along the elastic region and can be included in the expansion in powers of the variable $z(s)$. This shows that the general representation (48) reduces, when $t = m_\pi^2$, to the Omnès representation (27) derived in the previous subsection. For arbitrary values of t, however, the amplitude (48) has additional cuts in the elastic region of both the s and t channels.

The physical amplitude is obtained from (48) for $s = m_K^2$ and $t = m_\pi^2$. With our normalization, it depends on the S-wave phase shift of $\pi\pi$ scattering and the Taylor coefficients $c_n^{(I)}$ and $\tilde{c}_n^{(J)}$ (the S-wave phase shift of the πK scattering contribute only indirectly, through these coefficients). As proved in [25], by the conformal mapping the rate of convergence is improved, so we expect to obtain an accurate representation at low energies with a small number of terms in the expansions. In the next section we shall discuss how to determine the coefficients $c_n^{(I)}$ and $\tilde{c}_n^{(J)}$ using lattice results at unphysical points.

We end this section with two remarks: first we notice that in the above derivation the symmetry between the two final pions is not explicit. This symmetry can be easily imposed by writing down a dispersion relation symmetrical with respect to the interchange of t and u. Namely, instead of Eq. (16) we have, more generally

$$A_I(s,t,u) = A_s^I(s) + \frac{1}{2}\left[A_t^I(t) + A_u^I(u)\right] , \tag{53}$$

where $A_u^I(u)$ satisfies a dispersion relation similar to (45).

The second remark concerns the starting point used for the analytic extrapolation: in our analysis we considered the amplitude given by the expression (4), which was obtained by making the LSZ reduction of the kaon in the S-matrix element (3). Alternatively, by reducing first one pion instead of the kaon, we obtain the expression

$$A_I = \frac{1}{\sqrt{2k_{10}}}\langle\pi(k_2)|\,\eta_\pi(0)\,|K(p)\rangle\,. \tag{54}$$

By further reducing the K-meson, one obtains, instead of (6), the expression

$$A_I(s,t) = \frac{i}{\sqrt{4k_{10}p_0}}\int \mathrm{d}x e^{-ipx}\theta(x_0)\,\langle\pi(k_2)|[\eta_\pi(0),\eta_K(x)]|0\rangle\,, \tag{55}$$

which allows the analytic continuation with respect to the variables $s = p^2$ and $t = (p-k_2)^2$. It is easy to see that the spectral function can be written as in Eq. (7), with the corresponding terms similar to (8)-(9), excepted that the momentum conservation reads now $p_n = p$ in σ_s and $p_n = k_2 - p$ in σ_t. This implies that the significance of the various terms in the spectral functions are different in the two approaches: in Eq. (8) the c.m. energy which generates the intermediate states is yielded by the total momentum carried by the kaon

and the interaction hamiltonian, while in (9) this energy is provided by the unphysical mass of the pion. In the alternative approach mentioned above, the energy in the term σ_s is yielded by the unphysical mass of the kaon, while in σ_t it is provided by the momenta of the pion and of the hamiltonian. Despite this different interpretation of the matrix elements, it is easy to see that the Omnès representation is formally similar in the two approaches. The differences concern only the inelastic contributions, parametrized by the expansions in powers of the conformal mapping variables.

4 Discussion

The Omnès representations derived above generalize previous results obtained in the literature. In Refs. [8, 9] the authors write down an Omnès representation for the decay amplitude as a function of s at fixed $t = m_\pi^2$, using the formal analogy with the scalar pion form factor. The decay amplitude is identified, up to a constant, to the Omnès factor $\Omega_I(s)$ in Eq. (27), *i.e.* the inelastic singularities are neglected. The role of the inelastic FSI and ISI contributions in the dispersion relation was discussed in Ref. [12], assuming that the decay amplitude defined in Eq. (1) satisfies a Mandelstam representation.

A similar line is followed in Ref. [14], where the weak Hamiltonian $\mathcal{H}_w$ in Eq. (1) is identified with a field operator with the quantum numbers of the kaon, and is assumed to carry a nonzero momentum. In this approach the $K \to \pi\pi$ decay amplitude is obtained from a dispersion relation for the elastic processes $\bar{K}K \to \pi\pi$ (s-channel) and $\pi K \to \pi K$ (t-channel). Assuming that only S and P waves contribute to both channels, the amplitude is written in [14] as a sum of functions depending on a single variable, s (t), respectively. The interplay between the weak and the strong dynamics is however not apparent in this treatment: the weak $K \to \pi\pi$ decay amplitude is given formally by the amplitude of the strong scattering process $\bar{K}K \to \pi\pi$, evaluated at an unphysical point.

As we mentioned already, the dispersion relations for the $K \to \pi\pi$ decay require a continuation in the external momenta. Therefore, the interpretation of the dispersive variables was not very clear in the previous works. In ref. [8] the variable s in the dispersion relation for the amplitude $A_I(s,t)$ at fixed $t = m_\pi^2$ was identified with the momentum squared of an off-shell kaon. This raised the subsequent criticism [13], which emphasized the ambiguity of the

ChPT calculations for off-shell operators. We mention also that in Ref. [11] the same variable s was identified with the mass squared of the kaon.

In the present treatment, the LSZ reduction formula allows the analytic continuation of the amplitudes in the complex planes of the external momenta. No off-shell extrapolation of the operators is necessary and the meaning of the variables is clear, as discussed at the end of the previous section. If we use as starting point of the analytic continuation the matrix element (4), the variable s is defined as $s = (k_1 + k_2)^2$, and in unphysical configurations it may be different from the kaon momentum. Moreover, in this case the variable t is equal to k_1^2, and represents the mass squared of the external pion. On the other hand, if we use as starting point of the analytic continuation the expression (54), we have $s = p^2$, *i.e.* it represents the mass squared of the external kaon, while $t = (p - k_2)^2$ includes, in unphysical configurations, the momentum carried by the interaction hamiltonians. This interpretation allows us to incorporate in the dispersion relations, at least to a certain extent, the results of the lattice simulations. We consider for illustration the first definition of dispersive variables.

Most lattice calculations simulate matrix elements of the form $\langle \pi | O_j | K \rangle$, related to the matrix elements of interest $\langle \pi\pi | O_j | K \rangle$ by lowest order ChPT in the soft pion limit [3]. In the limit $k_1 \to 0$ we have $t = k_1^2 = 0$ and $s = (k_1 + k_2)^2 = m_\pi^2$. Therefore, the lattice simulations of this type provide the value of the amplitude $A_I(m_\pi^2, 0)$.

Direct simulations of the matrix elements $\langle \pi\pi | O_j | K \rangle$ are done only for special configurations, for instance when the kaon and one pion are at rest. Assuming that the reduced pion is at rest, $\mathbf{k_1} = 0$, we have $s = t + m_\pi^2 + 2\sqrt{t}\, E_\pi$, where E_π is the energy of the moving pion. At present the lattice simulations are done using values of the mass $\tilde{m}_\pi$ larger than the physical one. To match this situation we take $\tilde{t} = \tilde{m}_\pi^2$, which means that $\tilde{s} = \tilde{m}_\pi^2 + m_\pi^2 + 2\tilde{m}_\pi\, E_\pi$. We assume therefore that $A_I(\tilde{s}, \tilde{t})$ is known approximately from the lattice calculations[3]. Actually, $A_I(\tilde{s}, \tilde{t})$ does not correspond exactly to the configuration which is simulated on the lattice, since in the dispersion relation the unreduced pion has the physical mass, $k_2^2 = m_\pi^2$. The analytic continuation with respect to k_2^2 requires the reduction of the second pion in the relation (6), and is more complicated. In the present formalism, we can assume that the lattice situation is approached by using a dispersion relation

[3]In the alternative interpretation of the dispersive variables, discussed above, the corresponding points are $\tilde{s} = \tilde{m}_K^2$ and $\tilde{t} = \tilde{m}_K^2 + m_\pi^2 - 2\tilde{m}_K\, E_\pi$.

symmetrized with respect to the interchange of t and u, as in Eq. (53), where one pion has the physical mass and the other a different mass, $\tilde{m}_\pi$. This is still not identical with the lattice case, but we believe that even approximate numbers are welcome, since the dispersive approach includes correctly the FSI. Using a sufficient number of points $(\tilde{s}, \tilde{t})$, it is possible to determine the coefficients $c_n^{(I)}$ and $\tilde{c}_n^{(J)}$ of the Taylor expansions in powers of the conformal variables z and w, and to calculate then the physical amplitude using the expression (48) for $s = m_K^2$ and $t = m_\pi^2$.

5 Conclusions

In the present paper we derived Omnès representations for the $K \to \pi\pi$ amplitudes, which include elastic and inelastic contributions in both the direct and the crossed channels. We showed that the amplitude is decomposed as a sum of two functions, one depending only on s and the other depending only on t. This decomposition follows naturally from the LSZ formalism and hadronic unitarity and does not require additional assumptions. The elastic contributions are parametrized by Omnès factors according to Watson theorem, and the inelastic singularities are accounted for by the technique of conformal mappings. The treatment based on LSZ formalism allows a clear interpretation of the dispersion relations and the meaning of the dispersive variables. The unknown coefficients $c_n^{(I)}$ and $\tilde{c}_n^{(J)}$ entering the parametrization (48) of the amplitude can be determined, at least approximately, using information provided by lattice calculations at unphysical momenta. The numerical implementation of this program is a subject of a future work. We mention finally that the effects of isospin violation, which were discussed recently in Ref. [28], can be incorporated in the dispersive treatment by a suitable modification of the unitarity relation.

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