



HAL
open science

On Solutions of a Non linear Wave Equation Derived From Brain Activity Modeling

Mouhamad Jradeh, Maïtine Bergounioux

► **To cite this version:**

Mouhamad Jradeh, Maïtine Bergounioux. On Solutions of a Non linear Wave Equation Derived From Brain Activity Modeling. 2007. hal-00126996v2

HAL Id: hal-00126996

<https://hal.science/hal-00126996v2>

Preprint submitted on 31 Mar 2007 (v2), last revised 19 Apr 2008 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On Solutions of a Non linear Wave Equation Derived From Brain Activity Modeling

Mouhamad Jradeh and Maitine Bergounioux

*Laboratoire MAPMO- UMR 6628 - Fédération Denis Poisson - Université
d'Orléans - BP 6759 - 45067 Orléans cedex 02, France*

Abstract

In this work, we analyze a cerebral activity model proposed by Jirsa [8]. Using a Galerkin approximation scheme, we prove existence and stability of global weak solutions to a damped non-linear wave equation that generalizes the cerebral activity equation. We prove uniqueness as well.

Key words: Neural fields, nonlinear wave equation, Galerkin method, boundary value problem

1991 MSC: : 35L05, 35Q80

1 Introduction

Non-invasive techniques such as functional Magnetic Resonance Imaging, ElectroEncephaloGraphy (EEG) and MagnetoEncephaloGraphy (MEG) provide entry points to human brain dynamics for clinical purposes, as well as the study of human behavior and cognition. Each of these observation technologies provides spatiotemporal information about the on-going neural activity in the cortex, but unfortunately the results of measures are generally noisy, and it is difficult to identify the equation which governs the dynamics of neural activity. Several physicists [23,16,8,6] have formulated continuous models called neural fields to predict neural activity, using brain anatomy.

Jirsa's model [8], which we will analyze in section 4 of this paper, generalizes the models [23,16], and leads to the following evolution problem that we investigate in this work:

Email addresses: mouhamad.jradeh@univ-orleans.fr (Mouhamad Jradeh),
maitine.bergounioux@univ-orleans.fr (Maitine Bergounioux).

$$u_{tt} - \alpha \Delta u = a(u, p)u_t + b(u, p, p_t), \quad x \in \Omega, \quad \text{a.e. } t \geq 0 \quad (1.1)$$

$$u(0, \cdot) = u_0 \quad u_t(0, \cdot) = u_1. \quad (1.2)$$

$$u = 0, \quad \text{on } [0, +\infty) \times \partial\Omega \quad (1.3)$$

where Ω is an open bounded domain in \mathbb{R}^n ($n \leq 4$) with sufficiently smooth boundary $\partial\Omega$, $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. The notation u_t (respectively u_{tt}) stands for the first (respectively second) partial derivative of u with respect to time variable t and Δ is the Laplace operator. For simplicity, we assume that $\alpha = 1$.

Although the equation(1.1) has a simple form, to our knowledge, without additional conditions on a, b or initial condition, the existence and uniqueness of the solution problem is still opened. In addition the existence of **global** solution is not always ensured (see by example [4,28,25]).

Equation (1.1) belongs to a class of non-linear damped wave equations that have been widely studied by many mathematicians and engineers. Existence and uniqueness with (1.2), (1.3) condition has been considered using various methods, semi-group theory [17,13,27,2], or point fix method [9,1]. In [5,3,18,19,26] the authors prove existence and uniqueness of global solutions with conditions (1.2), (1.3) using Galerkin methods.

Zhou [28], has studied a particular case of equation (1.1), without external input p , a constant function and b defined by

$$b(u) = |u|^{m-1}u$$

with condition (1.2) and $\Omega = \mathbb{R}^n$; he proved that, if $1 < m < \frac{n+2}{n}$ the solution blows-up in finite time. Similarly authors in [17,24,28,29,10,11,21,22,4,15,25] discussed the case when the solution blows-up in finite time.

The closest equation to the one we investigate here, has been studied by Zhijian [26] who considered :

$$u_{tt} - \Delta u - \Delta u_t = \sum_{i=1}^n \frac{\partial}{\partial x_i} (\sigma_i(u_{x_i}) + \beta_i(u_{x_{it}})) + F(u, u_t, \nabla u, \nabla u_t) \quad \text{on } [0, \infty) \times \Omega \quad (1.4)$$

with initial and boundary conditions (1.2) (1.3). He proved that this problem has a unique classical solution, assuming what follows on the initial data :

$$\left\| \sum_{i=1}^n \frac{\partial}{\partial x_i} (\sigma_i(u_{x_i}(0)) + \beta_i(v_{x_i}(0))) + F(u(0), v(0), \nabla u(0), \nabla v(0)) \right\|_{H^k} < \delta \quad (1.5)$$

$$\|v(0)\|_{H^{k+1}}^2 + \|v(0)\|_{H^k} + \frac{3}{2} \|u(0)\|_{H^k} + \langle u(0), v(0) \rangle_{H^k} < \gamma \quad (1.6)$$

where δ and γ are two real numbers. In addition, it was supposed that

$$|D^\beta F| \leq B(|\lambda_1|^{\gamma_1} + |\lambda_2|^{\gamma_2} + |\tilde{\lambda}_1|^{\gamma_3} + |\tilde{\lambda}_2|^{\gamma_4} + 1) \text{ and } F(0) = 0 \quad (1.7)$$

The RHS of (1.1) is a particular case of RHS of (1.4), with an additional parameter (the input function p) that we suppose to be regular enough. We have to mention here that the presence of Δu_t in the LHS of equation (1.4), is crucial in the proof estimates, namely that the solution is bounded with respect to H^2 norm.

Now, the question is : if the initial data does not satisfy (1.5), (1.6) or (1.7) and no “regularizing” term Δu_t appears, does problem (1.1)-(1.3) admit a global solution? If yes, is the solution unique and/or stable with respect to initial data perturbations. In other terms, we look for the well-posedness of the problem.

In this paper, we prove that problem (1.1)-(1.3) has a unique global stable solution under “realistic” assumptions. Indeed, these assumptions are motivated by the underlying physical cerebral activity model that leads to an equation which is a particular case of equation (1.1). From the physical point of view, these assumptions mean that the brain activity remains “normal” if the initial data are “normal”.

The outline of the paper is the following : existence results are stated in Section 2. Uniqueness and stability are discussed in Section 3. Finally in Section 4, we present and analyze the cerebral activity model, and apply the general result.

2 Existence of weak global solutions

We denote $\|\cdot\|$ the $L^2(\Omega)$ norm, $\|\cdot\|_\infty$ the L^∞ -norm, $\|\cdot\|_1$ the $H^1(\Omega)$ norm, and (\cdot, \cdot) the L^2 - inner product.

Definition 2.1 *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $T > 0$ be given. We shall say that u is a weak solution to problem (1.1), (1.3) if*

$$u \in L^2([0, T], H_0^1(\Omega)), u_t \in L^2([0, T], H^1(\Omega)), u_{tt} \in L^2([0, T], L^2(\Omega))$$

and for almost every $t \geq 0$, $\forall \omega \in H_0^1(\Omega)$

$$(u_{tt}(t), \omega) + (\nabla u(t), \nabla \omega) = (a(u, p)(t)u_t(t) + b(u, p, p_t)(t), \omega) \quad (2.1)$$

and

$$u(0, \cdot) = u_0, \quad u_t(0, \cdot) = u_1 \quad (2.2)$$

From now, we do not indicate the dependence with respect to t to make the paper more readable. First we give an existence result of weak solutions to problem (1.1)-(1.3).

Theorem 2.1 Assume that for some $T > 0$

(A₁) $a, b \in C^1(\mathbb{R}^2, \mathbb{R})$ and $C^1(\mathbb{R}^3, \mathbb{R})$ respectively, and $a, \nabla a$ and ∇b are (uniformly) bounded functions.

(A₂) $|b(u, p, q)(t, x)| \leq C_1|u(t, x)| + C_2$ a.e. on $[0, T) \times \Omega$ where C_1 and C_2 are nonnegative constants.

(A₃) $p \in H^2([0, T), L^\infty(\Omega))$.

(A₄) $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$

Then for problem (1.1)-(1.3) admits (at least) a weak solution u on $[0, T)$ and

$$u \in H^2([0, T), L^2(\Omega)) \cap H^1([0, T), H_0^1(\Omega))$$

Proof - We need several steps to prove this result. We use the Galerkin method to construct a solution. Let $\{\lambda_k\}_{k=1}^\infty$ be a sequence of eigenvalues of $-\Delta$ in Ω . Let $\{\omega_k\}_{k=1}^\infty \in H_0^1(\Omega) \cap H^2(\Omega)$ be the associated eigenfunction such that $\{\omega_k\}_{k=1}^\infty$ a complete orthonormal system of $L^2(\Omega)$. We construct approximate solutions u^n as following

$$u^n(t) = \sum_{k=1}^n d^{n,k}(t)\omega_k$$

where $d^{n,k}$ functions are determined by the ordinary differential equations system

$$(u_{tt}^n, \omega_k) + (\nabla u^n, \nabla \omega_k) = (a(u^n, p)u_t^n + b(u^n, p, p_t), \omega_k) \quad k = 1, \dots, n. \quad (2.3)$$

$$d^{n,k}(0) = u_0^k, \quad d_t^{n,k}(0) = u_1^k \quad (2.4)$$

with $u_0, u_1 \in C_0^\infty$, such that $u_0^k \rightarrow u_0$ and $u_1^k \rightarrow u_1$ in $L^2(\Omega)$. Now, (2.3) and (2.4) are equivalent to:

$$d_{tt}^{n,k} + \lambda_k d^{n,k} = f(d^{n,k}, d_t^{n,k}, t) \quad k = 1, \dots, n, \quad (2.5)$$

$$d^{n,k}(0) = u_0^k, \quad d_t^{n,k}(0) = u_1^k \quad (2.6)$$

where,

$$f(d^{n,k}, d_t^{n,k}, t) := (a(u^n, p)u_t^n + b(u^n, p, p_t), \omega_k) \quad \text{a.e. } t \geq 0. \quad (2.7)$$

Since (2.5), (2.6) is a second order $n \times n$ Cauchy system of differential equations with continuous nonlinearities, it follows from the Cauchy-Peano Theorem [20], that for every $n \geq 1$, the system has at least one solution $d^{n,k}$ defined on $[0, T_n]$, for some $T_n > 0$. Moreover, for $1 \leq k \leq n$, $d^{n,k} \in C^2[0, T_n]$.

Lemma 2.1 Functions (u_n) satisfy

$$\|u_t^n(t)\| + \|u^n(t)\|_1 \leq M(T)$$

where $M(T)$ only depends on T . In particular $u^n(t)$ can be extended to $[0, T)$.

Proof - Multiply equation (2.3) by $d_t^{n,k}$, and sum over $k=1, \dots, n$, by assumption A_1 and A_2 we obtain

$$\begin{aligned} (u_{tt}^n, u_t^n) + (\nabla u^n, \nabla u_t^n) &= (a(u^n, p)u_t^n + b(u^n, p, p_t), u_t^n) \\ &\leq C(|u_t^n| + C_1|u^n| + C_2, |u_t^n|) \end{aligned}$$

Using Cauchy-Schwartz inequality we have

$$\begin{aligned} \frac{d}{dt}(\|u_t^n(t)\|^2 + \|\nabla u^n(t)\|^2) &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|\|u_t^n(t)\| + \|u_t^n(t)\|) \\ &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|^2 + \|u_t^n(t)\|^2 + \|u_t^n(t)\|^2 + 1) \\ &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|^2 + 1) \\ &\leq C(\|u_t^n(t)\|^2 + \|\nabla u^n(t)\|^2 + 1) \end{aligned}$$

by Poincaré inequality, where C denotes (here and in the sequel) a generic constant. Applying Gronwall inequality yields:

$$\|u_t^n(t)\| + \|\nabla u^n(t)\| \leq M(T); \quad (2.8)$$

then by Poincaré inequality

$$\|u_t^n(t)\| + \|u^n(t)\|_1 \leq M(T) \quad (2.9)$$

and

$$\begin{aligned} |f(d^{n,k}, d_t^{n,k}, t)| &= |(a(u^n, p)u_t^n + b(u^n, p, p_t), \omega_k)| \quad \text{a.e. } t \geq 0 \\ &\leq C(\|u_t^n(t)\|^2 + \|u^n(t)\|^2) \leq C \end{aligned}$$

So $f(d^{n,k}, d_t^{n,k}, \cdot)$ is a bounded function, and the solution can be extended to $[0, T)$ [20]. \square

Lemma 2.2 *The sequence of approximated solutions u^n satisfies the following:*

- (1) u^n is bounded in $L^\infty([0, T], H_0^1(\Omega))$.
- (2) u_t^n is bounded in $L^\infty([0, T], H_0^1(\Omega))$.
- (3) u_{tt}^n is bounded in $L^\infty([0, T], L^2(\Omega))$.

Proof - The first item is an immediate consequence of Lemma 2.1. Differentiating (2.3) with respect to t gives for $k = 1, \dots, n$

$$\begin{aligned}
(u_{ttt}^n, \omega_k) + (\nabla u_t^n, \nabla \omega_k) &= (a_u(u^n, p)(u_t^n)^2 + a_p(u^n, p)p_t u_t^n \\
&\quad + a(u^n, p)u_{tt}^n + b_u(u^n, p, p_t)u_t^n + b_p(u^n, p, p_t)p_t \\
&\quad + b_q(u^n, p, p_t)p_{tt}, \omega_k) \quad k = 1, \dots, n \\
&\leq C((u_t^n)^2 + |u_t^n| + |u_{tt}^n| + |u_t^n| + M, |\omega_k|),
\end{aligned}$$

with assumptions A_1 , A_2 and A_3 . Here a_u and a_p denote the partial derivatives of $a : (u, p) \mapsto a(u, p)$. Similarly, b_u , b_p and b_q are the partial derivatives of $b : (u, p, q) \mapsto b(u, p, q)$.

The method is similar to the one used in the previous Lemma. Replace now ω_k by $u_{tt}^n(t)$

$$\begin{aligned}
\frac{d}{dt}(\|u_{tt}^n(t)\|^2 + \|\nabla u_t^n(t)\|^2) &\leq C\left(\|(u_t^n(t))^2\| + \|u_t^n(t)\| + \|u_{tt}^n(t)\| + M\right)\|u_{tt}^n(t)\| \\
&\leq C(\|(u_t^n(t))^2\|\|u_{tt}^n(t)\| + \|u_t^n(t)\|\|u_{tt}^n(t)\| + \\
&\quad \|u_{tt}^n(t)\|^2 + \|u_{tt}^n(t)\|) \\
&\leq C(\|(u_t^n(t))^2\|^2 + \|u_{tt}^n(t)\|^2 + \|u_t^n(t)\|^2 + \\
&\quad \|u_{tt}^n(t)\|^2 + \|u_{tt}^n(t)\|^2 + \|u_{tt}^n(t)\|^2 + 1) \\
&\leq C(\|(u_t^n(t))^2\|^2 + \|u_{tt}^n(t)\|^2 + \|u_t^n(t)\|^2 + 1)
\end{aligned}$$

Using Sobolev-Poincaré inequality [14],

$$\|u_t^n(t)\|_{L^4} \leq C\|u_t^n(t)\|_1$$

and Poincaré inequality, we get

$$\begin{aligned}
\frac{d}{dt}(\|u_{tt}^n(t)\|^2 + \|\nabla u_t^n(t)\|^2) &\leq C(\|u_{tt}^n(t)\|^2 + \|u_t^n(t)\|_1^2 + 1) \\
&\leq C(\|u_{tt}^n(t)\|^2 + \|\nabla u_t^n(t)\|^2 + 1).
\end{aligned}$$

Then by Gronwall inequality we obtain

$$\|u_t^n(t)\|_1 + \|u_{tt}^n(t)\| \leq M(T) \quad \text{a.e. } t \in [0, T], \quad (2.10)$$

and the lemma is proved. \square

Thanks to estimations (2.9) and (2.10), we may extract subsequences of $\{u^n\}$ (denoted similarly in the sequel) such that

$$u^n \rightharpoonup u \quad \text{weakly star in } L^\infty([0, T], H_0^1(\Omega)) \text{ and weakly in } L^2([0, T], H_0^1(\Omega)) \quad (2.11)$$

$$u_t^n \rightharpoonup u_t \quad \text{weakly star in } L^\infty([0, T], H_0^1(\Omega)) \text{ and weakly in } L^2([0, T], H_0^1(\Omega)) \quad (2.12)$$

$$u_{tt}^n \rightharpoonup u_{tt} \quad \text{weakly star in } L^\infty([0, T], L^2(\Omega)) \text{ and weakly in } L^2([0, T], L^2(\Omega)) \quad (2.13)$$

With classical compactness results [12]

$$u^n \rightarrow u \quad \text{strongly in } L^2([0, T], L^2(\Omega)) \quad (2.14)$$

And then by (2.14), (2.12) we can extract subsequences $\{u^n\}$ such that

$$u^n(t, x) \rightarrow u(t, x) \text{ a.e. } (t, x) \in [0, T] \times \Omega \quad (2.15)$$

$$u^n(t, x) \leq g(t, x) \text{ a.e. } (t, x) \in [0, T] \times \Omega \quad (2.16)$$

$$u_t^n(t) \rightarrow u_t(t) \quad \text{weakly in } H_0^1(\Omega) \quad (2.17)$$

Now, by continuity of a and b, (A1), and (2.15) we obtain

$$a^4(u^n(t, x), p(t, x)) \rightarrow a^4(u(t, x), p(t, x)) \quad (2.18)$$

$$b(u^n(t, x), p(t, x), p_t(t, x)) \rightarrow b(u(t, x), p(t, x), p_t(t, x)) \quad (2.19)$$

for almost every $(t, x) \in [0, T] \times \Omega$. Using Lebesgue theorem by (2.18) and (A₁) we obtain

$$a(u^n(t), p(t)) \rightarrow a(u(t), p(t)) \text{ in } L^4(\Omega) \quad \text{a.e.t } \in [0, T] \quad (2.20)$$

and by (2.19), (2.16) and (A₂) we obtain

$$b(u^n(t), p(t), p_t(t)) \rightarrow b(u(t), p(t), p_t(t)) \text{ in } L^2(\Omega) \quad \text{a.e.t } \in [0, T] \quad (2.21)$$

On the other hand, for all $v \in L^2$, we have

$$\begin{aligned} |(a(u^n, p)u_t^n - a(u, p)u_t, v)| &= |(a(u^n, p)u_t^n - a(u, p)u_t^n + a(u, p)u_t^n - a(u, p)u_t, v)| \\ &\leq |(a(u^n, p) - a(u, p))u_t^n, v| + |(a(u, p)(u_t^n - u_t), v)| \\ &\leq \|a(u^n, p) - a(u, p)\|_{L^4} \|u_t^n\|_{L^4} \|v\| + |(u_t^n - u_t, a(u, p)v)| \\ &\leq C \|a(u^n, p) - a(u, p)\|_{L^4} \|u_t^n\|_1 \|v\| + |(u_t^n - u_t, a(u, p)v)| \end{aligned}$$

Using (2.20), (2.10) and (2.17) we obtain for almost every $t \in [0, T]$

$$a(u^n(t), p(t))u_t^n(t) \rightarrow a(u(t), p(t))u_t(t) \text{ weakly in } L^2(\Omega) \quad (2.22)$$

Then, using (2.21) and (2.22), and letting $n \rightarrow \infty$ in (2.3) we obtain that the limiting function $u \in H^2([0, T], L^2(\Omega)) \cap H^1([0, T], H_0^1(\Omega))$ and for almost every $t \in [0, T]$,

$$(u_{tt}, \omega) + (\nabla u, \nabla \omega) = (a(u, p)u_t + b(u, p, p_t), \omega) \quad \forall \omega \text{ in } H_0^1(\Omega) \quad (2.23)$$

$$u(0) = u_0 \text{ in } H_0^1(\Omega), \quad u_t(0) = u_1 \text{ in } L^2(\Omega) \quad (2.24)$$

□

Corollary 2.1 *If we suppose that conditions of theorem 2.1 are fulfilled, for every $T > 0$, then u is a globally weak solution of problem (1.1), (1.2), (1.3).*

3 Stability and uniqueness of 1D-solution

In this part we suppose $n=1$.

Theorem 3.1 *Assume $n = 1$ and Theorem 2.1 hypothesis for some $T > 0$. Let u, v be two solutions of problem (1.1)-(1.3) corresponding to initial data (u_0, u_1) and $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then for almost every $t \in [0, T]$*

$$\|u_t(t) - v_t(t)\|^2 + \|u(t) - v(t)\|_1^2 \leq M(T)(\|u_1 - v_1\|^2 + \|u_0 - v_0\|_1^2),$$

where, $M(T)$ is a positive constant depending on T .

Proof - Let u and v be two solutions of problem (1.1), (1.2) (1.3) corresponding to initial data (u_0, u_1) and (v_0, v_1) . We set $\omega = u - v$ and $\omega_i = u_i - v_i$, $i = 0, 1$; then ω satisfies for all $\varphi \in H_0^1(\Omega)$:

$$\begin{aligned} (\omega_{tt}, \varphi) + (\nabla \omega, \nabla \varphi) &= (a(u, p)u_t - a(v, p)v_t + b(u, p, p_t) - b(v, p, p_t), \varphi) \\ \omega &= 0, \text{ on } [0, T] \times \partial\Omega \\ \omega(0, x) &= \omega_0, \quad \omega_t(0, x) = \omega_1, \text{ in } \Omega \end{aligned}$$

Replacing φ by $2\omega_t$ (u_t and $v_t \in H_0^1(\Omega)$) gives

$$2(\omega_{tt}, \omega_t) + 2(\nabla \omega, \nabla \omega_t) = (a(u, p)u_t - a(v, p)v_t + b(u, p, p_t) - b(v, p, p_t), 2\omega_t)$$

and

$$\begin{aligned} \frac{d}{dt}(\|\omega_t(t)\|^2 + \|\nabla \omega(t)\|^2) &= \\ \left(\frac{\partial(a(u, p)u_t)}{\partial u}(\beta)\omega + \frac{\partial(a(u, p)u_t)}{\partial u_t}(\beta)\omega_t, 2\omega_t\right) &+ \left(\frac{\partial b}{\partial u}(\gamma)\omega, 2\omega_t\right), \end{aligned}$$

where $\beta(t, x) \mapsto (\beta_1(t, x), \beta_2(t, x), p(t, x))$ and γ are vector functions between (u, u_t, p) , (v, v_t, p) and (u, p, p_t) , (v, p, p_t) respectively. By (A_1) , (A_2) and Cauchy-Schwartz inequality

$$\frac{d}{dt}(\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2) \leq C(\|\omega_t(t)\|^2 + \|\omega_t(t)\|\|\omega(t)\|) + |(\beta_2\omega, 2\omega_t)|$$

$$\leq C(\|\omega_t(t)\|^2 + \|\omega_t(t)\|\|\omega(t)\| + \|\beta_2(t)\|\|\omega_t(t)\|\|\omega(t)\|_\infty)$$

On the other hand, as in the proof of (2.10) we have

$$\|u_t(t)\|_1 \quad \text{and} \quad \|v_t(t)\|_1 \leq M \quad \text{a.e. on} \quad [0, T] . \quad (3.1)$$

Therefore, as $\beta_2(t, x)$ is between $u_t(t, x)$ and $v_t(t, x)$ then $\|\beta_2(t)\| \leq M$ and we obtain by Sobolev embedding theorem ($\|\omega(t)\|_\infty \leq C\|\omega(t)\|_1$, since $n = 1$).

$$\frac{d}{dt}(\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2) \leq M(\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2), \quad \text{a.e. } t \in [0, T] . \quad (3.2)$$

Applying Gronwall inequality to (3.2) we get

$$\|\omega_t(t)\|^2 + \|\omega(t)\|_1^2 \leq M(T)(\|\omega_t(0, x)\|^2 + \|\omega(0, x)\|_1^2), \quad \text{a.e. } t \in [0, T] \quad (3.3)$$

This achieves the proof. \square

Corollary 3.1 *Assume theorem 2.1 assumptions are fulfilled and $n = 1$, problem (1.1)- (1.3) admits a unique weak solution $u \in H^2([0, T], L^2(\Omega)) \cap H^1([0, T], H_0^1(\Omega))$*

4 Application to cerebral activity model

Let us present now a cerebral activity model stated by V. Jirsa [8]. This model equation is a particular case of equation (1.1). We mathematically justify this model.

The elementary unit of the nervous system is the neuron (Fig 1), which is divided into three basic components [6]: dendrites, cell body and axons. A neuron communicates with others at synapses and there are mainly two kinds of synapses [6]: excitatory or inhibitory types. The information transfer between two neurones A and B can be described as follows

- a pulse (electrical signal) arrives to dendrite of neuron A (under potential action) and acquires neurotransmitters (chemical substances) so that it is transformed to a wave. According to [6] this conversion is a linear operation between neural sheets.
- then the waves reaches the so-called “trigger zone” (Figure 1.) and is converted to pulse again. Now the conversion law is sigmoidal .

Let us describe and justify the conversion operations at the synapses and the trigger zone of neural ensembles, and derive a nonlinear partial differential field equation describing the spatio-temporal behavior of brain activity.

The pulse and wave variables are classified in two sub-variables according

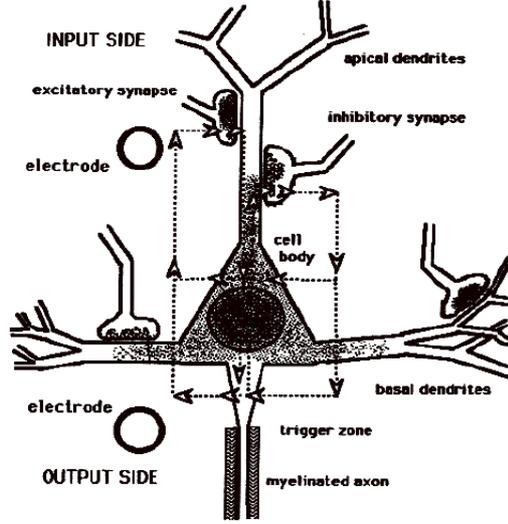


Fig. 1. Dendritic trees form thousands of synapses, excitatory current flows inwardly at excitatory synapses and outwardly at the trigger zone. Inhibitory loop current flows in the opposite direction (Figure from Freeman[6]).

to their excitatory or inhibitory characters: excitatory pulse $E(x, t)$ and inhibitory pulse $I(x, t)$, excitatory wave $\psi_e(x, t)$ and inhibitory wave $\psi_i(x, t)$. We can view the wave as:

$$\psi_e(x, t) = \int_{\Gamma} f_e(x, X) H_e(x, X, t) dX \quad (4.1)$$

$$\psi_i(x, t) = \int_{\Gamma} f_i(x, X) H_i(x, X, t) dX . \quad (4.2)$$

Functions $H_k(x, X, t)$ are conversion operations outputs and functions $f_k(x, X)$ are the corresponding depending on the spatial connectivity distributions. Γ denotes the neural sheet set (at the brain surface which supposed continuous). We assume that, the connectivity functions have the following form:

$$f_e(x, X) = \frac{1}{2\sigma_e} e^{-\frac{|x-X|}{\sigma_e}} \quad (4.3)$$

$$f_i(x, X) = \frac{1}{2\sigma_i} e^{-\frac{|x-X|}{\sigma_i}} \quad (4.4)$$

where σ_e and σ_i are excitatory and inhibitory connectivity parameters (respectively). Moreover $\sigma_i \ll 1$, since the corticocortical connections (connections between two distant neurons via corticocortical fibers) are only excitatory. On the other hand, the information propagation along corticocortical fibers, implies a delay $t = \frac{|x - X|}{v_j}$, where v_e and v_i are the propagation speed exci-

tatory and inhibitory respectively. Thus we obtain

$$H_e(x, X, t) = a_e E\left(X, t - \frac{|x - X|}{v_e}\right) \quad (4.5)$$

$$H_i(x, X, t) = a_i I\left(X, t - \frac{|x - X|}{v_i}\right) \quad (4.6)$$

where a_e and a_i are constants.

Replacing (4.5) and (4.6) in (4.1) and (4.2) respectively, we obtain:

$$\psi_e(x, t) = a_e \int_{\Gamma} f_e(x, X) E\left(X, t - \frac{|x - X|}{v}\right) dX \quad (4.7)$$

$$\begin{aligned} \psi_i(x, t) &= \int_{\Gamma} f_i(x, X) H_i(x, X, t) dX \\ &\approx f_i(x, x) H_i(x, x, t) = a_i I(x, t) \quad (\text{because } \sigma_i \ll 1) \end{aligned} \quad (4.8)$$

The pulse value is calculated from the wave value and the external input in the neural tissue (Fig 1); it has a sigmoidal form :

$$E(x, t) = S_e[\psi_e(x, t) - \psi_i(x, t) + p_e(x, t)] \quad (4.9)$$

$$I(x, t) = S_i[\psi_e(x, t) - \psi_i(x, t) + p_i(x, t)] \quad (4.10)$$

where $p_j(x, t)$ is the external input. The sigmoid function S_j is the following

$$S_j(n_j) = \frac{1}{1 + \exp(-v_j n_j)} - \frac{1}{2}$$

When n_j is small enough, we have a third order approximation of S_j :

$$S_j(n_j) \approx n_j - \frac{v_j^3 n_j^3}{48} \quad (4.11)$$

Inserting (4.9) in (4.7) and (4.10) in (4.8) we obtain

$$\begin{aligned} \psi_e(x, t) &= a_e \int_{\Gamma} f_e(x, X) \cdot S_e\left[\psi_e\left(X, t - \frac{|x - X|}{v}\right) - \psi_i\left(X, t - \frac{|x - X|}{v}\right) + p_e\left(X, t - \frac{|x - X|}{v}\right)\right] dX \end{aligned} \quad (4.12)$$

$$\psi_i(x, t) = a_i S_i[\psi_e(x, t) - \psi_i(x, t) + p_i(x, t)] \quad (4.13)$$

Because $\frac{v_e}{v_i} \approx 100$, we take only the linear part of (4.11) in consideration, replacing it in (4.13) we obtain:

$$\psi_i(x, t) \approx \frac{a_i v_i}{4 + a_i v_i} (\psi_e(x, t) + p_i(x, t)) \quad (4.14)$$

Inserting (4.14) in (4.12), we obtain the following equation for the dynamics of the excitatory synaptic activity:

$$\psi_e(x, t) = a_e \int_{\Gamma} f_e(x, X) \cdot S_e \left[\tilde{\rho} \psi_e \left(X, t - \frac{|x - X|}{v} \right) + p \left(X, t - \frac{|x - X|}{v} \right) \right] dX$$

where $p(x, t) = p_e(x, t) - \frac{a_i v_i}{4 + a_i v_i} \cdot p_i(x, t)$, and $\tilde{\rho} = 1 - \frac{a_i v_i}{4 + a_i v_i}$.

We suppose hereafter that Γ is one-dimensional, that is we connect all the neuron sheets by a line (see [7]). On the other hand, we may assume $\psi_e(0, t) = \psi_e(L, t) = 0$ as in [7], and $f_e(x, X)$ is an exponential function depending only on the distance between two neural sheets. Therefore

$$\psi_e(x, t) = a_e \int_{\mathbb{R}} f_e(x, X) \cdot S_e \left[\tilde{\rho} \psi_e \left(X, t - \frac{|x - X|}{v} \right) + p \left(X, t - \frac{|x - X|}{v} \right) \right] dX .$$

Let us set: $\rho(X, T) = a_e S_e \left(\tilde{\rho} \psi(X, T) + p(X, T) \right)$ and

$$\begin{aligned} \mu : L^1(\mathbb{R} \times \mathbb{R}) &\rightarrow \mathbb{R} \\ \varphi &\mapsto \langle \mu, \varphi \rangle = \int_{\mathbb{R}} h(x) \varphi(x, \frac{|x|}{v}) dx \end{aligned}$$

with $h(x) = e^{-\frac{|x|}{\sigma}}$

$$\begin{aligned} \mu * \rho(x, t) &= \int_{\mathbb{R}} h(X) g(X, \frac{|X|}{v}) dX \quad \text{with } g(X, T) = f(x - X, t - T) \\ &= \int_{\mathbb{R}} h(X) \rho(x - X, t - \frac{|x|}{v}) dX \\ &= \int_{\mathbb{R}} f_e(x, X) \rho(X, t - \frac{|x - X|}{v}) dX \end{aligned}$$

and

$$\mu * \rho(x, t) = \psi_e(x, t).$$

The extended Fourier transform gives: $\widehat{\psi}_e = \widehat{\mu} \widehat{\rho}$.

Now, we prove that $\widehat{\mu} = T_f$, $f \in L^1_{loc}(\mathbb{R} \times \mathbb{R})$. Let $\varphi \in D(\mathbb{R} \times \mathbb{R})$.

$$\begin{aligned} \langle \widehat{\mu}, \varphi \rangle &= \langle \mu, \widehat{\varphi} \rangle = \int_{\mathbb{R}} h(X) \widehat{\varphi}(X, \frac{|X|}{v}) dX \\ &= \int_{\mathbb{R}} h(X) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x, t) e^{-2i\pi(xX + t\frac{|X|}{v})} dx dt \right) dX \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(X) e^{-2i\pi(xX + t\frac{|X|}{v})} dX \right) \varphi(x, t) dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, t) \varphi(x, t) dx dt \end{aligned}$$

with $f(x, t) = \frac{1}{2\sigma_e} \int_{\mathbb{R}} e^{-\frac{|X|}{\sigma}} e^{-2i\pi(xX+t\frac{|X|}{v})} dX$.

Then:

$$\widehat{\psi}_e(x, t) = f(x, t)\widehat{\rho}(x, t) \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}. \quad (4.15)$$

As:

$$\int_0^{+\infty} e^{-(a+i\alpha)\zeta} d\zeta = \frac{1}{a+i\alpha} \quad (4.16)$$

supposing $\omega_0 = \frac{v}{\sigma}$; we obtain

$$\begin{aligned} f(x, t) &= \frac{1}{2} \left[\frac{1}{\frac{1}{\sigma_e} + i\left(\frac{t}{v} - x\right)} + \frac{1}{\frac{1}{\sigma_e} + i\left(\frac{t}{v} + x\right)} \right] \\ &= \frac{1}{2} \left[\frac{\omega_0}{\omega_0 + i(t - xv)} + \frac{\omega_0}{\omega_0 + i(t + xv)} \right] \\ &= \left[\frac{\omega_0^2 + i\omega_0 t}{(\omega_0 + it)^2 + x^2 v^2} \right] \in L_{loc}^1(\mathbb{R} \times \mathbb{R}) \end{aligned}$$

Equation (4.15) becomes

$$\widehat{\psi}_e(x, t) = \left[\frac{\omega_0^2 + i\omega_0 t}{(\omega_0 + it)^2 + x^2 v^2} \right] \widehat{\rho}(x, t) \quad (4.17)$$

Developing equation (4.17), we obtain

$$\omega_0^2 \widehat{\psi}_e(x, t) + 2\omega_0 \frac{\partial \widehat{\psi}_e}{\partial t}(x, t) + \frac{\partial^2 \widehat{\psi}_e}{\partial t^2}(x, t) - v^2 \frac{\partial^2 \widehat{\psi}_e}{\partial x^2}(x, t) = \widehat{\omega_0^2 + \omega_0 \frac{\partial}{\partial t}} \cdot \rho(x, t)$$

Applying the inverse Fourier Transform, we formally obtain the following partial derivative equation :

$$\frac{\partial^2 \psi_e}{\partial t^2} + (\omega_0^2 - v^2 \frac{\partial^2}{\partial x^2}) \psi_e + 2\omega_0 \frac{\partial \psi_e}{\partial t} = (\omega_0^2 + \omega_0 \frac{\partial}{\partial t}) \cdot \rho \quad (4.18)$$

with

$$\rho(x, t) = a_e S_e \left(\widetilde{\rho} \psi_e(x, t) + p(x, t) \right).$$

Developing (4.18) and summing the similar terms gives

$$\begin{aligned} \frac{\partial^2 \psi_e}{\partial t^2} - v^2 \frac{\partial^2 \psi_e}{\partial x^2} &= \left(a_e \omega_0^2 \widetilde{\rho} S_e'(\widetilde{\rho} \psi_e + p) - 2\omega_0 \right) \frac{\partial \psi_e}{\partial t} \\ &\quad - \omega_0^2 \psi_e + a_e \omega_0^2 S_e(\widetilde{\rho} \psi_e + p) + a_e \omega_0^2 S_e'(\widetilde{\rho} \psi_e + p) p_t. \end{aligned} \quad (4.19)$$

We complete this equation with $\psi_e(t, 0) = \psi_e(t, L) = 0$, and Cauchy initial condition.

Thanks to the previous sections results we have the following :

Theorem 4.1 Assume $p \in H^2([0, T], L^\infty(\Omega))$. Then equation (4.19) with $\psi_e(t, 0) = \psi_e(t, L) = 0$, and Cauchy initial condition with $(\psi_{e,0}, \psi_{e,1}) \in H_0^1(\Omega) \times L^2(\Omega)$ has a unique weak solution in $H^2([0, T], L^2(\Omega)) \cap H^1([0, T], H_0^1(\Omega))$.

Proof - Equation (4.19) is a particular case from equation (1.1), with

$$a(u, p) = a_e \omega_0^2 \tilde{\rho} S'_e(\tilde{\rho}u + p) - 2\omega_0,$$

and

$$b(u, p, p_t) = \omega_0^2 u + a_e \omega_0^2 S_e(\tilde{\rho}u + p) + a_e \omega_0^2 S'_e(\tilde{\rho}u + p) p_t.$$

If we suppose $p \in H^2([0, T], L^\infty(\Omega))$, it is easy to view that a and b functions verify Theorem 2.1 conditions and consequently by (2.23), (3.3) the above equation has a unique weak solution. \square

5 Conclusion

In this paper, we prove the existence and uniqueness of solution to a new form of wave equation motivated by brain activity modeling. We have proven the stability of solution with respect to the initial data : this means that the responses to small perturbations are small as well, which is coherent from a physical point of view. We have simplified the model assuming that:

- the geometry of brain area is 1D. The 2D geometry is more consistent but introduces additional difficulties,
- the connectivity function (4.3), (4.4) between the neural sheet is homogeneous. In fact more than 50 % of neural sheet have heterogeneous connections, so we should add another term representing the heterogeneity,
- the input p belongs to $H^2([0, T], L^\infty(\Omega))$. In reality p maybe less smooth or non local.

In our future work, we will try to generalize our model in order to satisfy the general constraints which are introduced above.

References

- [1] Nikolaos Bournaveas. Low regularity solutions for a class of nonlinear wave equations. *Proc. Amer. Math. Soc.*, 133(9):2721–2727 (electronic), 2005.
- [2] J. W. Cholewa and Tomasz Dlotko. Strongly damped wave equation in uniform spaces. *Nonlinear Anal.*, 64(1):174–187, 2006.
- [3] Yuki Yoshi Ebihara. On some nonlinear evolution equations with the strong dissipation. *J. Differential Equations*, 30(2):149–164, 1978.

- [4] Jorge A. Esquivel-Avila. The dynamics of a nonlinear wave equation. *J. Math. Anal. Appl.*, 279(1):135–150, 2003.
- [5] Eduard Feireisl. Global attractors for semilinear damped wave equations with supercritical exponent. *J. Differential Equations*, 116(2):431–447, 1995.
- [6] W.J Freeman. Tutorial on neurobiology: From single neurons to brain chaos. *J. Int. J. Bifurcation and chaos*, 2:451–482, 1992.
- [7] V.K Jirsa. Connectivity and dynamics of neural information processing. neuroinformatics. *Neuroinformatics*, 2:183–204, 2004.
- [8] V.K. Jirsa and H Haken. Field theory of electromagnetic brain activity. *Phys. Rev. Let.*, 77:960–963, 1996.
- [9] Tariel Kiguradze. On bounded and time-periodic solutions of nonlinear wave equations. *J. Math. Anal. Appl.*, 259(1):253–276, 2001.
- [10] Howard A. Levine, Sang Ro Park, and James Serrin. Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation. *J. Math. Anal. Appl.*, 228(1):181–205, 1998.
- [11] Howard A. Levine and Grozdna Todorova. Blow up of solutions of the Cauchy problem for a wave equation with nonlinear damping and source terms and positive initial energy. *Proc. Amer. Math. Soc.*, 129(3):793–805 (electronic), 2001.
- [12] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.
- [13] Julián López-Gómez. On the linear damped wave equation. *J. Differential Equations*, 134(1):26–45, 1997.
- [14] Tokio Matsuyama and Ryo Ikehata. On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms. *J. Math. Anal. Appl.*, 204(3):729–753, 1996.
- [15] Kenji Nishihara and Huijiang Zhao. Existence and nonexistence of time-global solutions to damped wave equation on half-line. *Nonlinear Anal.*, 61(6):931–960, 2005.
- [16] P.L. Nunez. The brain wave equation : A model for the eeg. *Mathematical Biosciences*, 21:279–297, 1974.
- [17] Rafael Orive and Enrique Zuazua. Long-time behavior of solutions to a nonlinear hyperbolic relaxation system. *J. Differential Equations*, 228(1):17–38, 2006.
- [18] Jong Yeoul Park and Jeong Ja Bae. On solutions of quasilinear wave equations with nonlinear damping terms. *Czechoslovak Math. J.*, 50(125)(3):565–585, 2000.

- [19] David R. Pitts and Mohammad A. Rammaha. Global existence and non-existence theorems for nonlinear wave equations. *Indiana Univ. Math. J.*, 51(6):1479–1509, 2002.
- [20] Hervé Queffelec and Claude Zuily. *élément d'analyse*. Dunod, Paris, second edition, 2002.
- [21] Grozdena Todorova. Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms. *Nonlinear Anal.*, 41(7-8, Ser. A: Theory Methods):891–905, 2000.
- [22] Grozdena Todorova and Borislav Yordanov. Critical exponent for a nonlinear wave equation with damping. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(7):557–562, 2000.
- [23] H.R. Wilson and J.D. Cowan. Excitatory and inhibitory interactions in localized populations of model neurons. *Biophysical Journal*, 12:1–24, 1972.
- [24] Karen Yagdjian. Global existence in the Cauchy problem for nonlinear wave equations with variable speed of propagation. In *New trends in the theory of hyperbolic equations*, volume 159 of *Oper. Theory Adv. Appl.*, pages 301–385. Birkhäuser, Basel, 2005.
- [25] Qi S. Zhang. A blow-up result for a nonlinear wave equation with damping: the critical case. *C. R. Acad. Sci. Paris Sér. I Math.*, 333(2):109–114, 2001.
- [26] Yang Zhijian. Initial boundary value problem for a class of non-linear strongly damped wave equations. *Math. Methods Appl. Sci.*, 26(12):1047–1066, 2003.
- [27] Shengfan Zhou and Xiaoming Fan. Kernel sections for non-autonomous strongly damped wave equations. *J. Math. Anal. Appl.*, 275(2):850–869, 2002.
- [28] Yong Zhou. A blow-up result for a nonlinear wave equation with damping and vanishing initial energy in \mathbb{R}^N . *Appl. Math. Lett.*, 18(3):281–286, 2005.
- [29] Yong Zhou. Global existence and nonexistence for a nonlinear wave equation with damping and source terms. *Math. Nachr.*, 278(11):1341–1358, 2005.