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Introduction

Relative duality is a useful tool in algebraic geometry and has been used several times. Here we prove a version of it in Voevodsky's triangulated category of geometric motives $DM_{gm}(k)$ [9], where k is a field which admits resolution of singularities.

Namely, let X be a smooth proper k-variety of pure dimension n and Y, Z two disjoint closed subsets of X. We prove in Theorem 3.1 an isomorphism

$$M(X - Z, Y) \simeq M(X - Y, Z)^*(n)[2n]$$

where $M(X-Z, Y)$ and $M(X-Y, Z)$ are relative Voevodsky motives, see Definition 1.1.

This isomorphism remains true after application of any $\otimes$-functor from $DM_{gm}(k)$, for example one of the realisation functors appearing in [8, I.VI.2.5.5 and I.V.2], [5] or [7]. In particular, taking the Hodge realisation, this makes the recourse to M. Saito's theory of mixed Hodge modules unnecessary in [1, Proof of 2.4.2].

The main tools in the proof of Theorem 3.1 are a good theory of extended Gysin morphisms, readily deduced from Déglise's work (Section 2) and Voevodsky's localisation theorem for motives with compact supports [9, 4.1.5]. This may be used for an alternative presentation of some of the duality results of [9, §4.3] (see Remark 4.7). The arguments seem axiomatic enough to be transposable to other contexts.

We assume familiarity with Voevodsky's paper [9], and use its notation throughout.

1. Relative motives and motives with supports

Definition 1.1. Let $X \in Sch/k$ and $Y \subseteq X$, closed. We set

$$M(X, Y) = \underline{C}_*(L(X)/L(Y))$$
$$M^Y(X) = \underline{C}_*(L(X)/L(X - Y)).$$

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Remark 1.2. This convention is different from the one of Déglise in [2, 3, 4] where what we denote by $M^Y(X)$ is written $M(X,Y)$ (and occasionally $M_Y(X)$ as well).

Note that $L(Y) \to L(X)$ and $L(X-Y) \to L(X)$ are monomorphisms, so that we have functorial exact triangles

$$\begin{gathered} M(Y) \to M(X) \to M(X,Y) \xrightarrow{+1} \\ M(X-Y) \to M(X) \to M^Y(X) \xrightarrow{+1} . \end{gathered} \tag{1}$$

We can mix the two ideas: for $Y, Z \subseteq X$ closed, define

$$M^Z(X,Y) = \underline{C}_*(L(X)/L(Y) + L(X-Z)).$$

Lemma 1.3. *If $Y \cap Z = \emptyset$, the obvious map $M^Z(X) \to M^Z(X,Y)$ is an isomorphism, and we have an exact triangle*

$$M(X-Z,Y) \to M(X,Y) \xrightarrow{\delta} M^Z(X) \xrightarrow{+1} . \quad \square$$

2. Extended Gysin

In the situation of Lemma 1.3, assume that Z is smooth of pure codimension c. F. Déglise has then constructed a purity isomorphism

$$p_{Z\subset X} : M^Z(X) \xrightarrow{\sim} M(Z)(c)[2c] \tag{2}$$

with the following properties:

(1) $p_{Z\subset X}$ coincides with Voevodsky's purity isomorphism of [9, 3.5.4] (see [4, 1.11]).
(2) If $f : X' \to X$ is transverse to Z in the sense that $Z' = Z \times_X X'$ is smooth of pure codimension c in X', then the diagram

$$\begin{tikzcd} M^{Z'}(X') \arrow[r, "p_{Z'\subset X'}"] \arrow[d, "(f,g)_*"'] & M(Z')(c)[2c] \arrow[d, "g_*"'] \\ M^Z(X) \arrow[r, "p_{Z\subset X}"] & M(Z)(c)[2c] \end{tikzcd}$$

commutes, where $g = f_{|Z'}$ ([2, Rem. 4] or [3, 2.4.5]).

(3) If $i : T \subset Z$ is a closed subset, smooth of codimension d in X, the diagram

$$\begin{tikzcd} M^Z(X) \arrow[r, "p_{Z\subset X}"] \arrow[dd, "i^*"] & M(Z)(c)[2c] \arrow[dr, "\alpha"] & \\ & & M^T(Z)(c)[2c] \arrow[dl, "p_{T\subset Z}"] \\ M^T(X) \arrow[r, "p_{T\subset X}"] & M(T)(d)[2d] & \end{tikzcd}$$

commutes, where α is the twist/shift of the boundary map in the triangle corresponding to (1) [4, proof of 2.3].

Definition 2.1. We set:

$$g^Y_{Z\subset X} = p_{Z\subset X} \circ \delta$$

where $p_{Z\subset X}$ is as in (2) and δ is the morphism appearing in Lemma 1.3.

In view of the properties of $p_{Z\subset X}$, these extended Gysin morphisms have the following properties:

Proposition 2.2. *a) Let $f : X' \to X$ be a morphism of smooth schemes. Let $Z' = f^{-1}(Z)$ and $Y' = f^{-1}(Y)$. If f is transverse to Z, the diagram*

$$\begin{tikzcd} M(X', Y') \arrow[r, "g^{Y'}_{Z'\subset X'}"] \arrow[d, "f_*"] & M(Z')(c)[2c] \arrow[d, "g_*"] \\ M(X, Y) \arrow[r, "g^Y_{Z\subset X}"] & M(Z)(c)[2c] \end{tikzcd}$$

commutes, with $g = f_{|Z}$.
b) Let $X \supset Z \supset Z'$ be a chain of smooth k-schemes of pure codimensions, and let $d = codim_Z Z'$. Let $Y \subset X$ be closed, with $Y \cap Z = \emptyset$. Then

$$g^Y_{Z'\subset X} = g_{Z'\subset Z}(d)[2d] \circ g^Y_{Z\subset X}.$$

3. Relative duality

In this section, X is a smooth proper variety purely of dimension n and Y, Z are two disjoint closed subsets of X. Consider the diagonal embedding of X into $X \times X$: its intersection with $(X - Y) \times (X - Z)$ is closed and isomorphic to $X - Y - Z$. The closed subset $(X - Y) \times$

$Y \cup Z \times (X - Z)$ is disjoint from $X - Y - Z$; from Definition 2.1 we get a extended Gysin map

$$M((X-Y)\times(X-Z),(X-Y)\times Y\cup Z\times(X-Z)) \\ \to M(X-Y-Z)(n)[2n].$$

Note that the left hand side is isomorphic to $M(X-Y,Z)\otimes M(X-Z,Y)$ by an explicit computation from the definition of relative motives. Composing with the projection $M(X-Y-Z)(n)[2n] \to \mathbf{Z}(n)[2n]$, we get a map

$$M(X-Y,Z)\otimes M(X-Z,Y) \to \mathbf{Z}(n)[2n]$$

hence a map

$$M(X-Z,Y) \xrightarrow{\alpha_X^{Y,Z}} M(X-Y,Z)^*(n)[2n]. \tag{3}$$

Theorem 3.1. *The map* (3) *is an isomorphism.*

The proof is given in the next section.

4. Proof of Theorem 3.1

Lemma 4.1. *If $Y = Z = \emptyset$ and X is projective, then* (3) *is an isomorphism.*

Proof. As pointed out in [9, p. 221], $\alpha_X^{\emptyset,\emptyset}$ corresponds to the class of the diagonal; then Lemma 4.1 follows from the functor of [9, 2.1.4] from Chow motives to $DM_{gm}(k)$. (This avoids a recourse to [9, 4.3.2 and 4.3.6].) □

The next step is when Z is empty. For any $U \in Sch/k$, write $M^c(U) := \underline{C}_*(L^c(U))$ [9, p. 224]. Since X is proper, by [9, 4.1.5] there is a canonical isomorphism

$$M(X,Y) \xrightarrow{\sim} M^c(X-Y)$$

induced by the map of Nisenvich sheaves

$$L(X)/L(Y) \to L^c(X-Y).$$

Therefore, from $\alpha_X^{Y,\emptyset}$, we get a map

$$\beta_X^Y : M^c(X-Y) \to M(X-Y)^*(n)[2n].$$

Lemma 4.2. *The map β_X^Y only depends on $X - Y$.*

Proof. Let $U = X - Y$. If X' is another smooth compactification of U, with $Y' = X' - U$, we need to show that $\beta_X^Y = \beta_{X'}^{Y'}$. By resolution of singularities, X and X' may be dominated by a third smooth compactification; therefore, without loss of generality, we may assume that the rational map $q : X' \to X$ is a morphism. The point is that, in the diagram

$$\begin{tikzcd}
M(X',Y') \arrow[dr] \arrow[drr, "\alpha_{X'}^{Y',\emptyset}"] \arrow[ddr, "\simeq"'] & & \\
& M(X,Y) \arrow[r, "\alpha_X^{Y,\emptyset}"'] \arrow[d, "\simeq"] & M(U)^*(n)[2n] \\
& M^c(U) &
\end{tikzcd}$$

both triangles commute. For the left one it is obvious, and for the upper one this follows from the naturality of the pairing (3). Indeed, the square

$$\begin{array}{ccc}
X' - Y' & \xrightarrow{\Delta'} & (X' - Y') \times X' \\
{\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q' \times q} \\
X - Y & \xrightarrow{\Delta} & (X - Y) \times X
\end{array}$$

is clearly transverse, where $q' = q_{|X'-Y'}$ (an isomorphism) and Δ, Δ' are the diagonal embeddings; therefore we may apply Proposition 2.2 a). □

From now on, we write β_{X-Y} for the map β_X^Y.

Lemma 4.3. *a) Let $U \in Sm/k$ of pure dimension n, $T \xrightarrow{i} U$ closed, smooth of pure dimension m and $V = U - T \xrightarrow{j} U$. Then the diagram*

$$\begin{array}{ccc}
M^c(T) & \xrightarrow{\beta_T} & M(T)^*(m)[2m] \\
{\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle g^*_{T\subset U}(n)[2n]} \\
M^c(U) & \xrightarrow{\beta_U} & M(U)^*(n)[2n] \\
{\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle j^*} \\
M^c(V) & \xrightarrow{\beta_V} & M(V)^*(n)[2n]
\end{array}$$

commutes.
b) Suppose that β_T is an isomorphism. Then β_U is an isomorphism if and only if β_V is.

Proof. a) The bottom square commutes by a trivial case of Proposition 2.2 a). For the top square, the statement is equivalent to the commutation of the diagram

$$\begin{tikzcd}
 & M^c(T)\otimes M(T)(c)[2c] \arrow[dr] & \\
M^c(T)\otimes M(U) \arrow[ur, "1\otimes g_{T\subset U}"] \arrow[dr, "i_*\otimes 1"'] & & \mathbf{Z}(n)[2n] \\
 & M^c(U)\otimes M(U) \arrow[ur] &
\end{tikzcd}$$

with $c = n - m$.

Take a smooth compactification X of U, and let $\bar{T}$ be a desingularisation of the closure of T in X. Let $q : \bar{T} \to X$ be the corresponding morphism, $Y = X - U$ and $W = \bar{T} - T$: we have to show that the diagram

$$\begin{tikzcd}
 & M(\bar{T},W)\otimes M(T)(c)[2c] \arrow[dr] & \\
M(\bar{T},W)\otimes M(U) \arrow[ur, "1\otimes g_{T\subset U}"] \arrow[dr, "q_*\otimes 1"'] & & \mathbf{Z}(n)[2n] \\
 & M(X,Y)\otimes M(U) \arrow[ur] &
\end{tikzcd}$$

or equivalently

$$\begin{tikzcd}
 & M(\bar{T}\times T, W\times T)(c)[2c] \arrow[dr] & \\
M(\bar{T}\times U, W\times U) \arrow[ur, "f\circ g^{W\times U}_{\bar{T}\times T\subset \bar{T}\times U}"] \arrow[dr, "(q\times 1)_*"'] & & \mathbf{Z}(n)[2n] \\
 & M(X\times U, Y\times U) \arrow[ur] &
\end{tikzcd}$$

commutes, where f is the map $M(\bar{T} \times T)(c)[2c] \to M(\bar{T} \times T, W \times T)(c)[2c]$. For this, it is enough to show that the diagram

$$\begin{tikzcd}
& M(\bar{T} \times T, W \times T)(c)[2c] \arrow[r, "g^{W\times T}_{T\subset \bar{T}\times T}(c)[2c]"] & M(T)(n)[2n] \arrow[dd, "i_*"] \\
M(\bar{T} \times U, W \times U) \arrow[ur, "f\circ g^{W\times U}_{\bar{T}\times T\subset \bar{T}\times U}"] \arrow[dr, "(q\times 1)_*"'] & & \\
& M(X \times U, Y \times U) \arrow[r, "g^{Y\times U}_{U\subset X\times U}"] & M(U)(n)[2n]
\end{tikzcd}$$

commutes. Since extended Gysin extends Gysin, Proposition 2.2 a) shows that this amounts to the commutatvity of

$$\begin{tikzcd}
M(\bar{T} \times U, W \times U) \arrow[r, "g^{W\times U}_{T\subset \bar{T}\times U}"] \arrow[d, "(q\times 1)_*"'] & M(T)(n)[2n] \arrow[d, "i_*"'] \\
M(X \times U, Y \times U) \arrow[r, "g^{Y\times U}_{U\subset X\times U}"] & M(U)(n)[2n]
\end{tikzcd}$$

which follows from the functoriality of the extended Gysin maps (Proposition 2.2 b)).

b) This follows immediately from a). □

Proposition 4.4. *β_U is an isomorphism for all smooth U.*

Proof. We argue by induction on $n = \dim U$, the case $n = 0$ being known by Lemma 4.1. In general, let V be an open affine subset of U and pick a smooth projective compactification X of V, with $Z = X - V$. Let $Z \supset Z_1 \supset \cdots \supset Z_r = \emptyset$, where Z_{i+1} is the singular locus of Z_i. Let also $T = U - V$ and define similarly $T \supset T_1 \subset \cdots \supset T_s = \emptyset$ (all Z_i and T_j are taken with their reduced structure). Let $V_i = X - Z_i$ and $U_j = U - T_j$. Then $V_i - V_{i-1}$ and $U_j - U_{j-1}$ are smooth for all i, j. Thus β_U is an isomorphism by Lemma 4.1 (case of β_X) and a repeated application of Lemma 4.3 b). □

Remark 4.5. We haven't tried to check whether β_U is the inverse of the isomorphism appearing in the proof of [9, 4.3.7]: we leave this interesting question to the interested reader.

End of proof of Theorem 3.1. By Lemma 1.3, the triangle $M(Z) \to M(X-Y) \to M(X-Y, Z) \xrightarrow{+1}$ and the duality pairings induce a map of triangles

$$\begin{array}{ccccc}
M(X-Y,Z)^*(n)[2n] & \longrightarrow & M(X-Y)^*(n)[2n] & \longrightarrow & M(Z)^*(n)[2n] \\
{\scriptstyle \alpha_X^{Y,Z}}\Big\uparrow & & {\scriptstyle \alpha_X^{Y,\emptyset}}\Big\uparrow & & {\scriptstyle \Phi}\Big\uparrow \\
M(X-Z,Y) & \longrightarrow & M(X,Y) & \longrightarrow & M^Z(X).
\end{array}$$

(The left square commutes by a trivial application of Proposition 2.2 a), and Φ is some chosen completion of the commutative diagram by the appropriate axiom of triangulated categories.)

Consider the following diagram (which is the previous diagram with $Y=\emptyset$):

$$\begin{array}{ccccc}
M(X,Z)^*(n)[2n] & \longrightarrow & M(X)^*(n)[2n] & \longrightarrow & M(Z)^*(n)[2n] \\
{\scriptstyle \alpha_X^{\emptyset,Z}}\Big\uparrow & & {\scriptstyle \alpha_X^{\emptyset,\emptyset}}\Big\uparrow & & {\scriptstyle \Phi}\Big\uparrow \\
M(X-Z) & \longrightarrow & M(X) & \longrightarrow & M^Z(X)
\end{array}$$

Note that $\alpha_X^{\emptyset,Z}$ is dual to $\alpha_X^{Z,\emptyset}$; therefore it is an isomorphism by Lemma 4.2 and Proposition 4.4. It follows that Φ is an isomorphism. Coming back to the first diagram and using Lemma 4.2 and Proposition 4.4 a second time, we get the theorem. □

Remark 4.6. It would be interesting to produce a canonical pairing

$$\cap_{(X,Z)} : M^Z(X)\otimes M(Z)\to \mathbf{Z}(n)[2n]$$

playing the rôle of Φ in the above proof, i.e., compatible with $\alpha_X^{Y,Z}$.

Remark 4.7. As explained in [6, App. B], resolution of singularities and the existence of the $\otimes$-functor of [9, 2.1.4] are sufficient to prove that the category $DM_{gm}(k)$ is rigid. Therefore, to apply the above arguments, one need only know that the motives of the form $M(X-Y,Z)$ belong to $DM_{gm}(k)$, which is a consequence of [9, 4.1.4].

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