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Partial observables

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I discuss the distinction between the notion of *partial* observable and the notion of *complete* observable. Mixing up the two is frequently a source of confusion. The distinction bears on several issues related to observability, such as (i) whether time is an observable in quantum mechanics, (ii) what are the observables in general relativity, (iii) whether physical observables should or should not commute with the Wheeler-DeWitt operator in quantum gravity. I argue that the *extended* configuration space has a direct physical interpretation, as the space of the partial observables. This space plays a central role in the structure of classical and quantum mechanics and the clarification of its physical meaning sheds light on this structure, particularly in the context of general covariant physics.

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I. INTRODUCTION

The notion of "observable quantity", or "observable", plays a central role in many areas of physics. Roughly, observable quantities are the quantities involved in physical measurements. They give us information on the state of a physical system and may be predicted by the theory. In quantum mechanics, observables are represented by self-adjoint operators. In gauge theory, we make the distinction between gauge-invariant quantities, which correspond to observables, and gauge-dependent quantities, which do not not.

The notion of observable, however, raises a certain number of issues, which have generated discussions in the literature. In particular: (i) Several papers discuss whether time is an observable in quantum theory. If time were an observable, it should be represented by a self-adjoint operator T. The spectrum of T should be the real line. A well known theorem [1] demands then its conjugate variable, which is the energy, to have unbounded spectrum. But energy is bounded from below. Therefore time cannot be an observable [2]. But if time is not an observable, how can we measure it? (ii) There are several discussions on observability in general relativity (see [3,4] and references therein). In the literature one finds contradictory statements. For instance, that the metric tensor $g_{\mu\nu}(x)$ is not observable but a curvature scalar $R(x)$ is observable; or that no local quantity such as $R(x)$ can be observable. (iii) Observability is a source of lively debates in quantum gravity [4,5]. Observables must be gauge invariant, therefore commute with the constraints, therefore, in particular, with the Wheeler-DeWitt operator, and therefore they have to be constant in the coordinate time t. Thus, no quantity that changes with t can be observable. This conclusion is considered unreasonable by some [6]. Others (including myself) [4,7–9] argue that the observables in quantum gravity are relative quantities expressing correlations between dynamical variables. But how can a correlation between two non-observable quantities be observable?

I believe that in many debates of this kind there is a confusion between two distinct notions of observability. Mixing up these two notions generates misunderstanding and conceptual mistakes. In this note, I try to clear up the source of this confusion.

The difference between the two notions of observability has to do with localization in time and in space. In a non-relativistic context, the spacetime structure of the world is assumed to be fixed and simple. Because of this, the distinction between the two notions of observability can be disregarded. More precisely, the distinction is replaced by the introduction of a fixed structure on the space of the observables, and then it is safely ignored. The fixed structure of the space of the observables reproduces the fixed structure of spacetime, as we shall see. In a generally relativistic context, on the other hand, the spacetime structure of the world is more complex, and we cannot trade the distinction between different notions of observability for a pre-established structure on the space of the observables. In such a context, ignoring the distinction between different meaning of "observable" leads to serious confusion.

Partial and complete observables are defined in Section II. The two notions are shown to be distinct and examples of the two are given. I then discuss the relevance of the distinction for different contexts: general relativity (in Section III), quantum mechanics (in Section IV), and quantum gravity (in Section V).

The space of the partial observables is the *extended* configuration space. This space, and its associated extended phase space, on which the hamiltonian constraint is defined, are often presented as devoid of direct physical significance. Instead, I argue in Section VI that the extended configuration space has indeed a direct physical interpretation: it is the space of the partial observables. This space plays a central role in the general structure of mechanics, both at the classical and at the quantum level. I illustrate this role and argue that it provides a unifying perspective that sheds light on the structure of

mechanics, especially of general covariant mechanics.

The distinction between partial and complete observables was discussed in [10]. The distinction is sometimes implicitly used, but I am not aware of any other explicit discussion on it in the literature.

II. PARTIAL OBSERVABLES AND COMPLETE OBSERVABLES

Let us start from the following two definitions*:

Partial observable: a physical quantity to which we can associate a (measuring) procedure leading to a number.

Complete observable: a quantity whose value can be predicted by the theory (in classical theory); or whose probability distribution can be predicted by the theory (in quantum theory).

At first sight, the two definitions might seem equivalent, but they are not. To see this, consider the following example. Imagine we have a bunch of cards in a box. Each card has an upper and a lower side (say, of different colors). On each side, there is a number. Denote the upper number as n and the lower number as N. We extract a certain number of cards from the box and we realize that there is law connecting the two numbers: say N is always a certain function of n. That is $N = N(n)$. The law $N = N(n)$ gives us a predictive theory for some observable quantities. What are the observables in this context? Clearly, both n and N are *partial observables*, according to the definition given above. However, neither of them is predictable, because at each new card we extract we do not know which particular value of n, or which particular value of N, will be found. What is predictable is the value of N on the back of a card marked with a certain n. Therefore we have is one "complete observable" $N(n)$ for each value of n. The "complete observables" are $N(1), N(2), N(3), \ldots$

The example may seem artificial and unrelated to the structure of realistic physical theories, but it is not. Indeed, realistic physical theories have a structure similar to the one of the example: the role of the "independent" partial observable n is played by the quantities giving the temporal localization or the spatio-temporal localization. Consider for instance a very simple physical system, a pendulum. Assume oscillations are small and described by the equation

$$\frac{d^2q(t)}{dt^2} = -\omega^2 q(t). \tag{1}$$

*The operational tone of the first definition does not imply any adherence to operationalism here [11]: the reference to measuring procedures is just instrumental for clarifying a distinction.

Now suppose we are in a (very simple) laboratory, and we want to check the correctness of (1). What do we need? Clearly we need *two* measuring instruments: one that gives us the pendulum position q and one that gives us the time t. The theory cannot predict the value of t. Nor can it predict the value of q, unless we specify that the value of q we are interested in is the one at a certain given time t. Therefore, there are two partial observables playing a role here: q and t. And there is one family of complete observables: the observables $q(t)$, for any real value of t. It is sufficient to know the actual value of a few of these complete observables (for instance $q(0)$ and $dq(t)/dt|_{t=0}$), in order to be able to predict the value of all the others.

This may seem a rather pedantic account of observability in the context of a non relativistic system. Indeed, one usually says that "q is observable", leaving implicit "yes, of course, one has to say at which time the observation is made". But, as mentioned, such carelessness in defining observability is then paid for at a high price in a generally relativistic context, where things are not simply evolving in a fixed external time t which can be measured by an external clock, as in non-relativistic physics. Let us therefore here clearly distinguish between (i) t and q (without specified time), which are partial observables, because there are measuring procedures specified for them, but they cannot be predicted, and (ii) the family of complete observables $q(t)$, which can be predicted.

Observe that the predictions of a mechanical theory can always be expressed as *relations between partial observables*. These relations depend on a certain number of parameters, which label the different possible histories of the system. For instance, in the case above the predictions of the theory are given by the following relation between t and q:

$$f(q, t; A, \phi) = q - A\sin(\omega t + \phi) = 0. \tag{2}$$

From this perspective, the partial observables q and t can be taken as being on the same footing. That is, they can be treated symmetrically in the theory. Observe, however, that in the example considered the two partial observables q and t are not entirely on the same footing. The predictions of the theory can certainly be expressed as a relation between the two, but this relation can be solved for q as a function of t, not for t as a function of q. Accordingly, we call t an *independent* partial observable and q a *dependent* partial observable. As we shall see, in a generally relativistic context such distinction between dependent and independent partial observables is lost.

In a non relativistic system with m degrees of freedom q^i, with $i = 1, \ldots, m$, there are in general $n = m+1$ partial observables: $q^a = (t, q^i)$ with $a = 1, \ldots, n$. The space of these form the *extended configuration space* of the system, which we denote $\mathcal{C}$. The predictions of classical mechanics can always be given as relations between the extended configuration space variables, as in (2). These relations depend on a certain number of parameters α^j

(A and ϕ in (2)), which label the different possible histories of the system.

$$f(q^a; \alpha^j) = 0. \tag{3}$$

Classical mechanics and quantum mechanics can be formulated in a very general and very clean form over the extended configuration space $\mathcal{C}$. Examples of such formulations are the Hamilton-Jacobi formalism, the extended phase space formalism, the path integral formalism and the propagator formalism [16]. These formulations stress the centrality of the notion of partial observable and show that mechanics treats all partial observables on the same ground. In Section VI, we shall discuss some of these formulations and their relation to partial observability.

Finally, consider a field theory, such as Maxwell electrodynamics. A dynamical variable is represented for instance by the Electric field $E(\vec{x}, t)$. The electric field at a given spacetime point $(\vec{x}, t)$ can be predicted, and therefore it is a complete observable of the theory. In order to measure $E(\vec{x}, t)$, we need *five* partial observables. Indeed, we may imagine that we have at our disposal five measuring devices: a clock measuring t, an electric field detector that measures E, and three distance measuring devices, giving the three components of $\vec{x}$. The complete observable $E(\vec{x}, t)$ is composed by these five partial observables.

III. GENERAL RELATIVITY

Let us now move to a generally relativistic context. For concreteness, let us consider general relativity coupled with N small bodies. For instance, these bodies may represent the planets and the satellites in a generally relativistic model of the solar system. The lagrangian variables can be taken to be the metric $g_{\mu\nu}(\vec{x}, t)$ and, say, the bodies' trajectories $X^\mu_{(n)}(\tau_n)$, with $n = 1, \ldots, N$ and orientations $E^{a\mu}_{(n)}(\tau_n)$ (a local tetrad on the $n-$th body, $a = 0, \ldots, 3$). As well known, the meaning of the coordinates $(\vec{x}, t)$ in general relativity is very different from their peaning in pre-general-relativistic (pre-GR) physics.† Indeed, the coordinates $(\vec{x}, t)$ do not represent observable quantities at all.‡ That is, the general relativistic coordinates $\vec{x}$ and t are neither partial observables nor complete observables.

The distinction between partial and complete observables, however, is still present. Consider some typical predictions of the theory. For instance, a prediction of the theory may be the following: tomorrow morning, when the Sun is 5 degrees over the horizon, Venus will be visible at 12 degrees over the horizon. This is a well defined prediction, and should thus refer to a complete observable. The complete observable is the angle α_V that Venus makes with the horizon, at the moment in which the angle α_S of the Sun with the horizon is 5 degrees. Clearly, to verify this prediction we need measuring procedures giving us the two angles. Therefore the two angles are partial observables. The complete observable is the value of $\alpha_V(\alpha_S)$ for $\alpha_S = 5^o$.

As a second example, we could replace α_S with the proper time τ measured on Earth by a clock that started ($\tau = 0$) at a certain specified event O (say, a certain eclipse). Then again, the proper time τ elapsed from the eclipse, or, equivalently, the length τ of the Earth's worldline since the eclipses, is a partial observable because it can be measured; but it is not a complete observable, because it cannot be predicted. Indeed, it is an observable quantity used for localizing a spacetime point.

The key difference between general relativity and pre-GR physics as far as observability is concerned is well illustrated by a third example. Consider the following (realistic!) experiment. A very accurate clock is mounted on a satellite. Say a satellite in the GPS system. The satellite broadcasts its local time and the signal is received by the launching base, and compared with the time of an equally accurate clock kept at the base. As well known, the discrepancy between the two due to generally relativistic effects is easily observable using current technology. Let τ_s and τ_b be the signal received from the satellite and the local clock reading. General relativity can be used to predict the relation between the two

$$f(\tau_s, \tau_b) = 0, \tag{4}$$

(once all the relevant initial data are known). Again, we are in a situation of two partial observables forming a complete observable. Now: which one of the two is the *independent* one? In general, (4) may not be solvable for either variable. One could say that τ_b has to be viewed as the "natural" independent variable, since this is "our" time. But one can equally well say that the τ_s is the "natural" independent variable, since it provides an accepted standard of time [13]... Clearly we are in a very different situation from one with the two partial observables q and t of the previous section. There, we had a clear distinction between an independent observable (t) and a dependent one (q). Here, τ_s and τ_b are truly on the same footing.

The key difference between general relativistic physics and pre-GR physics is the fact that in general relativistic physics the distinction between *dependent* and *independent* partial observables is lost. A pre-GR theory is formulated in terms of variables (such as q) evolving as functions of certain distinguished variables (such as t). General relativistic systems are formulated in terms of

†Einstein has described his 1912 to 1915 final struggle for general relativity as a dramatic effort to understand the new "meaning of the coordinates".

‡Unless one fixes a physically interpreted gauge, in which case the discussion of observability is a bit different, but the final conclusions are unchanged. See below.

variables (such as τ_b, τ_s, α_V, α_S) that evolve with respect to each other. General relativity expresses relations between these, but in general we cannot solve for one as function of the other. Partial observables are genuinely on the same footing.

What are the complete observables, in general, in this context? A complete observable is a quantity that can be predicted uniquely. Therefore it is a quantity which is well defined once we know the solution of the equations of motion, up to all gauges (that is, which not affected by the indetermination of the evolution). Such a quantity can be seen as a function on the space of the solutions modulo all gauges. This space is the physical phase space of the theory Γ. In the canonical formalism, Γ can be obtained as the space of the orbits generated by the constraints on the constraint surface. Any complete observable can thus be expressed as a function on Γ. Equivalently, it can be expressed as a function on the extended phase having vanishing Poisson brackets with all first class constraints, including, of course, the hamiltonian constraint. Vice versa, any function that commutes with all constraints defines, in principle, a complete observable.

Partial observables are hard to construct formally in general, but it is far easier to define and use them concretely. For a recent concrete construction of a complete set of partial and complete observables in GR, see [13].

I close the section with a note on gauge-fixed formulations of GR. One may fix the gauge by choosing coordinates that have a physical interpretation. More precisely, one may select a family of partial observables (curvature scalars, scalar fields, dust variables, GPS readings...) and fix the coordinate gauge by tying the coordinate system to these partial observables. Within a formulation of this kind, coordinates represent partial observables. Furthermore, they have a natural character of independent partial observables. However, this does imply that the independent partial observables are determined by the theory, because the same physical situation can be described by a different physical gauge choice, in which the role of dependent and independent partial observables is interchanged.

IV. QUANTUM THEORY

In quantum theory observables are represented by operators. Which observables are represented by operators: the partial or the complete observables? The answer is different in the Heisenberg picture (evolving operators) and in the Schrödinger picture (evolving states). Let us start from the Heisenberg picture. Here the operators are time dependent. For instance, in the quantum theory of a harmonic oscillator in the Heisenberg picture, there is no position operator Q, but only the operator $Q(t)$ that represents "position at time t". This is immediately recognized as the operator corresponding to the complete observable $q(t)$ discussed in Section II. In the Heisenberg picture operators are associated with complete observables.

In the Schrödinger picture, there is an operator Q associated with the partial observable q. However, specific predictions are not given just in terms of this operator: we need the state as well, and, in the Schrödinger picture, the state $\Psi(t)$ is time dependent. Thus, for instance, the expectation value $\overline{Q}(t) = \langle\Psi(t)Q\Psi(t)\rangle$, which is a prediction of the theory, is associated with the complete observable $q(t)$, as it should, not with the partial observable q. In order for the Schrödinger picture to be meaningful, we need the theory to be expressed in terms of a well defined independent partial observable t: "the external time". In a theory such as general relativity, where the dynamics expresses the relation between partial observables that are on equal footing, the Schrödinger picture is not viable. More precisely, it will be viable only in special circumstances, in which we can choose (arbitrarily) one of the partial observables as the independent one and solve the dynamical relations expressing the predictions of the theory in terms of this quantity. In general, no such quantity exists. On the other hand, the Heisenberg picture remains meaningful whatever the spacetime structure of the theory. Let us therefore return to the Heisenberg picture, which is far more general.

In the Heisenberg context, consider the problem of whether there should be a time operator in quantum theory. The time t is a partial observable, not a complete observable. Operators are associated with complete observables, not with partial observables. Therefore it is against the tenets of quantum theory to search for an operator corresponding to t. Operators correspond to quantities that are in principle predictable (such as $q(t)$), not to quantities (such as t) that serve only to localize the measurement of a predictable quantity in spacetime. A quantity that is described by an operator in quantum theory is a quantity such that there are states that diagonalize it, namely such that there are physical situations in which the outcome of a measurement of that quantity is certain: the time t, on the contrary, can never be predicted.

In other words, quantum theory deals with the relation between q and t, and not with q alone or t alone. Therefore it is meaningless to search for the quantum theory of the t variable alone.

Of course, the reading T of a clock *can* be predicted, but only *if* we first read *another* clock. If we know that the second clock indicates t, we can predict that the first clock will read a certain T. If we now take into account the fact that the clock is a physical mechanical system and it is subject to quantum fluctuations, then we can describe it in terms of an operator. This operator describes the complete observable $T(t)$. There will be quantum fluctuations described by generic states in the state space on which this operator acts. These fluctuations are not the quantum fluctuations of one independent time variable. They are the quantum fluctuations in the observ-

able correlation between *two* clock variables.

V. QUANTUM GRAVITY

In quantum gravity, operators corresponding to physical observables must commute with the Wheeler-DeWitt constraint operator. This operator is the generator of evolution in the coordinate time t. Thus, physical observables must be invariant under evolution in t. This fact has raised much confusion. How can observables invariant under evolution in t describe the evolution we observe? The question is ill posed, because it confuses evolution with respect to the coordinate time t and physical evolution. In Section III we have observed that in general relativity quantities like the proper times τ_b and τ_s are partial observables and their relative evolution is well defined. Let us fix a value $\tau_b = \tau$ of the first, and consider the corresponding value of τ_s. (If there are several such values, take the highest). Call this value T_τ. That is T_τ is the highest number for which

$$f(T_\tau, \tau) = 0. \tag{5}$$

where f is the function in (4). T_τ is a complete observable. It is the signal we receive from the satellite when our local proper time at the base is τ. It describes the change of the value of the received signal as the proper time at the base passes. This is a description of evolution. At the same time, this is a quantity independent of the coordinate t. To see this, recall that to calculate its value from a specific solution of the Einstein equations, we first find the dependence of τ_b and τ_s on the coordinate time t. Namely we compute the functions $\tau_b(t)$ and $\tau_s(t)$. The form of these two functions is gauge dependent: it changes if we use a different coordinate representation of the same four geometry. We then locally invert the second function and insert $t(\tau_s)$ in the first. The resulting $T_\tau \equiv \tau_s(\tau_b = \tau)$ is independent of the coordinate t chosen, and thus it is uniquely determined by the equivalence class of solutions of the field equations under diffeomorphisms. It is a well defined on the space of these equivalence classes, namely on Γ. Equivalently, it can be represented as a function on the extended phase space that commutes with all the constraints, including the hamiltonian constraint.

Let us now come to a main objection that we want to address in this paper, which is the following.

Objection: T_τ cannot be observable without τ_b and τ_s being individually observable. Thus τ_b and τ_s are observable. Observables must be represented by physical operators. τ_b and τ_s depend on t and do not commute with the hamiltonian constraint. Therefore in any quantum theory of gravity there should be physical operators representing observables that do not commute with the Wheeler-DeWitt constraint.

It should be clear at this point why this objection is wrong. It confuses partial and complete observables. τ_b and τ_s are partial observables, and partial observables are not associated with quantum operators in quantum theory (more precisely, in Heisenberg picture quantum theory, which is the only one viable in this context).

We close this section with an observation on the role of the coordinates in the formalism of quantum gravity. The general relativistic spacetime coordinates $(\vec{x}, t)$ have no direct physical interpretation. In a gauge fixed context, they can be tied to partial observables. In any case, however, they do not represent complete observables. It follows that the idea that the coordinates should be represented by quantum operators is not justified is the light of quantum theory and general relativity alone. Operators are attached to complete observables, while spacetime coordinates are –at best– partial observables: they cannot be predicted, they serve only to localize complete observables.

Quantum theory deals with the relation between partial observables. It can deal with the relation between physical variables and (gauge-fixed) coordinates $(\vec{x}, t)$. But not with the value of the coordinates alone. Therefore it is meaningless to search for the quantum theory of the $(\vec{x}, t)$ variables alone.

Non-commutative geometry approaches to quantum gravity search for a mathematics capable of promoting the spacetime coordinates $(\vec{x}, t)$ to a non-commuting operator algebra. This approach is sometimes motivated with the argument that quantum theory should require the coordinates $(\vec{x}, t)$ to be represented by operators. In the light of the discussion above, I think that this motivation mistakes partial observables and complete observables. Non-commutative approaches to quantum gravity are extremely interesting in my view, both mathematically and the physically. But I think that this particular motivation is naive and not tenable. Physical non-commutativity of quantities related to physical localization and geometry, on the other hand, is likely to follow from the fact such quantities should, in fact, be functions of the gravitational field, and therefore quantum dynamical variables.

VI. THE EXTENDED CONFIGURATION SPACE

In this section I discuss the role of the partial observables in the formal structure of mechanics. I focus here on theories with a finite number of degrees of freedom, leaving the extension to field theory to the reader.[§]

[§] Examples of generally relativistic systems with a finite number of degrees of freedom are provided for instance by cosmological models, or by models as in [8].

This discussion sheds light on the physical interpretation of certain structures, such as the extended phase space of the fully constrained systems. For a more complete treatement of this subject, see [14]; see also [9]. The central message of this section is double. First, that when the formalism is sufficiently general, partial observables are the main quantities mechanics deals with. Second, that, in general, mechanics makes no distinction between dependent and independent observables. The distinction between independent and dependent observables can be seen as an accident of the specific dynamics of non-relativistic theories. In the light of these two observations, I think that the interpretation of general relativistic theories becomes more transparent.

As observed in Section II, the partial observables of a mechanical system form the extended configuration space $\mathcal{C}$. Recall that we denote the partial observables by q^a, $a = 1, \ldots, n$. Dynamics can be given in terms of a first order partial differential equation on $\mathcal{C}$, the Hamilton-Jacobi equation

$$C\left(q^a, \frac{\partial S(q^a)}{\partial q^a}\right) = 0, \tag{6}$$

The function of $2n$ variables $C(q^a, p_a)$ determines the dynamics. One searches for an n-parameter family of solutions of this equation $S(q^a, Q^a)$, where Q^a are n constants, and the predictions of the theory are contained in equations (3), which are obtained as follows

$$f_a(q^a; Q^a, P_a) = \frac{\partial S(q^a, Q^a)}{\partial Q^a} - P_a = 0. \tag{7}$$

These form a $2n$-parameter family of n relations between the partial observables (not all independent). The parameters Q^a, P_a label the possible histories of the system: each history determines a set of relations among partial observables. These relations are the physical predictions of the theory. Notice that all partial observables q^a are treated on the same footing: the "time" partial observable, if present at all, is just a variable among the others. Notice also that the usual last step of the Hamilton-Jacobi prescription, which is to invert (7) for the dependent variables, is not necessary from this point of view.

Let Γ be the space of the histories. Generically, there is one history connecting any two points of $\mathcal{C}$. Therefore Γ has dimension $2n-2$. Since Q^a and P_a are $2n$ functions on Γ, they over-coordinatize Γ and there are 2 relations among them. Also, histories are one-dimensional, and therefore only n-1 of the n relations (7) are independent. The space Γ is the phase space of the system. A point in Γ is a "state" of the theory, in the sense of "Heisenberg state‘" [12]. It represents a possible history of the system, not a "state at a certain time".**

The function f in (7) is defined on the cartesian product of the space of the partial observables with the space of the states.

$$f : \mathcal{C} \times \Gamma \longmapsto I\!R^n. \tag{8}$$

The entire predictive content of a dynamical theory is in the surface $f = 0$ in the cartesian product of the space of the partial observables and the space of the states. For each point q^a in $\mathcal{C}$, the surface $f = 0$ determines the set of states compatible with the value q^a of the partial observables. For each state in Γ, the surface determines a relation among the partial observables in $\mathcal{C}$.

In the *special* case of a non-relativistic system, one of the partial observables q^a is the time t. Let it be, say, q^0, and call the other partial observables q^i with $i = 1, \ldots, m = n - 1$. In this case the function $C(q^a, p_a)$ has the *special* form

$$C(q^a, p_a) = p_0 + H(q^i, p_i). \tag{9}$$

Therefore in this special case the Hamilton-Jacobi equation takes the well known form

$$\frac{\partial S(q^i, t)}{\partial t} + H\left(q^i, \frac{\partial S(q^i, t)}{\partial q^i}\right) = 0. \tag{10}$$

The general Hamilton-Jacobi formalism has a nice geometrical interpretation in the canonical framework. Let us illustrate it, with the purpose of discussing the meaning of the structures of generally covariant hamiltonian systems.

Consider the cotangent bundle $T^*\mathcal{C}$ over the extended configuration space, with canonical coordinates (q^a, p_a). Call it the extended phase space. It carries the natural Poincaré one-form $\theta = p_a dq^a$, and the symplectic form $\omega = -d\theta$. The dynamics is coded in a relation on $T^*\mathcal{C}$:

$$C(q^a, p_a) = 0, \tag{11}$$

In the special case of a non-relativistic system, $q^a = (q^0, q^i)$ and Equation (11) has the form

$$C(q^a, p_a) = p_0 + H(q^i, p_i) = 0 \tag{12}$$

where H is the Hamiltonian. The variable $q^0 = t$ is the time variable, and its conjugate momentum $p_0 = -E$ is (minus) the energy. The dynamics of the system is then coded in the relation (12) which gives the energy as a function of the other coordinates and momenta.

**Dirac argued repeatedly that the Heisenberg notion of state is the good one, and the only one that makes sense in a relativistic context. See for instance Sec I.3 of the first edition of [12]. In later editions of this book Dirac shifted the emphasis to the Schrödinger states, explaining (in the Preface) that these, after all, are easier to work with in the non relativistic context, although "it seems a pity" to give the cleaner notion.

Equation (11) defines a surface Σ in $T^*\mathcal{C}$. Call ω_Σ the restriction of ω to this surface. The "presymplectic" two-form ω_Σ is degenerate and has a null direction. It is not difficult to see that the integral curves of this null direction are the solutions of the equations of motion of the system.[††] The space of these curves is the physical phase space of the system Γ and carries a unique symplectic two-form ω_Γ whose pull back to Σ under the natural projection $\pi : \Sigma \to \Gamma$ is ω_Σ. Let P_a and Q^a be coordinates that (over-)coordinatize[‡‡] Γ and define a one-form $\theta_\Gamma = P_a dQ^a$ such that $d\theta_\Gamma = -\omega_\Gamma$. Then (using the same notations for the forms and their pull back), since on Σ we have $\omega_\Gamma = \omega_\Sigma$, it follows that

$$d(p_a dq^a) = d(P_a dQ^a) \tag{13}$$

or

$$p_a dq^a = P_a dQ^a + dS \tag{14}$$

where S is a zero-form on Σ. But let us pull the coordinates Q^a back onto Σ and assume that the set (q^a, Q^a) (over-) coordinatizes Σ. Then (14) gives

$$\frac{\partial S(q^a, Q^a)}{\partial q^a} = p_a, \tag{15}$$

$$\frac{\partial S(q^a, Q^a)}{\partial Q^a} = P_a. \tag{16}$$

From (15) and (11) we obtain

$$C\left(q^a, \frac{\partial S(q^a, Q^i)}{\partial q^a}\right) = 0, \tag{17}$$

which is the Hamilton-Jacobi equation (6) and can be used to compute S; while (16) is the equation (7) giving the physical predictions from S.

What is the physical meaning of $S(q^a, Q^a)$? Without loss of generality, we can choose the integration constants so that

$$S(q^a, Q^a = q^a) = 0. \tag{18}$$

Fix a point p on Σ, and consider the trajectory that starts on p. Along this trajectory $dQ^a = 0$ and thus from (14) we have $dS = p_a dq^a$. Parametrize the trajectory with an arbitrary time parameter t and write $dS = p_a dq^a = p_a \dot{q}^a dt$. The canonical hamiltonian with respect to this parameter is null, and therefore $p_a \dot{q}^a - L = 0$. Therefore we have $dS = Ldt$ along each orbit $q^a(t)$ and

$$S(q^a, Q^a) = \int_{Q^a}^{q^a} L(q^a(t))dt. \tag{19}$$

That is, $S(q^a, Q^a)$ is the action, computed over the physical trajectory that joins the points with coordinates Q^a and q^a. In the case of a non-relativistic system, let $q^a = (q^i, t)$. Then $dS = p_a dq^a = p_i dq^i - Hdt$. Recall that $H = p_i \dot{q}^i - L$, where L is the Lagrangian. Therefore $dS = p_i dq^i - p_i dq^i + Ldt = Ldt$ along each orbit. Thus,

$$S(q^a, Q^a) = \int_{Q^i}^{q^i} L(q^i(t))dt. \tag{20}$$

where the trajectory starts at time t in q^i and ends at time T in Q^i. That is, $S(q^a, Q^a)$ is still the action, computed over the physical trajectory that joins the points with coordinates Q^a and q^a.

Notice that from this point of view Hamilton's principal function and Hamilton's characteristic function are identified. More precisely, $S(q^a, Q^a)$ is the principal function for the evolution in any partial observable identified as the time $q^0 = t$. But it is also the characteristic function of the evolution in an arbitrary parameter time along the histories. And it is also the principal function for the evolution in such a time, since the hamiltonian that generates this motion vanishes. This compactification of the formalism is quite remarkable.

In conclusion, the ingredients of mechanics can be taken to be solely the extended configuration space $\mathcal{C}$ and the function C on $T^*\mathcal{C}$. A mechanical system is determined by the pair $(\mathcal{C}, C)$. The kinematics of a specific theory is determined by the space of its partial observables $\mathcal{C}$; its dynamics is determined by the constraint $C(q^a, p_a) = 0$ on the associated phase space. There is no need to single out a specific partial variable as the time, nor to mention evolution. Mechanics is a theory of relations between partial observables. No distinction between dependent and independent partial observables is required. This distinction is an accident of non-relativistic theories, in which the constraint $C(q^a, p_a)$ happens to have the form (12).

Why do I stress this fact? Because generally relativistic theories are formulated in terms of constraints such as (11) over an extended configuration space. It is sometimes claimed that the theory can only be interpreted if one finds a way to "deparametrize" the theory, namely to select the independent variable, among the variables q^a. In the opposite camp, the statement is sometimes made that only variables on the physical phase space Γ have physical interpretation, and no interpretation should be associated with the variables of the extended configuration space $\mathcal{C}$. Instead, I have argued here that the variables of the extended configuration space have a physical interpretation as partial observables. In a sense, they are

[††] The coordinate form of the relation $Y(\omega_\Sigma) = 0$ between ω_Σ and its null vector field Y is given by the Hamilton equations.

[‡‡] $2(n-1)$ coordinates are sufficient to coordinatize Γ. For instance, one can take initial coordinate and momenta at $t = t_0$. We prefer to use here $2n$ coordinates for reasons that will be clear below. The extra coordinates can be seen as the initial time $t = t_0$ and the energy. A change in the first amounts to a relabeling of the meaning of the initial data. The second is constrained by C.

the quantities with the most direct physical interpretation in the theory.

Finally, consider quantum theory. The Schrödinger equation, as well as the Wheeler-DeWitt equation, are *both* partial differential equations on the extended configuration space $\mathcal{C}$. They can both be obtained in general from the constraint (11), with no need of distinguishing dependent from independent partial observables. Indeed, they are obtained as

$$C\left(q^a, -i\hbar\frac{\partial}{\partial q^a}\right)\psi(q^a) = 0. \tag{21}$$

The usual physical scalar product on an appropriate space of the solutions of this equation has an intrinsic meaning and does not need the time variable to be singled out in order to be defined – see for instance [15,16]. All relevant physical predictions of the theory can be extracted from the knowledge of the propagator $W(q^a, Q^a)$, which satisfies

$$C\left(q^a, -i\hbar\frac{\partial}{\partial q^a}\right)W(q^a, Q^a) = 0. \tag{22}$$

The propagator gives the probability amplitude for finding the combination of partial observables q^a if the combination of partial observables Q^a was previously observed. Virtually all predictions of quantum mechanics can be formulated in this covariant manner, on the extended configuration space. This is discussed in detail in [16]. As well known, in the limit of small $\hbar$ the Schrödinger equation (or the Wheeler-DeWitt equation) goes over into the Hamilton-Jacobi equation, and the propagator $W(q^a, Q^a)$ is given to first order just by the exponential of the action $W(q^a, Q^a) \sim \exp\{iS(q^a, Q^a)/\hbar\}$.

VII. CONCLUSIONS

I have observed that the notion of observable is ambiguous, and I have discussed the distinction between partial observables and complete observables. This distinction clarifies a certain number of issues related to observability. In particular, I have examined the role played by this distinction in general relativity, in quantum mechanics and in quantum gravity.

The partial observables form the extended configuration space $\mathcal{C}$. This space seems to be a natural home for classical and quantum mechanics. The two theories admit a clean formulation over this space, which is sufficiently general to deal naturally with general relativistic systems.

A mechanical system is a pair $(\mathcal{C}, C)$. The space of the partial observables $\mathcal{C}$ describes the kinematics of the theory. C is a function on $T^*\mathcal{C}$ that determines the dynamics. Classical dynamics is about relations between partial observables. These relations depend on a certain number of parameters, which label the (time independent) states of the system. The space of these states is the phase space Γ. The predictions of the theory are therefore given by a surface $f = 0$ on $\mathcal{C} \times \Gamma$. The surface $f = 0$, as well as Γ, are determined by the pair $(\mathcal{C}, C)$. (The general structure of classical and quantum mechanics in this language is discussed more in detail in [14].)

By fixing a subset of partial observables (one for a mechanical system, four for a field theory), the other partial observables are determined as functions on Γ. This defines the complete observables of the theory, whose value can be predicted uniquely if the state is known.

Quantum mechanics gives the probability amplitude $W(q^a, Q^a)$ for measuring the combination of partial observables q^a after having measured the combination Q^a. Alternatively, it gives the probability distribution for the different possible outcomes of a measurement of the complete observables. These are represented by self-adjoint operator over the (Heisenberg) state space.

No distinction between independent and dependent partial observable is required. The different partial observables can be viewed as being on the same footing. This formulation of mechanics does not require the notion of external time. It is therefore appropriate for general relativistic systems, which are not formulated in terms of evolution in time.

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