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# On Nearly Parallel $G_2$ -Structures

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**Abstract** - A nearly parallel  $G_2$ -structure on a 7-dimensional Riemannian manifold is equivalent to a spin structure with a Killing spinor. We prove general results about the automorphism group of such structures and we construct new examples. We classify all nearly parallel  $G_2$ -manifolds with large symmetry group and in particular all homogeneous nearly parallel  $G_2$ -structures.

**Keywords** -  $G_2$ -structure, Sasakian manifold, Killing spinor.

## 1 Introduction

A nearly parallel  $G_2$ -structure on a 7-dimensional manifold is a 3-form  $\omega^3$  of special algebraic type satisfying the differential equation

$$d\omega^3 = -8\lambda(*\omega^3)$$

for some constant  $\lambda \neq 0$ . The existence of  $\omega^3$  is equivalent to the existence of a spin structure with a Killing spinor, i.e. a spinor  $\psi$  satisfying

$$\nabla_X \psi = \lambda X \cdot \psi, \quad \forall X \in TM.$$

In case  $\lambda = 0$ ,  $\omega^3$  defines a *geometric*  $G_2$ -structure ( $d\omega^3 = 0, \delta\omega^3 = 0$ ). Excluding the case of the 7-dimensional sphere there are three types of nearly parallel  $G_2$ -structures depending on the dimension of the space  $KS$  of all Killing spinors. Nearly parallel  $G_2$ -structures with  $\dim(KS) = 3$  are 3-Sasakian manifolds and nearly parallel  $G_2$ -structures such that  $\dim(KS) = 2$  are Einstein-Sasakian spaces. There are examples of compact nearly parallel  $G_2$ -manifolds where the dimension of the space of all Killing spinors equals one and we call such spaces *proper  $G_2$ -manifolds*.

Recently D. Joyce [22] solved an open problem in holonomy theory, namely the existence problem of compact 7-dimensional Riemannian manifolds with  $G_2$ -holonomy. On the other hand, Boyer / Galicki / Mann [7] constructed new compact examples of 3-Sasakian manifolds and investigated the global geometry of this spaces. In dimension seven, 3-Sasakian manifolds are special nearly parallel  $G_2$ -structures and such manifolds have been studied a long time ago (see [20], [12]). However, during the last 10 years, these special Einstein manifolds appeared as Einstein spaces where the Dirac operator has the smallest possible eigenvalue and many compact examples are known since this time (see [11]). The aim of this paper is to revisit once again the results as well as the examples of compact nearly parallel  $G_2$ -structures known up to now. Moreover, starting from 3-Sasakian manifold we construct new manifolds with a nearly parallel  $G_2$ -structure. A 3-Sasakian manifold admits a second Einstein metric obtained from the given one by scaling the metric in the directions of the orbits of the the  $Spin(3)$ -action. It turns out that this Einstein metric is a proper  $G_2$ -structure and we obtain new nearly parallel  $G_2$ -structures from the examples of 3-Sasakian manifolds mentioned above.

Finally we investigate the automorphism group of a compact nearly  $G_2$ -manifold and we classify in particular all homogeneous  $G_2$ -manifolds. The automorphism group  $G = Aut(M^7, \omega^3)$  of a nearly parallel  $G_2$ -manifold has some special properties. In particular, if  $\dim(G) \geq 10$ ,  $G$  acts transitively on  $M^7$ . The zero set of infinitesimal automorphisms is either one- or three-dimensional and a four-dimensional orbit of this group-action is of special topological and geometric type. Moreover, the isotropy groups  $G(m)$  are subgroups of the exceptional  $G_2$  and one can list them explicitly. Combining all these informations we can classify the compact, nearly parallel  $G_2$ -manifolds with a large symmetry group.

## 2 The exceptional group $G_2$ .

The group  $G_2$  is a compact, simple and simply connected 14-dimensional Lie group. In this section we collect some basic algebraic facts about this group. In particular, we will define  $G_2$  as the isotropy group of a real  $Spin(7)$ -spinor. Since in dimension 7 these spinors correspond to the 3-forms  $\omega^3$  of general type in  $\Lambda^3(\mathbb{R}^7)$ , this definition of the group  $G_2$  is equivalent to the usual one as the subgroup of  $GL(7; \mathbb{R})$  preserving the 3-form in  $\mathbb{R}^7$

$$\begin{aligned} \omega_0^3 = & e_1 \wedge e_2 \wedge e_7 + e_1 \wedge e_3 \wedge e_5 - e_1 \wedge e_4 \wedge e_6 \\ & - e_2 \wedge e_3 \wedge e_6 - e_2 \wedge e_4 \wedge e_5 + e_3 \wedge e_4 \wedge e_7 + e_5 \wedge e_6 \wedge e_7. \end{aligned} \quad (1)$$

The advantage of this point of view is that a topological  $G_2$ -structure on a 7-dimensional manifold defines a Riemannian metric as well as a spinor field of constant length. We shall use the equivalence between topological  $G_2$ -structures and 3-forms of general type and between these and Riemannian metrics together with a unit spinor field many times in our investigations of  $G_2$ -structures of special geometrical type.

Let  $e_1, \dots, e_7$  be the standard orthonormal basis of the Euclidian vector space  $\mathbb{R}^7$  and denote by  $Cliff(\mathbb{R}^7)$  the real Clifford algebra. We will use the real representation of this algebra on  $\Delta_7 := \mathbb{R}^8$  given on its generators by

$$\begin{aligned} e_1 &= E_{18} + E_{27} - E_{36} - E_{45} \\ e_2 &= -E_{17} + E_{28} + E_{35} - E_{46} \\ e_3 &= -E_{16} + E_{25} - E_{38} + E_{47} \\ e_4 &= -E_{15} - E_{26} - E_{37} - E_{48} \\ e_5 &= -E_{13} - E_{24} + E_{57} + E_{68} \\ e_6 &= E_{14} - E_{23} - E_{58} + E_{67} \\ e_7 &= E_{12} - E_{34} - E_{56} + E_{78} , \end{aligned}$$

where  $E_{ij}$  is the standard basis of the Lie algebra  $\mathfrak{so}(8)$  :

$$E_{ij} = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & & -1 & \cdots & \\ \cdots & & & & \\ \cdots & 1 & \cdots & \cdots & \\ 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots \\ i & & & & j \end{pmatrix} \begin{matrix} \cdots & i \\ \cdots & j \end{matrix}$$

If we restrict this representation onto  $Spin(7) \subset Cliff(\mathbb{R}^7)$  we obtain the real spin representation  $\kappa : Spin(7) \rightarrow SO(\Delta_7)$ . The group  $Spin(7)$  acts transitively on the sphere

$$S(\Delta_7) = \{\|\psi\| = 1\} \subset \Delta_7 = \mathbb{R}^8.$$

We now define the group  $G_2$  as the subgroup of  $Spin(7)$  preserving the spinor  $\psi_0 := {}^t(1, 0, \dots, 0)$

$$G_2 = \{g \in Spin(7) \mid g\psi_0 = \psi_0\}.$$

Consequently the sphere  $S^7$  is diffeomorphic to the homogeneous space  $Spin(7)/G_2$  and we obtain from the exact homotopy sequence of this fibration

$$\pi_0(G_2) = 0, \quad \pi_1(G_2) = 0, \quad \pi_2(G_2) = 0, \quad \pi_3(G_2) = \mathbb{Z}.$$

Let us now calculate the Lie algebra  $\mathfrak{g}_2$  of  $G_2$ . We identify the Lie algebra of  $Spin(7)$  with  $\mathfrak{spin}(7) = \{\omega = \sum_{i < j} \omega_{ij} e_i e_j \mid \omega_{ij} \in \mathbb{R}\} \subset Cliff(\mathbb{R}^7)$ . The Lie algebra  $\mathfrak{g}_2$  is the subalgebra of this algebra containing all elements  $\omega$  satisfying  $\omega \cdot \psi_0 = 0$ . Let  $\omega = \sum_{i < j} \omega_{ij} e_i e_j$  be any element of  $\mathfrak{spin}(7)$ . Then  $\omega \cdot \psi_0 = 0$  holds iff

$$\begin{aligned} \omega_{12} + \omega_{34} + \omega_{56} &= 0, & -\omega_{13} + \omega_{24} - \omega_{67} &= 0, & -\omega_{14} - \omega_{23} - \omega_{57} &= 0, \\ -\omega_{16} - \omega_{25} + \omega_{37} &= 0, & \omega_{15} - \omega_{26} - \omega_{47} &= 0, & \omega_{17} + \omega_{36} + \omega_{45} &= 0, \\ \omega_{27} + \omega_{35} - \omega_{46} &= 0. \end{aligned}$$

We consider the universal covering  $Spin(7) \rightarrow SO(7)$  of the special orthogonal group  $SO(7)$ . Because of  $(-1) \notin G_2$ , there is an isomorphism from  $G_2$  onto a subgroup of  $SO(7)$ , which we also denote by  $G_2$ . We now describe this group. This will yield a second definition of  $G_2$  using 3-forms on  $\mathbb{R}^7$ . The key point is a special relation in dimension 7 between real spinors and generic 3-forms.

Let  $\psi \in \Delta_7$  a fixed spinor. Then the map

$$\mathbb{R}^7 \ni X \mapsto X\psi \in \Delta_7$$

is an isomorphism between  $\mathbb{R}^7$  and the orthogonal complement of  $\psi$  in  $\Delta_7$ . We observe that for  $X, Y \in \mathbb{R}^7$  the spinors  $\psi$  and  $YX\psi + \langle X, Y \rangle \psi$  are orthogonal to each other. Therefore we can define a (2,1)-tensor  $A_\psi$  by

$$YX\psi = -\langle X, Y \rangle \psi + A_\psi(Y, X)\psi. \quad (2)$$

$A_\psi$  has the following properties

1.  $A_\psi(X, Y) = -A_\psi(Y, X)$
2.  $\langle Y, A_\psi(Y, X) \rangle = 0$
3.  $A_\psi(Y, A_\psi(Y, X)) = -\|Y\|^2 X + \langle X, Y \rangle Y$ .

It defines a 3-form  $\omega_\psi^3$  by  $\omega_\psi^3(X, Y, Z) = \langle X, A_\psi(Y, Z) \rangle$ .

Vice versa, a (2,1)-tensor  $A$  on  $\mathbb{R}^7$  which has the properties 1, 2, 3 defines a 1-dimensional subspace  $E(A) = \{\psi \in \Delta_7 \mid YX\psi = -\langle X, Y \rangle \psi + A(Y, X)\psi\}$ . Consequently, we obtain a bijection from the projective space  $P(\Delta_7) = \mathbb{RP}^7$  onto the set of 3-forms  $\omega^3 \in \Lambda^3(\mathbb{R}^7)$  whose tensor  $A$  defined by  $\omega^3(X, Y, Z) = \langle X, A(Y, Z) \rangle$  has the above mentioned properties.

In particular, if  $\psi = \psi_0 := {}^t(1, 0, \dots, 0)$ , then a direct calculation yields  $\omega_{\psi_0}^3 = \omega_0^3$ , where  $\omega_0^3$  is given by (1).

Let  $g$  be an element of  $Spin(7)$  and  $\pi(g)$  the corresponding element in  $SO(7)$ . We compare the 3-forms associated to the spinors  $\psi$  and  $g\psi$  and obtain the equation

$$\omega_{g\psi}^3 = (\pi(g^{-1}))^* \omega_\psi^3.$$

The 3-form  $\omega_\psi^3$  defines the spinor  $\psi$  up to a real number. Hence, the image of the group  $G_2 \subset Spin(7)$  with respect to  $\pi : Spin(7) \rightarrow SO(7)$  equals

$$G_2 = \{A \in SO(7) \mid A^* \omega_{\psi_0}^3 = \omega_{\psi_0}^3\}.$$

However, the equation  $A^* \omega_0^3 = \omega_0^3$  for  $A \in GL(7)$  implies  $A \in SO(7)$ . See for a proof [8], [24]. Using this, we obtain

$$G_2 = \{A \in GL(7) \mid A^* \omega_0^3 = \omega_0^3\}.$$

**Remark 2.1** *Similarly, we can investigate the action of  $Spin(7)$  on the Stiefel manifolds  $V_2(\Delta_7)$  and  $V_3(\Delta_7)$  of orthonormal pairs and triples of spinors, respectively. This action is transitive, too. The isotropy group of a fixed pair of spinors is isomorphic to  $SU(3)$  and the one of a triple is isomorphic to  $SU(2)$ .*

**Remark 2.2** The  $GL(7)$ -orbit  $\Lambda_+^3(\mathbb{R}^7) := \{A^*\omega_0^3 \mid A \in GL(7)\}$  is an open subset of  $\Lambda^3(\mathbb{R}^7)$  since  $\dim \Lambda^3(\mathbb{R}^7) = 35$  and  $\dim GL(7) - \dim G_2 = 49 - 14 = 35$ . Let  $\alpha^3$  be an element of this orbit, i.e.  $\alpha^3 = A^*\omega^3$  for some  $A \in GL(7)$ . Then  $\alpha^3$  defines an inner product on  $\mathbb{R}^7$  by  $\langle \cdot, \cdot \rangle_\alpha := A^*\langle \cdot, \cdot \rangle$ , an orientation  $O_\alpha := A^*(e_1 \wedge \dots \wedge e_7)$  and a corresponding Hodge operator  $*_\alpha : \Lambda^p(\mathbb{R}^7) \mapsto \Lambda^{7-p}(\mathbb{R}^7)$ .

**Remark 2.3** Let  $\psi_1, \psi_2 \in \Delta_7$  be spinors of the same length and  $\xi \in \mathbb{R}^7$  such that  $\xi\psi_1 = \psi_2$ . Then we have for the induced 3-forms  $\omega_1^3 = \omega_{\psi_1}^3$  and  $\omega_2^3 = \omega_{\psi_2}^3$

$$\omega_2 = -\omega_1 + 2(\xi \lrcorner \omega_1) \wedge \xi.$$

*Proof.* We use the equations which define the tensors  $A_1 = A_{\psi_1}$  and  $A_2 = A_{\psi_2}$ . From  $YX\psi_2 = -\langle Y, X \rangle \psi_2 + A_2(Y, X)\psi_2$  it follows that  $YX\xi\psi_1 = -\langle Y, X \rangle \xi\psi_1 + A_1(Y, X)\xi\psi_1$ . By the definition of  $A_1$  this is equivalent to

$$\begin{aligned} -\langle X, \xi \rangle Y\psi_1 - \langle Y, A_1(X, \xi) \rangle \psi_1 + A_1(Y, A_1(X, \xi))\psi_1 &= \\ = -\langle Y, X \rangle \xi\psi_1 - \langle A_2(Y, X), \xi \rangle \psi_1 + A_1(A_2(Y, X), \xi)\psi_1, \end{aligned}$$

or to

$$\begin{aligned} (-\langle X, \xi \rangle Y + A_1(Y, A_1(X, \xi)) + \langle Y, X \rangle \xi - A_1(A_2(Y, X), \xi))\psi_1 &= \\ = (\langle Y, A_1(X, \xi) \rangle + \langle A_1(Y, X), \xi \rangle)\psi_1. \end{aligned}$$

Since the Clifford multiplication of real spinors by a vector is anti-symmetric we conclude that

$$A_1(Y, A_1(X, \xi)) + \langle Y, X \rangle \xi = A_1(A_2(Y, X), \xi) + \langle X, \xi \rangle Y \quad (3)$$

$$\langle Y, A_1(X, \xi) \rangle = \langle A_2(Y, X), \xi \rangle, \quad (4)$$

where (4) is equivalent to  $\omega_1(X, Y, \xi) = \omega_2(X, Y, \xi)$  and to  $A_1(X, \xi) = A_2(X, \xi)$ .

Let now  $X, Y, Z \in \mathbb{R}^7$  be vectors orthogonal to  $\xi$ . There exists an  $X \in \mathbb{R}^7, X \perp \xi$  such that  $Z = A_1(X, \xi) = A_2(X, \xi)$ . From equations (3) and (4) we conclude

$$\begin{aligned} \langle W, A_1(Y, Z) \rangle &= \langle W, A_1(Y, A_1(X, \xi)) \rangle = \langle W, A_1(A_2(Y, X), \xi) \rangle = \\ &= \langle W, A_2(A_2(Y, X), \xi) \rangle = -\langle W, A_2(\xi, A_2(Y, X)) \rangle, \end{aligned}$$

where the last equation holds because of property 3 of the (2,1)-tensor  $A_2$ . Consequently, we get  $\omega_1(W, Y, Z) = -\omega_2(W, Y, Z)$ . The assertion follows. ■

Now we recall the decomposition of  $\Lambda^p(\mathbb{R}^7)$  into irreducible components with respect to the action of  $G_2$ .

#### Proposition 2.4

1.  $\mathbb{R}^7 = \Lambda^1(\mathbb{R}^7) =: \Lambda_7^1$  is irreducible.

2.  $\Lambda^2(\mathbb{R}^7) = \Lambda_7^2 \oplus \Lambda_{14}^2$ , where

$$\begin{aligned}\Lambda_7^2 &= \{\alpha^2 \in \Lambda^2 \mid *(\omega^3 \wedge \alpha^2) = 2\alpha^2\} = \{X \lrcorner \omega^3 \mid X \in \mathbb{R}^7\} \\ \Lambda_{14}^2 &= \{\alpha^2 \in \Lambda^2 \mid *(\omega^3 \wedge \alpha^2) = -\alpha^2\} = \mathfrak{g}_2\end{aligned}$$

3.  $\Lambda^3(\mathbb{R}^7) = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3$ , where

$$\begin{aligned}\Lambda_1^3 &= \{t\omega^3 \mid t \in \mathbb{R}^1\} \\ \Lambda_7^3 &= \{*(\omega^3 \wedge \alpha^1) \mid \alpha^1 \in \Lambda_7^1\} \\ \Lambda_{27}^3 &= \{\alpha^3 \in \Lambda^3 \mid \alpha^3 \wedge \omega^3 = 0, \alpha^3 \wedge *\omega^3 = 0\}\end{aligned}$$

**Proposition 2.5** *The wedge product  $\omega^3 \wedge : \Lambda^3(\mathbb{R}^7) \longrightarrow \Lambda^6(\mathbb{R}^7)$  has the following properties with respect to the decomposition  $\Lambda^3(\mathbb{R}^7) = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3$ .*

1.  $\omega^3 \wedge (\Lambda_1^3 \oplus \Lambda_{27}^3) = 0$
2. If  $\eta^3 = *(\omega^3 \wedge \alpha^1) \in \Lambda_7^3$ , then  $\omega^3 \wedge \eta^3 = -4 * \alpha^1$ .

Similarly, the wedge product  $*\omega^3 \wedge : \Lambda^2(\mathbb{R}^7) \longrightarrow \Lambda^6(\mathbb{R}^7)$  has the following properties with respect to the decomposition  $\Lambda^2(\mathbb{R}^7) = \Lambda_7^2 \oplus \Lambda_{14}^2$ .

3.  $(*\omega^3) \wedge \Lambda_{14}^2 = 0$
4. If  $\alpha^2 = X \lrcorner \omega^3 \in \Lambda_7^2$ , then  $(*\omega^3) \wedge \alpha^2 = 3(*X)$ .

Next we study the action of the group  $G_2$  on the Grassmannian manifolds  $G_2(\mathbb{R}^7)$  and  $G_3(\mathbb{R}^7)$  of oriented 2- and 3-dimensional linear subspaces in  $\mathbb{R}^7$ .

**Proposition 2.6**

1.  $G_2$  acts transitively on  $G_2(\mathbb{R}^7)$ .
2.  $G_2$  acts on  $G_3(\mathbb{R}^7)$  with cohomogeneity one. The principal orbits have dimension 11 and there are two exceptional orbits of dimension 8.
3. For any  $E^3 \in G^3(\mathbb{R}^7)$  the inequality  $|\omega^3(E^3)| \leq 1$  holds. The 3-dimensional subspace  $E^3$  belongs to the exceptional orbit with respect to the  $G_2$ -action if and only if  $|\omega^3(E^3)| = 1$ .

*Proof.*  $G_2(\mathbb{R}^7) = SO(7)/[SO(2) \times SO(5)]$  is a 10-dimensional manifold. On the other hand, the intersection of the Lie algebras  $\mathfrak{g}_2$  and  $\mathfrak{so}(2) \times \mathfrak{so}(5)$  is the 4-dimensional subalgebra of  $\mathfrak{so}(7)$  defined by the equations

$$\omega_{1i} = \omega_{2i} = 0 \text{ for } i \geq 3, \quad \omega_{57} = \omega_{37} = \omega_{47} = 0$$

$$\omega_{36} + \omega_{45} = \omega_{35} - \omega_{46} = \omega_{24} - \omega_{67} = 0, \quad \omega_{12} + \omega_{34} + \omega_{56} = 0.$$

Hence the  $G_2$ -orbit of the standard 2-plane  $Span\{e_1, e_2\}$  has dimension 10. Since this orbit is a compact submanifold of  $G_2(\mathbb{R}^7)$ , it coincides with the Grassmannian manifold.

Fix a 3-dimensional subspace  $E^3$ . Since  $G_2$  acts transitively on  $G_2(\mathbb{R}^7)$  in the  $G_2$ -orbit through  $E^3$  there exists a 3-dimensional subspace containing the vectors  $e_1$  and  $e_2$ . For simplicity we denote this space by  $E^3$ , too. The isotropy group of the vectors  $e_1, e_2$  inside  $G_2$  is the group

$SU(2)$  acting on  $Span\{e_3, e_4, e_5, e_6\}$ . Therefore we may assume that the third vector of  $E^3$  is given by  $\cos(\varphi)e_3 + \sin(\varphi)e_7$ . Consequently, any  $G_2$ -orbit in  $G_3(\mathbb{R}^7)$  contains a subspace of the special form

$$E^3(\varphi) = Span\{e_1, e_2, \cos(\varphi)e_3 + \sin(\varphi)e_7\}.$$

The Lie algebra  $\mathfrak{h}(\varphi)$  of the isotropy group of  $E^3(\varphi)$  is the 9-dimensional subalgebra of  $\mathfrak{so}(7)$  given by the equations

$$\begin{aligned} \omega_{14} = \omega_{15} = \omega_{16} = \omega_{24} = \omega_{25} = \omega_{26} = 0, \quad \omega_{37} = 0, \\ \cos(\varphi)\omega_{34} + \sin(\varphi)\omega_{74} = \cos(\varphi)\omega_{35} + \sin(\varphi)\omega_{75} = \cos(\varphi)\omega_{36} + \sin(\varphi)\omega_{76} = 0, \\ \sin(\varphi)\omega_{13} - \cos(\varphi)\omega_{17} = \sin(\varphi)\omega_{23} - \cos(\varphi)\omega_{27} = 0. \end{aligned}$$

We calculate the intersection of the Lie algebras  $\mathfrak{g}_2$  and  $\mathfrak{h}(\varphi)$ . It turns out that

$$\dim [\mathfrak{g}_2 \cap \mathfrak{h}(\varphi)] = \begin{cases} 3 & \text{if } \cos(\varphi) \neq 0 \\ 6 & \text{if } \cos(\varphi) = 0. \end{cases}$$

Consequently, the  $G_2$ -orbit of the space  $E^3(\varphi)$  has dimension 11 (in case  $\cos(\varphi) \neq 0$ ), or dimension 8 (in case  $\cos(\varphi) = 0$ ). Moreover, we calculate the value  $\omega^3(E^3(\varphi))$  :

$$\omega^3(E^3(\varphi)) = \sin(\varphi).$$

■

**Remark 2.7** A 3-dimensional subspace  $E^3 \subset \mathbb{R}^7$  is said to be  $G_2$ -special if its  $G_2$ -orbit is an exceptional orbit. The following conditions are equivalent.

- (i)  $E^3$  is a special  $G_2$ -subspace.
- (ii)  $|\omega^3(E^3)| = 1$
- (iii) For any vectors  $X, Y \in E^3$  and  $Z \perp E^3$  the relation  $\omega^3(X, Y, Z) = 0$  holds.

### 3 Topological and Geometrical $G_2$ -Reductions.

Let  $M^7$  be a 7-dimensional manifold and  $R(M^7)$  the frame bundle of  $M^7$ . We define the bundle  $\Lambda_+^3(M^7)$  by

$$\Lambda_+^3(M^7) := R(M^7) \times_{GL(7)} \Lambda_+^3(\mathbb{R}^7) \subset R(M^7) \times_{GL(7)} \Lambda^3(\mathbb{R}^7) = \Lambda^3(M^7).$$

**Definition 3.1** A topological  $G_2$ -structure on  $M^7$  is a  $G_2$ -reduction of the frame bundle  $R(M^7)$ , i.e. a subbundle  $P_{G_2}$  satisfying

$$\begin{array}{ccc} G_2 & \hookrightarrow & GL(7) \\ \downarrow & & \downarrow \\ P_{G_2} & \hookrightarrow & R(M^7) \\ \searrow & & \swarrow \\ & M^7 & . \end{array}$$

Similarly we define topological  $SU(2)$ -,  $SU(3)$ - and  $Spin(7)$ -structures.



The fact that  $G_2$  is a subset of  $SO(7)$  and of  $Spin(7)$  implies that a  $G_2$ -structure  $P_{G_2}$  on  $M^7$  induces an orientation of  $M^7$  (i.e.  $\omega_1 = 0$ ), a Riemannian metric  $g$  on  $M^7$  such that the corresponding  $SO(7)$ -bundle equals  $P_{G_2} \times_{G_2} SO(7)$ , and a spin structure  $P_{G_2} \times_{G_2} Spin(7)$  (i.e.  $\omega_2 = 0$ ). Furthermore it defines the following nowhere vanishing spinor  $\psi \in \Gamma(S)$  in the real spinor bundle  $S = P_{G_2} \times_{G_2} \Delta_7$  of  $M^7$ . Since  $G_2 \subset Spin(7)$  is the isotropy group of  $\psi_0 \in \Delta_7$  the map  $\psi : P_{G_2} \rightarrow \Delta_7$ ,  $\psi(p) = \psi_0$ , has the property  $\psi(pg) = g^{-1}\psi$  for all  $g \in G_2$  and is therefore a section in  $S$ . Because of the  $G_2$  - invariance of  $\omega_0$  the  $G_2$ -structure defines in the same way a section  $\omega^3$  in  $\Lambda_+^3(M^7) = R(M^7) \times_{GL(7)} \Lambda_+^3(\mathbb{R}^7) = P_{G_2} \times_{G_2} \Lambda_+^3(\mathbb{R}^7)$ , by  $\omega^3 : P_{G_2} \rightarrow \Lambda_+^3(\mathbb{R}^7)$ ,  $\omega^3(p) = \omega_0^3$ . On the other hand the spinor  $\psi$  defines a (2,1)-tensor field  $A = A_\psi$  (see equation (2)) on  $M^7$  and we have  $\omega^3 = g(\cdot, A(\cdot, \cdot))$ .

**Proposition 3.2** *Let  $M^7$  be a compact 7-dimensional manifold. The following conditions are equivalent.*

- (i)  $M^7$  admits a topological  $SU(2)$ -structure.
- (ii)  $M^7$  admits a topological  $SU(3)$ -structure.
- (iii)  $M^7$  admits a topological  $G_2$ -structure.
- (iv)  $M^7$  admits a topological  $Spin(7)$ -structure.
- (v) The first and the second Stiefel - Whitney class of  $M^7$  vanish, i.e.  $\omega_1 = 0$  and  $\omega_2 = 0$

*Proof.* The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) and the equivalence (iv)  $\Leftrightarrow$  (v) are obvious. It remains to show that the existence of a topological  $Spin(7)$ -structure implies the existence of a topological  $SU(2)$  - structure. Let  $S$  be the real spinor bundle associated to the given  $Spin(7)$ -structure. Its dimension equals 8, the dimension of  $M^7$  equals 7. Thus, there exists section  $\psi$  of length 1 in  $\Gamma(S)$ . On the other hand, any 7-dimensional orientable compact manifold admits two linearly independent vector fields [26]. Denote these vector fields by  $X$  and  $Y$ . Then  $\psi, X\psi$  and  $Y\psi$  are spinor fields, which are linearly independent in any point of  $M^7$ . Thus  $M^7$  admits a triple  $(\psi_1, \psi_2, \psi_3)$  of spinor fields which are orthogonal in any point. The spinors  $\psi_i$  ( $i = 1, 2, 3$ ) are maps  $\psi_i : P_{Spin} \rightarrow \Delta_7$  satisfying  $\psi_i(pg) = g^{-1}\psi_i(p)$  for all  $g \in Spin(7)$ . Now we can define a  $SU(2)$ -structure on  $M^7$  by

$$P_{SU(2)} := \{p \in P_{Spin} \mid \psi_1(p) = {}^t(1, 0, \dots, 0), \psi_2(p) = {}^t(0, 1, 0, \dots, 0), \psi_3(p) = {}^t(0, 0, 1, 0, \dots, 0)\}.$$

■

Obviously the above mentioned map from the set of  $G_2$  - reductions of  $R(M^7)$  into the set of 3-forms is injective. Thus we obtain

**Proposition 3.3** *There is a one-to-one correspondence between the  $G_2$  - structures on  $M^7$  and the sections of  $\Lambda_+^3(M^7)$ .*

And, similarly

**Proposition 3.4** *There is a one-to-one correspondence between the  $G_2$ -structures on  $M^7$  and the 4-tupels  $(O, g, P_{Spin}, \psi)$ , where  $O$  is an orientation,  $g$  a metric,  $P_{Spin}$  a spin structure and  $\psi$  a spinor field of length 1 on  $M^7$ .*

Now we turn to geometrical  $G_2$ -structures.

**Definition 3.5** Let  $P_{G_2} \subset R(M^7)$  be a  $G_2$ -reduction and  $g$  the associated Riemannian metric. We denote by  $\nabla$  the Levi-Civita connection of  $g$ .  $P_{G_2}$  is said to be geometrical if one of the following equivalent conditions is satisfied.

- (i)  $\nabla$  reduces to  $P_{G_2}$ .
- (ii) The holonomy group  $Hol(M^7, g)$  of  $M^7$  is contained in  $G_2$ .
- (iii) The associated 3-form  $\omega^3$  is parallel, i.e.  $\nabla\omega^3 = 0$ .
- (iii) The associated spinor field  $\psi$  is parallel, i.e.  $\nabla\psi = 0$  where here  $\nabla$  is the induced covariant derivative on the spinor bundle  $S$ .

An immediate consequence is the following fact proved by E. Bonan in 1966 (see [5])

**Proposition 3.6** If  $g$  is the Riemannian metric of a geometrical  $G_2$ -structure on  $M^7$ , then  $(M^7, g)$  is Ricci-flat, i.e.  $Ric = 0$ .

*Proof.* Let  $\psi$  be the associated section of the spinor bundle  $S$  of  $M^7$ . Because of  $\nabla\psi = 0$  we obtain for the curvature tensor  $\mathfrak{R}^S$  of the induced connection  $\nabla$  on  $S$

$$\mathfrak{R}^S(X, Y)\psi = \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X, Y]}\psi = 0$$

for all vector fields  $X, Y$  on  $M^7$ . We recall that the Ricci tensor on  $M^7$  satisfies

$$Ric(X)\varphi = -2 \sum_{k=1}^7 s_k \mathfrak{R}^S(X, s_k)\varphi$$

for any vector field  $X$  and any spinor  $\varphi$  on  $M^7$ , where  $s_1, \dots, s_7$  is a local orthonormal frame (see [2]). Consequently,  $Ric(X)\psi = 0$  for all vector fields  $X$  and the assertion follows since  $\psi$  vanishes nowhere. ■

Now we can generalize the condition  $\nabla\psi = 0$  and obtain the notion of a nearly parallel  $G_2$ -structure.

**Definition 3.7** A topological  $G_2$ -structure on  $M^7$  is said to be nearly parallel if the associated spinor  $\psi$  is a Killing spinor, i.e. there exists a real number  $\lambda$  such that  $\psi$  satisfies the differential equation

$$\nabla_X \psi = \lambda X \psi$$

with respect to the Levi-Civita connection of the induced metric.

Differentiating the equation that defines the (2,1)-tensor  $A$  we obtain the following equivalent condition.

**Proposition 3.8** A topological  $G_2$ -structure on  $M^7$  is nearly parallel if and only if the associated tensor  $A$  satisfies

$$(\nabla_Z A)(Y, X) = 2\lambda\{g(Y, Z)X - g(X, Z)Y + A(Z, A(Y, X))\} \quad (5)$$

with respect to the Levi-Civita connection of the induced metric where  $\lambda$  is the same number as in Definition (3.7).

Now we translate this condition into a differential equation for the 3-form  $\omega^3$ .

**Proposition 3.9** *A topological  $G_2$ -structure on  $M^7$  is nearly parallel if and only if the associated 3-form  $\omega^3$  satisfies*

$$\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3)$$

with respect to the Levi-Civita connection of the induced metric where  $\lambda$  is the same number as in Definition (3.7).

*Proof.* The 3-form  $\omega^3$  is defined by  $\omega^3(X, Y, Z) = g(X, A(Y, Z))$ . Differentiating this equation we observe that the equation (5) is equivalent to

$$(\nabla_Z \omega^3)(W, Y, X) = 2\lambda g(Z, g(W, X)Y - g(W, Y)X - A(W, A(Y, X)))$$

for any vector field  $Z$ . For fixed  $Z$  the 3-form on the right hand side of this equation equals locally

$$\begin{aligned} 2\lambda \sum_{i < j < k} g(Z, g(s_i, s_k)s_j - g(s_i, s_j)e_k - A(s_i, A(s_j, s_k)))s_i \wedge s_j \wedge s_k = \\ = -2\lambda \sum_{i < j < k} g(Z, A(s_i, A(s_j, s_k)))s_i \wedge s_j \wedge s_k \end{aligned}$$

where  $s_1, \dots, s_7$  is a section of the  $G_2$ -structure on  $M^7$ . However, we obtain from

$$\begin{aligned} \omega^3 = & s_1 \wedge s_2 \wedge s_7 + s_1 \wedge s_3 \wedge s_5 - s_1 \wedge s_4 \wedge s_6 \\ & - s_2 \wedge s_3 \wedge s_6 - s_2 \wedge s_4 \wedge s_5 + s_3 \wedge s_4 \wedge s_7 + s_5 \wedge s_6 \wedge s_7. \end{aligned}$$

on one hand all  $A(s_i, A(s_j, s_k))$  and on the other hand  $*\omega^3$ . The assertion follows by comparing these terms. ■

In the same way as in the case of geometrical  $G_2$ -structures we prove

**Proposition 3.10** *If  $g$  is the Riemannian metric of a nearly parallel  $G_2$ -structure on  $M^7$ , then  $(M^7, g)$  is an Einstein space.*

*Proof.* The induced spinor  $\psi$  is a Killing spinor and we obtain from  $\nabla_X \psi = \lambda X \cdot \psi$

$$\mathfrak{R}^S(X, Y)\psi = \nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X, Y]}\psi = 2\lambda^2(Y \cdot X + g(X, Y)) \cdot \psi.$$

This yields for the Ricci tensor

$$Ric(X)\psi = -2 \sum_{k=1}^7 s_k \mathfrak{R}^S(X, s_k)\psi = -4\lambda^2 \sum_{k=1}^7 s_k(s_k X + g(X, s_k))\psi = 24\lambda^2 X\psi$$

Since  $\psi$  has no zeros,  $Ric(X) = 24\lambda^2 X$  and, therefore,  $(M^7, g)$  is an Einstein space of constant scalar curvature  $R = 7 \cdot 24\lambda^2$ . ■

Next we generalize the following theorem of Gray and Fernandez.

**Proposition 3.11** ([20], [21], [9]) *Let  $P_{G_2} \subset R(M^7)$  be a topological  $G_2$ -reduction,  $g$  its induced metric,  $\omega^3$  the induced 3-form and  $*$  the Hodge operator. Then the following conditions are equivalent.*

- (i)  $P_{G_2}$  is geometrical.
- (ii)  $\nabla\omega^3 = 0$
- (iii)  $d\omega^3 = 0, \quad d*\omega^3 = 0.$

We transfer the proof of this theorem given in ([9]) to the case of nearly parallel  $G_2$ -reductions and obtain

**Proposition 3.12** *Let  $P_{G_2} \subset R(M^7)$  be a topological  $G_2$ -reduction,  $g$  its induced metric,  $\omega^3$  the induced 3-form,  $\psi$  the induced spinor and  $*$  the Hodge operator. Then the following conditions are equivalent.*

- (i)  $P_{G_2}$  is nearly parallel, i.e. the spinor  $\psi$  satisfies  $\nabla_X\psi = \lambda X\psi.$
- (ii)  $\nabla_Z\omega^3 = -2\lambda(Z\lrcorner*\omega^3)$
- (iii)  $\delta\omega^3 = 0, \quad d\omega^3 = -8\lambda*\omega^3.$

*Proof.* The 3-form  $\omega^3$  defines the metric  $g$ . Let  $\Sigma_g \subset \Lambda_+^3(M^7)$  be the set of 3-forms that define this metric, too. The fibre of  $\Sigma_g$  equals the  $SO(7)$ -orbit of  $\omega^3$ , i.e.  $SO(7)/G_2$ . Its tangent space  $T(SO(7)/G_2)$  is  $G_2$ -invariant and 7-dimensional, therefore  $T(SO(7)/G_2) = S_{\omega^3} := \{X\lrcorner*\omega^3 \mid X \in TM^7\}$ . Since  $\omega^3$  is a section in  $\Sigma_g$  and  $\nabla$  is a covariant derivative in  $\Sigma_g$ , the covariant derivative  $\nabla\omega^3$  is a section of  $T^*M^7 \otimes S_{\omega^3}$ . We consider now the projection  $p_1$  defined by

$$p_1 : T^*M^7 \otimes S_{\omega^3} \ni \alpha^1 \otimes \alpha^3 \longmapsto \alpha^1 \wedge \alpha^3 \in \Lambda^4$$

and the contraction

$$p_2 : T^*M^7 \otimes S_{\omega^3} \longrightarrow \Lambda^2.$$

By comparing the decomposition of  $T^*M^7 \otimes S_{\omega^3}$  and  $\Lambda^4 \oplus \Lambda^2$  into irreducible  $G_2$ -subspaces we see that the sum of  $p_1$  and  $p_2$

$$p_1 \oplus p_2 : T^*M^7 \otimes S_{\omega^3} \longrightarrow \Lambda^4 \oplus \Lambda^2$$

is injective. Consequently,  $\nabla_X\psi = \lambda X\psi$  is equivalent to

$$p_1(\nabla\omega^3) = -2\lambda p_1(\cdot\lrcorner*\omega^3) = -2\lambda \sum_{i=1}^7 s_i \wedge s_{i\lrcorner}*\omega^3 = -8\lambda*\omega^3$$

$$p_2(\nabla\omega^3) = -2\lambda p_2(\cdot\lrcorner*\omega^3) = -2\lambda \sum_{i=1}^7 *\omega^3(s_i, s_i, \dots) = 0$$

The assertion now follows from  $p_1(\nabla\omega^3) = d\omega^3, \quad p_2(\nabla\omega^3) = \delta\omega^3.$

**Remark 3.13** *There is the following difference between the cases  $\lambda = 0$  and  $\lambda \neq 0$ . We proved that a  $G_2$ -structure is nearly parallel if and only if for the induced 3-form  $\omega^3$  the equations  $\delta\omega^3 = 0, \quad d\omega^3 = -8\lambda*\omega^3$  hold. In case  $\lambda = 0$ , the resulting equations  $d\omega^3 = 0$  and  $\delta\omega^3 = 0$  are independent. In case  $\lambda \neq 0$ , the condition  $d\omega^3 = -8\lambda*\omega^3$  implies  $\delta\omega^3 = 0.$*

## 4 Nearly Parallel $G_2$ -Structures, Killing Spinors and Contact Geometry.

We summarize now several results on nearly parallel  $G_2$ -structures. A general reference is the book [2]. In particular, we derive necessary geometric conditions for the underlying Riemannian metric and we introduce three types of nearly parallel  $G_2$ -structures depending on the number of Killing spinors. Finally we discuss the compact examples of each type known up to now.

Let  $(M^7, g)$  be a compact Riemannian spin manifold with a Killing spinor  $\psi$ ,

$$\nabla_X \psi = \lambda X \cdot \psi,$$

and denote by  $\omega^3$  the corresponding 3-form satisfying the differential equation

$$d\omega^3 = -8\lambda * \omega^3.$$

Then  $M^7$  is an Einstein manifold of positive scalar curvature  $R = 4 \cdot 7 \cdot 6 \cdot \lambda^2 = 168\lambda^2$  and, consequently, the fundamental group  $\pi_1(M)$  is finite. In case  $\lambda \neq 0$  the Riemannian manifold  $(M^7, g)$  is locally irreducible and not locally symmetric except if it has constant sectional curvature (see [2]). Using the associated nearly parallel  $G_2$ -structure we decompose the bundles of forms  $\Lambda^p(M^7)$  into the irreducible components mentioned above. The curvature tensor

$$\mathfrak{R} : \Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2 \quad \rightarrow \quad \Lambda_7^2 \oplus \Lambda_{14}^2 = \Lambda^2$$

splits into the scalar curvature and the Weyl tensor  $W$ :

$$\mathfrak{R} = W - \frac{R}{42}.$$

The Weyl tensor satisfies several algebraic equations. They can be formulated in the following way. For any 2-form  $\omega^2 \in \Lambda^2$  the Clifford product  $W(\omega^2) \cdot \psi$  vanishes, i.e.

$$W(\omega^2) \cdot \psi = 0$$

holds (see [2]). Since  $\Lambda_{14}^2$  is the Lie algebra of the group  $G_2$ , being the isotropy group of the spinor  $\psi$ , we conclude that the Weyl tensor has the form

$$W = \begin{pmatrix} 0 & 0 \\ 0 & W_{14} \end{pmatrix},$$

where  $W_{14} : \Lambda_{14}^2 \rightarrow \Lambda_{14}^2$  is a symmetric endomorphism. In case  $W_{14} \neq 0$ , the holonomy representation  $\text{Hol}^0 \rightarrow SO(7)$  is irreducible and we can apply Berger's Holonomy Theorem. Since  $\dim(M^7) = 7$ , there are two possibilities:  $\text{Hol}^0 = G_2$  or  $\text{Hol}^0 = SO(7)$ . The case of  $\text{Hol}^0 = G_2$  cannot occur since  $M^7$  is an Einstein space with positive scalar curvature ( $\lambda \neq 0$ ). Consequently, the Riemannian manifold  $M^7$  is - at least from the point of view of holonomy theory - of general type:  $\text{Hol}^0 = SO(7)$ . Since the fundamental group of  $M^7$  is finite we can without loss of generality assume that  $M^7$  is simply-connected,  $\pi_1(M^7) = 0$ . Furthermore, we exclude the case of the space of constant curvature, i.e.  $M^7 \neq S^7$ . Denote by  $KS(M^7, g)$  the space of all Killing spinors,

$$KS(M^7, g) = \{\psi \in \Gamma(S) : \nabla_X \psi = \lambda X \cdot \psi \text{ for all vectors } X \in T(M^7)\}$$

The dimension of  $KS(M^7, g)$  is bounded by three,  $\dim[KS(M^7, g)] \leq 3$  (see [2]). The nearly parallel  $G_2$ -structures split into three different types:

**nearly parallel  $G_2$ -structures of type 1:**  $\dim[KS] = 1.$  ( proper  $G_2$ -structures)

**nearly parallel  $G_2$ -structures of type 2:**  $\dim[KS] = 2.$

**nearly parallel  $G_2$ -structures of type 3:**  $\dim[KS] = 3.$

The nearly parallel  $G_2$ -structures of type 2 and 3 are described using the language of contact geometry. In fact, Th. Friedrich and I. Kath observed that a simply-connected 7-dimensional Riemannian spin manifold with scalar curvature  $R = 42$  admits at least

two Killing spinors  $\iff M^7$  is an Einstein-Sasakian manifold

three Killing spinors  $\iff M^7$  is a 3-Sasakian manifold

(see [16]; for the definition of a Sasakian manifold also see next section). For the  $G_2$ -structures of type 1 we also use the notion of a *proper  $G_2$ -structure*.

Examples of nearly parallel  $G_2$ -structures of type three (i.e. 3-Sasakian manifolds) are known. We have the sphere  $S^7$ , the space  $N(1, 1) = SU(3)/S^1$  and these are the only regular 3-Sasakian manifolds in dimension seven (see [16]). During the last years Boyer / Galicki / Mann obtained non-regular examples  $S(p_1, p_2, p_3)$ , (see [6], [7]). Up to now, strong topological conditions for a compact 7-dimensional manifold  $M^7$  in order to admit a 3-Sasakian structure are not known. For example, it seems to be an open question whether the manifold  $S^2 \times S^5$  possesses such a structure or not! This special question is interesting since a 7-dimensional manifold with 3-Sasakian structure and being the product of two lower-dimensional manifolds must be diffeomorphic to  $S^2 \times S^5$ .

### Examples of nearly parallel $G_2$ -structures of type 3 (3-Sasakian manifolds)

$M^7$	$\text{Iso}_o(M^7)$	$\dim[\text{Iso}]$
$N(1, 1)$	$SU(3) \times SU(2)$	11
$S(p_1, p_2, p_3)$	depends on $p_i$	$< 8$

Nearly parallel  $G_2$ -structures of type two (i.e. Einstein-Sasakian manifolds) can be obtained as principal  $S^1$ -bundles over 6-dimensional Kähler-Einstein manifolds with positive scalar curvature. Indeed, let  $X^6$  be a Kähler-Einstein manifold with positive scalar curvature and denote by  $c_1(X^6)$  its first Chern class. Let  $A > 0$  be the largest integer such that  $c_1(X^6)/A$  is an integral cohomology class. Consider the principal  $S^1$ -bundle  $S^1 \rightarrow M^7 \rightarrow X^6$  with Chern class

$c_1^* = c_1(X^6)/A$ . Then  $M^7$  is simply-connected and admits an Einstein-Sasakian structure. Using the described construction we obtain the following regular Einstein-Sasakian manifolds:

**Examples of nearly parallel  $G_2$ -structures of type 2 (Einstein-Sasakian manifolds)**

$X^6$	$M^7$	$\text{Iso}_o(M^7)$	$\text{dim}[\text{Iso}]$
$F(1, 2)$	$N(1, 1)$	$SU(3) \times SU(2)$	11
$S^2 \times S^2 \times S^2$	$Q(1, 1, 1)$	$SU(2) \times SU(2) \times SU(2) \times U(1)$	10
$\mathbb{C}P^2 \times S^2$	$M(3, 2)$	$SU(3) \times SU(2) \times U(1)$	12
$G_{5,2}$	$V_{5,2}$	$SO(5) \times U(1)$	11
$P_k \times S^2$	$M_k^7 (3 \leq k \leq 8)$	$SO(3) \times U(1)$	4

where  $P_k$  ( $3 \leq k \leq 8$ ) denotes one of the del Pezzo surfaces with a Kähler-Einstein metric of positive scalar curvature. The spaces  $N(1, 1)$ ,  $Q(1, 1, 1)$ ,  $M(3, 2)$  and the Stiefel manifold  $V_{5,2}$  are homogeneous spaces together with some invariant Einstein metric. The table contains also the isometry group of the Einstein-Sasakian manifold  $M^7$  as well as its dimension (see [11]).

There are three examples of nearly parallel  $G_2$ -structures of type 1, i.e. proper  $G_2$ -structures. The first example is the so-called squashed 7-sphere. Indeed, the standard sphere  $(S^7, g_{can})$  is a Riemannian submersion over the projective space  $\mathbb{H}P^1$  with fibre  $S^3$ . Scaling the canonical metric in the fibre  $S^3$ , there exists a second scaling factor such that the metric  $g_1$  on  $S^7$  is an Einstein metric. It turns out that  $(S^7, g_1)$  admits exactly one Killing spinor. The second example is the homogeneous space  $N(k, l) = SU(3)/S_{k,l}^1$  where the embedding of the group  $S^1 = U(1)$  into  $SU(3)$  is given by

$$S^1 \ni z \mapsto \text{diag}(z^k, z^l, z^{-(k+l)}) \in SU(3).$$

These spaces have two homogeneous Einstein metrics. In case  $(k, l) = (1, 1)$  one of these Einstein metrics is the 3-Sasakian structure mentioned above and the second Einstein metric admits one Killing spinor. In case  $(k, l) \neq (1, 1)$ , there exists only one Killing spinor for each of these two metrics, i.e. the nearly parallel  $G_2$ -structure is of type 1 (a proper  $G_2$ -structure). The third example is a special Riemannian metric on  $SO(5)/SO(3)$  with one Killing spinor (see [8]). The isotropy representation of this space is the unique 7-dimension irreducible representation of the group  $SO(3) \rightarrow G_2 \subset SO(7)$ .

## Examples of nearly parallel $G_2$ -structures of type 1

$M^7$	$\text{Iso}_o(M^7)$	$\dim[\text{Iso}]$
$(S^7, g_{squas})$	$Sp(2) \times Sp(1)$	13
$N(k, l), (k, l) \neq (1, 1)$	$SU(3) \times U(1)$	9
$SO(5)/SO(3)$	$SO(5)$	10

### Remark 4.1

As we mentioned before, strong topological obstructions for the existence of a 3-Sasakian metric on a compact 7-dimensional spin manifold are not known (very recently, the obstruction  $b_3(M^7) = 0$  was found, see [17]). The same situation happens in case of an Einstein-Sasakian metric with positive scalar curvature. This gives rise to the following question:

Do there exist compact, simply-connected spin manifolds  $M^7$  with a nearly parallel  $G_2$ -structure of type 1 (resp. 2) which cannot admit - for example for topological reasons - any Einstein-Sasakian (resp. 3-Sasakian) metric at all?

## 5 New Examples.

In this section we construct new examples of nearly parallel  $G_2$ -structures and show that they are of type 1, i.e. they are proper  $G_2$ -structures. Let us recall the definition of a Sasakian structure.

**Definition 5.1** *A vector field  $V$  on a Riemannian manifold  $(M, g)$  is called a Sasakian structure if the following conditions are satisfied:*

1.  $V$  is a Killing vector field of unit length;
2. The  $(1,1)$ -tensor  $\varphi$  defined by  $\varphi = -\nabla V$  is an almost complex structure on the distribution orthogonal to  $V$  ( $\varphi^2 = -1$  and  $\varphi = -\varphi^*$  on  $V^\perp$ );
3.  $(\nabla_X \varphi)Y = g(X, Y)V - g(V, Y)X$ , for all vectors  $X, Y$ .

**Definition 5.2** *A triple  $(V_1, V_2, V_3)$  is called a 3-Sasakian structure on  $M$  if the following conditions are satisfied:*

1. The vector  $V_i$  defines a Sasakian structure for each  $i = 1, 2, 3$ ;
2. The frame  $(V_1, V_2, V_3)$  is orthonormal;
3. For each permutation  $(i, j, k)$  of signature  $\delta$ , we have  $\nabla_{V_i} V_j = (-1)^\delta V_k$ ;
4. On the distribution orthogonal to  $(V_1, V_2, V_3)$ , the tensors  $\varphi_i = -\nabla V_i$  satisfy  $\varphi_i \varphi_j = (-1)^\delta \varphi_k$ .

Consider a Riemannian manifold  $M^7$  of dimension 7, admitting a 3-Sasakian structure. A vector is called *horizontal* if it is orthogonal to each  $V_i$  and *vertical* if it is a linear combination of  $V_i$ . Define, for  $s > 0$ , the metric  $g^s$  on  $M^7$  by

$$g^s(X, Y) = g(X, Y) \text{ if } X \text{ (or } Y) \text{ is horizontal, and } g^s(V, W) = s^2 g(V, W) \text{ for vertical } V, W.$$

A straightforward computation gives the following



**Lemma 5.3** *The manifold  $(M^7, g^s)$  is Einstein if and only if  $s = 1$  or  $s = \frac{1}{\sqrt{5}}$ .*

The 3-Sasakian manifold  $(M^7, g^1)$  admits, by definition, a nearly parallel  $G_2$ -structure of type 3. On the other hand, by Proposition 3.10, every nearly parallel  $G_2$ -structure on  $M$  defines an Einstein metric. Hence, the manifold  $(M^7, g^s)$  with  $s = \frac{1}{\sqrt{5}}$  is a natural candidate for a nearly parallel  $G_2$ -structure. Indeed, we have

**Theorem 5.4** *The manifold  $(M^7, g^s)$  admits a nearly parallel  $G_2$ -structure for  $s = \frac{1}{\sqrt{5}}$ .*

*Proof.* Fix  $s > 0$  and a local orthonormal frame  $X_1, \dots, X_4$  of the horizontal distribution. Let  $Z_a$  ( $a = 1, 2, 3$ ) be the vector  $Z_a := V_a/s$  and denote by  $\nabla$  the Levi-Civita connection of the metric  $g = g^1$ . We define a 3-form  $\omega$  by

$$\omega := F_1 + F_2$$

where  $F_1 := Z_1 \wedge Z_2 \wedge Z_3$ ,  $F_2 := \sum_a Z_a \wedge \omega_a$  and  $\omega_a := \frac{1}{2} \sum_i X_i \wedge \nabla_{X_i} V_a$ .

The 3-form  $\omega$  is clearly in  $\Lambda_+^3(M^7)$ . Denote by  $*$  the Hodge operator with respect to the metric  $g^s$ . Then we calculate the forms  $*F_1$  and  $*F_2$ :

$$6 * F_1 = \sum_a \omega_a \wedge \omega_a, \quad *F_2 = Z_1 \wedge Z_2 \wedge \omega_3 + Z_2 \wedge Z_3 \wedge \omega_1 + Z_3 \wedge Z_1 \wedge \omega_2.$$

A straightforward computation yields the formulas

$$dZ_1 = 2s\omega_1 - \frac{2}{s}Z_2 \wedge Z_3, \quad dZ_2 = 2s\omega_2 - \frac{2}{s}Z_3 \wedge Z_1, \quad dZ_3 = 2s\omega_3 - \frac{2}{s}Z_1 \wedge Z_2$$

and  $dF_1 = d(Z_1 \wedge Z_2 \wedge Z_3) = 2s(*F_3)$ .

Now we compute

$$d\omega_1 = d\left(\frac{1}{2s}dZ_1 + \frac{1}{s^2}Z_2 \wedge Z_3\right) = \frac{1}{s^2}(dZ_2 \wedge Z_3 - Z_2 \wedge dZ_3) = \frac{2}{s}(\omega_2 \wedge Z_3 - \omega_3 \wedge Z_2)$$

$$d\omega_2 = \frac{2}{s}(\omega_3 \wedge Z_1 - \omega_1 \wedge Z_3), \quad d\omega_3 = \frac{2}{s}(\omega_1 \wedge Z_2 - \omega_2 \wedge Z_1)$$

and  $dF_2 = \sum_a dZ_a \wedge \omega_a - \sum_a Z_a \wedge d\omega_a = 12s(*F_1) + \frac{2}{s}(*F_2)$ .

Finally we obtain  $d\omega = d(F_1 + F_2) = 12s(*F_1) + (2s + \frac{2}{s})(*F_2)$ . So  $d\omega$  is a scalar multiple of  $*\omega$  if and only if  $12s = 2s + \frac{2}{s}$ .

■

As remarked in Section 4, the only known examples of proper nearly parallel  $G_2$ -structures - up to now - are the squashed 7-sphere, the Wallach spaces  $N(k, l)$  and an Einstein metric on  $SO(5)/SO(3)$  related to the irreducible representation  $SO(3) \rightarrow G_2 \rightarrow SO(7)$ . The importance of Theorem 5.4 can thus be seen in the light of the following result:

**Theorem 5.5** *The nearly parallel  $G_2$ -structures constructed in Theorem 5.4 are proper.*

*Proof:* Suppose that a constant multiple  $k$  of the metric  $g^s$  on  $M^7$  admits an Einstein Sasakian structure given by the Killing vector field  $\xi$ . Denote by  $\mathfrak{R}$  the curvature tensor of  $(M^7, g)$  and by  $\mathfrak{R}^0$  the curvature tensor of  $(M^7, kg^s)$ . Then we obtain from Lemma 4 of [2], page 78:

$$g^s(\mathfrak{R}^0(X, Y)\xi, V_a) = k[g^s(Y, \xi)g^s(X, V_a) - g^s(X, \xi)g^s(Y, V_a)]. \quad (*)$$

Choosing  $X$  and  $Y$  horizontal we obtain

$$g^s(\mathfrak{R}^0(X, Y)\xi, V_a) = 0.$$

On the other hand, comparing the Levi-Civita connection  $\nabla$  of the metric  $g$  with the Levi-Civita connection  $\nabla^0$  of the metric  $g^s$  we calculate

$$\mathfrak{R}^0(X, Y)V_a = s^2\mathfrak{R}(X, Y)V_a + (s^2 - 1)\nabla_{[X, Y]^\vee}V_a = (s^2 - 1)\nabla_{[X, Y]^\vee}V_a, .$$

Here we applied the same lemma for  $V_a$  as Sasakian structure on  $(M^7, g)$ . Consequently,  $\xi$  is perpendicular to all vectors of the form  $\nabla_{[X, Y]^\vee}V_a$ . It is easy to see that the set of all these vectors is just the vertical distribution, so  $\xi$  is horizontal.

Next, taking  $X = V_1, a = 2$  and  $Y$  horizontal in the equation (\*), one obtains

$$g^s(\mathfrak{R}^0(V_1, Y)\xi, V_2) = 0. \quad (**)$$

The vector  $[Y, V_1]$  is a horizontal one and we can calculate

$$\begin{aligned} \mathfrak{R}^0(V_1, Y)V_2 &= \nabla_{V_1}^0 \nabla_Y^0 V_2 - \nabla_Y^0 \nabla_{V_1}^0 V_2 - \nabla_{[V_1, Y]}^0 V_2 = \\ &= s^2(\nabla_{V_1}^0 \nabla_Y V_2 - \nabla_{V_1} \nabla_Y V_2) + s^2\mathfrak{R}(V_1, Y)V_2 = s^2(\nabla_{V_1}^0 \nabla_Y V_2 - \nabla_{V_1} \nabla_Y V_2) \end{aligned}$$

by similar arguments. Now  $\nabla_Y V_2$  runs through all horizontal vector fields when  $Y$  is horizontal. Together with (\*\*) we obtain that  $\xi$  is perpendicular to all vectors of the form  $\nabla_{V_1}^0 Z - \nabla_{V_1} Z$ . The relation

$$\nabla_{V_1}^0 Z - \nabla_{V_1} Z = (\nabla_{V_1}^0 Z - [Z, V_1]) - (\nabla_{V_1} Z - [Z, V_1]) = (s^2 - 1)\nabla_Z V_1$$

shows that  $\xi$  is also perpendicular to all horizontal vectors, a contradiction. ■

Our new examples of nearly parallel  $G_2$ -structures are all proper. The recent work of C. Boyer, K. Galicki, B. Mann [7] provides a multitude of new examples of strongly inhomogeneous 7-manifolds admitting a 3-Sasakian structure. By our previous theorems, they generate the first examples of strongly inhomogeneous proper nearly parallel  $G_2$ -structures. However, these examples arise from a deformation of the 3-Sasakian structure and therefore they live on manifolds with 3-Sasakian metric.

As K. Galicki pointed out to us, he also proved the result of Theorem 5.4 in a joint paper with S. Salamon (in preparation).

## 6 The Automorphism Group of a Nearly Parallel $G_2$ -Structure.

We consider a compact, 7-dimensional manifold  $M^7$  with a nearly parallel  $G_2$ -structure and denote by  $\omega^3$  its 3-form. Then we have the differential equations

$$\nabla_X \omega^3 = -2\lambda(X \lrcorner * \omega^3), \quad d\omega^3 = -8\lambda * \omega^3, \quad \lambda \neq 0.$$

Let  $X$  be a vector field preserving the 3-form, i.e.

$$L_X \omega^3 = d(X \lrcorner \omega^3) + X \lrcorner d\omega^3 = d(X \lrcorner \omega^3) - 8\lambda(X \lrcorner * \omega^3) = 0.$$

In particular,  $X$  is a Killing vector field of the Riemannian metric  $g$  and  $\nabla X \in \Gamma(T \otimes T)$  is anti-symmetric and coincides - up to a multiple - with the exterior derivative of the 1-form  $X$ :

$$\nabla X = \frac{1}{2}dX.$$

We now calculate the form  $d(X \lrcorner \omega^3)$  using the differential equation for  $\omega^3$ :

$$\begin{aligned} d(X \lrcorner \omega^3)(\alpha, \beta, \gamma) &= (\nabla_\alpha \omega^3)(X, \beta, \gamma) - (\nabla_\beta \omega^3)(X, \alpha, \gamma) + (\nabla_\gamma \omega^3)(X, \alpha, \beta) + \\ &\quad + \omega^3(\nabla_\alpha X, \beta, \gamma) - \omega^3(\nabla_\beta X, \alpha, \gamma) + \omega^3(\nabla_\gamma X, \alpha, \beta) = \\ &= 6\lambda(*\omega^3)(X, \alpha, \beta, \gamma) + \omega^3(\nabla_\alpha X, \beta, \gamma) - \omega^3(\nabla_\beta X, \alpha, \gamma) + \omega^3(\nabla_\gamma X, \alpha, \beta). \end{aligned}$$

The equation  $d(X \lrcorner \omega^3) - 8\lambda(X \lrcorner * \omega^3) = 0$  becomes:

$$\begin{aligned} 2\lambda(X \lrcorner * \omega^3)(\alpha, \beta, \gamma) &= \omega^3(\nabla_\alpha X, \beta, \gamma) - \omega^3(\nabla_\beta X, \alpha, \gamma) + \omega^3(\nabla_\gamma X, \alpha, \beta) = \\ &= \frac{1}{2}\{\omega^3(\alpha \lrcorner dX, \beta, \gamma) - \omega^3(\beta \lrcorner dX, \alpha, \gamma) + \omega^3(\gamma \lrcorner dX, \alpha, \beta)\}. \end{aligned}$$

We apply now the following easy algebraic observation:

**Lemma 6.1** *Let  $\eta^2$  be a 2-form and denote by  $\pi_7(\eta^2)$  its  $\Lambda_7^2$ -component with respect to the decomposition  $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2$ . Suppose that  $\pi_7(\eta^2)$  is given by a vector  $Z$ , i.e.  $\pi_7(\eta^2) = Z \lrcorner \omega^3$ . Then*

$$\omega^3(\alpha \lrcorner \eta^2, \beta, \gamma) - \omega^3(\beta \lrcorner \eta^2, \alpha, \gamma) + \omega^3(\gamma \lrcorner \eta^2, \alpha, \beta) = 3(Z \lrcorner * \omega^3)(\alpha, \beta, \gamma).$$

The condition that the vector field  $X$  preserves the 3-form becomes equivalent to

$$2\lambda(X \lrcorner * \omega^3) = \frac{1}{2} \cdot 3(Z \lrcorner * \omega^3)$$

where  $\pi_7(dX) = Z \lrcorner \omega^3$ . This implies  $Z = \frac{4}{3} \cdot \lambda \cdot X$  and consequently we have proved

**Theorem 6.2** *A Killing vector field  $X$  preserves a nearly parallel  $G_2$ -structure  $\omega^3$  if and only if*

$$\pi_7(dX) = \frac{4}{3} \cdot \lambda \cdot (X \lrcorner \omega^3).$$

We now use Stokes Theorem as well as the identities given in the Propositions 2.4 and 2.5 in order to obtain the following relation between the  $L^2$ -norms  $|\cdot|$  of  $X$  and the  $\Lambda_{14}^2$ -part  $\pi_{14}(dX)$  of  $dX$ :

**Theorem 6.3** *Let  $X$  be a Killing vector field preserving a nearly parallel  $G_2$ -structure  $\omega^3$  on a closed manifold  $M^7$ . Then*

$$\frac{128}{3}\lambda^2|X|^2 = |\pi_{14}(dX)|^2.$$

*Proof:* We start with the algebraic identity  $dX \wedge dX \wedge \omega^3 = 2|\pi_7(dX)|^2 - |\pi_{14}(dX)|^2$  which is valid for any 2-form  $\eta(=dX)$ . By Stokes Theorem and Propositions 2.4 and 2.5 we obtain

$$\begin{aligned} \int dX \wedge dX \wedge \omega^3 &= \int X \wedge dX \wedge d\omega^3 = -8\lambda \int X \wedge dX \wedge (*\omega^3) \\ &= -8\lambda \int X \wedge \pi_7(dX) \wedge (*\omega^3) = -\frac{32}{3}\lambda^2 \int X \wedge (X \lrcorner \omega^3) \wedge (*\omega^3) \\ &= -32\lambda^2 \int X \wedge (*X) = -32\lambda^2|X|^2. \end{aligned}$$

Therefore we get

$$2|\pi_7(dX)|^2 - |\pi_{14}(dX)|^2 = -32\lambda^2|X|^2.$$

Using the equation,  $\pi_7(dX) = \frac{4}{3} \cdot \lambda \cdot (X \lrcorner \omega^3)$ , we have

$$|\pi_7(dX)|^2 = \frac{16}{3}\lambda^2|X|^2,$$

and the formula follows immediately. ■

Consider a component  $\Sigma \subset M^7$  of the zero set of  $X$ . Since  $X$  is a Killing vector field,  $\Sigma$  is a totally geodesic submanifold of even codimension. Suppose that  $\dim[\Sigma] = 5$ . Then at any point of  $\Sigma$  we obtain that  $0 \neq dX \in \Lambda_{14}^2$  has rank 2 ( $\pi_7(dX) = 0!$ ). This implies  $dX \wedge dX = 0$ . On the other hand, since  $dX \in \Lambda_{14}^2$  we have  $dX \wedge dX \wedge \omega^3 = -|dX|^2$  (see the definition of the space  $\Lambda_{14}^2$ ), a contradiction. This yields the

**Corollary 1** *Any connected component of the zero set of a Killing vector field  $X$  preserving a nearly parallel  $G_2$ -structure  $\omega^3$  has dimension one or three.*

We investigate now the geometry of the 3-dimensional components of the zero set  $\Sigma$ .

**Theorem 6.4** *Let  $\Sigma^3 \subset M^7$  be a three-dimensional component of the zero set of a Killing vector field preserving a nearly parallel  $G_2$ -structure. Then*

- (i) *the tangent spaces  $T(\Sigma^3) \subset T(M^7)$  are  $G_2$ -special, i.e. the restriction of  $\omega^3$  to  $\Sigma^3$  is the volume form of  $\Sigma^3$ .*
- (ii)  *$\Sigma^3$  is a space form of positive sectional curvature  $K = \frac{R}{42}$ .*

*Proof:* The equation  $X \lrcorner d\omega^3 + d(X \lrcorner \omega^3) = 0$  yields at any point  $m \in \Sigma^3$  and for any three vectors  $\alpha, \beta, \gamma \in T_m(M^7)$  the relation

$$0 = d(X \lrcorner \omega^3)(\alpha, \beta, \gamma) = \omega^3(\nabla_\alpha X, \beta, \gamma) - \omega^3(\nabla_\beta X, \alpha, \gamma) + \omega^3(\nabla_\gamma X, \alpha, \beta).$$

Let  $e_1, e_2, \dots, e_7$  be a local orthonormal frame in the  $G_2$ -bundle such that  $e_1(m), e_2(m)$  belong to the tangent space  $T_m(\Sigma^3)$ . There exists a frame with the required property since the group  $G_2$  acts transitively on the Grassmannian manifold  $G_2(R^7)$ . With respect to

$$\nabla_{e_1} X = \nabla_{e_2} X = 0$$

we obtain ( $\beta = e_1, \gamma = e_2$ )  $\omega^3(\nabla_\alpha X, e_1, e_2) = 0$ . The vectors  $\nabla_\alpha X$ ,  $\alpha \in T_m(M^7)$ , generate the normal space of  $T_m(\Sigma^3)$  and, therefore, the latter equation means that the subspace  $T_m(\Sigma^3) \subset T_m(M^7)$  is of special  $G_2$ -type (see Proposition 2.6). In particular, the vector  $e_7$  is the third vector tangent to  $\Sigma^3$  at the point  $m \in \Sigma^3$ . The 2-forms

$$e_2 \wedge e_7 + e_3 \wedge e_5 - e_4 \wedge e_6, \quad e_1 \wedge e_7 + e_3 \wedge e_6 + e_4 \wedge e_5, \quad e_1 \wedge e_2 + e_3 \wedge e_4 + e_5 \wedge e_6$$

are elements of  $\Lambda_7^2$ . The curvature tensor  $\mathfrak{R}$  of  $M^7$  acts on forms of the type  $\Lambda_7^2$  by the scalar multiplication by  $-\frac{R}{42}$  (see Section 4). This implies

$$\mathfrak{R}(e_2 \wedge e_7 + e_3 \wedge e_5 - e_4 \wedge e_6) = -\frac{R}{42}(e_2 \wedge e_7 + e_3 \wedge e_5 - e_4 \wedge e_6)$$

and, finally,  $R_{2772} = \frac{R}{42}$  because  $\Sigma^3$  is a totally geodesic submanifold (i.e.  $R_{3527} = R_{4627} = 0$  for example). Similarly we obtain  $R_{1771} = R_{1221} = \frac{R}{42}$  and, hence,  $\Sigma^3$  is a space form of positive sectional curvature  $K = \frac{R}{42}$ .

■

Let  $H \subset G = \text{Aut}(M^7, \omega^3)$  be a subgroup of the connected component  $G$  of the automorphism group of a nearly parallel  $G_2$ -structure ( $\lambda \neq 0$ ) and suppose that for some point  $m^* \in M^7$  the  $H$ -orbit  $N^4 = H \cdot m^*$  is a four-dimensional submanifold. Then  $(*\omega^3)$  is an  $H$ -invariant 4-form on  $N^4$ , i.e. a constant multiple of the volume form of  $N^4$ . On the other hand, we have

$$(-8\lambda) \int_{N^4} (*\omega^3) = \int_{N^4} d\omega^3 = 0$$

and, thus,  $*\omega^3$  vanishes on  $N^4$ . This implies that  $\omega^3$  vanishes on the normal bundle  $T^\perp(N^4)$ . Using Proposition 2.6 we obtain a local frame  $e_1, e_2, \dots, e_7$  in the  $G_2$ -bundle over  $N^4$  such that  $e_4, e_5, e_6, e_7$  Span the tangent space  $T(N^4)$  of  $N^4$ . Moreover, on  $N^4$  the formula

$$\omega^3|_{N^4} = e_5 \wedge e_6 \wedge e_7$$

holds, i.e.  $\omega^3|_{N^4}$  is a 3-form on  $N^4$  of length one. Denote by  $\xi$  the tangent vector field on  $N^4$  corresponding to this 3-form under the Hodge operator of  $N^4$  ( $\xi = e_4$ ). Then we have

$$\xi \lrcorner \omega^3 = 0, \quad dN^4 = \xi \wedge \omega^3.$$

The 3-form  $\omega^3$  is invariant under the flow of the vector field  $\xi$  on  $N^4$ :

$$L_\xi(\omega^3) = \xi \lrcorner d\omega^3 + d(\xi \lrcorner \omega^3) = -8\lambda(\xi \lrcorner * \omega^3) + 0 = 0.$$

We summarize the result in the following

**Theorem 6.5** *Let  $N^4 = H \cdot m^*$  be a four-dimensional orbit,  $H \subset G = \text{Aut}(M^7, \omega^3)$ . Then the restriction of  $\omega^3$  to  $N^4$  is a 3-form on  $N^4$  with length one. Moreover, there exists a vector field  $\xi$  such that*

$$(i) \quad \xi \lrcorner \omega^3 = 0, \quad dN^4 = \xi \wedge \omega^3;$$

$$(ii) \quad L_\xi(\omega^3) = 0.$$

*In particular, the Euler characteristic  $\chi(N^4)$  of  $N^4$  vanishes,  $\chi(N^4) = 0$ .*

**Corollary 2** *The isotropy representation  $G(m) \rightarrow GL(T_m(M^7))$  at any point  $m \in N^4$  of a four-dimensional orbit  $N^4$  decomposes into a 1-dimensional and two 3-dimensional representations.*

Denote by  $G = \text{Aut}(M^7, \omega^3)$  the connected component of the automorphism group of the nearly parallel  $G_2$ -manifold. The isotropy subgroup  $G(m)$  of any point  $m \in M^7$  is a subgroup of  $G_2$ . Thus, we obtain

$$\dim(G) - \dim(G(m)) \leq 7, \quad G(m) \subset G_2.$$

**Theorem 6.6** *Let  $(M^7, \omega^3)$  be a simply-connected, compact manifold with nearly parallel  $G_2$ -structure not isometric to the sphere  $S^7$ . Then the automorphism group  $G$  has dimension  $\leq 13$ .*

*Proof:* First we discuss the case of  $15 \leq \dim(G)$ . Then the isotropy subgroup  $G(m)$  is a subgroup of  $G_2$  with  $8 \leq \dim(G(m))$ . However, the group  $G_2$  contains only two subgroups satisfying this condition, namely  $G(m) = SU(3)$  and  $G_2$  (see [12]). If  $G(m) = G_2$  for any point  $m \in M^7$ , the Weyl tensor vanishes identically and the space  $M^7$  is the sphere  $S^7$ . Suppose that there exists a point  $m \in M^7$  such that  $G(m) = SU(3)$ . Then the group  $G$  acts transitively on  $M^7$ . Moreover,  $G$  is a simply-connected, compact group of dimension 15 containing a subgroup isomorphic to  $SU(3)$ . The classification of compact groups yields that there exists only one group with these properties, namely  $G = SU(4)$ . The Riemannian metric on  $M^7$  is given by an  $SU(3)$ -invariant scalar product of  $\mathbb{R}^7 = \mathbb{C}^3 \oplus \mathbb{R}^1$ . The family of  $SU(3)$ -invariant scalar products depends on one positive parameter, but only the usual scalar product in  $\mathbb{R}^7$  defines an Einstein metric on the homogeneous space  $M^7 = SU(4)/SU(3)$ . Consequently,  $M^7$  is isometric to the sphere  $S^7$ .

Next we study the case of  $\dim(G) = 14$ . Then  $7 \leq \dim(G(m))$  for any point  $m \in M^7$ . The group  $G_2$  does not contain a subgroup of dimension 7 (see [12]) and therefore we obtain again  $G(m) = SU(3)$  or  $G_2$ . The case  $G(m) = G_2$  for any point  $m \in M^7$  is impossible. Suppose  $G(m) = SU(3)$  for some point. Then  $G$  is a compact group of dimension 14 containing a subgroup isomorphic to  $SU(3)$ . Moreover,  $G$  acts on  $M^7$  with cohomogeneity one. Since  $M^7$  is simply-connected, there exists a point  $m_0 \in M^7$  such that  $G(m_0) = G$ . Then  $G$  is isomorphic to  $G_2$ . In a neighbourhood of this point the Einstein metrics is a warped product metric

$dr^2 \oplus f(r)g_0$ , where  $g_0$  is a  $G_2$ -invariant metric on the sphere  $G_2/SU(3) = S^6$ . Since the metric is regular at the point  $m_0$ ,  $M^7$  is a space of constant sectional curvature (see [4]).

■

**Theorem 6.7** *Let  $(M^7, \omega^3)$  be a simply-connected, compact manifold with nearly parallel  $G_2$ -structure not isometric to the sphere  $S^7$ . The group  $SU(3)$  cannot occur as an isotropy subgroup  $G(m) \subset \text{Aut}(M^7, \omega^3)$ .*

*Proof:* The isotropy group  $G(m)$  of an arbitrary point  $m \in M^7$  is a subgroup of  $G_2$ . Suppose that it is isomorphic to  $SU(3)$  for one point  $m \in M^7$ . The isotropy representation  $G(m) \rightarrow SO(T_m(M^7))$  is the standard representation of  $SU(3)$  in  $SO(7)$ . The possible dimensions of  $G(m)$ -invariant subspaces  $V \subset T_m(M^7)$  are 0, 1, 6 and 7. The tangent space  $T_m(N)$  of the orbit  $N = G \cdot m = G/G(m)$  defines a  $G(m)$ -invariant subspace. Consequently, we obtain four possibilities

- a)  $G = G(m) = SU(3)$ ,
- b)  $\dim(G) = 9$  and  $G(m) = SU(3)$ ,
- c)  $\dim(G) = 14$ ,
- d)  $\dim(G) = 15$ .

In case  $\dim(G) = 14$  or  $15$ ,  $M^7$  is isometric to the sphere  $S^7$ . If  $\dim(G) = 9$ , the automorphism group  $G$  is (locally) isomorphic to  $G = SU(3) \times U(1)$ . Denote by  $X$  the Killing vector field corresponding to the  $U(1)$ -action. Suppose that  $X$  has a zero point  $m^*$  and consider the orbit  $N$  through  $m^*$ . Then  $X$  vanishes at every point of  $N$  and therefore by Theorem 4,  $N$  is a 1- or 3-dimensional submanifold. The group  $SU(3)$  acts on  $N$  as a group of isometries and we obtain an isomorphism  $SU(3) \rightarrow \text{Iso}(N)$ . The compact group  $\text{Iso}(N)$  is isomorphic to  $U(1)$  (in case  $\dim(N) = 1$ ) or to  $SO(4)$  (in case  $\dim(N) = 3$ ). Since any two- or four-dimensional real representation of the group  $SU(3)$  is trivial, we conclude that  $G$  acts trivially on  $N$ . Hence,  $G$  is a 9-dimensional subgroup of  $G_2$ , a contradiction. Consequently, the Killing vector field  $X$  corresponding to the  $U(1)$ -action has constant length 1. Next we prove that the  $U(1)$ -action on  $M^7$  is a free action. Indeed, for any point  $m^*$  the isotropy subgroup  $G(m^*)$  of  $G = SU(3) \times U(1)$  has the dimension bounded by  $\dim(G) - 7 = 2 \leq \dim(G(m^*))$ . In case  $\dim(G(m^*)) = 2$ , the group  $G$  acts transitively on  $M^7$  and then the isotropy group  $G(m) = SU(3)$  cannot occur. Hence,  $G(m^*) = G_1 \times \mathbb{Z}_p \subset SU(3) \times U(1)$  is a group of dimension at least 3. There are only two possibilities:  $G_1 = SU(2)$  or  $G_1 = SU(3)$ . In both cases we get a  $\mathbb{Z}_p$ -action preserving the orientation on the 6-dimensional sphere  $S^6 = G_2/SU(3)$  commuting with the usual  $SU(3)$ -action on  $S^6$ . This means that the group  $\mathbb{Z}_p$  is trivial, i.e. the action of  $U(1)$  is free. This  $U(1)$ -action on  $M^7$  defines a compact 6-dimensional manifold  $K^6 = M^7/U(1)$  as well as a principal bundle  $\pi : M^7 \rightarrow K^6$ . Since  $M^7$  is an Einstein space of positive scalar curvature,  $K^6$  is an Einstein space of positive scalar curvature, too. The group  $SU(3)$  acts as a group of isometries on  $K^6$  and the isotropy subgroups of this action are  $SU(2)$  or  $SU(3)$ . Hence  $K^6$  is isometric to the projective space  $\mathbb{C}P^3$ . Moreover, the 2-form  $dX$  is a horizontal 2-form

$$\begin{aligned} (X \lrcorner dX)(Y) &= dX(X, Y) = X\langle X, Y \rangle - Y\langle X, X \rangle - \langle X, [X, Y] \rangle = \\ &= \langle \nabla_X X, Y \rangle + \langle X, \nabla_Y X \rangle = 0, \end{aligned}$$

i.e.  $X$  is a connection in the principal  $U(1)$ -fibre bundle  $\pi : M^7 \rightarrow K^6$  with curvature form  $dX$ . Finally it turns out that  $M^7$  is the 7-dimensional sphere.

It remains to discuss the case of  $\dim(G) = 8$ . In this case, the group  $G$  coincides with  $G(m) = SU(3)$  and acts on  $M^7$  with cohomogeneity two. The subgroups of  $SU(3)$  and their dimensions are

$SO(3)$	3
$S(U(2) \times U(1))$	4
$SU(2)$	3
$U(1) \times U(1)$	2
$U(1)$	1

The orbit  $G/G(m^*)$  for any point  $m^* \in M^7$  is therefore either a point or at least a 4-dimensional submanifold. The group  $G(m^*) = S(U(2) \times U(1))$  cannot occur since the Euler characteristic of  $G/G(m^*) = SU(3)/S(U(2) \times U(1)) = \mathbb{C}P^2$  is not zero (Theorem 6.5). On the other hand, near the point  $m \in M^7$  all orbits are of type  $SU(3)/SU(2)$ . Since the set of all principal orbits of the  $G$ -action is dense, the type of the principal orbit is  $G(m^*) = SU(2)$ . Consequently, we see that  $G = SU(3)$  acts on  $M^7$  with two orbit types only. There is a finite set  $\gamma_1, \dots, \gamma_k$  of closed geodesics in  $M^7$  such that  $G(m_i) = SU(3)$  ( $m_i \in \gamma_i$ ) and any other orbit is of type  $SU(3)/SU(2) = S^5$ . There exists only one geodesic  $\gamma$ . Indeed,  $Y = M^7/SU(3)$  is a 2-dimensional manifold with  $k$  boundary components and

$$M^7 - \{\gamma_1, \dots, \gamma_k\} \longrightarrow \text{Int}(Y)$$

is an  $S^5$ -fibration. On the other hand, we have

$$0 = \pi_1(M^7) = \pi_1(M^7 - \{\gamma_1, \dots, \gamma_k\}) = \pi_1(\text{Int}(Y))$$

and  $Y = D^2$  has only one boundary component. Consequently,  $M^7$  is a 7-dimensional Einstein manifold with isometry group  $SU(3)$  and the principal orbits are of type  $SU(3)/SU(2)$ ; there exists only one exceptional orbit - the fixed point set of  $SU(3)$ . It turns out that  $M^7$  is isometric to the standard 7-dimensional sphere  $S^7$  ( $M^7$  is topologically the sphere and the metric is an  $SU(3)$ -invariant Einstein metric with respect to the usual action of  $SU(3) \subset SO(6) \subset SO(8)$  see [3]).

■

**Corollary 3** *Let  $(M^7, \omega^3)$  be a simply-connected, compact manifold with nearly parallel  $G_2$ -structure not isometric to the sphere  $S^7$ . Then*

- (i) *the dimension of the automorphism  $G = \text{Aut}(M^7, \omega^3)$  has dimension  $\leq 13$ .*
- (ii) *any isotropy subgroup  $G(m)$  has dimension  $\leq 6$ .*



## 7 Nearly Parallel $G_2$ -Structures with Large Symmetry Group.

In this section we will classify all seven-dimensional compact, simply-connected manifolds with a nearly parallel  $G_2$ -structure and symmetry group of dimension at least 10. This classification includes in particular the classification of compact, simply-connected homogeneous nearly parallel  $G_2$ -structures.

Let  $(M^7, g)$  be a compact, simply-connected 7-dimensional nearly parallel  $G_2$ -manifold different from the sphere  $S^7$ . Let  $G$  be the connected component of the automorphism group of the  $G_2$ -structure. We already know that

$$\dim(G) \leq 13 \quad \text{and} \quad \dim(G(m)) \leq 6 \quad \text{for any point } m \in M^7$$

holds. We will discuss the spaces case by case depending on the dimension of the group  $G$ .

**1. case:**  $\dim(G) = 13$ .

In this case the dimension of the isotropy group  $G(m)$  is 6 for any point  $m \in M^7$  and the group  $G$  acts transitively on  $M^7 = G/G(m)$ . There exists only one connected six-dimensional subgroup of the group  $G_2$  (see [12]), namely the isotropy group of the exceptional orbit of the  $G_2$ -action on the Grassmannian manifold  $G_3(\mathbb{R}^7)$  (see Proposition 2.6). The Lie algebra of this subgroup is defined by the relations:

$$\omega_{12} + \omega_{34} + \omega_{56} = 0, \quad \omega_{17} + \omega_{36} + \omega_{45} = 0, \quad \omega_{27} + \omega_{35} - \omega_{46} = 0$$

$$\omega_{13} = \omega_{14} = \omega_{15} = \omega_{16} = \omega_{23} = \omega_{24} = \omega_{25} = \omega_{26} = \omega_{37} = \omega_{47} = \omega_{57} = \omega_{67} = 0$$

and the subgroup is isomorphic to  $G(m) = SO(4) = [SU(2) \times SU(2)]/\{\pm 1\}$ . Denote by  $G^*$  and  $G^*(m)$  the 2-fold covering of the group  $G$  respectively of the group  $G(m)$ . Then  $G^*$  is a compact, simply-connected 13-dimensional Lie group containing a subgroup isomorphic to  $G^*(m) = Sp(1) \times Sp(1)$ . Using the classification of simple Lie groups we deduce that  $G^*$  is isomorphic to  $G^* = Sp(2) \times Sp(1)$ . Consequently, the homogeneous Einstein manifold  $M^7$  is of type  $M^7 = [Sp(2) \times Sp(1)]/[Sp(1) \times Sp(1)]$  and therefore  $M^7$  is isometric either to the standard sphere  $S^7$  or to the squashed sphere  $S_{squas}^7$ . ■

**2. case:**  $\dim(G) = 12$ .

In this case the dimension of any isotropy group  $G(m)$  is bounded by  $5 \leq \dim(G(m)) \leq 6$ . Since the group  $G_2$  does not contain a subgroup of dimension 5 we obtain that any isotropy group  $G(m)$  has dimension 6, i.e. any isotropy group is a 6-dimensional subgroup of  $G_2$  containing the group  $SO(4)$  described above:

$$SO(4) \subset G(m) \subset G_2.$$

It is a matter of fact that such a subgroup of  $G_2$  coincides with  $SO(4)$ . Indeed, consider the covering  $G_2/SO(4) \rightarrow G_2/G(m)$ . Any deck transformation  $g \in G_2$  is homotopic to the identity map and therefore its Lefschetz number coincides with the Euler characteristic  $\chi(G_2/SO(4)) > 0$ , a contradiction. Consequently, the group  $G$  acts on  $M^7$  with one orbit type only and  $M^7$  is the total space of a fibration over  $S^1$  with the fibre  $F = G/SO(4)$ . On the other hand, the exact homotopy sequence of this fibration yields

$$\cdots \rightarrow \pi_1(F) \rightarrow \pi_1(M^7) = 1 \rightarrow \pi(S^1) = \mathbb{Z} \rightarrow \pi_0(F) = 1,$$

a contradiction. Finally we see that the case  $\dim(G) = 12$  is impossible. ■

### 3. case: $\dim(G) = 11$ .

In this case the dimension of any isotropy group  $G(m)$  is bounded by  $4 \leq \dim(G(m)) \leq 6$ .

Suppose that  $\dim(G(m)) = 4$  for one point  $m \in M^7$ . Then  $G$  acts transitively,  $M^7 = G/G(m)$ , and the isotropy group  $G(m) \subset G_2$  is connected. Using the list of all connected subgroups of the exceptional group  $G_2$  (see [12]) we obtain two possibilities:

- a)  $G(m)$  is the subgroup  $[SU(2) \times U(1)]/\{\pm 1\}$  of  $SU(3)$ . This is in fact the group  $SU(3) \cap SO(4)$  and its Lie algebra is given by the equations:

$$\begin{aligned} \omega_{12} + \omega_{34} + \omega_{56} &= 0, & \omega_{36} + \omega_{45} &= 0, & \omega_{35} - \omega_{46} &= 0 \\ \omega_{13} = \omega_{14} = \omega_{15} = \omega_{16} = \omega_{17} = \omega_{23} = \omega_{24} = \omega_{25} = \omega_{26} = \omega_{27} &= 0 \\ \omega_{37} = \omega_{47} = \omega_{57} = \omega_{67} &= 0. \end{aligned}$$

The representation of  $G(m)$  in  $\mathbb{R}^7$  splits into a 1-, 2- and 4-dimensional invariant subspace,

$$\mathbb{R}^7 = E^1 \oplus E^2 \oplus E^4$$

where  $E^2 = \text{Span}(e_1, e_2)$ ,  $E^4 = \text{Span}(e_3, e_4, e_5, e_6)$  and  $E^1 = \text{Span}(e_7)$ .

- b)  $G(m)$  is the subgroup  $[U(1) \times SU(2)]/\{\pm 1\}$  of  $SO(4) = [SU(2) \times SU(2)]/\{\pm 1\}$ . The Lie algebra of this group is given by the equations:

$$\begin{aligned} \omega_{13} = \omega_{14} = \omega_{15} = \omega_{16} = \omega_{23} = \omega_{24} = \omega_{25} = \omega_{26} = \omega_{37} = \omega_{47} = \omega_{57} = \omega_{67} &= 0. \\ \omega_{12} + \omega_{34} + \omega_{56} &= 0, \\ \omega_{17} + \omega_{36} + \omega_{45} &= 0, & \omega_{36} = \omega_{45}, & \omega_{27} + \omega_{35} - \omega_{46} &= 0, & \omega_{35} = -\omega_{46}. \end{aligned}$$

The representation of  $G(m)$  in  $\mathbb{R}^7$  splits into a 3- and 4-dimensional invariant subspace,

$$\mathbb{R}^7 = F^3 \oplus F^4$$

where  $F^3 = \text{Span}(e_1, e_2, e_7)$  and  $F^4 = \text{Span}(e_3, e_4, e_5, e_6)$ .

First we consider the case that  $\pi_1(G)$  is a finite group. Denote by  $G^*$  its universal covering and lift the isotropy subgroup to  $G^*(m) = Sp(1) \times U(1)$ . Then  $G^*$  is a simply-connected Lie group of dimension 11 containing the two-dimensional torus  $T^2 \subset Sp(1) \times U(1)$ . Moreover, since the Euler characteristic of  $M^7 = G^*/G^*(m)$  vanishes we conclude that the rank of  $G^*$  is greater or equal to 3,

$$\text{rank}(G^*) \geq 3, \quad \dim(G^*) = 11, \quad \pi_1(G^*) = 1.$$

The classification of all compact Lie groups yields that  $G^*$  is isomorphic to  $SU(3) \times SU(2)$ . In case a) the isotropy group  $G(m)$  is contained in  $SU(3)$  and consequently the space  $M^7$  admits two real Killing spinors (the  $G_2$ -structure is of type 2). On the other hand, the automorphism group of the  $G_2$ -structure of the manifold  $M(3,2)$  with two Killing spinors described in Section 4 is isomorphic to  $SU(3) \times SU(2)$ , this group acts transitively on  $M^7$  and the isotropy representation coincides with the representation of  $G(m)$  in case a). Hence, in case a)  $M^7$  is isometric to  $M(3,2)$ . In a similar way we can handle the case b). The manifold  $N(1,1)$  admits a  $G_2$ -structure of type 1 (not the 3-Sasakian metric!, see Sections 4 and 5) and the automorphism group of this  $G_2$ -structure coincides obviously with the isometry group  $SU(3) \times SU(2)$ . A calculation of the isotropy representation yields that it coincides with the representation of case b) and consequently  $M^7$  is isometric to  $N(1,1)$ .

Suppose now that  $\pi_1(G)$  is not a finite group. The exact homotopy sequence

$$\dots \rightarrow \pi_2(M^7) \rightarrow \pi_1(G(m)) = \mathbb{Z} \rightarrow \pi_1(G) \rightarrow 1$$

yields that  $\pi_1(G(m)) = \pi_1(G) = \mathbb{Z}$ . Consider a finite covering  $G^*$  of  $G$  such that  $G^*$  splits into  $G^* = U(1) \times G_1$ , where  $G_1$  is a simply-connected group of dimension 10. Then  $G_1$  is isomorphic to  $Spin(5)$ . The decomposition  $G^* = U(1) \times Spin(5)$  defines a Killing vector field  $X$  on  $M^7$  invariant with respect to the action of  $Spin(5)$ . Consequently,  $X$  has a constant length. In particular, at the point  $m \in M^7$  the isotropy group  $G(m)$  preserves the vector  $X(m)$ , i.e. the group  $G(m)$  is of type  $G(m) = SU(3) \cap SO(4)$  and the isotropy representation splits into

$$T_m(M^7) = E^1 \oplus E^2 \oplus E^4.$$

On the other hand, the embedding  $\Phi : G^*(m) = U(1) \times Spin(3) \rightarrow U(1) \times Spin(5) = G^*$  is given by two injective homomorphisms

$$i : Spin(3) \rightarrow Spin(5), \quad j : U(1) \rightarrow U(1)$$

( $\pi_1(G^*/G^*(m)) = 1!$ ) and by one homomorphism  $k : U(1) \rightarrow Spin(5)$ ,

$$\Phi(z, g) = (j(z), k(z) \cdot i(g)).$$

Therefore the isotropy representation of the space  $G^*/\Phi(G^*(m))$  considered only as an  $Spin(3)$ -representation is isomorphic to the isotropy representation of the space  $Spin(5)/i(Spin(3))$ . There are only two injective homomorphisms  $i_1, i_2 : Spin(3) \rightarrow Spin(5)$ . The first of them  $i_1$  is related to the 5-dimensional irreducible representation of  $SO(3)$  and  $i_2$  is the usual inclusion of  $SO(3)$  into  $SO(5)$ . In case of  $i_1$  we obtain that the isotropy representation of the homogeneous

space is irreducible and in case of  $i_2$  we obtain the isotropy representation of the Stiefel manifold  $V_{5,2}$  which splits into the irreducible subspaces  $E^1 \oplus E^3 \oplus E^3$ . This contradicts the mentioned decomposition of  $T_m(M^7)$  and finally the case  $G^* = U(1) \times Spin(5)$  is not possible.

We discuss now the case that any isotropy group  $G(m)$  is a 6-dimensional group, i.e.  $G(m) = SO(4) \subset G_2$ . Since  $\dim(G) = 11$ , any orbit  $N = G/G(m) \subset M^7$  has dimension 5 and its tangents space  $T_m(N) \subset T_m(M^7)$  defines a  $G(m) = SO(4)$ -invariant subspace of  $T_m(M^7) = \mathbb{R}^7$ . The representation of the group  $SO(4) \subset G_2 \subset SO(7)$  splits into two  $SO(4)$ -irreducible parts, namely  $\mathbb{R}^7 = F^3 \oplus F^4$  where  $F^3 = Span(e_1, e_2, e_7)$  and  $F^4 = Span(e_3, e_4, e_5, e_6)$ , a contradiction. Consequently, this case is impossible. ■

#### 4. case: $\dim(G) = 10$ .

In this case the dimension of any group  $G(m)$  is bounded by  $3 \leq \dim(G(m)) \leq 6$ .

Suppose that  $\dim(G(m)) = 3$  for one point  $m \in M^7$ . Then  $G$  acts transitively,  $M^7 = G/G(m)$ , and the isotropy group  $G(m) \subset G_2$  is connected. Using the list of all connected subgroups of the exceptional group  $G_2$  (see [12]) we obtain four possibilities. In any case,  $G(m)$  is isomorphic to  $SO(3)$  or to  $SU(2)$ . Since  $\pi_1(M^7) = 1$  we obtain  $\pi(G) = \pi_1(G(m)) = 0$  or  $\mathbb{Z}_2$ . Consider the universal coverings  $G^*$  and  $G^*(m) = Spin(3)$ . Then  $G^*$  is a simply-connected Lie group of dimension 10. Moreover, since the Euler characteristic of  $M^7 = G/G^*(m)$  vanishes we conclude that the rank of  $G^*$  is greater or equal to 2,

$$rank(G^*) \geq 2, \quad \dim(G^*) = 10, \quad \pi_1(G^*) = 1.$$

The classification of all compact Lie groups yields that  $G^*$  is isomorphic to  $Spin(5)$  and the manifold  $M^7$  is isometric to the Stiefel manifold  $V_{5,2}$  or to the spaces  $SO(5)/SO(3)$  described in Section 4.

Suppose now that the isotropy group  $G(m)$  is a four-dimensional subgroup for one point  $m \in M^7$ . Then  $G(m)$  is one of the two subgroups of  $G_2$  considered in the discussion of the case  $\dim(G) = 11$ . In particular,  $G(m)$  is a connected subgroup. The orbit  $G \cdot m$  through  $m$  is a 6-dimensional manifold, but only the group  $G(m) = SU(3) \cap SO(4) \subset G_2$  has a 6-dimensional invariant subspace. Consequently,  $G/G(m)$  is the principal orbit of the  $G$ -action on  $M^7$  and there are no other orbits of dimension 6. But exceptional orbits do not exist at all. Indeed, since  $SU(3)$  cannot occur as an isotropy subgroup, an exceptional orbit must be of type  $O^4 = G/SO(4)$ . However, the isotropy representation of  $SO(4)$  is  $F^3 \oplus F^4$ , a contradiction to the Corollary 2 in Section 6. Finally the  $G$ -action defines a fibration  $M^7 \rightarrow M^7/G = S^1$  and the exact homotopy sequence yields that  $M^7$  cannot be simply-connected.

It remains to discuss the situation where any orbit is a four-dimensional manifold and every isotropy group  $G(m)$  coincides with  $SO(4)$ . In this situation we can apply the same argument as before and we obtain again a contradiction to Corollary 2. ■

In particular we proved the following

**Theorem 7.1** *Any compact nearly parallel  $G_2$ -manifold with automorphism group of dimension  $\dim(G) \geq 10$  is homogeneous.*

Probably there exist non-homogeneous nearly parallel  $G_2$ -manifolds admitting an automorphism group of dimension 9, 8,  $\dots$ . However, explicit non-homogeneous examples with a 9- or 8-dimensional automorphism group up to now are not known.

On the other hand, using similar arguments as before one can finish the classification of compact, homogeneous nearly parallel  $G_2$ -manifolds. It turns out that in case  $\dim(G) \leq 9$  the space is isometric to  $Q(1, 1, 1)$  or to one of the manifolds  $N(k, l)$ .

**Theorem 7.2** *Any compact, simply-connected, homogeneous nearly parallel  $G_2$ -manifold is one of the spaces described in the three tables of Section 4.*

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