Spin^c Manifolds and Complex Contact Structures

Andrei Moroianu[1]

Abstract - *In this paper we extend our notion of projectable spinors ([9], Ch.1) to the framework of Spin^c manifolds and deduce the basic formulas relating spinors on the base and the total space of Riemannian submersions with totally geodesic one-dimensional fibres. Some geometric applications concerning positive Kähler-Einstein complex contact manifolds (e.g. their characterisation as twistor spaces over positive quaternionic Kähler manifolds) are also given.*

1 Introduction

Projectable spinors for Riemannian submersions of spin manifolds arose in a quite natural way ([9], Ch.1) and have led to important geometric applications, as the classification of Kähler manifolds admitting Kählerian Killing spinors ([8]) or results on the spectrum of the Dirac operator for certain classes of Riemannian manifolds ([10]).

In this paper we introduce projectable spinors for Riemannian submersions of Spin^c manifolds, motivated by the following facts: K.-D. Kirchberg and U. Semmelmann discovered that every complex contact manifold of complex dimension $4l+3$ admitting a Kähler–Einstein metric of positive scalar curvature carries a canonical spin structure with Kählerian Killing spinors [4]. Using this together with the results in [8], we were able to prove the following characterisation of twistor spaces over positive quaternionic Kähler manifolds in half of the possible dimensions:

Theorem A. (cf. [12]) *Let M be a compact spin Kähler manifold of positive scalar curvature and complex dimension $4l+3$. Then the following statements are equivalent:*

(i) *M is the twistor space of some quaternionic Kähler manifold;*

(ii) *M is Kähler-Einstein and admits a complex contact structure;*

(iii) *M admits Kählerian Killing spinors.*

By different methods, C. LeBrun independently obtained the following

Theorem B. (cf. [7]) *Let Z be a Fano contact manifold. Then Z is a twistor space iff it admits a Kähler–Einstein metric.*

[1]Supported by the SFB 288 "Differentialgeometrie und Quantenphysik" of the DFG.

In complex dimensions $4l+3$, this is a direct corollary of Theorem A. The reasons for which our Theorem A fails to hold in complex dimensions $4l+1$ are of a topological nature: neither the twistor spaces, nor the complex contact manifolds of complex dimensions $4l+1$ are spin (with one exception: the complex projective space). On the other hand, each Kähler manifold admits a natural Spin^c structure; it is thus natural to try to extend the above notions to the framework of Spin^c structures, and to generalise the results in [12] to this case.

In order to keep the computations as simple as possible, we do not construct here the whole theory of projectable spinors on Spin^c manifolds, and restrict ourselves to a particular situation which is of special interest to us. Generalisations of the constructions described below can be easily obtained.

The author is very indebted to Jean Pierre Bourguignon for the careful reading of a preliminary version of this paper, and to Uwe Semmelmann for many helpful discussions.

2 Preliminaries

Definition 2.1 *A $Spin^c$ structure on an oriented Riemannian manifold (M^n, g) is given by a $U(1)$ principal bundle $P_{U(1)}M$ and a $Spin^c_n$ principal bundle $P_{Spin^c_n}M$ together with a projection $\theta : P_{Spin^c_n}M \to P_{SO(n)}M \times P_{U(1)}M$ ($P_{SO(n)}M$ means the $SO(n)$ principal bundle of oriented orthonormal frames on M), satisfying*

$$\theta(\tilde{u}\tilde{a}) = \theta(\tilde{u})\xi(\tilde{a}),$$

for every $\tilde{u} \in P_{Spin^c_n}M$ and $\tilde{a} \in Spin^c_n$, where ξ is the canonical 2–fold covering of $Spin^c_n$ over $SO(n) \times U(1)$.

Recall that $\mathrm{Spin}^c_n = \mathrm{Spin}_n \times_{\mathbf{Z}_2} \mathrm{U}(1)$, and that ξ is given by $\xi([u,a]) = (\phi(u), a^2)$, where $\phi : \mathrm{Spin}_n \to \mathrm{SO}(n)$ is the canonical 2–fold covering.

If M has a Spin^c structure, we denote by ΣM the associated complex spinor bundle and by L the complex line bundle associated to $P_{U(1)}M$, which is called the auxiliary bundle. On ΣM there is a canonical Hermitian product $(.,.)$, with respect to which the Clifford multiplication by vectors is skew–Hermitian:

$$(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi), \quad \forall X \in TM,\ \psi, \varphi \in \Sigma M. \tag{1}$$

Every connection form A on $P_{U(1)}M$ defines, together with the Levi-Civita connection of M, a covariant derivative on ΣM denoted by ∇^A. Correspondingly, we define the Dirac operator as the composition $\gamma \circ \nabla^A$, where γ denotes the Clifford contraction. The Dirac operator can be expressed using a local orthonormal frame $\{e_1, \cdots, e_n\}$ as

$$D = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}.$$

Suppose now that (M^{2m}, g, J) is a Kähler manifold. We define the twisted Dirac operator $\tilde{D}$ by

$$\tilde{D} = \sum_{i=1}^{2m} J(e_i) \cdot \nabla_{e_i} = -\sum_{i=1}^{2m} e_i \cdot \nabla_{J(e_i)},$$

which satisfies

$$\tilde{D}^2 = D^2 \quad \text{and} \quad \tilde{D}D + D\tilde{D} = 0. \tag{2}$$

We also define the complex Dirac operators $D_\pm := \frac{1}{2}(D \mp i\tilde{D})$, and (2) becomes

$$D_-^2 = D_+^2 = 0 \quad \text{and} \quad D^2 = D_+D_- + D_-D_+. \tag{3}$$

Consider a local orthonormal frame $\{X_\alpha, Y_\alpha\}$ such that $Y_\alpha = J(X_\alpha)$. Then $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$, and $D_\pm$ can be expressed as

$$D_+ = 2\sum_{\alpha=1}^{m} Z_\alpha \cdot \nabla^A_{Z_{\bar{\alpha}}} \quad , \quad D_- = 2\sum_{\alpha=1}^{m} Z_{\bar{\alpha}} \cdot \nabla^A_{Z_\alpha}. \tag{4}$$

A k-form ω acts on ΣM by

$$\omega \cdot \Psi := \sum_{i_1<\cdots<i_k} \omega(e_{i_1}, \cdots, e_{i_k})\, e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \Psi \ ,$$

where $\{e_i\}$ is a local orthonormal frame on M. With respect to this action, the Kähler form Ω (defined by $\Omega(X,Y) = g(X, JY)$) satisfies

$$\Omega = \frac{1}{2}\sum_{i=1}^{2m} J(e_i) \cdot e_i = -\frac{1}{2}\sum_{i=1}^{2m} e_i \cdot J(e_i). \tag{5}$$

For later use let us note that

$$\sum_{\alpha=1}^{m} Z_\alpha \cdot Z_{\bar{\alpha}} = -\frac{i}{2}\Omega - \frac{m}{2} \quad , \quad \sum_{\alpha=1}^{m} Z_{\bar{\alpha}} \cdot Z_\alpha = \frac{i}{2}\Omega - \frac{m}{2}, \tag{6}$$

where Z_α and $Z_{\bar{\alpha}}$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ as before.

The action of Ω on ΣM yields an orthogonal decomposition

$$\Sigma M = \oplus_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ is the eigenbundle associated to the eigenvalue $i\,\mu_r = i\,(m - 2\,r)$ of Ω. If we define $\Sigma_{-1}M = \Sigma_{m+1}M = \{0\}$, then

$$D_\pm \Gamma(\Sigma_r M) \subset \Gamma(\Sigma_{r\pm1} M). \tag{7}$$

The complex volume element

$$\omega_{\mathbf{C}} := i^m e_1 \cdot \ldots \cdot e_{2m}$$

acts on ΣM by Clifford multiplication and its square is the identity. We denote by $\Sigma^\pm M$ the eigenbundles corresponding to the eigenvalues ± 1, and it is easy to see that $\Sigma_r M \subset \Sigma^+ M$ ($\Sigma_r M \subset \Sigma^- M$) iff r is even (respectively odd). If, with respect to the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, a spinor ψ is written as $\psi = \psi_+ + \psi_-$, then we define its conjugate $\bar{\psi} := \psi_+ - \psi_-$.

3 Projectable spinors to Spin^c-Manifolds

As in the case of spin manifolds, projectable spinors may be defined for arbitrary Riemannian submersions of Spin^c manifolds with 1–dimensional totally geodesic fibres, but for the sake of simplicity we treat only a particular case here.

Let $P_{U(1)}M$ be the principal U(1) bundle associated to a Spin^c structure on a Riemannian manifold (M^n, g) of even dimension and suppose that on $P_{U(1)}M$ a connection form A is given. Denote by $\bar{M} := P_{U(1)}M$ and by π the canonical bundle projection. It is well-known that there exists an unique 2-form α on M whose pull-back is just i times the curvature form dA of the connection A (note that A and dA are imaginary-valued forms on $\bar{M}$). Let T be the (1,1) tensor on M associated to α (defined by $\alpha(X,Y) = g(X,TY)$).

The manifold $\bar{M}$ carries a canonical 1–parameter family of Riemannian metrics g_t which make the bundle projection $\pi : \bar{M} \to M$ into a Riemannian submersion with totally geodesic fibres. These metrics are given by

$$g_t(X,Y) = g(\pi_* X, \pi_* Y) - t^2 A(X)A(Y), \quad \forall x \in \bar{M},\ X, Y \in T_x\bar{M},$$

and we denote by ∇^t the covariant derivative of the Levi-Civita connection of g_t and by V the unit vertical vector field on $(\bar{M}, \bar{g})$ satisfying $A(V) = i/t$. This choice of V fixes an orientation for $\bar{M}$.

Before proceeding, we mention here a simple result relating spin and Spin^c structures, that will be used in a moment.

Lemma 3.1 *A Spin^c structure with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection A of the auxiliary bundle L is flat, then by this identification ∇^A corresponds to ∇ on the spinor bundles.*

Proof. One first remarks that since the U(1) bundle associated to L is trivial, we can exhibit a global section of it, that we will call σ. Denote by $P_{Spin_n}M$ the inverse image by θ of $P_{SO(n)}M \times \sigma$. It is straightforward to check that this defines a spin structure on M, and that the connection on $P_{Spin^c_n}M$ restricts to the Levi-Civita connection on $P_{Spin_n}M$ if σ can be chosen to be parallel, i.e. if A defines a flat connection.

Q.E.D.

Proposition 3.1 *Every Spin^c structure on M induces a canonical spin structure on $\bar{M}$.*

Proof. By enlargement of the structure groups, the two-fold covering

$$\theta : P_{Spin^c_n}M \to P_{SO(n)}M \times P_{U(1)}M,$$

gives a two-fold covering

$$\theta : P_{Spin^c_{n+1}}M \to P_{SO(n+1)}M \times P_{U(1)}M,$$

which, by pull-back through π, gives rise to a Spin^c structure on $\bar{M}$

$$\begin{array}{ccc} P_{Spin^c_{n+1}}\bar{M} & \stackrel{\pi}{\longrightarrow} & P_{Spin^c_{n+1}}M \\ \pi^*\theta \downarrow & & \theta \downarrow \\ P_{SO(n+1)}\bar{M} \times P_{U(1)}\bar{M} & \stackrel{\pi}{\longrightarrow} & P_{SO(n+1)}M \times P_{U(1)}M \\ \downarrow & & \downarrow \\ \bar{M} & \stackrel{\pi}{\longrightarrow} & M \end{array}$$

Using Lemma 3.1 we see that this construction actually yields a *spin* structure on $\bar{M}$. Indeed, the pull back of a G principal bundle ($P_{U(1)}M$, in our situation) with respect to its own projection map is always trivial:

$$\begin{array}{ccc} \pi^*P \simeq P \times G & \stackrel{\pi}{\longrightarrow} & P \\ \pi^*\pi \downarrow & & \pi \downarrow \\ P & \stackrel{\pi}{\longrightarrow} & M \end{array}$$

Nevertheless, we will continue to call this spin structure the induced Spin^c structure on $\bar{M}$.

Q.E.D.

The next step is to relate the covariant derivatives of spinors on M and $\bar{M}$. We point out an important detail here: since we are actually interested in $\bar{M}$ as *spin* manifold, the connection on $P_{U(1)}\bar{M}$ (which defines the covariant derivative of spinors on $\bar{M}$) that we consider, will not be the pull-back connection, but the *flat connection* on the canonically trivial bundle $P_{U(1)}\bar{M}$. The following result relates an arbitrary connection on a principal bundle $\pi : P \to M$ and the flat connection on $\pi^*P \to P$.

Lemma 3.2 *The connection form A_0 of the flat connection on π^*P can be related to an arbitrary connection A on P by*

$$A_0((\pi^*s)_*(U)) = -A(U), \tag{8}$$

$$A_0((\pi^*s)_*(X^*)) = A(s_*X), \tag{9}$$

where U is a vertical vector field on P, X^* is the horizontal lift (with respect to A) of a vector field X on M, and s is a local section of $P \to M$.

PROOF. The identification $P \times \mathrm{U}(1) \simeq \pi^* P$ is given by $(u, a) \mapsto (u, ua)$, for all $(u, a) \in P \times \mathrm{U}(1)$. For some fixed $u \in P$, take a path u_t in the fiber over $x := \pi(u)$ such that $u_0 = u$ and $\dot{u}_0 = U$. We define $a_t \in \mathrm{U}(1)$ by $u_t = s(x)a_t$, so via the above identification we have

$$(\pi^* s)(u_t) = (u_t, s(x)) = (u_t, (a_t)^{-1}),$$

and thus

$$A_0((\pi^* s)_*(U)) = -a_0^{-1}\dot{a}_0 = -A(\dot{u}_0) = -A(U).$$

Similarly, for $x \in M$ and $X \in T_x M$, take a path x_t in M such that $x_0 = x$ and $\dot{x}_0 = X$. Let $u \in \pi^{-1}(x)$ and u_t the horizontal lift of x_t such that $u_0 = u$. We define $a_t \in \mathrm{U}(1)$ by $s(x_t) = u_t a_t$, which by derivation gives $s_*(X) = R_{a_0}\dot{u}_0 + u_0\dot{a}_0$. Then

$$(\pi^* s)(u_t) = (u_t, s(x_t)) = (u_t, a_t),$$

and thus, using the fact that $\dot{u}_0$ is horizontal,

$$A_0((\pi^* s)_*(X^*)) = a_0^{-1}\dot{a}_0 = A(s_*(X)).$$

Q.E.D.

Recall that the complex Clifford representation Σ_n can be made into a $Cl(n+1)$–representation by defining

$$\mu(e_j) \cdot \psi = \begin{cases} e_j \cdot \psi & \text{for } j \leq n \\ i\bar{\psi} & \text{for } j = n+1 \end{cases}$$

Corresponding to this, we obtain an identification of the pull back $\pi^*\Sigma M$ with $\Sigma\bar{M}$, and with respect to this identification, if X is a vector and ψ a spinor on M, then

$$X^* \cdot \pi^*\psi = \pi^*(X \cdot \psi), \tag{10}$$

$$V \cdot \pi^*\psi = \pi^*(i\bar{\psi}), \tag{11}$$

where V is the unit vertical vector field defined at the beginning of this section.

Definition 3.1 *The sections of $\Sigma\bar{M}$ which can be written as pull-back of sections of ΣM are called projectable spinors.*

We now compute the covariant derivative of projectable spinors on $\bar{M}$ in terms of the covariant derivative of spinors on M.

Proposition 3.2 *The covariant derivative ∇^t on $\Sigma\bar{M}$ induced by the Levi-Civita connection on $(\bar{M}, g_t)$ and the flat connection on $\pi^* P_{U(1)}M$ satisfies*

$$\nabla^t_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X\psi - i\frac{t}{4}T(X) \cdot \bar{\psi}) \qquad \forall X \in TM, \tag{12}$$

$$\nabla^t_V \pi^*\psi = -\pi^*(\frac{t}{4}\alpha \cdot \psi + \frac{i}{2t}\psi). \tag{13}$$

PROOF. Recall that the curvature form dA of the principal U(1) bundle $\bar{M} \to M$ satisfies

$$dA = -i\pi^*\alpha. \tag{14}$$

The metric g_t is given by

$$g_t(X,Y) = g(\pi_* X, \pi_* Y) - t^2 A(X)A(Y),\ \forall X, Y \in T\bar{M}. \tag{15}$$

If V denotes as before the unit vertical vector field, then $A(V) = i/t$, and we obtain

$$\begin{aligned} g_t(\nabla^t_{X^*} V, Y^*) &= -\frac{1}{2} g_t(V, [X^*, Y^*]) = \frac{t^2}{2} A(V)A([X^*, Y^*]) \\ &= \frac{it}{2} A([X^*, Y^*]) = -\frac{it}{2} dA(X^*, Y^*) \\ &= -\frac{t}{2}\alpha(X,Y) = \frac{t}{2} g_t(T(X)^*, Y^*), \end{aligned}$$

so finally

$$\nabla^t_{X^*} V = \frac{t}{2} T(X)^*. \tag{16}$$

Consider the pull-back $\pi^*\psi$ of a spinor field $\psi = [\sigma, \xi]$, where $\xi : U \subset M \to \Sigma_n$ is a vector valued function, and σ is a local section of $P_{Spin^c_n} M$ whose projection onto $P_{SO(n)}M$ is a local orthonormal frame $(X_1, ..., X_n)$ and whose projection onto $P_{U(1)}M$ is a local section s. Then $\pi^*\psi$ can be expressed as $\pi^*\psi = [\pi^*\sigma, \pi^*\xi]$, and it is easy to see that the projection of $\pi^*\sigma$ onto $P_{SO(n+1)}\bar{M}$ is the local orthonormal frame $(X_1^*, ..., X_n^*, V)$ and its projection onto $P_{U(1)}\bar{M}$ is just $\pi^* s$.

Using Lemma 3.2 and (16) we obtain

$$\begin{aligned} \nabla^t_{X^*}\pi^*\psi &= [\pi^*\sigma, X^*(\pi^*\xi)] + \frac{1}{2}\sum_{j<k} g_t(\nabla^t_{X^*} X_j^*, X_k^*) X_j^* \cdot X_k^* \cdot \pi^*\psi \\ &\quad + \frac{1}{2}\sum_j g_t(\nabla^t_{X^*} X_j^*, V) X_j^* \cdot V \cdot \pi^*\psi + \frac{1}{2} A_0((\pi^* s)_* X^*)\pi^*\psi \\ &= [\pi^*\sigma, \pi^*(X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k)\pi^*(X_j \cdot X_k \cdot \psi) \\ &\quad - \frac{1}{2}\frac{t}{2} i \sum_j g(T(X), X_j)\pi^*(X_j \cdot \bar{\psi}) + \frac{1}{2} A(s_* X)\pi^*\psi \\ &= \pi^*\Big([\sigma, (X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k) X_j \cdot X_k \cdot \psi \\ &\quad - i\frac{t}{4} T(X) \cdot \bar{\psi} + \frac{1}{2} A(s_* X)\psi\Big) \\ &= \pi^*(\nabla^A_X \psi - i\frac{t}{4} T(X) \cdot \bar{\psi}). \end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla^t_V \pi^*\psi &= [\pi^*\sigma, V(\pi^*\xi)] + \frac{1}{2}\sum_{j<k} g_t(\nabla^t_V X^*_j, X^*_k) X^*_j \cdot X^*_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_j g_t(\nabla^t_V X^*_j, V) X^*_j \cdot V \cdot \pi^*\psi + \frac{1}{2} A_0((\pi^* s)_* V)\pi^*\psi \\
&= \frac{t}{2}\frac{1}{2}\sum_{j<k} g(T(X_j), X_k)\pi^*(X_j \cdot X_k \cdot \psi) - \frac{1}{2}A(V)\pi^*\psi \\
&= -\frac{t}{4}\pi^*(\alpha\cdot\psi) - \frac{i}{2t}\pi^*\psi \\
&= -\pi^*\Big(\frac{t}{4}\alpha\cdot\psi + \frac{i}{2t}\psi\Big).
\end{aligned}$$

Q.E.D.

We now particularise the above results to the case where M is a Kähler-Einstein manifold (M^n, g, J) of positive scalar curvature, and the auxiliary bundle L of the Spin^c structure on M is a root of the canonical bundle K, i.e. $L^{\otimes r} = K$ for some $r \in \mathbf{N}^*$. The canonical connection on K, whose curvature form is just $-i\rho$ (ρ is the Ricci form), induces then a connection A on L, whose curvature form ω satisfies $\omega = -i\rho/r$. As before, we denote by $\bar{M}$ the U(1) principal bundle associated to L. By rescaling the metric on M if necessary, we can suppose that the scalar curvature of M is equal to $n(n+2)$, and thus $\rho = (n+2)\Omega$. The 2-form α on M defined at the beginning of this section is given in this case by

$$\alpha = \frac{n+2}{r}\Omega, \tag{17}$$

so the above proposition becomes

Proposition 3.3 *If M is a Spin^c Kähler-Einstein manifold of positive scalar curvature and the auxiliary bundle L of the Spin^c structure on M is a r–root of the canonical bundle K, then the covariant derivative ∇^t on $\Sigma\bar{M}$ induced by the Levi-Civita connection on $(\bar{M}, g_t)$ and the flat connection on $\pi^* P_{U(1)}M$ satisfies*

$$\nabla^t_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X\psi - i\frac{t(n+2)}{4r}J(X)\cdot\bar{\psi}) \qquad \forall X \in TM, \tag{18}$$

$$\nabla^t_V \pi^*\psi = -\pi^*(\frac{t(n+2)}{4r}\Omega\cdot\psi + \frac{i}{2t}\psi). \tag{19}$$

The formula (16) allows us to compute the Ricci tensor Ric^t of the Riemannian manifold $(\bar{M}, g_t)$. If we denote by $a := \frac{t(n+2)}{2r}$, then

$$\mathrm{Ric}^t(V,V) = na^2 \quad , \quad \mathrm{Ric}^t(X^*, V) = 0, \tag{20}$$

$$\mathrm{Ric}^t(X^*, Y^*) = (n + 2 - 2a^2)g(X, Y). \tag{21}$$

Let us take $t_0 = \frac{2r}{n+2}$ and denote $\bar{g} := g_{t_0}$, $\bar{\nabla} := \nabla^{t_0}$. The vertical vector field V defines then an Einstein–Sasakian structure on the manifold $(\bar{M}, \bar{g})$ (cf. [2]). We can synthetise the above results in the following

Theorem 3.1 *Let (M^n, g, J) be a Kähler-Einstein manifold with scalar curvature $R = n(n+2)$, $L := K^{\frac{1}{r}}$ a root of the canonical bundle K and $\bar{M}$ the associated $U(1)$ principal bundle with connection form A, induced by the Levi-Civita connection on K. Then the following hold:*

(i) *There is a canonical metric $\bar{g}$ on $\bar{M}$ making the bundle projection $\pi : \bar{M} \to M$ into a Riemannian submersion with totally geodesic fibres, and satisfying $\bar{\nabla}_{X^*} V = J(X)^*$.*

(ii) *With respect to the metric $\bar{g}$, V defines a regular Einstein–Sasakian structure on $\bar{M}$. The length of the fibres of the corresponding S^1–action is constant and equal to $\frac{4\pi r}{n+2}$.*

(iii) *The Spin^c structure on M defined by (L, A) induces a canonical spin structure on $\bar{M}$ and every spinor field on M induces a projectable spinor field $\pi^*\psi$ on $\bar{M}$, satisfying*

$$\bar{\nabla}_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X \psi - \frac{i}{2} J(X) \cdot \bar{\psi}) \qquad \forall X \in TM, \tag{22}$$

$$\bar{\nabla}_V \pi^*\psi = -\frac{1}{2}\pi^*(\Omega \cdot \psi + \frac{i(n+2)}{2r}\psi). \tag{23}$$

4 Complex contact structures

Definition 4.1 (cf. [5]) *Let M^{2m} be a complex manifold of complex dimension $m = 2k + 1$. A complex contact structure is a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:*

(i) *$\{U_i\}$ is an open covering of M.*

(ii) *ω_i is a holomorphic 1-form on U_i.*

(iii) *$\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0} M)$ is different from zero at every point of U_i.*

(iv) *$\omega_i = f_{ij}\omega_j$ in $U_i \cap U_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$.*

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex contact structure. Then there exists an associated holomorphic line sub–bundle $L_{\mathcal{C}} \subset \Lambda^{1,0}(M)$ with transition functions $\{f_{ij}^{-1}\}$ and local sections ω_i. It is easy to see that

$$\mathcal{D} := \{Z \in T^{1,0}M \mid \omega(Z) = 0, \ \forall \omega \in L_{\mathcal{C}}\}$$

is a codimension 1 maximally non–integrable holomorphic sub–bundle of $T^{1,0}M$, and conversely, every such bundle defines a complex contact structure. From condition (iii) immediately follows the isomorphism $L_{\mathcal{C}}^{k+1} \cong K$, where $K = \Lambda^{m,0}(M)$ denotes the canonical bundle of M.

From now on, M will denote a Kähler–Einstein manifold of odd complex dimension $m = 4l + 1$ with positive scalar curvature, admitting a complex contact structure $\mathcal{C}$. The manifold M is compact, by Myers' Theorem. By rescaling the metric on M if necessary, we can suppose that the scalar curvature of M is equal to $2m(2m + 2)$, and thus the Ricci form ρ and the Kähler form Ω are related by $\rho = (2m + 2)\Omega$. The main objective of this section is to construct the analogues of Kählerian Killing spinors ([3], [4], [8]) for a certain Spin^c structure on M, determined by $\mathcal{C}$. This is done just as in [4].

The collection $(U_i, \omega_i \wedge (\partial\omega_i)^l)$ defines a holomorphic line bundle $L_l \subset \Lambda^{2l+1,0}M$, and from the definition of $\mathcal{C}$ we easily obtain

$$L_l \cong L_{\mathcal{C}}^{l+1}. \tag{24}$$

We now fix some $(U, \omega) \in \mathcal{C}$ and define a local section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$ by

$$\psi_{\mathcal{C}}|_U := |\xi_\tau|^{-2}\bar{\tau} \otimes \xi_\tau, \tag{25}$$

where $\tau := \omega \wedge (\partial\omega)^l$ and ξ_τ is the element corresponding to τ through the isomorphism (24). The fact that $\psi_{\mathcal{C}}$ does not depend of the element $(U, \omega) \in \mathcal{C}$ shows that it actually defines a global section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$.

We now recall ([6], Appendix D) that $\Lambda^{0,*}M$ is just the spinor bundle associated to the canonical Spin^c structure on M, whose auxiliary line bundle is K^{-1}, so that $\Lambda^{0,*}M \otimes L_{\mathcal{C}}^{l+1}$ is actually the spinor bundle associated to the Spin^c structure on M with auxiliary bundle $L = K^{-1} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}^{-(2l+1)} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}$. The section $\psi_{\mathcal{C}}$ is thus a spinor lying in $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1} \cong \Sigma_{2l+1}M$, so

$$\Omega \cdot \psi_{\mathcal{C}} = -i\psi_{\mathcal{C}}. \tag{26}$$

Proposition 4.1 *The spinor field $\psi_{\mathcal{C}}$ satisfies $\nabla_Z \psi_{\mathcal{C}} = 0$, $\forall Z \in T^{1,0}M$ (in particular $D_-\psi_{\mathcal{C}} = 0$), and*

$$D^2\psi_{\mathcal{C}} = D_-D_+\psi_{\mathcal{C}} = \frac{l+1}{2l+1}\left(\frac{1}{2}R\,\psi_{\mathcal{C}} - i\rho \cdot \psi_{\mathcal{C}}\right), \tag{27}$$

where R is the scalar curvature of M.

Proof. This is actually a variant of Proposition 5 from [4], the only difference being that ξ_τ (Ψ_ω in their notations) is not any more a section of $K^{1/2}$, but of $K^{(l+1)/(2l+1)}$, so the coefficients $1/2$ in formulas (8) and (9) of [4] have to be replaced by $\frac{l+1}{2l+1}$.

Q.E.D.

Using (26), (27) and the fact that $\rho = \frac{1}{8l+2} R\,\Omega = (8l+4)\Omega$, we obtain

Corollary 4.1 *The spinor field $\psi_{\mathcal{C}}$ is an eigenspinor of D^2 with respect to the eigenvalue $16l(l+1)$.*

Let us introduce some notations

$$\psi_- := \psi_{\mathcal{C}} \in \Gamma(\Sigma_{2l+1}M) \qquad , \qquad \psi_+ := \frac{1}{4l+4} D\psi_{\mathcal{C}} \in \Gamma(\Sigma_{2l+2}M). \tag{28}$$

By integration over M we immediately obtain from the above Corollary

$$|\psi_-|^2_{L^2} = \frac{l+1}{l} |\psi_+|^2_{L^2}. \tag{29}$$

Proposition 4.2 *The following relations hold*

$$\nabla_Z \psi_- = 0, \ \forall Z \in T^{1,0}M, \tag{30}$$

$$\nabla_{\bar{Z}} \psi_- + \bar{Z} \cdot \psi_+ = 0, \ \forall \bar{Z} \in T^{0,1}M, \tag{31}$$

$$\nabla_{\bar{Z}} \psi_+ = 0, \ \forall \bar{Z} \in T^{0,1}M, \tag{32}$$

$$\nabla_Z \psi_+ + Z \cdot \psi_- = 0, \ \forall Z \in T^{1,0}M. \tag{33}$$

Proof. The first relation is part of Proposition 4.1. In order to prove (31), let us consider the local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ introduced in Section 2: $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$, where $Y_\alpha = J(X_\alpha)$, and $\{X_\alpha, Y_\alpha\}$ is a local orthonormal frame of TM. From (30) we find $\nabla_{Z_{\bar{\alpha}}}\psi_- = \nabla_{X_\alpha}\psi_- = i\nabla_{Y_\alpha}\psi_-$, so using (6) and (28) gives

$$\begin{aligned}
0 \ &\leq \ \sum_{\alpha=1}^{m} |\nabla_{Z_{\bar{\alpha}}}\psi_- + Z_{\bar{\alpha}} \cdot \psi_+|^2 \\
&= \ \sum_{\alpha=1}^{m} |\nabla_{X_\alpha}\psi_-|^2 - 2\Re e \sum_{\alpha=1}^{m} (\psi_+, Z_\alpha \cdot \nabla_{Z_{\bar{\alpha}}}\psi_-) - \sum_{\alpha=1}^{m} (\psi_+, Z_\alpha \cdot Z_{\bar{\alpha}} \cdot \psi_+) \\
&= \ \frac{1}{2}|\nabla\psi_-|^2 - \Re e(\psi_+, D_+\psi_-) - \frac{1}{2}(\psi_+, (-i\Omega - m)\psi_+) \\
&= \ \frac{1}{2}|\nabla\psi_-|^2 - (4l+4)|\psi_+|^2 + \frac{1}{2}(4l+4)|\psi_+|^2.
\end{aligned}$$

The last expression is by construction a positive function, say $|F|^2$, on M. Integrating over M and using the generalised Lichnerowicz formula ([6], Appendix

D), Corollary 4.1 and (29), we obtain

$$\begin{aligned}
|F|^2_{L^2} &= \frac{1}{2}(\nabla^*\nabla\psi_-,\psi_-)_{L^2} - (4l+4)|\psi_+|^2_{L^2} + \frac{1}{2}(4l+4)|\psi_+|^2_{L^2} \\
&= \frac{1}{2}(D^2\psi_- - \frac{1}{4}R\psi_- + \frac{i}{2}\frac{1}{2l+1}\rho\cdot\psi_-,\psi_-)_{L^2} - (2l+2)|\psi_+|^2_{L^2} \\
&= |\psi_-|^2_{L^2}\Big(8l(l+1) - \frac{(8l+2)(8l+4)}{8} + \frac{i}{4}\frac{-i(8l+4)}{2l+1} - 2l\Big) = 0,
\end{aligned}$$

thus proving that $F=0$ and consequently (31). In order to check the last two equations one has to make use of the operator $\tilde{D}$. From $D_-\psi_- = 0$ we find

$$0 = \frac{1}{4l+4}D_+^2\psi_- = D_+\psi_+, \tag{34}$$

so

$$\tilde{D}\psi_+ = -iD\psi_+. \tag{35}$$

We take a local orthonormal frame e_i and write (using (1), (5), (28) and (35))

$$\begin{aligned}
0 &\leq \sum_{j=1}^n |\nabla_{e_j}\psi_+ + \frac{1}{2}(e_j - iJ(e_j))\cdot\psi_-|^2 \\
&= |\nabla\psi_+|^2 - \Re e((D+i\tilde{D})\psi_+,\psi_-) \\
&\quad -\frac{1}{4}\sum_{j=1}^n((e_j+iJ(e_j))\cdot(e_j-iJ(e_j))\cdot\psi_-,\psi_-) \\
&= |\nabla\psi_+|^2 - 2\Re e(D\psi_+,\psi_-) + ((m-i\Omega)\cdot\psi_-,\psi_-) \\
&= |\nabla\psi_+|^2 - 8l|\psi_-|^2 + 4l|\psi_-|^2 := |G|^2
\end{aligned}$$

Just as before, we compute the integral over M of the positive function $|G|^2$, namely

$$\begin{aligned}
|G|^2_{L^2} &= |\nabla\psi_+|^2_{L^2} - 4l|\psi_-|^2_{L^2} \\
&= (\nabla^*\nabla\psi_+,\psi_+)_{L^2} - 4l|\psi_-|^2_{L^2} \\
&= (D^2\psi_+ - \frac{1}{4}R\psi_+ + \frac{i}{2}\frac{1}{2l+1}\rho\cdot\psi_+,\psi_+)_{L^2} - 4l|\psi_-|^2_{L^2} \\
&= |\psi_+|^2_{L^2}\Big(16l(l+1) - \frac{(8l+2)(8l+4)}{4} + \frac{i}{2}\frac{-3i(8l+4)}{2l+1} - 4(l+1)\Big) = 0,
\end{aligned}$$

thus proving $G=0$. Consequently $\nabla_X\psi_+ + \frac{1}{2}(X - iJ(X))\cdot\psi_- = 0,\ \forall X\in TM$, which is equivalent to (32) and (33).

Q.E.D.

The above proposition motivates the following

Definition 4.2 *A section ψ of the spinor bundle of a given Spin^c structure on a Kähler manifold (M^{8l+2}, g, J) satisfying*

$$\nabla^A_X \psi = \frac{1}{2} X \cdot \psi + \frac{i}{2} JX \cdot \bar{\psi}, \quad \forall X \in TM \tag{36}$$

is called a Kählerian Killing spinor.

Defining $\psi := \psi_+ - \psi_-$ we immediately obtain the

Corollary 4.2 *Let $\mathcal{C}$ be a complex contact structure on a Kähler–Einstein manifold (M^{8l+2}, g, J). Then the Spin^c structure on M with auxiliary bundle $L_{\mathcal{C}}$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$.*

5 Geometric consequences

We can now state the main application of the above results:

Theorem 5.1 *Let M be a compact Kähler manifold of positive scalar curvature and complex dimension $4l+1$. Then the following statements are equivalent:*

(i) *M is the twistor space of some quaternionic Kähler manifold;*

(ii) *M is Kähler-Einstein and admits a complex contact structure;*

(iii) *There exist a Spin^c structure on M with auxiliary bundle L and spinor bundle ΣM such that $L^{\otimes(2l+1)} \cong \Lambda^{4l+1,0}M$ and ΣM carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$.*

Proof. The implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) follow directly from [13] and Corollary 4.2 respectively.

Suppose now that (iii) holds. The proof of (iii)$\Longrightarrow$(i) parallels that of [8]. We first show that M is Kähler–Einstein. Let $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$ be a spinor field on M which satisfies (36). Taking the covariant derivative with respect to an arbitrary vector field Y we obtain

$$\nabla^A_Y \nabla^A_X \psi = \frac{1}{4}(X \cdot Y + JX \cdot JY) \cdot \psi + \frac{i}{4}(X \cdot JY - JX \cdot Y) \cdot \bar{\psi} + \nabla^A_{\nabla_Y X}\psi, \tag{37}$$

which easily implies

$$\mathcal{R}^A_{Y,X}\psi = \frac{1}{2}(X \cdot Y + JX \cdot JY + 2g(X,Y)) \cdot \psi - ig(X, JY)\bar{\psi}. \tag{38}$$

A local computation shows that the curvature operator $\mathcal{R}^A$ on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2}\omega, \tag{39}$$

where $i\omega := -\frac{i}{2l+1}\rho$ is the curvature form of the auxiliary bundle L, and

$$\mathcal{R}_{X,Y} = \frac{1}{2}\sum_{j<k} R(X,Y,e_j,e_k)\, e_j \cdot e_k \cdot \tag{40}$$

in a local orthonormal frame $\{e_1, ..., e_n\}$. Using the first Bianchi identity for the curvature tensor one obtains ([2], p.16)

$$\sum_i e_i \cdot \mathcal{R}_{e_i,X} = \frac{1}{2}\mathrm{Ric}(X), \tag{41}$$

so, by (39) and (41),

$$\sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi = \sum_j e_j \cdot (\mathcal{R}_{e_j,X}\psi + \frac{i}{2}\omega(e_j,X)\psi) = \frac{1}{2}\mathrm{Ric}(X)\cdot\psi - \frac{i}{2}X \lrcorner\, \omega \cdot \psi. \tag{42}$$

On the other hand, a straightforward computation using (38) and the fact that $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$ yields

$$\begin{aligned}\sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi &= (4l+2)X\cdot\psi + iJX\cdot\bar{\psi} + JX\cdot\Omega\cdot\psi \\ &= (4l+2)X\cdot\psi - 2iJX\cdot\psi,\end{aligned}$$

which, together with (42), gives

$$\Big(\frac{1}{2}\mathrm{Ric}(X) - (4l+2)X\Big)\cdot\psi = \frac{i}{2l+1}J\Big(\frac{1}{2}\mathrm{Ric}(X) - (4l+2)X\Big)\cdot\psi. \tag{43}$$

As ψ never vanishes, if the equality $A\cdot\psi = iB\cdot\psi$ holds for some real vectors $A,\ B$, then $|A| = |B|$. The above formula thus shows that $\mathrm{Ric}(X) = (8l+4)X,\ \forall X \in TM$, so M is Kähler–Einstein with scalar curvature $R = (8l+2)(8l+4)$.

From Theorem 3.1 we deduce that the principal U(1) bundle $\bar{M}$ associated to L admits a canonical metric $\bar{g}$ and a canonical spin structure such that the spinor $\pi^*\psi$ induced by ψ satisfies

$$\bar{\nabla}_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X\psi - \frac{i}{2}J(X)\cdot\bar{\psi}) = \pi^*(\frac{1}{2}X\cdot\psi) \qquad \forall X \in TM, \tag{44}$$

$$\bar{\nabla}_V\pi^*\psi = -\frac{1}{2}\pi^*(\Omega\cdot\psi + \frac{i(8l+4)}{2(2l+1)}\psi) = \pi^*(\frac{i}{2}\bar{\psi}), \tag{45}$$

and (10), (11) show that $\pi^*\psi$ is a Killing spinor on $\bar{M}$.

The spinor field $\pi^*\psi$ induces then a parallel spinor Ψ on the cone $C\bar{M}$ over $\bar{M}$, which is a Kähler manifold (cf. [1], [8], [11]). Moreover, using (45) we can compute the action of the Kähler form of $C\bar{M}$ on Ψ, and obtain that $\Psi \in \Sigma_{2l+3}C\bar{M}$. From C. Bär's classification [1] we know that the restricted holonomy group of $C\bar{M}$ is one of the following: $\mathrm{SU}(4l+2)$, $\mathrm{Sp}(2l+1)$ or 0. The fixed points of the spin representation of $\mathrm{SU}(4l+2)$ ly in Σ_0 and Σ_{4l+2}, so as Ψ is a parallel spinor in $\Sigma_{2l+3}C\bar{M}$, the restricted holonomy group of $C\bar{M}$ cannot be equal to $\mathrm{SU}(4l+2)$. This implies that the universal covering of $C\bar{M}$ is hyperkähler, and thus that the universal covering of $\bar{M}$ is 3–Sasakian (see [1]).

Let us denote by $\bar{M}'$ the $\mathrm{U}(1)$ bundle associated to some maximal root of L. Using the Gysin exact sequence we deduce that $\bar{M}'$ is simply connected (see [2], p.85). Moreover, there exists a canonical covering projection $\bar{M}' \to \bar{M}$, thus proving that $\bar{M}'$ is the universal covering of $\bar{M}$. Consequently, $(\bar{M}', \bar{g}')$ is a 3–Sasakian manifold, where $\bar{g}'$ is the metric induced from $\bar{g}$ via the covering projection. On the other hand, the unit vertical vector field V' on $\bar{M}'$ defines a Sasakian structure, since this is true for its projection V on $\bar{M}$. It is well known that any Sasakian structure on a 3–Sasakian manifold P^{4k-1} of non-constant sectional curvature belongs to the 2–sphere of Sasakian structures. Indeed, the cone CP over P has restricted holonomy $\mathrm{Sp}(k)$, and since the centraliser of $\mathrm{Sp}(k)$ in $\mathrm{U}(2k)$ is just $\mathrm{Sp}(1)$, every Kähler structure on CP must belong to the 2–sphere of Kähler structures of CP, which is equivalent to our statement.

Now, $\bar{M}'$ is regular in the direction of V', so an old result of Tanno implies that it is actually a regular 3–Sasakian manifold (cf. [14]). It is then well known that the quotient of $\bar{M}'$ by the corresponding $\mathrm{SO}(3)$ action is a quaternionic Kähler manifold of positive scalar curvature, say N, and that the twistor space over N is biholomorphic to the quotient of $\bar{M}'$ by each of the S^1 actions given by the Sasakian vector fields, so in particular to M, which is the quotient of $\bar{M}'$ by the S^1 action generated by V'.

Q.E.D.

From Theorem A and Theorem 5.1 we immediately obtain the result of LeBrun mentioned in Section 1:

Corollary 5.1 *Let Z be a Fano contact manifold. Then Z is a twistor space iff it admits a Kähler–Einstein metric.*

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Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Ziegelstr. 13a
D–10099 Berlin