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# A MUSIC-type Algorithm for Detecting Internal Corrosion from Electrostatic Boundary Measurements \*

Habib Ammari<sup>†</sup>   Hyeonbae Kang<sup>‡</sup>   Eunjoo Kim<sup>‡</sup>   Kaouthar Louati<sup>§</sup>  
Michael Vogelius<sup>¶</sup>

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## Abstract

We establish an asymptotic representation formula for the steady state currents caused by internal corrosive parts of small Hausdorff measures. Based on this formula we design a non-iterative method of MUSIC (multiple signal classification) type for localizing the corrosive parts from voltage-to-current observations.

**Mathematics subject classification (MSC2000):** 35R30

**Keywords:** corrosion, asymptotic representation formula, reconstruction, layer potentials

## 1 Introduction

In the field of nondestructive evaluation, new and improved techniques are constantly being sought to facilitate the detection of hidden corrosion in pipelines. Hidden corrosion and defects can cause serious problems and is responsible for millions of dollars annually in cost of replacement infrastructure and lost production, and is a dangerous threat to safety and the environment. It is of great importance to detect and to quantify most unseen potential hazards before they become problems.

Corrosion occurs in many different forms and several different models can be encountered in the literature (see, for example, Kaup and Santosa [16], Kaup et al. [17], Vogelius and Xu [21], Inglese [12], Luong and Santosa [13], Banks et al. [3] and references therein).

In this paper, we adopt the potential model: Laplace's equation holds in the pipeline and the effect of corrosion is described by means of Robin boundary conditions. The study of a model like this is motivated by a number of favorable indications. A first indication is

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<sup>†</sup>Laboratoire Ondes et Acoustique, CNRS & ESPCI, 10 rue Vauquelin, 75231 Paris Cedex 05, France (habib.ammari@polytechnique.fr).

<sup>‡</sup>School of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea (hkang@math.snu.ac.kr, kej@math.snu.ac.kr).

<sup>§</sup>Centre de Mathématiques Appliquées, Ecole Polytechnique, 91128 Palaiseau Cedex, France (louati@cmmapx.polytechnique.fr).

<sup>¶</sup>Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA (vogelius@math.rutgers.edu).

based on the observation that corrosion tends to roughen a surface: in fact, this effect can be modelled by the introduction of a thin coating characterized by rapid oscillations. In the limit where the thickness of the coating goes to zero and the rapidity of the oscillations diverges, the arising of Robin boundary conditions has been observed by Buttazzo and Kohn [5]. On the other hand, the study of electrochemical corrosion processes can be based on Faraday's law which says that the mass loss which is a measure of corrosion is proportional to the normal current flux. In Vogelius and Xu [21] a potential model of this kind of process is proposed. If we linearize with respect to the transfer coefficient the nonlinear boundary conditions in [21], we get Robin boundary conditions.

The inverse problem of corrosion detection consists of the determination of the corrosion damage of an inaccessible part of the surface of a specimen when the available data are a finite number of voltage-to-current pairs on the accessible part. Difficulties of this inverse problem result from its inherent ill-posedness and nonlinearity. Most of the techniques for detecting the corrosion are based on iterative algorithms: least-square algorithms and Newton-type iteration schemes. In these methods, one needs tremendous computational costs and time to get a close image to the true solution, since these iterative algorithms may not converge to an approximate solution.

The purpose of this work is to design a direct (non-iterative) technique for detecting corrosion in pipelines from voltage-to-current observations. Our new algorithm is of MUSIC-type (multiple signal classification) and is based on an accurate asymptotic representation formula for the steady state current perturbations.

To set up our problem mathematically, we consider a simply connected bounded  $\mathcal{C}^2$  domain  $U$  in  $\mathbb{R}^2$ , and a simply connected  $\mathcal{C}^2$  domain  $D$  compactly contained in  $U$ . Let  $\Omega = U \setminus \overline{D}$  represent the specimen to be inspected. We define  $\Gamma_e = \partial U$  and  $\Gamma_i = \partial D$  so that  $\partial\Omega = \Gamma_i \cup \Gamma_e$ . Suppose that the inaccessible surface  $\Gamma_i$  contains some corrosive parts  $I_s$ ,  $s = 1, \dots, m$ . The parts  $I_s$  are well-separated and the reciprocal of the surface impedance (the corrosion coefficient) of each  $I_s$ ,  $s = 1, \dots, m$ , is  $\gamma_s \geq 0$ , not identically zero. We assume that each  $\gamma_s \in \mathcal{C}^1(I_s)$ . Let

$$\gamma(x) = \sum_{s=1}^m \gamma_s \chi_s(x), \quad x \in \Gamma_i, \quad (1.1)$$

where  $\chi_s$  denotes the characteristic function on  $I_s$ . The domain  $\Omega$  in two dimensions may be considered as a cross section of a pipe inside which there are corrosive parts. A typical shape of  $\Omega$  is an annulus. We assume that the one-dimensional Hausdorff measures of  $I_s$  are small:

$$|I_s| = O(\epsilon), \quad s = 1, \dots, m, \quad (1.2)$$

where  $\epsilon$  is a small parameter representing the common order of magnitude of  $I_s$ . Here and throughout this paper  $|\cdot|$  denotes the one-dimensional Hausdorff measure. Then for each  $p \geq 1$ , we have

$$\|\gamma\|_{L^p(\Gamma_i)} \leq C\epsilon^{1/p}. \quad (1.3)$$

The voltage potential  $u_\epsilon$  generated by a voltage  $f$  applied on  $\Gamma_e$  satisfies

$$\begin{cases} \Delta u_\epsilon = 0 & \text{in } \Omega, \\ -\frac{\partial u_\epsilon}{\partial \nu} + \gamma u_\epsilon = 0 & \text{on } \Gamma_i, \\ u_\epsilon = f & \text{on } \Gamma_e, \end{cases} \quad (1.4)$$

where  $\nu$  is the outward unit normal to  $\Omega$  on  $\Gamma_e$  and inward on  $\Gamma_i$ . Since  $f$  may have a variable sign,  $u_\epsilon$  may vanish somewhere on  $\Gamma_i$  and thus, the formula  $\gamma = (1/u_\epsilon) \partial u_\epsilon / \partial \nu$  may be undetermined or highly unstable. The Cauchy data continuation technique as described in the paper by Yang, Choulli, and Cheng [22], which is well justified for the heat equation since the assumption of a non-negative prescribed boundary data is realistic, fails for the electrostatic model for detecting corrosion.

The aim of this paper is to detect the well-separated corrosive parts  $I_s$ , in particular, their locations  $z_s \in I_s$ ,  $s = 1, \dots, m$ , from measurements of the boundary perturbations  $u_\epsilon - u_0$  on  $\Gamma_e$ , where  $u_0$  is the solution in absence of the corrosion, *i.e.*, the solution to the problem

$$\begin{cases} \Delta u_0 = 0 & \text{in } \Omega, \\ -\frac{\partial u_0}{\partial \nu} = 0 & \text{on } \Gamma_i, \\ u_0 = f & \text{on } \Gamma_e. \end{cases} \quad (1.5)$$

The paper is organized as follows. We review in the next section some basic facts on the layer potentials. In Section 3 we establish a representation formula for the unique solution to (1.4). This formula generalizes the formula proved by Kang and Seo in [14, 15]. Our aim in Section 4 is to rigorously derive an asymptotic expansion of  $u_\epsilon - u_0$  on  $\Gamma_e$ . Section 5 is devoted to the imaging of the corrosive parts  $I_s$ . We present a non-iterative method of MUSIC type that allows us to reconstruct their locations  $z_s \in I_s$  from measurements of current boundary perturbations.

## 2 Layer Potentials

Let us first review some well-known properties of the layer potentials for the Laplacian and prove some useful identities. The theory of layer potentials has been developed in relation to boundary value problems.

Let  $\Phi(x)$  be the fundamental solution of the Laplacian  $\Delta$ ,

$$\Phi(x) = \frac{1}{2\pi} \ln|x|, \quad x \neq 0.$$

Let  $D$  be a bounded Lipschitz domain  $D$  in  $\mathbb{R}^2$ , and let  $\Gamma := \partial D$ . Let  $H^1(D)$  denote the set of functions  $f \in L^2(D)$  such that  $\nabla f \in L^2(D)$ . Further, we define  $H^2(D)$  as the space of functions  $u \in H^1(D)$  such that  $\partial^2 u \in L^2(D)$  and the space  $H^{3/2}(D)$  as the interpolation space  $[H^1(D), H^2(D)]_{1/2}$ . Let  $H^{1/2}(\partial D)$  be the set of traces of functions in  $H^1(D)$  and  $H^{-1/2}(\partial D) = (H^{1/2}(\partial D))^*$ . Finally, let  $H^1(\partial D)$  denote the set of functions  $f \in L^2(\partial D)$  such that  $\partial f / \partial T \in L^2(\partial D)$ , where  $\partial / \partial T$  is the tangential derivative.

We will denote the single and double layer potentials of a function  $\varphi \in L^2(\Gamma)$  as  $\mathcal{S}_\Gamma[\varphi]$  and  $\mathcal{D}_\Gamma[\varphi]$ , respectively, where

$$\mathcal{S}_\Gamma[\varphi](x) := \int_\Gamma \Phi(x-y) \varphi(y) ds(y), \quad x \in \mathbb{R}^2, \quad (2.1)$$

$$\mathcal{D}_\Gamma[\varphi](x) := \int_\Gamma \frac{\partial}{\partial \nu_y} \Phi(x-y) \varphi(y) ds(y), \quad x \in \mathbb{R}^2 \setminus \Gamma. \quad (2.2)$$

For a function  $u$  defined on  $\mathbb{R}^2 \setminus \Gamma$ , we denote

$$\frac{\partial u}{\partial \nu} \Big|_{\pm} (x) := \lim_{t \rightarrow 0^+} \langle \nabla u(x \pm t\nu_x), \nu_x \rangle, \quad x \in \Gamma,$$

if the limit exists. Here  $\nu$  is the outward unit normal to  $\Gamma = \partial D$  at  $x$ , and  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{R}^2$ .  $u|_{\pm}$  is defined likewise.

Throughout this paper we assume that the domains under consideration have  $\mathcal{C}^2$  boundaries. It is just for the simplicity and all the results of this paper are valid even if the domains are  $\mathcal{C}^{1,\alpha}$  for some positive  $\alpha$ .

It is well-known, see [11] for example, that for a  $\mathcal{C}^2$ -domain that for  $\varphi \in L^2(\Gamma)$

$$\frac{\partial(\mathcal{S}_{\Gamma}[\varphi])}{\partial \nu} \Big|_{\pm} (x) = \left( \pm \frac{1}{2}I + \mathcal{K}_{\Gamma}^* \right) [\varphi](x), \quad x \in \Gamma, \quad (2.3)$$

$$\mathcal{D}_{\Gamma}[\varphi] \Big|_{\pm} (x) = \left( \mp \frac{1}{2}I + \mathcal{K}_{\Gamma} \right) [\varphi](x), \quad x \in \Gamma, \quad (2.4)$$

where  $\mathcal{K}_{\Gamma}$  is defined by

$$\mathcal{K}_{\Gamma}[\varphi](x) = \frac{1}{2\pi} \int_{\Gamma} \frac{\langle y-x, \nu_y \rangle}{|x-y|^2} \varphi(y) ds(y)$$

and  $\mathcal{K}_{\Gamma}^*$  is the  $L^2$ -adjoint of  $\mathcal{K}_{\Gamma}$ , i.e.,

$$\mathcal{K}_{\Gamma}^*[\varphi](x) = \frac{1}{2\pi} \int_{\Gamma} \frac{\langle x-y, \nu_x \rangle}{|x-y|^2} \varphi(y) ds(y).$$

Since  $\Gamma$  is a  $\mathcal{C}^2$  curve, there is a constant  $C$  such that

$$\frac{|\langle x-y, \nu_x \rangle|}{|x-y|^2} \leq C, \quad x, y \in \Gamma \quad (2.5)$$

Therefore  $\mathcal{K}_{\Gamma}^*$  is a compact operator on  $L^p(\Gamma)$  for any  $p \geq 1$ . Let  $L_0^p(\Gamma) := \{\varphi \in L^p(\Gamma) : \int_{\Gamma} \varphi ds = 0\}$ . A proof of the following lemma can be found in [11].

**Lemma 2.1** *If  $p \geq 1$ , then the operator  $\frac{1}{2}I + \mathcal{K}_{\Gamma}$  is invertible on  $L^p(\Gamma)$  as well as on  $\mathcal{C}^k(\Gamma)$  for  $k = 1, 2$ .*

If  $\Gamma$  is a circle of radius  $r$ , then for  $x, y \in \Gamma$ ,

$$\frac{\langle \nu_x, x-y \rangle}{|x-y|^2} = \frac{1}{2r},$$

and hence

$$\mathcal{K}_{\Gamma}^*[\varphi](x) = \frac{1}{4\pi r} \int_{\Gamma} \varphi(y) ds(y). \quad (2.6)$$

It then follows from (2.3) and (2.6) that

$$\mathcal{S}_{\Gamma}[1](x) = \begin{cases} r \log r & \text{if } |x| \leq r, \\ r \log |x| & \text{if } |x| \geq r. \end{cases} \quad (2.7)$$

Finally, we recall the following mapping properties of the single layer potentials for later use. If  $p > 1$ , there is a constant  $C_p$  such that

$$\|\mathcal{S}_\Gamma[\varphi]\|_{L^\infty(\Gamma)} \leq C_p \|\varphi\|_{L^p(\Gamma)} \quad (2.8)$$

for all  $\varphi \in L^p(\Gamma)$ . In fact, (2.8) can be proved using the Hölder inequality. The following property is well-known:

$$\|\mathcal{S}_\Gamma[\varphi]\|_{H^1(D)} \leq C \|\varphi\|_{L^2(\Gamma)} \quad (2.9)$$

for any  $\varphi \in L^2(\Gamma)$ .

### 3 Representation Formula

Recall that  $\Omega = U \setminus \overline{D}$  and where  $U$  and  $D$  are bounded domains with  $C^2$  boundaries. Let  $\Gamma_e = \partial U$  and  $\Gamma_i = \partial D$ . For  $f \in H^1(\Gamma_e)$ , let  $u_0$  be the solution in absence of the corrosion, *i.e.*, the solution to the problem (1.5). We seek to represent the solution  $u_0$  to (1.5) in the following form:

$$u_0 = \mathcal{D}_{\Gamma_e}[\varphi_0] + \mathcal{S}_{\Gamma_i}[\psi_0] \quad \text{in } \Omega$$

for some functions  $\varphi_0 \in H^1(\Gamma_e)$  and  $\psi_0 \in L^2(\Gamma_i)$ . Then by the boundary conditions in (1.5) and jump relations (2.4) and (2.3), the pair  $(\varphi_0, \psi_0)$  should satisfy

$$\begin{pmatrix} \frac{1}{2}I + \mathcal{K}_{\Gamma_e} & \mathcal{S}_{\Gamma_i} \\ -\frac{\partial}{\partial \nu_i} \mathcal{D}_{\Gamma_e} & -\frac{1}{2}I - \mathcal{K}_{\Gamma_i}^* \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \in H^1(\Gamma_e) \times L^2(\Gamma_i), \quad (3.1)$$

where  $\nu_e$  and  $\nu_i$  indicate the normal derivatives on  $\Gamma_e$  and  $\Gamma_i$  in the direction outward to  $U$  and  $D$ , respectively.

**Lemma 3.1** *For  $p > 1$ , let  $X_p := L^p(\Gamma_e) \times L^p(\Gamma_i)$  and*

$$A_0 := \begin{pmatrix} \frac{1}{2}I + \mathcal{K}_{\Gamma_e} & \mathcal{S}_{\Gamma_i} \\ -\frac{\partial}{\partial \nu_i} \mathcal{D}_{\Gamma_e} & -\frac{1}{2}I - \mathcal{K}_{\Gamma_i}^* \end{pmatrix}. \quad (3.2)$$

*Then  $A_0$  is invertible on  $X_p$  as well as on  $H^1(\Gamma_e) \times L^2(\Gamma_i)$ .*

*Proof.* Since there is a distance between  $\Gamma_e$  and  $\Gamma_i$ , the operator  $A_0$  is a compact perturbation of

$$\begin{pmatrix} \frac{1}{2}I + \mathcal{K}_{\Gamma_e} & 0 \\ 0 & -\frac{1}{2}I - \mathcal{K}_{\Gamma_i}^* \end{pmatrix},$$

which is known to be invertible on  $X_p$  (Lemma 2.1). Therefore, it suffices, by applying the Fredholm alternative, to show that the operator  $A_0$  is injective.

Suppose that  $(\varphi, \psi) \in X_p$  satisfies

$$A_0 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = 0.$$

Since  $(\frac{1}{2}I + \mathcal{K}_{\Gamma_e})[\varphi] = -\mathcal{S}_{\Gamma_i}[\psi]$  on  $\Gamma_e$  and  $\mathcal{S}_{\Gamma_i}[\psi]$  is  $\mathcal{C}^2$  in  $\Gamma_e$ , we have in particular  $\varphi \in H^1(\Gamma_e)$ . Likewise we can show that  $\psi \in L^2(\Gamma_i)$ . Therefore, the function  $u$  defined by  $u = \mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi]$  in  $\Omega$  is the solution in  $H^{3/2}(\Omega)$  to (1.5) with  $f = 0$ . Since such a solution to (1.5) is unique, we have

$$\mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi] = 0 \quad \text{in } \Omega,$$

and hence

$$\mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi] = 0 \quad \text{in } U.$$

It then follows from (2.3) that

$$\begin{aligned} \psi &= \frac{\partial}{\partial \nu} \mathcal{S}_{\Gamma_i}[\psi] \Big|_+ - \frac{\partial}{\partial \nu} \mathcal{S}_{\Gamma_i}[\psi] \Big|_- \\ &= \frac{\partial}{\partial \nu} (\mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi]) \Big|_+ - \frac{\partial}{\partial \nu} (\mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi]) \Big|_- = 0 \quad \text{on } \Gamma_i. \end{aligned}$$

Now we have  $\mathcal{D}_{\Gamma_e}[\varphi] = 0$  in  $U$ . Since  $\frac{1}{2}I + \mathcal{K}_{\Gamma_e}$  is invertible on  $L^2(\Gamma_e)$ , it follows from (2.4) that  $\varphi = 0$ .

Note that  $A_0$  maps  $H^1(\Gamma_e) \times L^2(\Gamma_i)$  into itself. Moreover if  $A_0 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in H^1(\Gamma_e) \times L^2(\Gamma_i)$ , then  $(\frac{1}{2}I + \mathcal{K}_{\Gamma_e})[\varphi] + \mathcal{S}_{\Gamma_i}[\psi] \in H^1(\Gamma_e)$ . Since  $\mathcal{S}_{\Gamma_i}[\psi] \in H^1(\Gamma_e)$ , it follows that  $\psi \in H^1(\Gamma_e)$ . Hence  $A_0$  is invertible on  $H^1(\Gamma_e) \times L^2(\Gamma_i)$ . This completes the proof.  $\square$

As an immediate consequence of Lemma 3.1, we obtain the following theorem.

**Theorem 3.2** *The solution  $u_0$  to (1.5) can be represented as*

$$u_0 = \mathcal{D}_{\Gamma_e}[\varphi_0] + \mathcal{S}_{\Gamma_i}[\psi_0] \quad \text{in } \Omega,$$

where  $(\varphi_0, \psi_0)$  is the unique solution to (3.1).

We now derive a similar representation for  $u_\epsilon$ , the solution to (1.4). The method of derivation is basically the same as before. In this case the relevant integral operator is

$$A_\gamma := \begin{pmatrix} \frac{1}{2}I + \mathcal{K}_{\Gamma_e} & \mathcal{S}_{\Gamma_i} \\ -\frac{\partial}{\partial \nu_i} \mathcal{D}_{\Gamma_e} + \gamma \mathcal{D}_{\Gamma_e} & -\frac{1}{2}I - \mathcal{K}_{\Gamma_i}^* + \gamma \mathcal{S}_{\Gamma_i} \end{pmatrix}.$$

Observe that

$$A_\gamma = A_0 + M_\gamma B, \tag{3.3}$$

where  $M_\gamma$  is the multiplier by  $\gamma$  and

$$B = \begin{pmatrix} 0 & 0 \\ \mathcal{D}_{\Gamma_e} & \mathcal{S}_{\Gamma_i} \end{pmatrix}.$$

Then we have

$$\begin{aligned} \left\| M_\gamma B \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{X_p} &\leq \|\gamma\|_{L^p(\Gamma_i)} \|\mathcal{D}_{\Gamma_\epsilon}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi]\|_{L^\infty(\Gamma_i)} \\ &\leq C\epsilon^{1/p} \left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{X_p}, \end{aligned}$$

for some  $C$  independent of  $\epsilon$  provided that  $p > 1$ . In other words, we have a bound for the operator norm  $\|M_\gamma B A_0^{-1}\|_p$  of  $M_\gamma B A_0^{-1}$  on  $X_p$ :

$$\|M_\gamma B A_0^{-1}\|_p \leq C\epsilon^{1/p}. \quad (3.4)$$

Thus, if  $\epsilon$  is sufficiently small, then  $A_\gamma$  is invertible on  $X_p$  and we have the following theorem.

**Theorem 3.3** *The solution  $u_\epsilon$  to (1.4) can be represented as*

$$u_\epsilon = \mathcal{D}_{\Gamma_\epsilon}[\varphi_\epsilon] + \mathcal{S}_{\Gamma_i}[\psi_\epsilon] \quad \text{in } \Omega,$$

where  $(\varphi_\epsilon, \psi_\epsilon)$  is the unique solution in  $X_2$  to

$$A_\gamma \begin{pmatrix} \varphi_\epsilon \\ \psi_\epsilon \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (3.5)$$

We now derive a complete expansion for  $u_\epsilon$  in terms of  $u_0$ . We first note that because of (3.3)

$$A_\gamma = (I + M_\gamma B A_0^{-1})A_0,$$

and hence

$$A_\gamma^{-1} = A_0^{-1}(I + M_\gamma B A_0^{-1})^{-1}. \quad (3.6)$$

Note that

$$M_\gamma B \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma(\mathcal{D}_{\Gamma_\epsilon}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi]) \end{pmatrix}. \quad (3.7)$$

By expanding (3.6) in terms of the Neumann series, we have

$$A_\gamma^{-1} = A_0^{-1} + \sum_{n=1}^{+\infty} (-1)^n A_0^{-1} (M_\gamma B A_0^{-1})^n.$$

Let

$$\begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} := A_0^{-1} (M_\gamma B A_0^{-1})^n \begin{pmatrix} f \\ 0 \end{pmatrix}, \quad n = 0, 1, 2, \dots \quad (3.8)$$

Then we have

$$\begin{pmatrix} \varphi_\epsilon \\ \psi_\epsilon \end{pmatrix} = \sum_{n=0}^{+\infty} (-1)^n \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}.$$

Moreover, one can see from (3.8) that the following recursive relation holds:

$$\begin{pmatrix} \varphi_{n+1} \\ \psi_{n+1} \end{pmatrix} = A_0^{-1} M_\gamma B \begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix}, \quad n = 0, 1, 2, \dots \quad (3.9)$$



Let

$$u_n := \mathcal{D}_{\Gamma_e}[\varphi_n] + \mathcal{S}_{\Gamma_i}[\psi_n] \quad \text{in } \Omega, \quad n = 0, 1, 2, \dots, \quad (3.10)$$

so that

$$u_\epsilon = \sum_{n=0}^{+\infty} (-1)^n u_n \quad \text{in } \Omega.$$

Then  $u_0$  is the solution to (1.5). By (3.7) and (3.9), we have

$$\begin{pmatrix} \varphi_n \\ \psi_n \end{pmatrix} = A_0^{-1} \begin{pmatrix} 0 \\ \gamma(\mathcal{D}_{\Gamma_e}[\varphi_{n-1}] + \mathcal{S}_{\Gamma_i}[\psi_{n-1}]) \end{pmatrix} = A_0^{-1} \begin{pmatrix} 0 \\ \gamma u_{n-1}|_{\Gamma_i} \end{pmatrix},$$

and hence  $u_n$ ,  $n \geq 1$ , is the solution to

$$\begin{cases} \Delta u_n = 0 & \text{in } \Omega, \\ -\frac{\partial u_n}{\partial \nu} = \gamma u_{n-1} & \text{on } \Gamma_i, \\ u_n = 0 & \text{on } \Gamma_e. \end{cases} \quad (3.11)$$

So far we obtain the following result:

**Theorem 3.4** *Let  $u_\epsilon$  be the solution to (1.4) for a given  $f \in H^1(\Gamma_e)$ . Then*

$$u_\epsilon = \sum_{n=0}^{+\infty} (-1)^n u_n \quad \text{in } \Omega, \quad (3.12)$$

where  $u_0$  is the solution to (1.5) and  $u_n$  is the solution to (3.11). The series converges in  $H^{3/2}(\Omega)$ .

## 4 Asymptotic Expansion

We now derive an asymptotic expansion of  $\frac{\partial}{\partial \nu}(u_\epsilon - u_0)$  on  $\Gamma_e$  as  $\epsilon \rightarrow 0$ , on which our detection algorithm will be based. Let us begin with investigating a regularity property of the boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ -\frac{\partial u}{\partial \nu} = \gamma g & \text{on } \Gamma_i, \\ u = 0 & \text{on } \Gamma_e. \end{cases} \quad (4.1)$$

We have shown that the solution  $u$  to (4.1) can be represented as

$$u := \mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi] \quad \text{in } \Omega,$$

where  $(\varphi, \psi)$  is the solution to

$$A_0 \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \gamma g \end{pmatrix},$$

namely,

$$\begin{cases} \left(\frac{1}{2}I + \mathcal{K}_{\Gamma_e}\right) [\varphi] + \mathcal{S}_{\Gamma_i}[\psi] = 0 & \text{on } \Gamma_e, \\ -\frac{\partial}{\partial \nu} \mathcal{D}_{\Gamma_e}[\varphi] - \left(\frac{1}{2}I + \mathcal{K}_{\Gamma_i}^*\right) [\psi] = \gamma g & \text{on } \Gamma_i. \end{cases} \quad (4.2)$$

Since  $A_0$  is invertible on  $X_p$ , we have in particular that

$$\|\psi\|_{L^p(\Gamma_i)} \leq \|\gamma g\|_{L^p(\Gamma_i)} \leq C\epsilon^{1/p} \|g\|_{L^\infty(\Gamma_i)}. \quad (4.3)$$

Let  $2\delta = \text{dist}(\Gamma_i, \Gamma_e)$  and  $\Omega_\delta := \{x \in \Omega | \text{dist}(x, \Gamma_e) \leq \delta\}$ . Then, we have

$$\|\mathcal{S}_{\Gamma_i}[\psi]\|_{C^2(\Omega_\delta)} \leq C\|\psi\|_{L^p(\Gamma_i)} \leq C\epsilon^{1/p} \|g\|_{L^\infty(\Gamma_i)}.$$

It then follows from the first equation in (4.2) and Lemma 2.1 that

$$\begin{aligned} \|\varphi\|_{C^2(\Gamma_e)} &= \left\| \left(\frac{1}{2}I + \mathcal{K}_{\Gamma_e}\right)^{-1} [(\mathcal{S}_{\Gamma_i}\psi)|_{\Gamma_i}] \right\|_{C^2(\Gamma_e)} \\ &\leq C \|\mathcal{S}_{\Gamma_i}[\psi]\|_{C^2(\Gamma_e)} \\ &\leq C_p \epsilon^{1/p} \|g\|_{L^\infty(\Gamma_i)}. \end{aligned} \quad (4.4)$$

Therefore, we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{C^1(\Gamma_e)} \leq \|\mathcal{D}_{\Gamma_e}[\varphi] + \mathcal{S}_{\Gamma_i}[\psi]\|_{C^2(\Omega_\delta)} \leq C_p \epsilon^{1/p} \|g\|_{L^\infty(\Gamma_i)}.$$

We have proved the following lemma.

**Lemma 4.1** *Let  $u$  be the solution to (4.1). Then, for each  $p > 1$  there is a constant  $C_p$  independent of  $g$  such that*

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{C^1(\Gamma_e)} \leq C_p \epsilon^{1/p} \|g\|_{L^\infty(\Gamma_i)}. \quad (4.5)$$

We now have from (3.11), (3.12), and (4.5) that

$$\sum_{n=2}^{+\infty} \left\| \frac{\partial u_n}{\partial \nu} \right\|_{C^1(\Gamma_e)} \leq C_p \sum_{n=2}^{+\infty} \epsilon^{1/p} \|u_{n-1}\|_{L^\infty(\Gamma_i)}.$$

We also get from (3.10) that

$$\|u_{n-1}\|_{L^\infty(\Gamma_i)} = \|\mathcal{D}_{\Gamma_e}[\varphi_{n-1}] + \mathcal{S}_{\Gamma_i}[\psi_{n-1}]\|_{L^\infty(\Gamma_i)} \leq C_p \left\| \begin{pmatrix} \varphi_{n-1} \\ \psi_{n-1} \end{pmatrix} \right\|_{X_p}.$$

It then follows from (3.4) and (3.8) that

$$\|u_{n-1}\|_{L^\infty(\Gamma_i)} \leq C_p \epsilon^{\frac{n-1}{p}} \|f\|_{L^p(\Gamma_e)},$$

and hence for  $k = 1, 2, \dots$ ,

$$\sum_{n=k}^{+\infty} \left\| \frac{\partial u_n}{\partial \nu} \right\|_{C^1(\Gamma_e)} \leq C_p \sum_{n=k}^{+\infty} \epsilon^{n/p} \|f\|_{L^p(\Gamma_e)} \leq C_p (\epsilon^{k/p} / (1 - \epsilon^{1/p})) \|f\|_{L^p(\Gamma_e)}.$$

Thus we obtain the following theorem.

**Proposition 4.2** For all  $p > 1$ , there is a constant  $C_p$  such that

$$\left\| \frac{\partial u_\epsilon}{\partial \nu} - \frac{\partial u_0}{\partial \nu} \right\|_{C^1(\Gamma_\epsilon)} \leq C_p \epsilon^{1/p} \|f\|_{L^p(\Gamma_\epsilon)}. \quad (4.6)$$

We now derive an approximation of the leading order term of  $\frac{\partial u_\epsilon}{\partial \nu} - \frac{\partial u_0}{\partial \nu}$ . We have

$$\frac{\partial u_\epsilon}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) - \frac{\partial u_1}{\partial \nu}(x) + O(\epsilon^{2/p})$$

uniformly in  $x \in \Gamma_\epsilon$ . Let us now derive a further approximation of  $\frac{\partial u_1}{\partial \nu}$ . For that purpose, let  $G(x, y)$  be the Green's function for the problem (4.1), i.e., for each  $x \in \Omega$ ,  $G(x, y)$  is the solution to

$$\begin{cases} \Delta_y G(x, y) = -\delta_x & \text{in } \Omega, \\ \frac{\partial}{\partial \nu_y} G(x, y) = 0, & y \in \Gamma_i, \\ G(x, y) = 0, & y \in \Gamma_e. \end{cases} \quad (4.7)$$

Then the solution  $u$  to (4.1) is given by

$$u(x) = \int_{\Gamma_i} G(x, y) \gamma(y) g(y) ds(y), \quad x \in \Omega.$$

For each  $x \in \Gamma_e$  we have

$$\frac{\partial u_1}{\partial \nu}(x) = \int_{\Gamma_i} \frac{\partial}{\partial \nu_x} G(x, y) \gamma(y) u_0(y) ds(y).$$

Since  $\frac{\partial}{\partial \nu_x} G(x, y)$  is a  $C^2$  function in  $y \in \Gamma_i$  as long as  $x \in \Gamma_e$ , we have

$$\frac{\partial}{\partial \nu_x} G(x, y) \gamma(y) u_0(y) = \sum_{s=1}^m \left[ \frac{\partial}{\partial \nu_x} G(x, z_s) u_0(z_s) + O(\epsilon) \right] \gamma(y) \chi_s(y),$$

where  $\chi_s$  is the characteristic function of  $I_s$ . Thus we obtain

$$\frac{\partial u_1}{\partial \nu}(x) = \sum_{s=1}^m \frac{\partial}{\partial \nu_x} G(x, z_s) u_0(z_s) \int_{I_s} \gamma ds + O(\epsilon^2).$$

Put

$$\langle \gamma \rangle_s := \int_{I_s} \gamma ds.$$

Then we have obtained the following approximation formula.

**Theorem 4.3** The following formula holds uniformly for  $x \in \Gamma_e$ :

$$\frac{\partial u_\epsilon}{\partial \nu}(x) = \frac{\partial u_0}{\partial \nu}(x) - \sum_{s=1}^m \langle \gamma \rangle_s u_0(z_s) \frac{\partial}{\partial \nu_x} G(x, z_s) + O(\epsilon^{1+\alpha}) \quad (4.8)$$

for some  $\alpha > 0$ .

## 5 MUSIC type algorithm for reconstruction

We now apply the asymptotic expansion (4.8) to design a MUSIC (MULTiple Signal Classification) algorithm for locating small internal corrosive parts from boundary measurements. MUSIC is generally used in signal processing problems as a method for estimating the individual frequencies of multiple-harmonic signals [20]. The MUSIC algorithm makes use of the eigenvalue structure of the multistatic response matrix for the Helmholtz equation and of the spectral structure of the boundary map for the Laplace operator, corresponding to a zero wavenumber. The eigenvectors corresponding to significant eigenvalues span some kind of signal subspace in the sense that they contain nearly all the information about the corrosive parts which can be extracted from the boundary map. The others span some kind of noise subspace. The aim of the MUSIC type of algorithm is to use the eigensystem analysis of a discrete version of the boundary map to determine the location and estimate the size of the small corrosive parts from the signal space.

It is worth mentioning that this algorithm is related to the linear sampling method of Colton and Kirsch [9]. We refer to Cheney [7] and Kirsch [18] for detailed discussions of the connection between the MUSIC algorithm and the linear sampling method.

Define the (voltage-to-current) map  $\Lambda_\gamma$  from  $H^{1/2}(\Gamma_e)$  into  $H^{-1/2}(\Gamma_e)$  by

$$\Lambda_\gamma(f) = \frac{\partial u_\epsilon}{\partial \nu} \Big|_{\Gamma_e}, \quad (5.1)$$

where  $u_\epsilon$  is the solution to (1.4). Let  $\Lambda_0$  be the Dirichlet-to-Neumann map for the case when no corrosion is present. We seek to use  $\Lambda_\gamma - \Lambda_0$  to determine the corrosive parts.

The estimate (4.6) yields that

$$\|(\Lambda_\gamma - \Lambda_0)(f)\|_{C^1(\Gamma_e)} \leq C_p \epsilon^{1/p} \|f\|_{L^2(\Gamma_e)}, \quad (5.2)$$

for all  $p$  with  $1 < p \leq 2$ . The above estimate shows that the operator  $\Lambda_\gamma - \Lambda_0$  originally defined on  $H^{1/2}(\Gamma_e)$  can be extended to an operator on  $L^2(\Gamma_e)$  and it is a compact operator. Thus we have the following lemma.

**Lemma 5.1**  $\Lambda_\gamma - \Lambda_0$  is self-adjoint, positive, and compact on  $L^2(\Gamma_e)$ .

The identity (4.8) shows that

$$(\Lambda_\gamma - \Lambda_0)(f)(x) = - \sum_{s=1}^m \langle \gamma \rangle_s u_0(z_s) \frac{\partial}{\partial \nu_x} G(x, z_s) + O(\epsilon^{1+\alpha}), \quad x \in \Gamma_e. \quad (5.3)$$

where  $1 + \alpha = 2/p$  and the remainder term  $O(\epsilon^{1+\alpha})$  is bounded by  $C_p \epsilon^{1/2} \|f\|_{L^2(\Gamma_e)}$ . Define the operator  $T$  on  $H^{1/2}(\Gamma_e)$  by

$$(Tf)(x) = - \sum_{s=1}^m \langle \gamma \rangle_s u_0(z_s) \frac{\partial}{\partial \nu_x} G(x, z_s), \quad x \in \Gamma_e, \quad (5.4)$$

so that

$$\Lambda_\gamma - \Lambda_0 = T + O(\epsilon^{1+\alpha}). \quad (5.5)$$

Since  $u_0$  depends linearly on  $f$ ,  $T$  is linear. We first establish the following.

**Lemma 5.2** *The operator  $T$  can be extended to a compact, self-adjoint, positive semi-definite operator on  $L^2(\Gamma_e)$ .*

*Proof.* We first observe that  $T$  is a finite-dimensional operator and hence, it is compact. In order to prove that  $T$  is self-adjoint it suffices to show that it is symmetric. Let  $g$  and  $h$  be in  $H^{1/2}(\Gamma_e)$  and denote  $u_0$  and  $v_0$  the solutions to (1.5) corresponding respectively to  $g$  and  $h$ . Then  $v_0$  is given by

$$v_0(y) = - \int_{\Gamma_e} \frac{\partial}{\partial \nu_x} G(x, y) h(x) ds(x), \quad y \in \Omega, \quad (5.6)$$

and hence we have

$$\int_{\Gamma_e} T(g)h = \sum_{s=1}^m \langle \gamma \rangle_s u_0(z_s) v_0(z_s).$$

Consequently, we infer that  $T$  is self-adjoint, positive semi-definite.  $\square$

Introduce now the linear operator  $\mathcal{G} : L^2(\Gamma_e) \rightarrow \mathbb{R}^m$  defined by

$$\mathcal{G}f = (u_0(z_1), \dots, u_0(z_m)), \quad (5.7)$$

where  $u_0$  is the solution to (1.4). Endowing  $\mathbb{R}^m$  with the standard Euclidean inner product, we then obtain

$$\langle \mathcal{G}f, a \rangle = \sum_{s=1}^m a_s u_0(z_s) = - \int_{\Gamma_e} \left( \sum_{s=1}^m a_s \frac{\partial}{\partial \nu_x} G(x, z_s) \right) f(x) ds(x),$$

for arbitrary  $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ . Therefore, the adjoint  $\mathcal{G}^* : \mathbb{R}^m \rightarrow L^2(\Gamma_e)$  is given by

$$\mathcal{G}^* a = - \sum_{s=1}^m a_s \frac{\partial}{\partial \nu_x} G(x, z_s). \quad (5.8)$$

Following Brühl et al. [4], the following characterization of the range of the operator  $T$  can be obtained.

**Lemma 5.3** (i)  $\mathcal{G}^*$  is injective;

(ii)  $\mathcal{G}$  is surjective;

(iii)  $T = \mathcal{G}^* \mathcal{M} \mathcal{G}$ , where

$$\mathcal{M}a = \left( \gamma_1 a_1, \dots, \gamma_m a_m \right), \quad a = (a_1, \dots, a_m) \in \mathbb{R}^m;$$

(iv)  $\text{Range}(T) = \text{span} \left\{ \frac{\partial}{\partial \nu_x} G(x, z_s); s = 1, \dots, m \right\}$ .

*Proof.* Suppose that  $\mathcal{G}^* a = 0$ . Then  $\sum_{s=1}^m a_s v(z_s) = 0$  for any harmonic function  $v$  in  $\Omega$  such that  $\frac{\partial v}{\partial \nu} = 0$  on  $\Gamma_i$ . Since there exist a harmonic function  $v$  in  $\mathbb{R}^2 \setminus \overline{D}$  such that  $\frac{\partial v}{\partial \nu} = 0$  and  $v = \varphi$  on  $\Gamma_i$  for any  $\varphi$  smooth on  $\Gamma_i$ , it follows that  $a_s = 0$  for  $s = 1, \dots, m$ , and thus assertion (i) holds. Assertion (ii) follows from (i) and the well-known relation between the

ranges and the null spaces of adjoint finite-dimensional operators:  $\text{Range}(\mathcal{G}) = \text{Ker}(\mathcal{G}^*)^\perp$ . Using (5.4), (5.7), and (5.8), it is easy to see that (iii) holds. Now according to (iii), we have  $\text{Range}(T) = \text{Range}(\mathcal{G}^* \mathcal{M} \mathcal{G}) = \text{Range}(\mathcal{G}^*)$ , since  $\mathcal{M}$  and  $\mathcal{G}$  are surjective. This yields (iv), and the proof is complete.  $\square$

Now we present the main tool for the identification of the locations  $z_s$  of the small corrosive parts.

**Theorem 5.4** *A point  $z \in \Gamma_i$  belongs to the set  $\{z_s : s = 1, \dots, m\}$  if and only if  $\frac{\partial}{\partial \nu_x} G(\cdot, z)|_{\Gamma_e} \in \text{Range}(T)$ .*

*Proof.* Assume that  $\frac{\partial}{\partial \nu_x} G(\cdot, z)|_{\Gamma_e} \in \text{Range}(T)$ . As a consequence of (iv) of Lemma 5.3,  $g_z$  may be represented as

$$\frac{\partial}{\partial \nu_x} G(x, z) = \sum_{s=1}^m a_s \frac{\partial}{\partial \nu_x} G(x, z_s), \quad \text{for } x \in \Gamma_e. \quad (5.9)$$

Since  $G(x, z) = 0$  for any  $x \in \Gamma_e$  and  $z \in \Omega$ , we have

$$G(x, z) = \sum_{s=1}^m a_s G(x, z_s), \quad \text{for } x \in \Omega. \quad (5.10)$$

This is only possible if  $z \in \{z_s : s = 1, \dots, m\}$ , and so we have established the necessity of this condition. The sufficiency follows immediately from (iv) in Lemma 5.3.  $\square$

We are now ready to present the MUSIC type algorithm for detection of internal corrosion. The finite-dimensional self-adjoint operator  $T$  has the spectral decomposition

$$T = \sum_{p=1}^m \kappa_p v_p \otimes v_p, \quad \|v_p\|_{L^2(\Gamma_e)} = 1, \quad (5.11)$$

where  $\kappa_p$  are (non-zero) eigenvalues of  $T$  and  $v_p$  is the corresponding eigenfunction. Here we assume that  $\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m > 0$ . Let  $P_p : L_0^2(\Gamma_e) \rightarrow \text{span}\{v_1, \dots, v_p\}$ ,  $p = 1, \dots, m$ , be the orthogonal projector  $P_p = \sum_{q=1}^p v_q \otimes v_q$ . It then follows from Theorem 5.4 that

$$z \in \{z_s : s = 1, \dots, m\} \quad \text{iff} \quad (I - P_m) \left( \frac{\partial}{\partial \nu_x} G(\cdot, z)|_{\Gamma_e} \right) = 0, \quad (5.12)$$

or equivalently,

$$z \in \{z_s : s = 1, \dots, m\} \quad \text{iff} \quad \cot \theta(z) = +\infty, \quad (5.13)$$

where the angle  $\theta(z) \in [0, \pi/2)$  is defined by

$$\cot \theta(z) = \frac{\left\| P_m \left( \frac{\partial}{\partial \nu_x} G(\cdot, z)|_{\Gamma_e} \right) \right\|_{L^2(\Gamma_e)}}{\left\| (I - P_m) \left( \frac{\partial}{\partial \nu_x} G(\cdot, z)|_{\Gamma_e} \right) \right\|_{L^2(\Gamma_e)}}. \quad (5.14)$$

On the other hand, since  $\Lambda_\gamma - \Lambda_0$  is a self-adjoint, positive, and compact operator on  $L^2(\Gamma_e)$ , it admits, the spectral decomposition

$$\Lambda_\gamma - \Lambda_0 = \sum_{p=1}^{+\infty} \kappa_p^\epsilon v_p^\epsilon \otimes v_p^\epsilon, \quad \|v_p^\epsilon\|_{L^2(\Gamma_e)} = 1, \quad (5.15)$$

with  $\kappa_1^\epsilon \geq \kappa_2^\epsilon \geq \dots \geq \kappa_m^\epsilon \geq \dots \geq 0$ . Let  $P_p^\epsilon : L^2(\Gamma_\epsilon) \rightarrow \text{span} \{v_1^\epsilon, \dots, v_p^\epsilon\}$ ,  $p = 1, 2, \dots$ , be the orthogonal projector  $P_p^\epsilon = \sum_{q=1}^p v_q^\epsilon \otimes v_q^\epsilon$ . Because of (5.5) where the  $O(\epsilon^{1+\alpha})$  term is bounded by  $C\epsilon^{1+\alpha}\|f\|_{L^2(\Gamma_\epsilon)}$ , standard arguments from perturbation theory for linear operators give (after appropriate enumeration of  $\kappa_p^\epsilon, p = 1, \dots, m$ )

$$\kappa_p^\epsilon = \kappa_p + O(\epsilon^{1+\alpha}) \quad \text{for } p = 1, 2, \dots, \quad (5.16)$$

where we have set  $\kappa_p = 0$  for  $p > m$ , and

$$P_p^\epsilon = P_m + O(\epsilon^\alpha) \quad \text{for } p \geq m. \quad (5.17)$$

Now in view of (5.16) the number  $m$  of inclusions may be estimated by looking for a gap in the set of eigenvalues of  $\Lambda_\gamma - \Lambda_0$ . In order to recover the locations  $z_s, s = 1, \dots, m$ , one can estimate, using (5.17), the  $\cot \theta_p(z)$  by

$$\cot \theta_p(z) = \frac{\left\| P_m^\epsilon \left( \frac{\partial}{\partial \nu_x} G(\cdot, z) \Big|_{\Gamma_\epsilon} \right) \right\|_{L^2(\Gamma_\epsilon)}}{\left\| (I - P_m^\epsilon) \left( \frac{\partial}{\partial \nu_x} G(\cdot, z) \Big|_{\Gamma_\epsilon} \right) \right\|_{L^2(\Gamma_\epsilon)}}. \quad (5.18)$$

If one plots  $\cot \theta_m(z)$  as a function of  $z$ , we may see large values  $\cot \theta_m(z)$  for  $z$  which are close to the locations  $z_s$ .

Once the locations  $z_s$  are found, we can estimate corrosion impedance  $\langle \gamma \rangle_s$ . Our procedure for doing this is the following. Define  $u_s, s = 1, \dots, m$ , to be the solution to

$$\begin{cases} \Delta u_s = 0 & \text{in } \Omega, \\ \frac{\partial u_s}{\partial \nu} = 0 & \text{on } \Gamma_i, \\ u_s(x) = \frac{\partial}{\partial \nu_x} G(x, z_s) & x \in \Gamma_\epsilon. \end{cases} \quad (5.19)$$

It then follows from (5.3) and (5.15) that

$$\begin{aligned} - \sum_{s=1}^m \langle \gamma \rangle_s u_{s'}(z_s) \frac{\partial}{\partial \nu_x} G(\cdot, z_s) &\approx (\Lambda_\gamma - \Lambda_0) \left( \frac{\partial}{\partial \nu_x} G(\cdot, z_{s'}) \right) \\ &\approx \sum_{p=1}^m \kappa_p^\epsilon \left\langle v_p, \frac{\partial}{\partial \nu_x} G(\cdot, z_{s'}) \right\rangle v_p. \end{aligned}$$

By integrating both sides of the above formula against  $v_{s'}$ , we obtain

$$- \sum_{s=1}^m u_{s'}(z_s) \left\langle v_{s'}, \frac{\partial}{\partial \nu_x} G(\cdot, z_s) \right\rangle \langle \gamma \rangle_s \approx \kappa_{s'}^\epsilon \left\langle v_{s'}, \frac{\partial}{\partial \nu_x} G(\cdot, z_{s'}) \right\rangle, \quad (5.20)$$

for  $s' = 1, \dots, m$ . Therefore the values of  $\langle \gamma \rangle_s, s = 1, \dots, m$ , can be calculated by solving a linear system. For the simplest case  $m = 1$  the formula reads

$$\langle \gamma \rangle_1 \approx - \frac{\kappa_1^\epsilon}{u_1(z_1)}.$$

## 5.1 Numerical Results

This section presents some results of numerical experiments of finding the internal corrosive parts,  $I_s \subset \mathbb{R}^2$ ,  $s = 1, 2, \dots, m$ , and illustrate the viability of the MUSIC-type algorithms we have designed. In the following,  $U$  and  $D$  in  $\mathbb{R}^2$  are assumed to be the disks centered at  $(0, 0)$  with radii  $r_e$  and  $r_i$ , respectively. And let  $\Omega = U \setminus \overline{D}$  as before.

We take the Dirichlet-to-Neumann map  $\Lambda_\gamma - \Lambda_0$  on  $\Gamma_e$  as our measurements for the reconstruction. To obtain this simulation data, we solve direct problems (1.4) and (1.5) for  $f(x) = \frac{\partial G}{\partial \nu_x}(x, y)$  for  $x \in \Gamma_e$  and  $y \in \Gamma_i$ . This is done by solving integral equations (3.1) and (3.5).

Let us briefly explain how we compute  $\frac{\partial G}{\partial \nu_x}(x, y)$ . Let  $\tilde{G}$  be the Green's function for  $U$ , that is, for each  $x \in U$ ,  $\tilde{G}(x, y)$  is the solution to

$$\begin{cases} \Delta_y \tilde{G} = 0 & \text{in } U, \\ \frac{\partial}{\partial \nu_y} \tilde{G}(x, y) = 0 & \text{on } \Gamma_e. \end{cases} \quad (5.21)$$

Since  $U$  is a disk, one can compute  $\tilde{G}(x, y)$  explicitly. We then compute  $\frac{\partial G}{\partial \nu_x}(x, y)$  by solving (5.22) in the following lemma.

**Lemma 5.5** *For any  $x \in \Gamma_e$  and  $y \in \Gamma_i$ , let  $G_y(x) := G(x, y)$  and  $\tilde{G}_y(x) = \tilde{G}(x, y)$ . Then the following holds:*

$$\left( I + 2\mathcal{K}_{\Gamma_e}^* - 4 \frac{\partial \mathcal{S}_{\Gamma_i}}{\partial \nu_e} \frac{\partial \mathcal{S}_{\Gamma_e}}{\partial \nu_i} \right) \left[ \frac{\partial G_y}{\partial \nu_e} \right] (x) = 2 \frac{\partial \tilde{G}_y}{\partial \nu_e}(x), \quad x \in \Gamma_e, \quad (5.22)$$

where  $\nu_e$  and  $\nu_i$  denote normal vectors on  $\Gamma_i$  and  $\Gamma_e$ , respectively.

*Proof.* Similarly to (3.1), it is easy to prove that

$$(G_z - \tilde{G}_z)(x) = -\mathcal{S}_{\Gamma_e}[G_z - \tilde{G}_z](x) + \mathcal{S}_{\Gamma_i}[\psi](x), \quad x \in \Omega,$$

where the density  $\psi$  is given by

$$\psi = -2 \frac{\partial \tilde{G}_z}{\partial \nu} + 2 \frac{\partial \mathcal{S}_{\Gamma_e}}{\partial \nu_i} \left[ \frac{\partial (G_z - \tilde{G}_z)}{\partial \nu_e} \right], \quad \text{on } \Gamma_i.$$

Since  $\tilde{G}_y(x) + \mathcal{S}_{\Gamma_e} \left[ \frac{\partial \tilde{G}_y}{\partial \nu} \right] (x) = -\Phi(x, y)$  for  $x \in U$  which was proved in [1], (5.22) follows.  $\square$

For computation, we discretize  $\Gamma_e$  and  $\Gamma_i$  given by

$$\Gamma_e = \{r_e(\cos \theta_n, \sin \theta_n) | \theta_n = 2\pi(n-1)/N, n = 0, 1, \dots, N-1\}$$

and

$$\Gamma_i = \{r_i(\cos \theta_n, \sin \theta_n) | \theta_n = 2\pi(n-1)/N, n = 0, 1, \dots, N-1\},$$

with  $N = 256$ . Put  $x_n := r_e(\cos \theta_n, \sin \theta_n)$  and  $y_n := r_i(\cos \theta_n, \sin \theta_n)$  for  $n = 0, 2, \dots, 255$ . We then compute  $256 \times 256$  matrices  $(\Lambda_\gamma - \Lambda_0) \left( \frac{\partial G}{\partial \nu} \right)$  and  $T \left( \frac{\partial G}{\partial \nu} \right)$  where  $(\Lambda_\gamma - \Lambda_0) \left( \frac{\partial G}{\partial \nu} \right)$



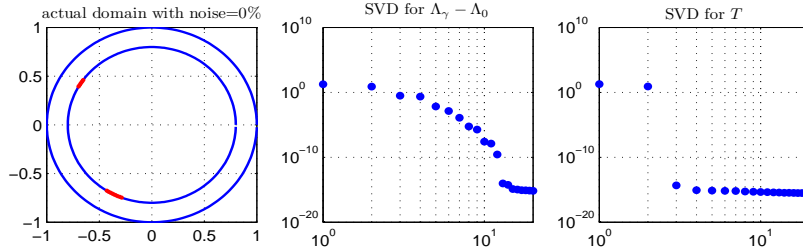


Figure 1: Pipe with two internal corrosive parts and SVD of  $\Lambda_\gamma - \Lambda_0$  and  $T$ .

denotes  $((\Lambda_\gamma - \Lambda_0)(\frac{\partial G}{\partial \nu_x}(\cdot, y_m))(x_n))_{m,n=1}^{256}$  and  $T(\frac{\partial G}{\partial \nu})$  is defined likewise. We then compute the singular value decomposition (SVD) of these two matrices.

Figure 1 is the SVD of  $\Lambda_\gamma - \Lambda_0$  and  $T$  when there are two internal corrosion. It shows that the SVD of  $T$  exhibits the clear drop of the singular values after two significant ones, from which we can conclude that there are two corrosive parts. On the other hand the SVD of  $\Lambda_\gamma - \Lambda_0$  does not have this clear drop of the singular values. It has many additional significant singular values. This is due to the  $O(\epsilon^2)$  discrepancy between  $\Lambda_\gamma - \Lambda_0$  and  $T$ . It means that we cannot determine the number of corrosive parts using the singular values of  $\Lambda_\gamma - \Lambda_0$ . However, this difficulty can be easily remedied.

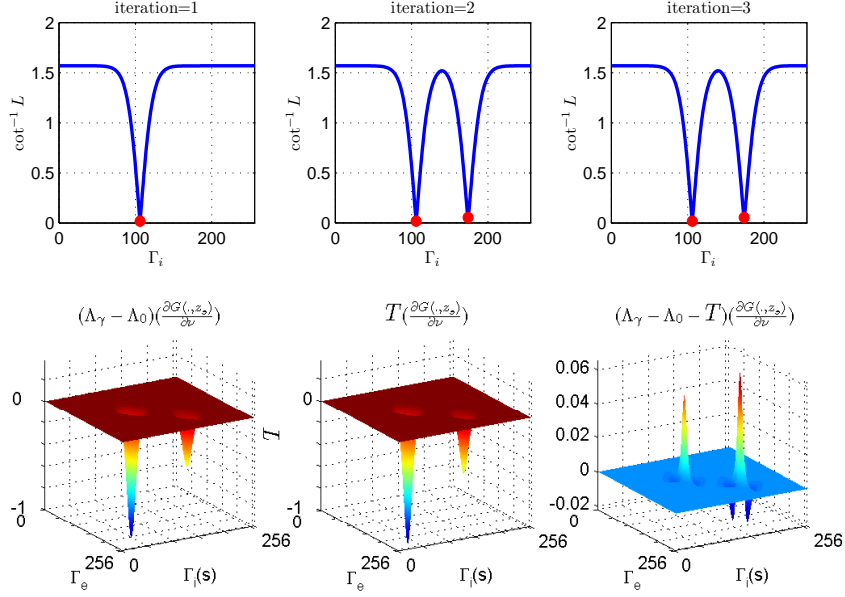
Let  $\lambda_1 \geq \lambda_2 \geq \dots$  be the singular values of  $\Lambda_\gamma - \Lambda_0$ . Using the first  $m$  singular values, we compute the minimal values of  $\cot^{-1} L$  where

$$L := \frac{\|P_m(\frac{\partial G(\cdot, z_s)}{\partial \nu})\|_{L^2(\Gamma_\epsilon)}}{\|(I - P_m)(\frac{\partial G(\cdot, z_s)}{\partial \nu})\|_{L^2(\Gamma_\epsilon)}}. \quad (5.23)$$

We do this process iteratively starting from the largest singular value. The iteration stops if the minimal value of  $\cot^{-1} L$  stabilizes.

**Example 1.** In this example, the outer radius  $r_e = 1$  and the inner one  $r_i = 0.8$  and there are two corrosive parts. Figure 1 shows the locations of the corrosion and the SVD of  $\Lambda_\gamma - \Lambda_0$  and  $T$ . Figure 2 clearly shows that where there are two corrosive parts, the minimum of  $\cot^{-1} L$  stabilizes after two iterations, from which we can conclude that there are two corrosive parts. The minimal points of  $\cot^{-1} L$  are the locations of the corrosive parts. By solving (5.20), we compute the impedance of the corrosion:  $\langle \gamma \rangle_1^\epsilon = 0.3846$ ,  $\langle \gamma \rangle_2^\epsilon = 0.1660$ . The actual impedance are  $\langle \gamma \rangle_1 = 0.3927$ ,  $\langle \gamma \rangle_2 = 0.1767$ .

**Example 2.** In this example we consider the case where there are five corrosive parts. The actual data of the configuration is summarized in the top table of Table 1. The numbering of the corrosive parts is given counter-clockwise starting from the zero angles. Note that the first three corrosive parts have low corrosion coefficients while the other two have relatively high ones. The computational results with 1% noise are summarized in Table 1 and Figure 3. It is interesting to note that the corrosion 2 and 3, which are close to each other and have low corrosion coefficients, are detected as a single one. On the other hand, the corrosion 4 and 5, which have high corrosion coefficients, are clearly detected at early stages of the



The results for iteration

| iteration( $k$ ) | $m$ | $a_k$  | $ a_k - a_{k+1} $ | $z_s^c$            | $ z_s - z_s^c $ |
|------------------|-----|--------|-------------------|--------------------|-----------------|
| 1                | 1   | 0.0175 |                   | (-0.6759, 0.4280)  |                 |
| 2                | 2   | 0.0174 | 2.0746e-008       | (-0.6759, 0.4280)  | 0               |
|                  |     | 0.0547 | 6.0178e-012       | (-0.3597, -0.7146) | 0               |
| 3                | 2   | 0.0174 | 9.7484e-012       | (-0.6759, 0.4280)  | 0               |
|                  |     | 0.0547 | 2.6916e-008       | (-0.3597, -0.7146) | 0               |

Figure 2: The top figure shows  $\cot^{-1} L$  on each iteration. The second line of figures show  $(\Lambda_\gamma - \Lambda_0)(\frac{\partial G}{\partial v})$ ,  $T(\frac{\partial G}{\partial v})$ , and  $(\Lambda_\gamma - \Lambda_0 - T)(\frac{\partial G}{\partial v})$ . In the table,  $m$  is the number of computed corrosive parts,  $a_k := \min(\cot^{-1} L)$  on each iteration step  $k$ ,  $z_s$  and  $z_s^c$  denote the actual locations and computed locations of the corrosive parts.

iteration.

**Example 3.** Figure 4 shows the computational results with various degree of noise. The information of the location and the corrosion coefficients is summarized in Table 2. Observe that the first two corrosion have low corrosion coefficients. They are detected as a single one under 5% noise. The other two corrosive parts, which have high corrosion coefficients, are detected very well even under high ratio noise.

## 6 Conclusion

We have designed a non-iterative algorithm of MUSIC-type for detecting small internal corrosion with multiple parts from boundary measurements. Our methods is based on an

| $m$ | $\gamma_s$ | $z_s$              | $\langle \gamma \rangle_s$ |
|-----|------------|--------------------|----------------------------|
| 5   | 0.1        | (0.0686, 0.6966)   | 0.0120                     |
|     | 0.5        | (-0.1197, 0.6897)  | 0.0430                     |
|     | 0.3        | (-0.4307, 0.5518)  | 0.0206                     |
|     | 1.0        | (0.2519, - 0.6531) | 0.1203                     |
|     | 1.2        | (0.5723, - 0.4031) | 0.1031                     |

| detected data |                    |                              |                 |   |
|---------------|--------------------|------------------------------|-----------------|---|
| $m^c$         | $z_s^c$            | $\langle \gamma \rangle_s^c$ | $ z_s - z_s^c $ | $ \langle \gamma \rangle_s - \langle \gamma \rangle_s^c $ |
| 4             | (-0.1027, 0.6924)  | 0.0514                       | 0.0172          | 0.0084  |
|               | (-0.4170, 0.5622)  | 0.0212                       | 0.0172          | 0.0006  |
|               | (0.2519, - 0.6531) | 0.1181                       | 0               | 0.0021  |
|               | (0.5723, - 0.4031) | 0.1026                       | 0               | 0.0005  |

| result for iteration |       |        |                   |                     |                 |
|----------------------|-------|--------|-------------------|---------------------|-----------------|
| iteration( $k$ )     | $m^c$ | $a_k$  | $ a_k - a_{k+1} $ | $z_s^c$             | $ z_s - z_s^c $ |
| 1                    | 1     | 0.3888 |                   | (0.3889, - 0.5820)  |                 |
| 2                    | 2     | 0.2190 |                   | (-0.1366, + 0.6865) |                 |
|                      |       | 0.3888 |                   | (0.3889, - 0.5820)  |                 |
| 3                    | 3     | 0.2177 |                   | (-0.1366, + 0.6865) |                 |
|                      |       | 0.0165 |                   | (0.2519, - 0.6531)  |                 |
|                      |       | 0.0143 |                   | (0.5723, - 0.4031)  |                 |
| 4                    |       | 0.0742 | 0.0270            | (-0.0686, + 0.6966) | 0.0515          |
|                      |       | 0.1037 | 0.0502            | (-0.4031, + 0.5723) | 0.0344          |
|                      |       | 0.0164 | 7.9909e-011       | (0.2519, - 0.6531)  | 0               |
|                      |       | 0.0141 | 2.0082e-005       | (0.5723, - 0.4031)  | 0               |
| 5                    | 4     | 0.0472 | 3.9986e-005       | (-0.1027, + 0.6924) | 0.0172          |
|                      |       | 0.0535 | 1.3118e-005       | (-0.4170, + 0.5622) | 0.0172          |
|                      |       | 0.0164 | 1.6488e-005       | (0.2519, - 0.6531)  | 0               |
|                      |       | 0.0141 | 6.6178e-005       | (0.5723, - 0.4031)  | 0               |
| 6                    | 4     | 0.0471 | 9.8377e-006       | (-0.1027, + 0.6924) | 0.0172          |
|                      |       | 0.0535 | 3.7071e-006       | (-0.4170, + 0.5622) | 0.0172          |
|                      |       | 0.0164 | 6.9044e-006       | (0.2519, - 0.6531)  | 0               |
|                      |       | 0.0140 | 4.6509e-007       | (0.5723, - 0.4031)  | 0               |

Table 1: Summary of computational results for five corrosive parts. Two corrosive parts with low corrosion coefficient are detected as a single one.

| $m$ | $\gamma_s$ | $z_s$              | $\langle \gamma \rangle_s$ |
|-----|------------|--------------------|----------------------------|
| 4   | 0.1        | (0.5657, + 0.5657) | 0.0137                     |
|     | 0.05       | (0.3597, + 0.7146) | 0.0049                     |
|     | 3.0        | (0.6307, - 0.4922) | 0.2945                     |
|     | 3.0        | (0.6307, - 0.4922) | 0.2945                     |

Table 2: Pipe with four corrosive parts.

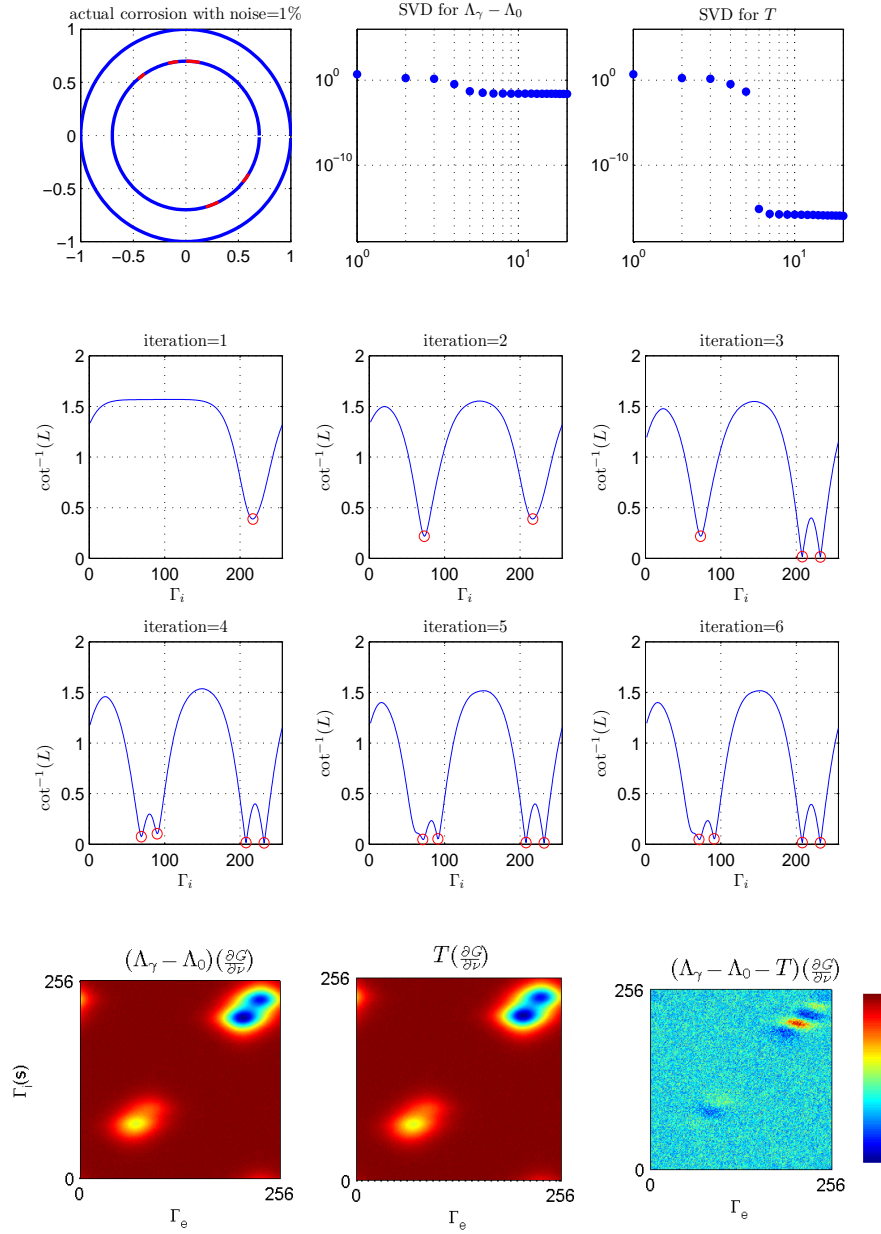


Figure 3: The computational results with noise 1%

asymptotic representation formula for the steady state currents caused by internal corrosive parts of small size. We perform numerical experiment to test viability of the algorithm using  $\Lambda_\gamma - \Lambda_0$  as measured data, which obtained by solving direct problem. The numerical simulation clearly demonstrate that the algorithm works very well even under presence of

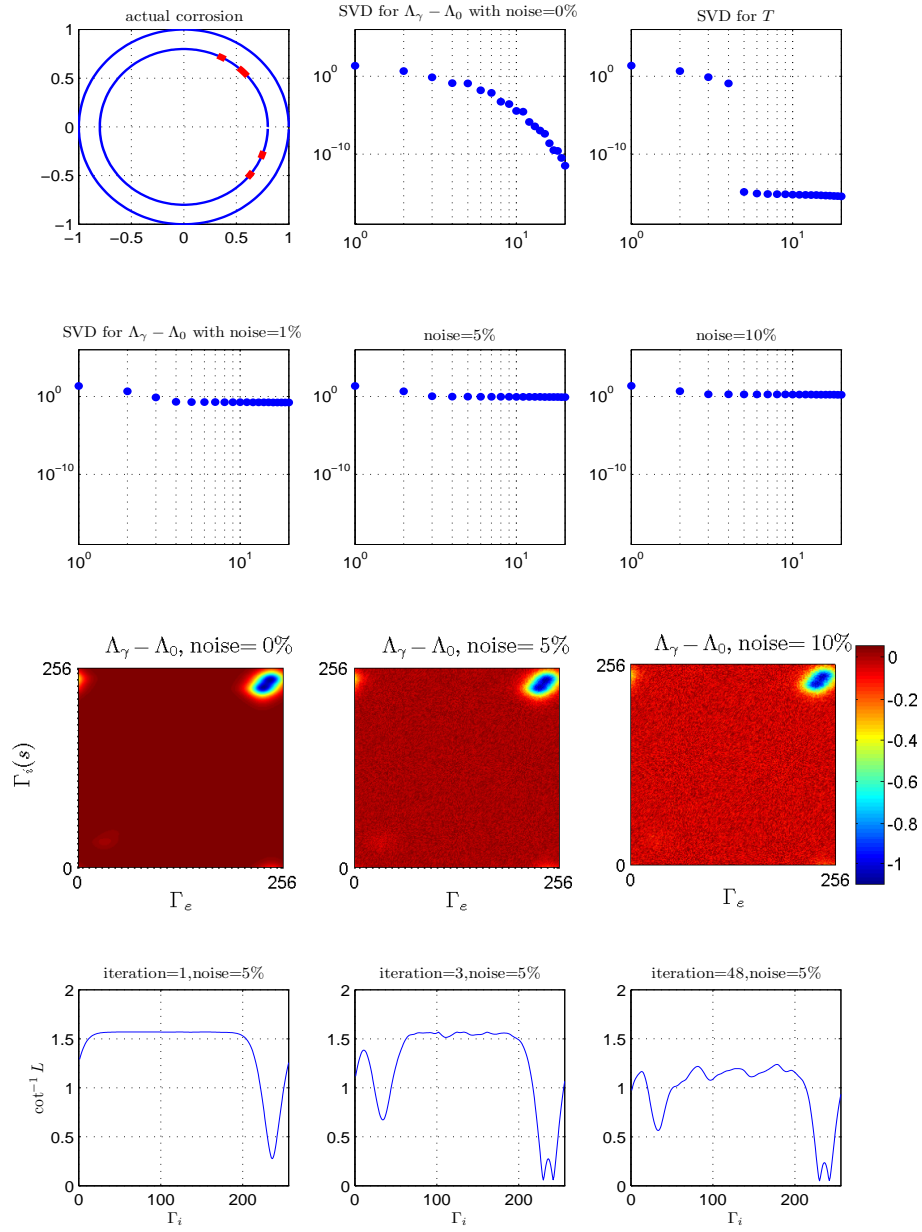


Figure 4: The computational results with noise 0, 1, 5, 10%

high ratio noise, provided that the corrosion coefficient is large enough. This is reasonably since those corrosion with high coefficients can cause significant changes in currents. It is also practically reasonable to detect those with high coefficients since only they can cause

serious damage to the material (or the pipe). It is worth noticing the fact that it is impossible to extract the size of the corrosive parts. We can only reconstruct  $\langle \gamma \rangle_s$  for  $s = 1, \dots, m$ . It is likely that from a certain level of signal-to-noise ratio, higher-order asymptotic expansions of the boundary perturbations yield such important information.

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