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# An Asymptotic Formalism for Reconstructing Small Perturbations of Scatterers from Electric or Acoustic Far-Field Measurements

Mikyoung Lim <sup>\*</sup>      Kaouthar Louati <sup>\*</sup>      Habib Zribi <sup>†</sup>

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## Abstract

In this paper we consider the problem of determining the boundary perturbations of an object from far-field electric or acoustic measurements. Assuming that the unknown object boundary is a small perturbation of a circle, we develop a linearized relation between the far-field data that result from fixed Dirichlet boundary conditions, entering as parameters, and the shape of the object, entering as variables. This relation is used to find the Fourier coefficients of the perturbation of the shape and makes use of an expansion of the Dirichlet-to-Neumann operator.

**Mathematics subject classification (MSC2000):** 35R30

**Keywords:** Small boundary perturbations, conductor, asymptotic expansions, Dirichlet-to-Neumann map, boundary integral method, reconstruction, Laplace equation, Helmholtz equation

## 1 Introduction

The field of inverse shape problems has been an active area of research for several decades. Several related scalar problems belong to this field: electric and acoustic scattering form two large classes. In direct problems one wants to calculate the field outside a given object. In two common situations, one knows either the values of the field on the object (the Dirichlet problem), or the values of the normal derivative of the field on the boundary (the Neumann problem). Direct problems are usually well posed. Inverse shape problems involve reconstructing the shape of an object from measurements of the electric or acoustic field. These problems are ill posed: the solution has an unstable dependence on the input data.

The formulation of the electric scattering problem is based on the quasi-static approximation and the related Laplace equation for the electric scalar potential. When a perfect conductor is exposed to extremely low-frequency electric fields the problem is equivalent to the Dirichlet boundary value problem for the Laplace operator.

The sound-soft acoustic scattering problem is characterized by the condition that the total field vanishes on the boundary of the scatterer. Thus, acoustic scattering is equivalent to the Dirichlet boundary value problem for the Helmholtz operator, with the scattered field equal to the negative of the known incident field.

These two problems are frequently solved by methods of potential theory. The single- and double-layer potentials relate a charge density on the boundary of the object to the

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limiting values of the field and its normal derivative. The resulting integral equations are then solved in an appropriate function space, a common choice being the Lebesgue space  $L^2$ .

In this paper, assuming that the unknown object boundary is a small perturbation of a unit circle, we develop for both electric and acoustic problems a linearized relation between the far-field data that result from fixed Dirichlet boundary conditions, entering as parameters, and the shape of the object, entering as variables. This relation is used to find the Fourier coefficients of the perturbation. Suppose that the angular oscillations in the perturbation are less than  $1/n$ . In order to detect that perturbation, it turns out that one needs to use the first  $n$  eigenvectors of the Dirichlet-to-Neumann operator corresponding to the unperturbed shape as Dirichlet boundary data. We may think that this result is quite general. Our asymptotic formulae for the Dirichlet-to-Neumann operator in terms of the small perturbations of the shape of the scatterer follow the expansions of Dirichlet-to-Neumann operators for rough non-periodic surfaces [8, 4] and for periodic interfaces [9].

Our approach relies on asymptotic expansions of the far-field data with respect to the perturbations in the boundary, in much the same spirit as the recent work [2] and the text [1]. It makes use of an expansion of the Dirichlet-to-Neumann operator. We consider only the two-dimensional case, the extension to three dimensions being obvious. In connection with our work, we should also mention the paper by Kaup and Santosa [6] on detecting corrosion from steady-state voltage boundary perturbations and the work by Tolmasky and Wiegmann [10] on the reconstruction of small perturbations of an interface for the inverse conductivity problem.

## 2 Formulation of the Electric Problem

Let  $D$  be a unit disk in  $\mathbb{R}^2$ . When the boundary value is given by  $\Psi \in C^\infty(\partial D)$  satisfying  $\int \Psi ds = 0$ , the voltage potential outside the disk is given by the harmonic function  $u_0$  which satisfies the following:

$$\begin{cases} \Delta u_0 = 0, & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\ u_0 = \Psi, & \text{on } \partial D, \\ u_0(x) \longrightarrow 0, & \text{as } |x| \longrightarrow +\infty. \end{cases} \quad (2.1)$$

We consider the boundary perturbation of  $D$  given by a Lipschitz function  $f$  and a small number  $\epsilon$ . Define  $\tilde{D}$  by

$$\partial \tilde{D} (= \partial D + \epsilon f \nu) := \left\{ (1 + \epsilon f(\theta)) e_\theta, \quad \theta \in [0, 2\pi] \right\},$$

where  $\nu$  is the outward unit normal vector to  $\partial D$  and  $e_\theta = (\cos \theta, \sin \theta)$ .

Let  $u$  be the voltage potential outside the perturbed domain  $\tilde{D}$  with the fixed Dirichlet data  $\Psi$  on the boundary, i.e.,

$$\begin{cases} \Delta u = 0, & \text{in } \mathbb{R}^2 \setminus \tilde{D}, \\ u(x_\theta) = \Psi, & \text{on } \partial \tilde{D}, \\ u(x) \longrightarrow 0, & \text{as } |x| \longrightarrow +\infty, \end{cases} \quad (2.2)$$

where  $x_\theta = (1 + \epsilon f(\theta)) e_\theta$ .

## 2.1 Representation Formula

It is known that a fundamental solution to the Laplacian in  $\mathbb{R}^2$  is given by

$$\Gamma(x) = \frac{1}{2\pi} \ln|x|.$$

Using Green's theorem, the solution  $u$  to (2.2) can be expressed as

$$u(x) = \int_{\partial\bar{D}} \Gamma(x-y) \frac{\partial u}{\partial \nu_y}(y) d\sigma_y - \int_{\partial\bar{D}} \frac{\partial \Gamma}{\partial \nu_y}(x-y) u(y) d\sigma_y, \quad \text{for } x \in \mathbb{R}^2 \setminus \bar{D}. \quad (2.3)$$

We change the variable from  $y$  to  $y = (1 + \epsilon f(\theta))e_\theta$ . The outward normal vector  $\nu_y$  is

$$\nu_y = \frac{N_\theta}{|N_\theta|}, \quad (2.4)$$

where

$$N_\theta = (1 + \epsilon f(\theta))e_\theta - \epsilon \dot{f} \tau_\theta, \quad \tau_\theta = (-\sin \theta, \cos \theta). \quad (2.5)$$

Here  $\dot{f}$  is the derivative of  $f$  with respect to  $\theta$ . Note that since  $f$  is Lipschitz continuous, it is differentiable almost everywhere. Moreover,

$$|\dot{f}| \leq M,$$

where  $M$  is the Lipschitz constant of  $f$ . We also have

$$d\sigma_y = |N_\theta| d\theta.$$

Now we define the operator  $\mathcal{N}_{\epsilon f}$  using polar coordinates by

$$\mathcal{N}_{\epsilon f}(\Psi)(\theta) := \frac{\partial u}{\partial N_\theta}(1 + \epsilon f(\theta), \theta). \quad (2.6)$$

Then (2.3) becomes

$$u(x) = \frac{1}{2\pi} \int_0^{2\pi} \left[ \ln|x - (1 + \epsilon f(\theta))e_\theta| \mathcal{N}_{\epsilon f}(\Psi) - \frac{\langle (1 + \epsilon f(\theta))e_\theta - x, N_\theta \rangle}{|(1 + \epsilon f(\theta))e_\theta - x|^2} \Psi \right] d\theta. \quad (2.7)$$

Letting  $f = 0$ , we have

$$u_0(x) = \frac{1}{2\pi} \int_0^{2\pi} \left[ \ln|x - e_\theta| \mathcal{N}_0(\Psi) - \frac{\langle e_\theta - x, e_\theta \rangle}{|x - e_\theta|^2} \Psi \right] d\theta, \quad (2.8)$$

where

$$\mathcal{N}_0(\Psi)(\theta) = \partial_r u_0(1, \theta).$$

Note that  $\mathcal{N}_0$  is the *Dirichlet-to-Neumann operator* of the unit disk in  $\mathbb{R}^2$ .

In the next section, we expand  $\mathcal{N}_{\epsilon f}$  in terms of  $\epsilon$ ,  $f$  and  $\Psi$ .

## 2.2 Expansion of $\mathcal{N}_{\epsilon f}$ using the field expansions method

In this section, we use the polar coordinates. We start by the simple computation:

$$\begin{aligned} \mathcal{N}_{\epsilon f}(\Psi)(\theta) &= \left( \begin{array}{c} \frac{\partial u}{\partial r} \\ \frac{1}{r} \frac{\partial u}{\partial \theta} \end{array} \right) \Big|_{r=1+\epsilon f(\theta)} \cdot \left( \begin{array}{c} (1 + \epsilon f(\theta)) \\ -\epsilon \dot{f}(\theta) \end{array} \right) \\ &= (1 + \epsilon f(\theta)) \frac{\partial u}{\partial r} \Big|_{r=1+\epsilon f(\theta)} - \frac{\epsilon \dot{f}(\theta)}{1 + \epsilon f(\theta)} \frac{\partial u}{\partial \theta} \Big|_{r=1+\epsilon f(\theta)} \\ &= (1 + \epsilon f(\theta)) \frac{\partial u}{\partial r} \Big|_{r=1+\epsilon f(\theta)} - \epsilon \dot{f} \frac{\partial u}{\partial \theta} \Big|_{r=1+\epsilon f(\theta)} + O(\epsilon^2) \\ &=: A(\epsilon, f) + O(\epsilon^2). \end{aligned}$$

Here  $O(\epsilon^2)$  depends on  $\|f\|_\infty$  and the Lipschitz constant of  $f$ .

To derive an asymptotic expansion of  $A(\epsilon, f)$ , we apply the method of field expansion (F.E) (see [9]). Firstly, we expand  $u$  only in powers of  $\epsilon$ , i.e.,

$$u(r, \theta) = \sum_{n=0}^{\infty} u_n(r, \theta) \epsilon^n, \quad \mathcal{N}_{\epsilon f}(\Psi) = \sum_{n=0}^{\infty} \mathcal{N}_f^n(\Psi)(r, \theta) \epsilon^n.$$

Now expanding in terms of  $r$  and evaluating the derivatives of  $u$  at  $r = 1 + \epsilon f$ , we obtain that

$$\left. \frac{\partial u}{\partial r} \right|_{r=1+\epsilon f(\theta)} = \sum_{n=0}^{\infty} \sum_{l=0}^n \partial_r^{l+1} u_{n-l}(1, \theta) \frac{f^l}{l!} \epsilon^n,$$

and

$$\left. \frac{\partial u}{\partial \theta} \right|_{r=1+\epsilon f(\theta)} = \sum_{n=0}^{\infty} \sum_{l=0}^n \partial_r^l \partial_\theta u_{n-l}(1, \theta) \frac{f^l}{l!} \epsilon^n.$$

Putting together these expansions, we arrive at

$$\begin{aligned} A(\epsilon, f) &= \sum_{n=0}^{\infty} \sum_{l=0}^n \partial_r^{l+1} u_{n-l}(1, \theta) \frac{f^l(\theta)}{l!} \epsilon^n + f(\theta) \sum_{n=0}^{\infty} \sum_{l=0}^n \partial_r^{l+1} u_{n-l}(1, \theta) \frac{f^l(\theta)}{l!} \epsilon^{n+1} \\ &\quad - \dot{f}(\theta) \sum_{n=0}^{\infty} \sum_{l=0}^n \partial_r^l \partial_\theta u_{n-l}(1, \theta) \frac{f^l(\theta)}{l!} \epsilon^{n+1}, \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{N}_{\epsilon f}(\Psi)(\theta) &= \partial_r u_0(1, \theta) + \epsilon \left( \partial_r u_1(1, \theta) \right. \\ &\quad \left. + \partial_r^2 u_0(1, \theta) f(\theta) + \partial_r u_0(1, \theta) \dot{f}(\theta) - \partial_\theta u_0(1, \theta) \dot{f}(\theta) \right) + O(\epsilon^2). \end{aligned} \quad (2.9)$$

Here  $O(\epsilon^2)$  depends on  $\|f\|_\infty$  and the Lipschitz constant of  $f$ .

We express the equation (2.9) in terms of  $\Psi$  and  $f$ . At first, we have

$$\partial_\theta u_0(1, \theta) = \dot{\Psi}. \quad (2.10)$$

To change  $\partial_r u_1(1, \theta)$ ,  $\partial_r^2 u_0(1, \theta)$  and  $\partial_r u_0(1, \theta)$ , we start from the Fourier series of  $\Psi$ :

$$\Psi(\theta) = \sum_{n=1}^{+\infty} [\hat{a}_n(\Psi) \cos(n\theta) + \hat{b}_n(\Psi) \sin(n\theta)],$$

where

$$\hat{a}_n(\Psi) = \frac{1}{\pi} \int_0^{2\pi} \Psi(\theta) \cos(n\theta) d\theta, \quad \hat{b}_n(\Psi) = \frac{1}{\pi} \int_0^{2\pi} \Psi(\theta) \sin(n\theta) d\theta. \quad (2.11)$$

By the uniqueness of a solution to (2.1),  $u_0$  satisfies

$$u_0(r, \theta) = \sum_{n=1}^{\infty} \left[ \hat{a}_n(\Psi) \frac{\cos(n\theta)}{r^n} + \hat{b}_n(\Psi) \frac{\sin(n\theta)}{r^n} \right].$$

Hence the *Dirichlet-to-Neumann operator*  $\mathcal{N}_0$  of  $D$  is given by

$$\mathcal{N}_0(\Psi)(\theta) (= \partial_r u_0(1, \theta)) = - \sum_{n=1}^{+\infty} [n \hat{a}_n(\Psi) \cos(n\theta) + n \hat{b}_n(\Psi) \sin(n\theta)]. \quad (2.12)$$

We also define the operator  $\mathcal{D}_0$  by

$$\mathcal{D}_0(\Psi)(\theta) := - \sum_{n=1}^{+\infty} [(n+1)\hat{a}_n(\Psi) \cos(n\theta) + (n+1)\hat{b}_n(\Psi) \sin(n\theta)], \quad (2.13)$$

to write

$$\partial_r^2 u_0(1, \theta) = \mathcal{D}_0 \mathcal{N}_0(\Psi)(\theta). \quad (2.14)$$

**Lemma 2.1** *For a given Lipschitz function  $f$  and  $\Psi \in C^\infty([0, 2\pi])$ , we have*

$$\mathcal{N}_{\epsilon f}(\Psi) = \mathcal{N}_f^0(\Psi) + \epsilon \mathcal{N}_f^1(\Psi) + O(\epsilon^2),$$

where

$$\mathcal{N}_f^0(\Psi) = \mathcal{N}_0(\Psi),$$

and

$$\mathcal{N}_f^1(\Psi) = \mathcal{D}_0 \mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) + \mathcal{N}_0(\Psi) f - f \dot{\Psi}. \quad (2.15)$$

Here the remainder  $O(\epsilon^2)$  depends on  $\|f\|_\infty$  and the Lipschitz constant of  $f$ .

*Proof.* From (2.10), (2.12) and (2.13), we are left to calculate  $\partial_r u_1(1, \theta)$ . From the field expansion method, we have

$$u(r, \theta) \Big|_{r=1+\epsilon f(\theta)} = \sum_{n=0}^{\infty} \sum_{l=0}^n \partial_r^l u_{n-l}(1, \theta) \frac{f^l}{l!} \epsilon^n.$$

Therefore

$$u(1 + \epsilon f(\theta), \theta) = u_0(1, \theta) + (u_1(1, \theta) + \partial_r u_0(1, \theta) f) \epsilon + O(\epsilon^2).$$

Since  $u(1 + \epsilon f(\theta), \theta) = u_0(1, \theta) = \Psi(\theta)$ , we deduce that  $u_1(r, \theta)$  satisfies

$$\begin{cases} \Delta u_1 = 0, & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ u_1(1, \theta) = -\mathcal{N}_0(\Psi)(\theta) f(\theta), & \text{on } \partial \tilde{D}, \\ u_1(r, \theta) \longrightarrow 0, & \text{as } r \longrightarrow +\infty. \end{cases}$$

Therefore we have

$$\partial_r u_1(1, \theta) = -\mathcal{N}_0(\mathcal{N}_0(\Psi) f), \quad (2.16)$$

which completes the proof.  $\square$

It is worth mentioning, in connection with our asymptotic expansion in Lemma 2.1, the expansions of Dirichlet-to-Neumann operators for rough non-periodic surfaces [8, 4] and for periodic interfaces [9].

### 2.3 Far Field Expansion

Define  $\tilde{\Omega} := \{x \in \mathbb{R}^2, \text{dist}(x, D) > d_0\}$  and suppose that  $\|f\|_\infty < c_0$ . For  $x \in \tilde{\Omega}$ , we have

$$|x - e_\theta - \epsilon f(\theta) e_\theta|^2 = |x - e_\theta|^2 \left( 1 - 2\epsilon f(\theta) \frac{\langle x - e_\theta, e_\theta \rangle}{|x - e_\theta|^2} \right) + O(\epsilon^2).$$

Using this, we obtain

$$\ln |x - e_\theta - \epsilon f(\theta) e_\theta| = \ln |x - e_\theta| - \epsilon f(\theta) \frac{\langle x - e_\theta, e_\theta \rangle}{|x - e_\theta|^2} + O(\epsilon^2), \quad (2.17)$$

and

$$\begin{aligned}
\frac{\langle e_\theta - x + \epsilon f(\theta)e_\theta, N_\theta \rangle}{|x - e_\theta - \epsilon f(\theta)e_\theta|^2} &= \frac{\langle e_\theta - x, e_\theta \rangle}{|x - e_\theta|^2} \\
&+ \epsilon f(\theta) \frac{\langle e_\theta - x, e_\theta \rangle}{|x - e_\theta|^2} + \epsilon \frac{f(\theta)}{|x - e_\theta|^2} \\
&- \epsilon \dot{f}(\theta) \frac{\langle e_\theta - x, \tau_\theta \rangle}{|x - e_\theta|^2} - 2\epsilon f(\theta) \frac{(\langle e_\theta - x, e_\theta \rangle)^2}{|x - e_\theta|^4} + O(\epsilon^2).
\end{aligned} \tag{2.18}$$

Here  $O(\epsilon^2)$  is bounded by  $c\epsilon^2$  where  $c$  depends only on  $d_0$ ,  $c_0$  and the Lipschitz constant of  $f$ .

From Lemma 2.1, (2.7) and (2.8), we have

$$\begin{aligned}
u(x) - u_0(x) &= \frac{\epsilon}{2\pi} \int_0^{2\pi} \left[ \ln|x - e_\theta| \mathcal{N}_f^1(\Psi) + \frac{\langle e_\theta - x, e_\theta \rangle}{|x - e_\theta|^2} \mathcal{N}_0(\Psi) f \right. \\
&\quad \left. - \frac{x \cdot \tau_\theta}{|x - e_\theta|^2} \Psi \dot{f} + \left( \frac{x \cdot e_\theta - 2}{|x - e_\theta|^2} + \frac{2(\langle e_\theta - x, e_\theta \rangle)^2}{|x - e_\theta|^4} \right) \Psi f \right] d\theta + O(\epsilon^2).
\end{aligned}$$

Integration by parts yields the following lemma.

**Lemma 2.2** *For  $x \in \tilde{\Omega} (= \{x \in \mathbb{R}^2, \text{dist}(x, D) > d_0\})$ , we have*

$$\begin{aligned}
u(x) - u_0(x) &= \frac{\epsilon}{2\pi} \int_0^{2\pi} \ln|x - e_\theta| \left( \mathcal{D}_0 \mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) + \mathcal{N}_0(\Psi) f \right) d\theta \\
&+ \frac{\epsilon}{2\pi} \int_0^{2\pi} \frac{\langle e_\theta - x, e_\theta \rangle}{|x - e_\theta|^2} \mathcal{N}_0(\Psi) f d\theta + \frac{\epsilon}{2\pi} \int_0^{2\pi} \ln|x - e_\theta| \ddot{\Psi} f d\theta \\
&+ \frac{\epsilon}{2\pi} \int_0^{2\pi} \left( -\frac{2}{|x - e_\theta|^2} + \frac{2(x \cdot \tau_\theta)^2 + 2(\langle e_\theta - x, e_\theta \rangle)^2}{|x - e_\theta|^4} \right) \Psi f d\theta + O(\epsilon^2),
\end{aligned} \tag{2.19}$$

where  $O(\epsilon^2)$  is bounded by  $c\epsilon^2$  with  $c$  depending on  $d_0$  and  $\|f\|_\infty$ .

Rewrite (2.19) as

$$u(x) - u_0(x) = \epsilon \mathcal{H}(x) + O(\epsilon^2).$$

By a change of coordinates from Cartesian to polar in (2.19), we get the following asymptotic formula for  $\mathcal{H}$  when  $|x| \gg 1$ .

**Lemma 2.3** *For  $x = re_\varphi$ ,  $r \gg 1$ , we have*

$$\begin{aligned}
\mathcal{H}(x) &= -\frac{1}{2r} \hat{a}_1 \left[ \mathcal{D}_0 \mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) + 2\mathcal{N}_0(\Psi) f + \ddot{\Psi} f \right] \cos(\varphi) \\
&\quad - \frac{1}{2r} \hat{b}_1 \left[ \mathcal{D}_0 \mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) + 2\mathcal{N}_0(\Psi) f + \ddot{\Psi} f \right] \sin(\varphi) + O\left(\frac{1}{r^2}\right),
\end{aligned} \tag{2.20}$$

where  $\hat{a}_1$  and  $\hat{b}_1$  are defined by (2.11).

*Proof.* Since

$$|re_\varphi - e_\theta| = r \left( 1 - \frac{1}{r} \langle e_\varphi, e_\theta \rangle + O\left(\frac{1}{r^2}\right) \right),$$

we have

$$\ln|re_\varphi - e_\theta| = \ln(r) - \frac{1}{r} \langle e_\varphi, e_\theta \rangle + O\left(\frac{1}{r^2}\right), \tag{2.21}$$

and

$$\frac{1}{|re_\varphi - e_\theta|} = \frac{1}{r} + O\left(\frac{1}{r^2}\right). \quad (2.22)$$

Substituting these equations into (2.19), we obtain

$$\mathcal{H}(r, \varphi) = -\frac{1}{2\pi r} \int_0^{2\pi} \langle e_\varphi, e_\theta \rangle \left( \mathcal{D}_0 \mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) + 2\mathcal{N}_0(\Psi) f + \ddot{\Psi} f \right) d\theta + O\left(\frac{1}{r^2}\right),$$

as desired.  $\square$

Now we rewrite (2.20) in terms of the Fourier coefficients of  $\Psi$  and  $f$ .

**Lemma 2.4** *We have*

$$\sum_{k=0}^{\infty} \begin{pmatrix} -\hat{b}_{k+1}(\dot{\Psi}) - \hat{b}_{k-1}(\dot{\Psi}) & \hat{a}_{k+1}(\dot{\Psi}) + \hat{a}_{k-1}(\dot{\Psi}) \\ \hat{a}_{k+1}(\dot{\Psi}) - \hat{a}_{k-1}(\dot{\Psi}) & \hat{b}_{k+1}(\dot{\Psi}) - \hat{b}_{k-1}(\dot{\Psi}) \end{pmatrix} \begin{pmatrix} \hat{a}_k(f) \\ \hat{b}_k(f) \end{pmatrix} = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix},$$

where

$$c_1(\Psi) := 2 \lim_{r \rightarrow +\infty} \hat{a}_1(r\mathcal{H}(r, \theta)),$$

$$d_1(\Psi) := 2 \lim_{r \rightarrow +\infty} \hat{b}_1(r\mathcal{H}(r, \theta)).$$

*Proof.* Note that  $\hat{a}_1(\mathcal{N}_0(\mathcal{N}_0(\Psi) f)) = -\hat{a}_1(\mathcal{N}_0(\Psi) f)$ . From (2.12) and (2.13), we get

$$\begin{aligned} \hat{a}_k \left( \mathcal{D}_0 \mathcal{N}_0(\Psi) + 3\mathcal{N}_0(\Psi) + \ddot{\Psi} \right) &= (k(k+1) - 3k - k^2) \hat{a}_k(\Psi) \\ &= -2k \hat{a}_k(\Psi), \end{aligned}$$

$$\hat{b}_k \left( \mathcal{D}_0 \mathcal{N}_0(\Psi) + 3\mathcal{N}_0(\Psi) + \ddot{\Psi} \right) = -2k \hat{b}_k(\Psi).$$

Moreover, we have

$$k \hat{a}_k(\Psi) = -\hat{b}_k(\dot{\Psi}) \text{ and } k \hat{b}_k(\Psi) = \hat{a}_k(\dot{\Psi}).$$

Letting  $\hat{a}_{-n} = 0$  for  $n \geq 1$  and  $\hat{b}_{-n} = 0$  for  $n \geq 0$ , we observe that

$$\hat{a}_1(FG) = \sum_{k=0}^{\infty} \frac{\hat{a}_k(F)}{2} [\hat{a}_{k+1}(G) + \hat{a}_{k-1}(G)] + \sum_{k=0}^{\infty} \frac{\hat{b}_k(F)}{2} [\hat{b}_{k+1}(G) + \hat{b}_{k-1}(G)],$$

$$\hat{b}_1(FG) = \sum_{k=0}^{\infty} \frac{\hat{a}_k(F)}{2} [\hat{b}_{k+1}(G) - \hat{b}_{k-1}(G)] + \sum_{k=0}^{\infty} \frac{\hat{b}_k(F)}{2} [\hat{a}_{k-1}(G) - \hat{a}_{k+1}(G)].$$

This completes the proof.  $\square$

## 2.4 Algorithm for the Inverse Shape Problem

Define  $\Psi_n$  and  $\Phi_n$ , for  $n \in \mathbb{N}$ , by

$$\Psi_n(\theta) = \frac{1}{n} \sin(n\theta), \Phi_n(\theta) = -\frac{1}{n} \cos(n\theta).$$

From Lemma 2.4, it follows that

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{n-1}(f) \\ \hat{b}_{n-1}(f) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \hat{a}_{n+1}(f) \\ \hat{b}_{n+1}(f) \end{pmatrix} = \begin{pmatrix} c_1(\Psi_n) \\ d_1(\Psi_n) \end{pmatrix}, \quad (2.23)$$



$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_{n-1}(f) \\ \hat{b}_{n-1}(f) \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \hat{a}_{n+1}(f) \\ \hat{b}_{n+1}(f) \end{pmatrix} = \begin{pmatrix} c_1(\Phi_n) \\ d_1(\Phi_n) \end{pmatrix}. \quad (2.24)$$

Thus we obtain that

$$\hat{b}_{n-1}(f) + \hat{b}_{n+1}(f) = c_1(\Psi_n), \quad \hat{a}_{n-1}(f) - \hat{a}_{n+1}(f) = d_1(\Psi_n),$$

and

$$\hat{b}_{n-1}(f) - \hat{b}_{n+1}(f) = d_1(\Phi_n), \quad -\hat{a}_{n-1}(f) - \hat{a}_{n+1}(f) = c_1(\Phi_n).$$

Therefore, we arrive at

$$\hat{b}_{n-1}(f) = \frac{c_1(\Psi_n) + d_1(\Phi_n)}{2},$$

and

$$\hat{a}_{n-1}(f) = \frac{d_1(\Psi_n) - c_1(\Phi_n)}{2}, \quad n \geq 1.$$

This simple calculation shows that in order to detect a perturbation that has oscillations of order  $1/n$ , one needs to use the first  $n$  eigenvectors  $(e^{il\theta}, l = 1, \dots, n)$  of the Dirichlet-to-Neumann operator  $\mathcal{N}_0$  as Dirichlet boundary data. This is a relatively simple but quite deep observation. We conjecture that this result holds for general domains. Another observation is that our asymptotic formula is in fact a low-frequency expansion which holds for fixed  $n$  as  $\epsilon$  goes to zero. It would be interesting to derive an expansion which is valid for high-frequencies, not just for finite  $n$ .

### 3 Formulation of the Acoustic Problem

We study the Helmholtz problem analogously to the Laplacian one. Let  $D$  be a unit disk in  $\mathbb{R}^2$  centered at 0, and  $u_0$  be the solution to the Helmholtz problem with prescribed boundary Dirichlet data  $\Psi \in C^\infty(\partial D)$ , i.e.,

$$\begin{cases} \Delta u_0 + k^2 u_0 = 0, & \text{in } \mathbb{R}^2 \setminus \bar{D}, \\ u_0(1, \theta) = \Psi(\theta), & \text{on } \partial D, \\ \frac{\partial}{\partial r} u_0(r, \theta) - ik u_0(r, \theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty. \end{cases} \quad (3.1)$$

Here we used the polar coordinates  $x = r(\cos \theta, \sin \theta)$ . As before we consider the solution  $u$  corresponding to the perturbed domain  $\tilde{D}$ :

$$\begin{cases} \Delta u + k^2 u = 0, & \text{in } \mathbb{R}^2 \setminus \tilde{D}, \\ u(1 + \epsilon f(\theta), \theta) = \Psi(\theta), & \text{on } \partial \tilde{D}, \\ \frac{\partial}{\partial r} u(r, \theta) - ik u(r, \theta) = o(r^{-\frac{1}{2}}), & r \longrightarrow +\infty, \end{cases} \quad (3.2)$$

where

$$\partial \tilde{D} (= \partial D + \epsilon f\nu) := \{(1 + \epsilon f(\theta))e_\theta, \theta \in [0, 2\pi]\}.$$

#### 3.1 Representation Formula

The outgoing fundamental solution to the Helmholtz operator  $(\Delta + k^2)$  in  $\mathbb{R}^2$  is given by

$$\Phi_k(x, y) = -\frac{i}{4} H_0^{(1)}(k|x - y|),$$

for  $x \neq 0$ , where  $H_0^{(1)}$  is the Hankel function of the first kind of order 0. In other words,  $\Phi_k$  satisfies

$$(\Delta + k^2)\Phi_k(\cdot, y) = \delta_y \quad \text{in } \mathbb{R}^2,$$

and

$$\frac{\partial}{\partial r}\Phi_k(\cdot, y) - ik\Phi_k(\cdot, y) = o(r^{-\frac{1}{2}}) \quad \text{as } r \rightarrow \infty.$$

Using Green's formula, we have

$$u(x) = \int_{\partial\bar{D}} \Phi_k(y, x) \frac{\partial u}{\partial \nu_y}(y) d\sigma_y - \int_{\partial\bar{D}} \frac{\partial \Phi_k}{\partial \nu_y}(y, x) u(y) d\sigma_y, \quad \text{for } x \in \mathbb{R}^2 \setminus \bar{D}. \quad (3.3)$$

Note that

$$\frac{d}{dz} H_0^{(1)}(z) = -H_1^{(1)}(z). \quad (3.4)$$

From (2.4) and (2.5), we get

$$u(x) = -\frac{i}{4} \int_0^{2\pi} \left[ k \frac{\langle (1 + \epsilon f(\theta))e_\theta - x, N_\theta \rangle}{|(1 + \epsilon f(\theta))e_\theta - x|} H_1^{(1)}(k|(1 + \epsilon f(\theta))e_\theta - x|) \Psi(\theta) \right. \\ \left. + H_0^{(1)}(k|(1 + \epsilon f(\theta))e_\theta - x|) \mathcal{N}_{\epsilon f}(\Psi)(\theta) \right] d\theta, \quad (3.5)$$

and

$$u_0(x) = -\frac{i}{4} \int_0^{2\pi} \left[ k \frac{\langle e_\theta - x, e_\theta \rangle}{|e_\theta - x|} H_1^{(1)}(k|e_\theta - x|) + H_0^{(1)}(k|e_\theta - x|) \mathcal{N}_0(\Psi)(\theta) \right] d\theta, \quad (3.6)$$

where

$$\mathcal{N}_{\epsilon f}(\Psi)(\theta) := \frac{\partial u}{\partial N_\theta}(1 + \epsilon f(\theta), \theta),$$

and

$$\mathcal{N}_0(\Psi)(\theta) := \partial_r u_0(1, \theta).$$

### 3.2 Expansion of $\mathcal{N}_{\epsilon f}$

We parametrize the unit circle  $S$ , which is the boundary of  $D$ , by  $\theta \in [0, 2\pi]$  and expand the  $2\pi$ -periodic function  $\Psi(= u|_S)$  as

$$\Psi(\theta) = \sum_{n \in \mathbb{Z}} \hat{c}_n e^{in\theta}.$$

Now we consider the *Dirichlet-to-Neumann operator*  $\mathcal{N}_0$  with respect to  $D$ :

$$\mathcal{N}_0 : u_0|_S \rightarrow \partial_r u_0|_S.$$

By the uniqueness of a solution to (3.1), we find that

$$u_0(r, \theta) = \sum_{n \in \mathbb{Z}} \frac{H_{|n|}^{(1)}(kr)}{H_{|n|}^{(1)}(k)} \hat{c}_n(\Psi) e^{in\theta}.$$

Therefore, in a pseudodifferential fashion,  $\mathcal{N}_0$  can be written as follows (see [7]):

$$\mathcal{N}_0(\Psi)(\theta) = \sum_{n \in \mathbb{Z}} \sigma_1(n, k) \hat{c}_n e^{in\theta}, \quad (3.7)$$

where the so-called *discrete symbol*  $\sigma_1$  of  $\mathcal{N}_0$  is given by

$$\sigma_1 = k \frac{H_{|n|}^{(1)'}(k)}{H_{|n|}^{(1)}(k)} = -k \frac{H_{|n+1|}^{(1)}(k)}{H_{|n|}^{(1)}(k)} + |n|.$$

We also define

$$\mathcal{D}_0(\Psi)(\theta) := \sum_{n \in \mathbb{Z}} \sigma_2(n, k) \hat{c}_n(\Psi) e^{in\theta}, \quad (3.8)$$

with

$$\sigma_2(n, k) = k \frac{H_{|n|}^{(1)''}(k)}{H_{|n|}^{(1)'}(k)}.$$

With this notation, we have the following asymptotic expansion for  $\mathcal{N}_{\epsilon f}$ .

**Lemma 3.1** *For any  $2\pi$ -periodic function  $\Psi \in C^\infty([0, 2\pi])$ , we have that*

$$\mathcal{N}_{\epsilon f}(\Psi) = \mathcal{N}_f^0(\Psi) + \epsilon \mathcal{N}_f^1(\Psi) + O(\epsilon^2),$$

where

$$\mathcal{N}_f^0(\Psi) = \mathcal{N}_0(\Psi),$$

and

$$\mathcal{N}_f^1(\Psi) = \mathcal{D}_0 \mathcal{N}_0(\Psi) f - \mathcal{N}_0(\mathcal{N}_0(\Psi) f) + \mathcal{N}_0(\Psi) f - f \ddot{\Psi}.$$

*Proof.* Note that (2.9), (2.10) and (2.16) are also valid in the case of the Helmholtz problem. We finish the proof by computing  $\partial_r^2 u_0(1, \theta)$ :

$$\begin{aligned} \partial_r^2 u_0(1, \theta) &= \sum_{n \in \mathbb{Z}} k^2 \frac{H_{|n|}^{(1)''}(k)}{H_{|n|}^{(1)'}(k)} \hat{c}_n(\Psi) e^{in\theta} \\ &= \sum_{n \in \mathbb{Z}} \sigma_2(n, k) \sigma_1(n, k) \hat{c}_n(\Psi) e^{in\theta} = \mathcal{D}_0 \mathcal{N}_0(\Psi)(\theta). \end{aligned}$$

□

### 3.3 Far-Field Asymptotic Formula

We define  $\tilde{\Omega} := \{x \in \mathbb{R}^2, \text{dist}(x, D) > d_0\}$  and assume that  $\|f\|_\infty < c_0$ .

**Lemma 3.2** *As  $\epsilon$  tends to 0 and  $x \in \tilde{\Omega}$ , we have*

$$u(x) - u_0(x) = \frac{i\epsilon}{4} \mathcal{H}(x) + O(\epsilon^2),$$

where

$$\begin{aligned} \mathcal{H}(x) &= \int_0^{2\pi} H_0^{(1)}(k|x - e_\theta|) \left[ -(\mathcal{D}_0 \mathcal{N}_0(\Psi) + \mathcal{N}_0(\Psi) + \ddot{\Psi}) f + \mathcal{N}_0(\mathcal{N}_0(\Psi) f) \right](\theta) d\theta \\ &\quad - k^2 \int_0^{2\pi} H_0^{(1)}(k|x - e_\theta|) \frac{(x \cdot \tau_\theta)^2 + (1 - x \cdot e_\theta)^2}{|x - e_\theta|^2} \Psi(\theta) f(\theta) d\theta \\ &\quad + k \int_0^{2\pi} H_1^{(1)}(k|x - e_\theta|) \frac{1 - x \cdot e_\theta}{|x - e_\theta|} \mathcal{N}_0(\Psi)(\theta) f(\theta) d\theta \\ &\quad - k \int_0^{2\pi} H_1^{(1)}(k|x - e_\theta|) \left( \frac{2}{|x - e_\theta|} - 2 \frac{(x \cdot \tau_\theta)^2 + (1 - x \cdot e_\theta)^2}{|x - e_\theta|^3} \right) \Psi(\theta) f(\theta) d\theta. \end{aligned}$$

Here  $O(\epsilon^2)$  is bounded by  $c\epsilon^2$  with  $c$  depending only on  $c_0$  and  $d_0$ .

*Proof.* For any  $x \in \widetilde{\Omega}$ , we have that

$$\begin{aligned} H_0^{(1)}(k|x - e_\theta - \epsilon f(\theta)e_\theta|) &= H_0^{(1)}(k|x - e_\theta|) \\ &\quad + \epsilon k f(\theta) \frac{\langle x - e_\theta, e_\theta \rangle}{|x - e_\theta|} H_1^{(1)}(k|x - e_\theta|) + O(\epsilon^2), \end{aligned}$$

and

$$\begin{aligned} H_1^{(1)}(k|x - e_\theta - \epsilon f(\theta)e_\theta|) &= H_1^{(1)}(k|x - e_\theta|) \\ &\quad - \epsilon f(\theta) \frac{\langle x - e_\theta, e_\theta \rangle}{|x - e_\theta|} \left[ k H_0^{(1)}(k|x - e_\theta|) - \frac{H_1^{(1)}(k|x - e_\theta|)}{|x - e_\theta|} \right] + O(\epsilon^2). \end{aligned}$$

From (3.5), (3.6), and Lemma 3.1, we obtain that

$$\begin{aligned} \mathcal{H}(x) &= - \int_0^{2\pi} H_0^{(1)}(k|x - e_\theta|) \mathcal{N}_f^1(\Psi)(\theta) d\theta \\ &\quad + k \int_0^{2\pi} \frac{\langle e_\theta - x, e_\theta \rangle}{|x - e_\theta|} H_1^{(1)}(k|x - e_\theta|) (\mathcal{N}_0 - I) \Psi(\theta) f(\theta) d\theta \\ &\quad - k \int_0^{2\pi} \frac{1}{|x - e_\theta|} H_1^{(1)}(k|x - e_\theta|) \Psi(\theta) f(\theta) d\theta + k \int_0^{2\pi} \frac{\langle e_\theta - x, \tau_\theta \rangle}{|x - e_\theta|} H_1^{(1)}(k|x - e_\theta|) \Psi(\theta) \dot{f}(\theta) d\theta \\ &\quad - k \int_0^{2\pi} \frac{(\langle e_\theta - x, e_\theta \rangle)^2}{|x - e_\theta|^2} \left( k H_0^{(1)}(k|x - e_\theta|) - \frac{1}{|x - e_\theta|} H_1^{(1)}(k|x - e_\theta|) \right) \Psi(\theta) f(\theta) d\theta \\ &\quad + k \int_0^{2\pi} \frac{(\langle e_\theta - x, e_\theta \rangle)^2}{|x - e_\theta|^3} H_1^{(1)}(k|x - e_\theta|) \Psi(\theta) f(\theta) d\theta. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} &\int_0^{2\pi} H_0^{(1)}(k|x - e_\theta|) \dot{\Psi}(\theta) \dot{f}(\theta) d\theta \\ &= \int_0^{2\pi} \left[ H_1^{(1)}(k|x - e_\theta|) \frac{-kx \cdot \tau_\theta}{|x - e_\theta|} \dot{\Psi} - H_0^{(1)}(k|x - e_\theta|) \ddot{\Psi} \right] f d\theta, \end{aligned}$$

and

$$\begin{aligned} &k \int_0^{2\pi} \frac{\langle e_\theta - x, \tau_\theta \rangle}{|x - e_\theta|} H_1^{(1)}(k|x - e_\theta|) \Psi(\theta) \dot{f}(\theta) d\theta \\ &= - \int_0^{2\pi} H_0^{(1)}(k|x - e_\theta|) \frac{k^2(x \cdot \tau_\theta)^2}{|x - e_\theta|^2} \Psi f d\theta + \int_0^{2\pi} H_1^{(1)}(k|x - e_\theta|) \frac{kx \cdot \tau_\theta}{|x - e_\theta|} \dot{\Psi} f d\theta \\ &\quad - \int_0^{2\pi} H_1^{(1)}(k|x - e_\theta|) \left[ \frac{kx \cdot e_\theta}{|x - e_\theta|} - \frac{2k(x \cdot \tau_\theta)^2}{|x - e_\theta|^3} \right] \Psi f d\theta. \end{aligned}$$

Here we have used (3.4) together with the identity

$$\frac{d}{dz} H_1^{(1)}(z) = H_0^{(1)}(z) - \frac{1}{z} H_1^{(1)}(z).$$

□

**Lemma 3.3** *For any  $x = re_\varphi$  with  $r \gg 1$ , we have*

$$\frac{i}{4} \mathcal{H}(r, \varphi) = \frac{e^{i\frac{\pi}{4}} e^{ikr}}{\sqrt{8\pi kr}} \widetilde{\mathcal{H}}(\varphi) + O\left(\frac{1}{r}\right),$$

where

$$\begin{aligned} \tilde{\mathcal{H}}(\varphi) = \int_0^{2\pi} e^{-ik\langle e_\theta, e_\varphi \rangle} \Big[ & -(\mathcal{D}_0 \mathcal{N}_0(\Psi) + \mathcal{N}_0(\Psi) + \dot{\Psi})(\theta) f(\theta) + \mathcal{N}_0(\mathcal{N}_0(\Psi) f)(\theta) \\ & + ik\langle e_\varphi, e_\theta \rangle \mathcal{N}_0(\Psi)(\theta) f(\theta) - k^2 \Psi(\theta) f(\theta) \Big] d\theta. \end{aligned} \quad (3.9)$$

*Proof.* It is known that, for a fixed  $n$ , the Hankel function of the first kind satisfies

$$H_n^{(1)}(x) = \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{\pi}{4} - n\frac{\pi}{2})} + O(|x|^{-1}), \quad x \gg n.$$

We refer to [3] for more properties of the Hankel function.

Putting  $re_\varphi$  instead of  $x$ , we have

$$H_0^{(1)}(k|re_\varphi - e_\theta|) = \sqrt{\frac{2}{\pi k|re_\varphi - e_\theta|}} e^{i(k|re_\varphi - e_\theta| - \frac{\pi}{4})} + O\left(\frac{1}{r}\right).$$

From the fact that

$$|re_\varphi - e_\theta| = r - \langle e_\varphi, e_\theta \rangle + O\left(\frac{1}{r}\right)$$

and

$$\frac{1}{|re_\varphi - e_\theta|} = \frac{1}{r} + O\left(\frac{1}{r^2}\right),$$

we can obtain

$$\frac{i}{4} H_0^{(1)}(k|re_\varphi - e_\theta|) = \frac{e^{i\frac{\pi}{4}} e^{ikr}}{\sqrt{8\pi kr}} e^{-ik\langle e_\varphi, e_\theta \rangle} + O\left(\frac{1}{r}\right).$$

Similarly, we have

$$\frac{i}{4} H_1^{(1)}(k|re_\varphi - e_\theta|) = (-i) \frac{e^{i\frac{\pi}{4}} e^{ikr}}{\sqrt{8\pi kr}} e^{-ik\langle e_\varphi, e_\theta \rangle} + O\left(\frac{1}{r}\right).$$

Substituting these approximations into the integral expression of  $\mathcal{H}$  in Lemma 3.2, we arrive at the desired result.  $\square$

### 3.4 Algorithm for the Inverse Shape Problem

In this subsection we use the field  $\tilde{\mathcal{H}}(\theta)$  to find the shape of  $\partial\tilde{D}$ .

From the Jacobi-Anger Expansion (see for example [5], Section 3.4):

$$e^{ik \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(k) e^{in\theta},$$

where  $J_n$  is the Bessel function of the first kind, we have

$$e^{-ik\langle e_\theta, e_\varphi \rangle} = \sum_{n=-\infty}^{\infty} (-1)^n J_n(k) e^{in\varphi} e^{-in\theta}. \quad (3.10)$$

From (3.9) and (3.10), we have

$$\tilde{\mathcal{H}}(\varphi) = \sum_{n \in \mathbb{Z}} (-1)^n J_n(k) A_n(\varphi) e^{in\varphi}, \quad (3.11)$$

where

$$A_n(\varphi) = \int_0^{2\pi} e^{-in\theta} \left[ -(\mathcal{D}_0 \mathcal{N}_0(\Psi) + \mathcal{N}_0(\Psi) + \ddot{\Psi})(\theta) f(\theta) + \mathcal{N}_0(\mathcal{N}_0(\Psi) f)(\theta) + ik \langle e_\varphi, e_\theta \rangle \mathcal{N}_0(\Psi)(\theta) f(\theta) - k^2 \Psi(\theta) f(\theta) \right] d\theta.$$

**Lemma 3.4** *We have*

$$\tilde{\mathcal{H}}(\varphi) = \sum_{n \in \mathbb{Z}} (-1)^n J_n(k) \hat{c}_n(\tilde{\mathcal{H}}) e^{in\varphi},$$

where

$$\hat{c}_n(\tilde{\mathcal{H}}) = 2\pi \sum_{p \in \mathbb{Z}} \mathcal{M}_{n,p} \hat{c}_p(\Psi) \hat{c}_{n-p}(f),$$

and

$$\mathcal{M}_{n,p} = \left[ -\sigma_1(p, k) \sigma_2(p, k) + \sigma_1(n, k) \sigma_1(p, k) - \sigma_1(p, k) + p^2 - k^2 \right] - in \sigma_1(p, k).$$

*Proof.* Let

$$\begin{aligned} A_n(\varphi) &= \int_0^{2\pi} e^{-in\theta} \left[ -\mathcal{D}_0 \mathcal{N}_0(\Psi) f + \mathcal{N}_0(\mathcal{N}_0(\Psi) f) - \mathcal{N}_0(\Psi) f - \ddot{\Psi} f - k^2 \Psi f \right] d\theta \\ &\quad + \int_0^{2\pi} e^{-in\theta} ik \langle e_\varphi, e_\theta \rangle \mathcal{N}_0(\Psi) f d\theta \\ &=: I(n) + II(n). \end{aligned}$$

From (3.7) and (3.8), we have

$$\begin{aligned} &\hat{c}_p \left( -\mathcal{D}_0 \mathcal{N}_0(\Psi) - \mathcal{N}_0(\Psi) - \ddot{\Psi} - k^2 \Psi \right) \\ &= [-\sigma_1(p, k) \sigma_2(p, k) - \sigma_1(p, k) + p^2 - k^2] \hat{c}_p(\Psi). \end{aligned}$$

Using the fact that

$$\hat{c}_n(fg) = \sum_{p \in \mathbb{Z}} \hat{c}_p(f) \hat{c}_{n-p}(g),$$

we can obtain

$$\begin{aligned} \hat{c}_n \left[ \mathcal{N}_0(\mathcal{N}_0(\Psi) f) \right] &= \sigma_1(n, k) \hat{c}_n \left[ \mathcal{N}_0(\Psi) f \right] \\ &= \sum_{p \in \mathbb{Z}} \sigma_1(n, k) \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n-p}(f) \end{aligned}$$

and

$$I(n) = 2\pi \sum_{p \in \mathbb{Z}} [-\sigma_1(p, k) \sigma_2(p, k) + \sigma_1(n, k) \sigma_1(p, k) - \sigma_1(p, k) + p^2 - k^2] \hat{c}_p(\Psi) \hat{c}_{n-p}(f). \quad (3.12)$$

Note that

$$\langle e_\varphi, e_\theta \rangle = \frac{e^{i\varphi} e^{-i\theta}}{2} + \frac{e^{-i\varphi} e^{i\theta}}{2}.$$

We have

$$\begin{aligned} II(n) &= ik\pi e^{i\varphi} \hat{c}_{n+1}[\mathcal{N}_0(\Psi)f] + ik\pi e^{-i\varphi} \hat{c}_{n-1}[\mathcal{N}_0(\Psi)f] \\ &= ik\pi e^{i\varphi} \sum_{p \in \mathbb{Z}} \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n+1-p}(f) + ik\pi e^{-i\varphi} \sum_{p \in \mathbb{Z}} \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n-1-p}(f). \end{aligned}$$

Therefore

$$\begin{aligned} & \sum_{n \in \mathbb{Z}} (-1)^n J_n(k) II(n) e^{in\varphi} \\ &= ik\pi \sum_{n \in \mathbb{Z}} (-1)^n J_n(k) e^{i(n+1)\varphi} \sum_{p \in \mathbb{Z}} \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n+1-p}(f) \\ & \quad + ik\pi \sum_{n \in \mathbb{Z}} (-1)^n J_n(k) e^{i(n-1)\varphi} \sum_{p \in \mathbb{Z}} \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n-1-p}(f) \\ &= -ik\pi \sum_{n \in \mathbb{Z}} (-1)^n \left[ J_{n-1}(k) + J_{n+1}(k) \right] e^{in\varphi} \sum_{p \in \mathbb{Z}} \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n-p}(f) \\ &= -ik\pi \sum_{n \in \mathbb{Z}} (-1)^n \frac{2n}{k} J_n(k) e^{in\varphi} \sum_{p \in \mathbb{Z}} \sigma_1(p, k) \hat{c}_p(\Psi) \hat{c}_{n-p}(f). \end{aligned} \tag{3.13}$$

Here we used recurrence relations of Bessel function, i.e.,

$$J_{n-1}(k) + J_{n+1}(k) = \frac{2n}{k} J_n(k).$$

From (3.12) and (3.13), we prove the lemma. □

Suppose now that

$$\Psi = e^{ip\theta}.$$

Then from Lemma 3.4, for each  $n$ , we obtain the coefficient  $\mathcal{M}_{n,p} \hat{c}_{n-p}(f)$  from the far-field measurements. This yields to stable reconstruction of the Fourier coefficients  $\hat{c}_{n-p}(f)$  for all  $n, p, p \leq n$ , such that  $\mathcal{M}_{n,p}$  is not too small.

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