



Non-compact groups, Coherent States, Relativistic Wave equations and the Harmonic Oscillator.

Diego Cirilo-Lombardo

► To cite this version:

Diego Cirilo-Lombardo. Non-compact groups, Coherent States, Relativistic Wave equations and the Harmonic Oscillator.. 2006. hal-00125558v1

HAL Id: hal-00125558

<https://hal.science/hal-00125558v1>

Preprint submitted on 21 Jan 2007 (v1), last revised 31 May 2007 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Non-compact groups, Coherent States, Relativistic Wave equations and the Harmonic Oscillator.

Diego Julio Cirilo-Lombardo

*Bogoliubov Laboratory of Theoretical Physics and
Joint Institute for Nuclear Research,
141980, Dubna, Russian Federation.**

(Date textdate; Received textdate; Revised textdate; Accepted textdate; Published textdate)

Abstract

Relativistic geometrical action for a quantum particle in the superspace is analyzed from theoretical group point of view. To this end an alternative technique of quantization outlined by the authors in a previous work and that is based in the correct interpretation of the square root Hamiltonian, is used. The obtained spectrum of physical states and the Fock construction consist of Squeezed States which correspond to the representations with the lowest weights $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ with four possible (non-trivial) fractional representations for the group decomposition of the spin structure. From the theory of semi-groups the analytical representation of the radical operator in the superspace is constructed, the conserved currents are computed and a new relativistic wave equation is proposed and explicitly solved for the time dependent case. The relation with the Relativistic Schrödinger equation and the Time-dependent Harmonic Oscillator is analyzed and discussed.

PACS: 03.65.-w, 11.30.Pb, 42.50.-p

*Electronic address: diego@thsun1.jinr.ru ; diego77jcl@yahoo.com

Contents

| | |
|---|----|
| I. Introduction and summary | 1 |
| II. The superparticle model | 3 |
| III. Hamiltonian treatment in Lanczo's formulation | 6 |
| IV. Quantization | 9 |
| V. Mass spectrum and square root of a bispinor | 10 |
| VI. Square root Hamiltonian and the Theory of Semigroups | 15 |
| VII. Relation with the relativistic Schrödinger equation: compatibility conditions and probability currents | 19 |
| VIII. Relativistic wave equation | 22 |
| IX. Concluding remarks | 26 |
| X. Acknowledgements | 28 |
| XI. References | 28 |

I. INTRODUCTION AND SUMMARY

The quantum behaviour of a relativistic particle in the superspace, besides to be a useful tool for certain studies and applications of Quantum Field Theory (QFT), is of notable importance in many physical contexts. Time-dependent Landau systems and the electron-monopole system are described naturally by the Super-Heisenberg-Weyl and $OSP(1/2)$ algebras [1-3]. If several more or less well known physical systems are intrinsically supersymmetric in nature an obvious following question was: Can any supersymmetric toy model give us a good picture of not so well known physical systems? Part of the purpose of this paper is to demonstrate the positive answer to this question showing that a relativistic particle in the superspace can describe particles with fractionary spin for the which not concrete action is known.

On the other hand, the Time-Dependent Harmonic Oscillator (TDHO) was demonstrated to be a powerful tool to describe systems with more complicated dynamics in closed form. From the famous reports of Ermakov [4] and Husimi [5] we can see that if any physical problem with a complicated or involved dynamics can be represented faithfully or "mapped" to a TDHO system, this complicated dynamics admits a Coherent State (CS) or Squeezed States (SS) realization [6]. It is clearly important that the model proposed here admits a Coherent State and Squeezed State realization. Most notably, squeezed states have been used in the context of quantum optics [7] and in the context of gravitational wave detection [8]. The correct choice for the realization of the physical states, however, will depend on the symmetry group that defines in some meaning the particular physical system under study. Another part of this work will be devoted to discuss this point and what happens when different algebras can characterize the same physical problem.

In a previous paper we considered the simple model of superparticle of Volkov and Pashnev [9], that is type G4 in the description of Casalbuoni [10,11], in order to quantize it and to obtain the spectrum of physical states with the Hamiltonian remaining in the natural square root form. To this end, we used the Hamiltonian formulation described by Lanczos in [12] and the inhomogeneous Lorentz group as a representation for the obtained physical states [13,14,15]. The quantization of this model was performed completely and the obtained spectrum of physical states, with the Hamiltonian operator in its square root form, was compared with the spectrum obtained with the Hamiltonian in the standard form (i.e.: quadratic in momenta). We showed that the square root Hamiltonian can operate in a natural manner with the states with correspond to the representations with the lowest weights $\lambda_{1,2} = \frac{1}{4}$ and $\lambda_{1,2} = \frac{3}{4}$ and that there are four possible (non-trivial) fractional representations for the group decomposition of the spin structure from the square root Hamiltonian, instead of $(1/2,0)$ and $(0,1/2)$, as the case when the Hamiltonian is quadratic in momentum (e.g. Ref.[9]). This result was a consequence of the geometrical Hamiltonian taken in its natural square root form and the Sannikov-Dirac oscillator representation for the generators of the Lorentz group $SO(3,1)$. In this manner, we also showed that the superparticle relativistic actions as of Ref.[9] are a good geometrical and natural candidate to describe quartionic states [16,17,18] (semions). In this paper we will complete the previous work giving now in more explicit form, how the states can be faithfully represented and realized from the geometrical point of view and from the dynamics of the group manifold.

In this work, strongly motivated for the several fundamental reasons described above, we considered the same simple model of superparticle of Volkov and Pashnev [9] to find the link with the TDHO problem, and for instance, with the CS and SS representations of the physical states obtained in [19] with the Hamiltonian remaining in its square root form. The plan of this paper is as follows: in order to make this work self-contained, in Sections 2, 3 and 4 we borrow from reference [19] the geometrical description, the Hamiltonian treatment and quantization of the superparticle model. Section 5 is devoted to describe the process of quantization and the obtaining of the mass spectrum of the superparticle model under consideration emphasizing the relation between the group representation of the physical states and their CS or SS realizations. In Section 6 from the theory of semigroups we construct a general analytic representation of the square root Hamiltonian. In Sections 7 and 8 the relation of the model with the relativistic Schrödinger equation is discussed and a new relativistic wave equation is proposed. Finally, some conclusions and remarks are given in Section 9.

II. THE SUPERPARTICLE MODEL

In the superspace the coordinates are given not only by the space-time x_μ coordinates, but also for anticommuting spinors θ^α and $\bar{\theta}^{\dot{\alpha}}$. The resulting metric [9,19,20] must be invariant to the action of the Poincare group, and also invariant to the supersymmetry transformations

$$x'_\mu = x_\mu + i \left(\theta^\alpha (\sigma)_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} - \xi^\alpha (\sigma)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \right), \quad \theta'^\alpha = \theta^\alpha + \xi^\alpha, \quad \bar{\theta}'^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \bar{\xi}^{\dot{\alpha}}$$

The simplest super-interval that obeys the requirements of invariance given above, is the following (our choice for the metric tensor is $g_{\mu\nu} = (+ - - -)$)

$$ds^2 = \omega^\mu \omega_\mu + a \omega^\alpha \omega_\alpha - a^* \omega^{\dot{\alpha}} \omega_{\dot{\alpha}} \quad (1)$$

where (to simplify notation from here we avoid the contracted indexes between the spin-tensors $(\sigma)_{\alpha\dot{\beta}}$ and the anticommuting spinors θ^α and $\bar{\theta}^{\dot{\alpha}}$, as usual)

$$\omega_\mu = dx_\mu - i (d\theta \sigma_\mu \bar{\theta} - \theta \sigma_\mu d\bar{\theta}), \quad \omega^\alpha = d\theta^\alpha, \quad \omega^{\dot{\alpha}} = d\bar{\theta}^{\dot{\alpha}}$$

are the Cartan forms of the group of supersymmetry [20].

The spinorial indexes are related as follows

$$\theta^\alpha = \varepsilon^{\alpha\beta} \theta_\beta, \quad \theta_\alpha = \theta^\beta \varepsilon_{\beta\alpha}, \quad \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}, \quad \varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}, \quad \varepsilon_{12} = \varepsilon^{12} = 1$$

and of analog manner for the spinors with punctuated indexes. The complex constants a and a^* in the line element (1) are arbitrary. This arbitrariness for the choice of a and a^* are constrained by the invariance and reality of the interval (1).

As we have extended our manifold to include fermionic coordinates, it is natural to extend also the concept of trajectory of point particle to the superspace. To do this we take the coordinates $x(\tau)$, $\theta^\alpha(\tau)$ and $\bar{\theta}^{\dot{\alpha}}(\tau)$ depending on the evolution parameter τ . Geometrically, the function action that will describe the world-line of the superparticle, is

$$S = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{\omega}_\mu^\circ \dot{\omega}^\mu + a \dot{\theta}^\alpha \dot{\theta}_\alpha - a^* \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}} = \int_{\tau_1}^{\tau_2} d\tau L(x, \theta, \bar{\theta}) \quad (2)$$

where $\dot{\omega}_\mu^\circ = \dot{x}_\mu - i \left(\dot{\theta}^\alpha \sigma_\mu \bar{\theta} - \theta^\alpha \sigma_\mu \dot{\bar{\theta}} \right)$ and the upper point means derivative with respect to the parameter τ , as usual.

The momenta, canonically conjugated to the coordinates of the superparticle, are

$$\begin{aligned} \mathcal{P}_\mu &= \partial L / \partial \dot{x}^\mu = (m^2/L) \dot{\omega}_\mu^\circ \\ \mathcal{P}_\alpha &= \partial L / \partial \dot{\theta}^\alpha = i \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \dot{\bar{\theta}}^{\dot{\beta}} + (m^2 a/L) \dot{\theta}_\alpha \\ \mathcal{P}_{\dot{\alpha}} &= \partial L / \partial \dot{\bar{\theta}}^{\dot{\alpha}} = i \mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} - (m^2 a^*/L) \dot{\bar{\theta}}_{\dot{\alpha}} \end{aligned} \quad (3)$$

It is difficult to study this system in the Hamiltonian formalism framework because of the constraints and the nullification of the Hamiltonian. As the action (2) is invariant under reparametrizations of the evolution parameter

$$\tau \rightarrow \tilde{\tau} = f(\tau)$$

one way to overcome this difficulty is to make the dynamic variable x_0 the time. For this, it is sufficient to introduce the concept of integration and derivation in supermanifolds as we introduce in Section 6 to write the action in the form

$$S = -m \int_{\tau_1}^{\tau_2} \dot{x}_0 d\tau \sqrt{[1 - iW_{,0}^0]^2 - [x_{,0}^i - W_{,0}^i]^2 + \dot{x}_0^{-2} \left(a \dot{\theta}_\alpha \dot{\theta}^\alpha - a^* \dot{\bar{\theta}}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}} \right)}$$

where the $W_{,0}^\mu$ was defined by

$$\dot{\omega}^0 = \dot{x}^0 [1 - iW_{,0}^0]$$

[1] We take the Berezin convention for the Grassmannian derivatives: $\delta F(\theta) = \frac{\partial F}{\partial \theta} \delta \theta$

$$\dot{\omega}^i = \dot{x}^0 [x_{,0}^i - iW_{,0}^i]$$

whence $x_0(\tau)$ turns out to be the evolution parameter

$$S = -m \int_{x_0(\tau_1)}^{x_0(\tau_2)} dx_0 \sqrt{[1 - iW_{,0}^0]^2 - [x_{,0}^i - W_{,0}^i]^2 + a\dot{\theta}^\alpha \dot{\theta}_\alpha - a^* \dot{\bar{\theta}}^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}}} \equiv \int dx_0 L$$

Physically, this parameter (we call it the dynamical parameter) is the time measured by an observer's clock in the rest frame.

Therefore, the invariance of a theory with respect to the invariance of the coordinate evolution parameter means that one of the dynamic variables of the theory ($x_0(\tau)$ in this case) becomes the observed time with the corresponding non-zero Hamiltonian

$$\begin{aligned} H &= \mathcal{P}_\mu \dot{x}^\mu + \Pi^\alpha \dot{\theta}_\alpha + \Pi^{\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}} - L \\ &= \sqrt{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} \end{aligned} \quad (4)$$

where

$$\begin{aligned} \Pi_\alpha &= \mathcal{P}_\alpha + i \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \\ \Pi_{\dot{\alpha}} &= \mathcal{P}_{\dot{\alpha}} - i \mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \end{aligned}$$

That gives the well known mass shell condition and losing, from the quantum point of view, the operatibility of the Hamiltonian.

In the paper[9], where this type of superparticle action was explicitly presented, the problem of nullification of Hamiltonian was avoided in the standard form. This means that the analog to a mass shell condition (4) in superspace was introduced by means of a multiplier (einbein) to obtain a new Hamiltonian

$$H = \frac{\varkappa}{2} \left\{ m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right) \right\} \quad (5)$$

(here \varkappa , as in Ref. [9], is a constant with the dimensions of the square of a length). With this Hamiltonian it is clear that in order to perform the quantization of the superparticle the problems disappear: \mathcal{P}_0 is restored into the new Hamiltonian, and the square root is eliminated. The full spectrum from this Hamiltonian was obtained in [9] where the quantum Hamiltonian referred to the center of mass was

$$H_{cm} = \frac{\varkappa}{2} \left\{ m^2 - M^2 + \frac{2^{3/2} M}{|a|} \left[1 - (\sigma_0)_{\alpha\dot{\beta}} \bar{s}^{\dot{\beta}} s^\alpha \right] \right\} \quad (6a)$$

with the mass distribution of the physical states being the following : two scalar supermultiplets $M_{1s} = \frac{2^{1/2}}{|a|} + \sqrt{\frac{2}{|a|} + m^2}$ and $M_{2s} = \sqrt{\frac{2}{|a|} + m^2} - \frac{2^{1/2}}{|a|}$; and one vector supermultiplet $M_v = m$. The Fock's construction in the center of mass for the eq.(6a) (Hamiltonian quadratic in momenta) consists of the following vectors:

$$\begin{aligned}
S_1 &= |0\rangle e^{iMt} & \Xi_{1\alpha} &= \bar{d}_{\dot{\alpha}} |0\rangle e^{iMt} & P_1 &= \bar{d}^{\dot{\beta}} \bar{d}_{\dot{\beta}} |0\rangle e^{iMt} \\
\Xi_{2\alpha} &= \bar{s}_{\dot{\alpha}} |0\rangle e^{iMt} & V_{\alpha\beta} &= \bar{s}_{\dot{\alpha}} \bar{d}_{\dot{\beta}} |0\rangle e^{iMt} & \Xi_{3\alpha} &= \bar{s}_{\dot{\alpha}} \bar{d}^{\dot{\beta}} \bar{d}_{\dot{\beta}} |0\rangle e^{iMt} \\
P_2 &= \bar{s}^{\dot{\alpha}} \bar{s}_{\dot{\alpha}} |0\rangle e^{iMt} & \Xi_{4\alpha} &= \bar{d}_{\dot{\alpha}} \bar{s}^{\dot{\beta}} \bar{s}_{\dot{\beta}} |0\rangle e^{iMt} \\
S_2 &= \bar{d}^{\dot{\beta}} \bar{d}_{\dot{\beta}} \bar{s}^{\dot{\alpha}} \bar{s}_{\dot{\alpha}} |0\rangle e^{iMt}
\end{aligned} \tag{6b}$$

where operators s_{α} and d_{α} acting on the vacuum give zero: $s_{\alpha} |0\rangle = d_{\alpha} |0\rangle = 0$.

We will show in this report that it is possible, in order to quantize the superparticle action, to remain the Hamiltonian in the square root form. As it is very obvious, in the form of square root the Hamiltonian operator is not linearly proportional with the operator $n_s = \bar{s}^{\dot{\beta}} s^{\alpha}$. The Fock construction for the Hamiltonian into the square root form agrees formally with the description given above for reference[9], but the operability of this Hamiltonian is over basic states with lowest helicities $\lambda = 1/4, 3/4$. This means that the superparticle Hamiltonian preserving the square root form operates over physical states of particles with fractional quantum statistics and fractional spin (quartions).

III. HAMILTONIAN TREATMENT IN LANCZO'S FORMULATION

In order to solve our problem from the dynamical and quantum mechanical point of view, we will use the formulation given in[12,21]. This Hamiltonian formulation for dynamical systems was proposed by C. Lanczos and allows us to preserve the square root form in the new Hamiltonian. We start from expression (4)

$$H = \sqrt{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^{\alpha} \Pi_{\alpha} - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)}$$

if

$$\frac{dt}{d\tau} \equiv \frac{dx^0}{d\tau} = g(\mathcal{P}_0, \mathcal{P}_i, \Pi_{\alpha}, \Pi_{\dot{\alpha}}, x_0, x_i, \theta_{\alpha}, \bar{\theta}_{\dot{\alpha}})$$

with the arbitrary function g given by

$$g = \frac{\left(\sqrt{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} - \mathcal{P}_0 \right) \sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)}}{m^2 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right) - \mathcal{P}_0^2} \quad (7)$$

the new Hamiltonian \mathcal{H} takes the required "square root" form

$$\mathcal{H} \equiv g(H + \mathcal{P}_0) = \sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} \quad (8)$$

where we shall set $\mathcal{H} = 0$ (now depending on $2n + 2$ canonical variables), and the variable \mathcal{P}_0 is clearly identified by the dynamical expression

$$\frac{d\mathcal{P}_0}{d\tau} = -g \frac{\partial \mathcal{H}}{\partial x^0} \quad \text{or} \quad \frac{d\mathcal{P}_0}{d\tau} = -\frac{\partial \mathcal{H}}{\partial t} \quad (9)$$

This means that $\mathcal{P}_0 = -H + \text{const.}$ Concerning the equations of motion, the integration and derivatives are consistent with the geometrical treatment of supermanifolds that we will describe with some detail in Section 6.

In order to make an analysis of the dynamics of our problem, we can compute the Poisson brackets between all the canonical variables and their conjugate momentum [9,10,11]

$$\dot{\mathcal{P}}_\mu = \{\mathcal{P}_\mu, \mathcal{H}\}_{pb} = 0 \quad (10)$$

$$\dot{\theta}^\alpha = \{\theta^\alpha, \mathcal{H}\}_{pb} = \frac{1}{a} \frac{\Pi^\alpha}{\mathcal{H}} \quad (11)$$

$$\dot{\bar{\theta}}^{\dot{\alpha}} = \left\{ \bar{\theta}^{\dot{\alpha}}, \mathcal{H} \right\}_{pb} = -\frac{1}{a^*} \frac{\Pi^{\dot{\alpha}}}{\mathcal{H}} \quad (12)$$

$$\dot{x}_\mu = \{x_\mu, \mathcal{H}\}_{pb} = \frac{1}{\mathcal{H}} \left\{ \mathcal{P}_\mu + \frac{i}{a} \Pi^\alpha (\sigma_\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} + \frac{i}{a^*} \theta^\alpha (\sigma_\mu)_{\alpha\dot{\beta}} \Pi^{\dot{\beta}} \right\} \quad (13)$$

$$\dot{\Pi}_\alpha = \{\Pi_\alpha, \mathcal{H}\}_{pb} = \frac{2i}{a^* \mathcal{H}} \mathcal{P}_{\alpha\dot{\beta}} \Pi^{\dot{\beta}} \quad (14)$$

$$\dot{\Pi}_{\dot{\alpha}} = \{\Pi_{\dot{\alpha}}, \mathcal{H}\}_{pb} = \frac{-2i}{a \mathcal{H}} \Pi^\beta \mathcal{P}_{\beta\dot{\alpha}} \quad (15)$$

where $\mathcal{P}_{\alpha\dot{\beta}} \equiv \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}}$. From the above expressions the set of classical equations to solve is easily seen

$$\ddot{\Pi}_\alpha = - \left(\frac{4\mathcal{P}^2}{|a|^2 \mathcal{H}^2} \right) \dot{\Pi}_\alpha \quad (16)$$

$$\ddot{\Pi}_{\dot{\alpha}} = - \left(\frac{4\mathcal{P}^2}{|a|^2 \mathcal{H}^2} \right) \dot{\Pi}_{\dot{\alpha}} \quad (17)$$

Assigning $\frac{4\mathcal{P}^2}{|a|^2 \mathcal{H}^2} \equiv \omega^2$, and having account for $\Pi_{\alpha}^+ = -\Pi_{\dot{\alpha}}$, the solution to the equations (16) and (17) takes the form

$$\Pi_{\alpha} = \xi_{\alpha} e^{i\omega\tau} + \eta_{\alpha} e^{-i\omega\tau}$$

$$\Pi_{\dot{\alpha}} = -\bar{\eta}_{\dot{\alpha}} e^{i\omega\tau} - \bar{\xi}_{\dot{\alpha}} e^{-i\omega\tau} \quad (18)$$

By means of the substitution of above solutions into (14) and (15), we find the relation between ξ_{α} and η_{α}

$$\eta_{\alpha} = \left(\frac{2}{a^* \mathcal{H} \omega} \right) \mathcal{P}_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}$$

From eqs. (18) and above we obtain

$$\Pi_{\alpha} = \xi_{\alpha} e^{i\omega\tau} + \left(\frac{2}{a^* \mathcal{H} \omega} \right) \mathcal{P}_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} e^{-i\omega\tau} \quad (19)$$

$$\Pi_{\dot{\alpha}} = - \left(\frac{2}{a \mathcal{H} \omega} \right) \xi^{\beta} \mathcal{P}_{\beta\dot{\alpha}} e^{i\omega\tau} - \bar{\xi}_{\dot{\alpha}} e^{-i\omega\tau} \quad (20)$$

where we used the fact that the constant two-component spinors ξ_{α} verify $\bar{\xi}_{\dot{\alpha}} = \xi_{\alpha}^+$. Integrating expressions (11) and (12), we obtain explicitly the following

$$\theta_{\alpha} = \zeta_{\alpha} - \frac{i}{a \mathcal{H} \omega} \left[\xi_{\alpha} e^{i\omega\tau} - \frac{2}{a^* \mathcal{H} \omega} \mathcal{P}_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}} e^{-i\omega\tau} \right] \quad (21)$$

$$\bar{\theta}_{\dot{\alpha}} = \bar{\zeta}_{\dot{\alpha}} + \frac{i}{a^* \mathcal{H} \omega} \left[-\frac{2}{a \mathcal{H} \omega} \xi^{\beta} \mathcal{P}_{\beta\dot{\alpha}} e^{i\omega\tau} + \bar{\xi}_{\dot{\alpha}} e^{-i\omega\tau} \right] \quad (22)$$

where ζ_{α} and $\bar{\zeta}_{\dot{\alpha}} = \zeta_{\alpha}^+$ are two-component constant spinors.

Analogically, from expression (13), we obtain x_{μ} in explicit form

$$\begin{aligned} x_{\mu} = & q_{\mu} - \frac{1}{\mathcal{H}} \left[\mathcal{P}_{\mu} - \frac{\omega \mathcal{H}}{\mathcal{P}^2} (\xi \sigma_{\mu} \bar{\xi}) \right] \tau + \frac{1}{\mathcal{H} \omega} \left[\frac{1}{a} e^{i\omega\tau} (\xi \sigma_{\mu} \bar{\zeta}) + \frac{1}{a^*} e^{-i\omega\tau} (\zeta \sigma_{\mu} \bar{\xi}) \right] + \\ & + \frac{\mathcal{P}_{\mu}}{2\mathcal{P}^2} \left[\zeta^{\alpha} \xi_{\alpha} e^{i\omega\tau} - \bar{\zeta}^{\dot{\alpha}} \bar{\xi}_{\dot{\alpha}} e^{-i\omega\tau} \right] \end{aligned} \quad (23)$$

IV. QUANTIZATION

Because of the correspondence between classical and quantum dynamics, the Poisson brackets between coordinates and canonical impulses are transformed into quantum commutators and anti-commutators

$$\begin{aligned} [x_\mu, \mathcal{P}_\mu] &= i \{x_\mu, \mathcal{P}_\mu\}_{pb} = -ig_{\mu\nu} \\ \{\theta^\alpha, \mathcal{P}_\beta\} &= i \{\theta^\alpha, \mathcal{P}_\beta\}_{pb} = -i\delta_\beta^\alpha \\ \{\theta^{\dot{\alpha}}, \mathcal{P}_{\dot{\beta}}\} &= i \{\theta^{\dot{\alpha}}, \mathcal{P}_{\dot{\beta}}\}_{pb} = -i\delta_{\dot{\beta}}^{\dot{\alpha}} \end{aligned} \quad (24)$$

and the new Hamiltonian (8) operates quantically as follows

$$\sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} |\Psi\rangle = 0 \quad (25)$$

where $|\Psi\rangle$ are the physical states. From the (anti)commutation relations (24) it is possible to obtain easily the commutators between the variables $\xi_\alpha, \bar{\xi}_{\dot{\alpha}}, \zeta_\alpha, \bar{\zeta}_{\dot{\alpha}}, q_\mu, \mathcal{P}_\mu$

$$\{\xi_\alpha, \bar{\xi}_{\dot{\alpha}}\} = -\mathcal{P}_{\alpha\dot{\alpha}} \quad \{\zeta_\alpha, \bar{\zeta}_{\dot{\alpha}}\} = -\left(\frac{1}{2\mathcal{P}^2}\right) \mathcal{P}_{\alpha\dot{\alpha}} \quad [q_\mu, \mathcal{P}_\mu] = -ig_{\mu\nu} \quad (26)$$

To obtain the physical spectrum we use the relations given by (26) into (25) and the Hamiltonian \mathcal{H} takes the following form

$$\mathcal{H} = \sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \mathcal{P}_i \mathcal{P}^i - \frac{2^{3/2} \sqrt{(\mathcal{P}_\mu)^2}}{|a|} - \frac{2^{3/2}}{|a| \sqrt{(\mathcal{P}_\mu)^2}} \xi^\alpha \mathcal{P}_{\alpha\dot{\beta}} \bar{\xi}^{\dot{\beta}}} \quad (27)$$

Passing to the center of mass of the system, and defining new operators $s_\alpha = (1/\sqrt{M})\xi_\alpha$, $\bar{s}_{\dot{\alpha}} = (1/\sqrt{M})\bar{\xi}_{\dot{\alpha}}$, $d_\alpha = \sqrt{2M}\zeta_\alpha$, $\bar{d}_{\dot{\alpha}} = \sqrt{2M}\bar{\zeta}_{\dot{\alpha}}$ (where $M = \mathcal{P}_0$), the \mathcal{H}_{cm} is

$$\mathcal{H}_{cm} = \sqrt{m^2 - M^2 + \frac{2^{3/2}M}{|a|} \left[1 - (\sigma_0)_{\alpha\dot{\beta}} \bar{s}^{\dot{\beta}} s^\alpha \right]} \quad (28)$$

being the anti-commutation relations of the operators $s_\alpha, \bar{s}_{\dot{\alpha}}, d_\alpha, \bar{d}_{\dot{\alpha}}$:

$$\{s_\alpha, \bar{s}_{\dot{\alpha}}\} = -(\sigma_0)_{\alpha\dot{\alpha}} \quad \{d_\alpha, \bar{d}_{\dot{\alpha}}\} = -(\sigma_0)_{\alpha\dot{\alpha}} \quad (29)$$

Now the question is: how does the square-root \mathcal{H} Hamiltonian given by expression (28) operate on a given physical state? The problem of locality and interpretation of the operator

like (25) is very well known. Several attempts to avoid these problems were given in the literature [22,23]. The main characteristic of all these attempts is to eliminate the square root of the Hamiltonian: e.g. imposing constraints. In this manner, the set of operators into the square root operates freely on the physical states, paying the price to lose the concept of locality and quantum interpretation of the spectrum of a well posed field theory.

Our plan is "to take " the square root to a bispinor in order to introduce the physical state into the square root Hamiltonian. In the next section we will perform the square root of a bispinor and obtain the mass spectrum given by the Hamiltonian \mathcal{H} .

V. MASS SPECTRUM AND SQUARE ROOT OF A BISPINOR

The square root from a spinor was extracted in 1965 by the soviet scientist S. S. Sannikov from Kharkov (Ukraine) [14] and the analysis of the structure of the Hilbert space containing such "square root " states was worked out by E. C. G. Sudarshan, N. Mukunda and C. C. Chiang in 1981 [24]. Taking the square root from a spinor was performed also by P.A.M. Dirac [15] in 1971.

We know that the group $SL(2, \mathbb{C})$ is locally isomorph to $SO(3,1)$, and $SL(2, \mathbb{R})$ is locally isomorph to $SO(2,1)$. For instance, the generators of the group $SO(3,1)$ for our case can be constructed from the usual operators a, a^+ (or q and p) in the following manner: we start from an irreducible unitary infinite dimensional representation of the Heisenberg-Weyl group, which is realized in the Fock space of states of one-dimensional quantum oscillator [13,17,18]. Creation operators and annihilation operators of these states obey the conventional commutation relations $[a, a^+] = 1$ $[a, a] = [a^+, a^+] = 0$. To describe this representation to the Lorentz group one may also use the coordinate-momentum realization $(q, p = -i\frac{\partial}{\partial q})$ of the Heisenberg algebra, which relates to the a, a^+ realization by the formulas

$$a = \frac{q+ip}{\sqrt{2}} \quad a^+ = \frac{q-ip}{\sqrt{2}} \quad (30)$$

as usual. Let us introduce the spinors

$$L_\alpha = \begin{pmatrix} a_1 \\ a_1^+ \end{pmatrix} \quad L_{\dot{\alpha}} = \begin{pmatrix} a_2 \\ a_2^+ \end{pmatrix} \quad (31)$$

The commutation relations take the form

$$[L_\alpha, L_\beta] = i\varepsilon_{\alpha\beta} ; \quad [L_{\dot{\alpha}}, L_{\dot{\beta}}] = i\varepsilon_{\dot{\alpha}\dot{\beta}} ; \quad [L_{\dot{\alpha}}, L_\beta] = 0 \quad (32)$$

The generators of $SL(2, \mathbb{C})$ are easily constructed [18] from L_α and $L_{\dot{\alpha}}$

$$S_{\alpha\beta} \equiv iS_{1i}(\sigma^i)_{\alpha\beta} = \frac{1}{4} \{L_\alpha, L_\beta\}$$

$$S_{\dot{\alpha}\dot{\beta}} \equiv iS_{2i}(\sigma^i)_{\dot{\alpha}\dot{\beta}} = \frac{1}{4} \{L_{\dot{\alpha}}, L_{\dot{\beta}}\} \quad (33)$$

where the index $i = 1, 2, 3$ and the six vectors S_{ai} ($a, b = 1, 2; a \neq b$), characteristics of the representation of $SL(2, \mathbb{C}) \approx SO(3, 1)$, satisfy the commutation relations

$$[S_{ai}, S_{aj}] = -i\varepsilon_{ijk}S_a^k, \quad [S_{bi}, S_{bj}] = -i\varepsilon_{ijk}S_b^k, \quad [S_{ai}, S_{bj}] = 0 \quad (34)$$

Notice that the above construction obeys the described decomposition of $SL(2, \mathbb{C}) \approx SO(3, 1)$

Then the quantities

$$\Phi_\alpha \equiv \langle \Psi | L_\alpha | \Psi \rangle, \quad \bar{\Phi}_{\dot{\alpha}} \equiv \langle \bar{\Psi} | L_{\dot{\alpha}} | \bar{\Psi} \rangle, \quad (35)$$

are the two-components of a bispinor

$$\Phi \equiv \langle \hat{\Psi} | L | \hat{\Psi} \rangle = \begin{pmatrix} \Phi_\alpha \\ \bar{\Phi}_{\dot{\alpha}} \end{pmatrix}$$

where we define $|\hat{\Psi}\rangle \equiv \begin{pmatrix} |\Psi\rangle \\ |\bar{\Psi}\rangle \end{pmatrix}$. Notice that $|\Psi\rangle$ and $|\bar{\Psi}\rangle$ are the square root of each component of this bispinor and can have the same form (given the isomorphism between the generators L_α and $L_{\dot{\alpha}}$), that is very easy to verify. In terms of q the basic vectors of the representation can be written as [13,14,17]

$$\langle q | n \rangle = \varphi_n(q) = \pi^{-1/4} (2^n n!)^{-1/2} H_n(q) e^{-q^2/2} \quad (36)$$

$$\int dq \varphi_m^*(q) \varphi_n(q) = \delta_{mn} \quad (37)$$

(where $H_n(q)$ are the Hermite polynomials) and form a unitary representation of $SO(3, 1)$, and

$$|n\rangle = (n!)^{-1/2} (a^+)^n |0\rangle \quad (38)$$

the normalized basic states where the vacuum vector is annihilated by a . The Casimir operator, that is $S_{ai}S_a^i$, has the eigenvalue $\lambda(\lambda-1) = -\frac{3}{16}$ (for each subgroup $\text{ISO}(2,1)$ given by eqs.(33)) and indeed corresponds to the representations with the lowest weights $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$. The wave functions which transform as linear irreducible representation of $\text{ISO}(2,1)$, subgroup of $\text{ISO}(3,1)$ generated by operators (33) are

$$\Psi_{1/4}(x, \theta, q) = \sum_{k=0}^{+\infty} f_{2k}(x, \theta) \varphi_{2k}(q) \quad (39)$$

$$\Psi_{3/4}(x, \theta, q) = \sum_{k=0}^{+\infty} f_{2k+1}(x, \theta) \varphi_{2k+1}(q) \quad (40)$$

(analogically for the $\bar{\Psi}_{1/4}$ and $\bar{\Psi}_{3/4}$ states with contrary helicity). We can easily see that the Hamiltonian \mathcal{H} (28) operates over the states $|\hat{\Psi}\rangle$, which become into \mathcal{H} as its square Φ_α and $\Phi_{\dot{\alpha}}$. It is natural to associate, up to a proportional factor, the spinors d_α and $\bar{d}_{\dot{\alpha}}$ with

$$d_\alpha \rightarrow (\Phi_{1/4})_\alpha \equiv \langle \Psi_{1/4} | \mathbb{L}_\alpha | \Psi_{1/4} \rangle \quad \bar{d}_{\dot{\alpha}} \rightarrow (\bar{\Phi}_{1/4})_{\dot{\alpha}} \equiv \langle \bar{\Psi}_{1/4} | \mathbb{L}_{\dot{\alpha}} | \bar{\Psi}_{1/4} \rangle \quad (41)$$

and, analogically, the spinors s_α and $\bar{s}_{\dot{\alpha}}$ with

$$s_\alpha \rightarrow (\Phi_{3/4})_\alpha \equiv \langle \Psi_{3/4} | \mathbb{L}_\alpha | \Psi_{3/4} \rangle \quad \bar{s}_{\dot{\alpha}} \rightarrow (\bar{\Phi}_{3/4})_{\dot{\alpha}} \equiv \langle \bar{\Psi}_{3/4} | \mathbb{L}_{\dot{\alpha}} | \bar{\Psi}_{3/4} \rangle \quad (42)$$

where the new spinors \mathbb{L}_α ($\mathbb{L}_{\dot{\alpha}}$) are defined as

$$\begin{aligned} \mathbb{L}_\alpha &= \begin{pmatrix} a_1 a_1 \\ a_1^+ a_1^+ \end{pmatrix} \\ \mathbb{L}_{\dot{\alpha}} &= \begin{pmatrix} a_2 a_2 \\ a_2^+ a_2^+ \end{pmatrix} \end{aligned} \quad (42a)$$

The reason to this choice is the following: as it was shown in Ref. [25] the Hilbert space for each subgroup $\text{ISO}(2,1) \approx \text{SU}(1,1)$ [26] can be decomposed as direct sum of two independent subspaces characterized for the states of helicity $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ respectively. Each subspace is composed by the even ($\lambda = \frac{1}{4}$) and odd states ($\lambda = \frac{3}{4}$) given by eqs.(39-40). These "cat" states admit (after a convenient choice for the functions $f_{2k}(x, \theta)$ and $f_{2k+1}(x, \theta)$) a coherent state realization being eigenvectors not of the ladder operator a of the Heisenberg-Weyl algebra, but those of the quadratic ladder operator aa of the $\text{SU}(1,1)$ algebra defined in general by

$$K_+ = \frac{1}{2} a^+ a^+, \quad K_- = \frac{1}{2} aa, \quad K_0 = \frac{1}{4} (a^+ a + aa^+), \quad (42b)$$

That means that when we are in the full Hilbert space the algebra is Heisenberg-Weyl and the states $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\Psi_{1/4}\rangle + |\Psi_{3/4}\rangle)$ are eigenvectors of the operator a , and when we pass to the decomposed space (by means of a suitable unitary transformation) the algebra becomes the $SU(1,1)$ algebra with the quadratic ladder operators given by expression (42b).

The relations (41) and (42) give a natural link between the spinors $\xi_\alpha(\dot{\xi}_\alpha)$ and $\zeta_\alpha(\dot{\zeta}_\alpha)$, solutions of the dynamical problem, with the only physical states that can operate freely with the Hamiltonian \mathcal{H} : the "square root" states $|\Psi\rangle, |\bar{\Psi}\rangle$ from the bispinor Φ . Notice that there are four (non-trivial) representations for the group decomposition of the bispinor Φ , as follows

$$\begin{aligned}\Phi_1 &= \begin{pmatrix} \Phi_{1/4} \\ \bar{\Phi}_{3/4} \end{pmatrix} \rightarrow (1/4, 0) \oplus (0, 3/4) \\ \Phi_2 &= \begin{pmatrix} \Phi_{3/4} \\ \bar{\Phi}_{1/4} \end{pmatrix} \rightarrow (3/4, 0) \oplus (0, 1/4) \\ \Phi_3 &= \begin{pmatrix} \Phi_{1/4} \\ \bar{\Phi}_{1/4} \end{pmatrix} \rightarrow (1/4, 0) \oplus (0, 1/4) \\ \Phi_4 &= \begin{pmatrix} \Phi_{3/4} \\ \bar{\Phi}_{3/4} \end{pmatrix} \rightarrow (3/4, 0) \oplus (0, 3/4)\end{aligned}$$

This result is a consequence of the geometrical Hamiltonian taken in its natural square root form and the Sannikov-Dirac oscillator representation for the generators of the Lorentz group $SO(3,1)$.

Commutation relations (29) obey the Clifford's algebra for spinorial creation-annihilation operators. In this manner, the square root of the operators s_α and d_α in the representation given by the associations (41) and (42) acting on the vacuum give zero, symbolically :

$$\begin{aligned}\sqrt{s_\alpha} &\rightarrow (\Phi_{3/4})_\alpha |0\rangle = \mathbb{L}_\alpha^{rs} \Psi_{r\ 3/4} \Psi_{s\ 3/4} |0\rangle = 0 \\ \sqrt{d_\alpha} &\rightarrow (\Phi_{1/4})_\alpha |0\rangle = \mathbb{L}_\alpha^{rs} \Psi_{r\ 1/4} \Psi_{s\ 3/4} |0\rangle = 0\end{aligned}$$

where here we introduce $r, s, t...$ latin indexes to design the fractional spin states. The Fock's construction in the center of mass of the system consists now, in contrast to the construction (6b), of the following vectors:

$$\hat{S}_1 = |0\rangle e^{\frac{iMt}{2}}, \quad \Xi_{1r} = \bar{\Psi}_{r\ 1/4} |0\rangle e^{\frac{iMt}{2}}, \quad \hat{P}_1 = \bar{\Psi}_{1/4}^r \bar{\Psi}_{r\ 1/4} |0\rangle e^{\frac{iMt}{2}},$$

$$\Xi_{2r} = \bar{\Psi}_{r\ 3/4} |0\rangle e^{\frac{iMt}{2}}, \quad V_{rs} = \bar{\Psi}_{r\ 3/4} \bar{\Psi}_{s\ 1/4} |0\rangle e^{\frac{iMt}{2}}, \quad \Xi_{3r} = \bar{\Psi}_{r\ 3/4} \bar{\Psi}_{1/4}^s \bar{\Psi}_{s\ 1/4} |0\rangle e^{\frac{iMt}{2}}, \quad (43)$$

$$\begin{aligned} \hat{P}_2 &= \bar{\Psi}_{3/4}^r \bar{\Psi}_{r\ 3/4} |0\rangle e^{\frac{iMt}{2}}, \quad \Xi_{4r} = \bar{\Psi}_{r\ 1/4} \bar{\Psi}_{3/4}^s \bar{\Psi}_{s\ 3/4} |0\rangle e^{\frac{iMt}{2}}, \\ \hat{S}_2 &= \bar{\Psi}_{1/4}^r \bar{\Psi}_{r\ 1/4} \bar{\Psi}_{3/4}^s \bar{\Psi}_{s\ 3/4} |0\rangle e^{\frac{iMt}{2}} \end{aligned}$$

Notice that the vectors given above are the only states that can operate into the square root operator given by expression (28), and not that constructed directly with the operators s_α and d_α . Schematically we have, e.g. for Ξ_{4r} , the following operability :

$$\begin{aligned} &\sqrt{m^2 - M^2 + \frac{2^{3/2}M}{|a|} \left[1 - (\sigma_0)_{\alpha\dot{\beta}} \bar{s}^{\dot{\beta}} s^\alpha \right]} \bar{\Psi}_{r\ 1/4} \bar{\Psi}_{3/4}^s \bar{\Psi}_{s\ 3/4} |0\rangle e^{\frac{iMt}{2}} \equiv \\ &\equiv \sqrt{\left[m^2 - M^2 + \frac{2^{3/2}M}{|a|} \left[1 - (\sigma_0)_{\alpha\dot{\beta}} \bar{s}^{\dot{\beta}} s^\alpha \right] \right]} \bar{d}_{\dot{\gamma}} \bar{s}^{\dot{\beta}} \bar{s}_{\dot{\beta}} e^{iMt} |0\rangle \end{aligned}$$

From expression (38) and taking into account that the number operator is $\bar{s}^{\dot{\beta}} s^\alpha \equiv n_s$, because $\bar{s}^{\dot{\beta}}$ and s^α work as creation-annihilation operators, we can easily obtain the mass for the different "square root" or fractional supermultiplets :

- i) $n_s = 0 \rightarrow M_I = \sqrt{-\frac{2^{1/2}}{|a|} + \sqrt{\frac{2}{|a|^2} + m^2}}$; Fractional supermultiplet I: $(\hat{S}_1, \Xi_{1r}, \hat{P}_1)$
- ii) $n_s = 1 \rightarrow M_{II} = \sqrt{m}$; Fractional supermultiplet II: $(\Xi_{2r}, V_{rs}, \Xi_{3r})$.
- iii) $n_s = 2 \rightarrow M_{III} = \sqrt{\sqrt{\frac{2}{|a|^2} + m^2} + \frac{2^{1/2}}{|a|}}$; Fractional supermultiplet III: $(\hat{P}_2, \Xi_{4r}, \hat{S}_2)$.

We emphasize now that the computations and algebraic manipulations given above were with $\bar{d}_{\dot{\alpha}} \rightarrow (\Phi_{1/4})_{\dot{\alpha}}$ and $\bar{s}_{\dot{\alpha}} \rightarrow (\Phi_{3/4})_{\dot{\alpha}}$ (the square of the true states) into the square root Hamiltonian. That means that the physical states are constrained by the explicit form of the Hamiltonian operator. Notice from expressions (35), (41) and (42) that the physical states for the Hamiltonian in the square root form are one half the number of physical states for the Hamiltonian quadratic in momenta.

Another important point is that the link between the new Hamiltonian \mathcal{H} given by expression (8) and the relativistic Schrödinger equation (e.g. ref.²⁷) can be given through the relation between the conserved currents of the fermionic "square" states and the para-states. This important issue will be analyzed in Section VII.

It is interesting to note that the arbitrary c-parameters a and a^* generate a deformation of the usual line element for a superparticle in proper time, and this deformation is responsible,

in any meaning, for the multiplets given above. This is not a casuality: one can easily see how the quantum Hamiltonian (28) is modified in the center of mass of the system by the c-parameters a and a^* . The implications of this type of superparticle actions with deformations of the quantization will be analyzed in a future paper [28].

VI. SQUARE ROOT HAMILTONIAN AND THE THEORY OF SEMIGROUPS

From the Hamiltonian eq.(8)

$$\mathcal{H} = \sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)}$$

where

$$\Pi_\alpha = \mathcal{P}_\alpha + i \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}}$$

$$\Pi_{\dot{\alpha}} = \mathcal{P}_{\dot{\alpha}} - i \mathcal{P}_\mu \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}}$$

(\mathcal{P}_α and \mathcal{P}_μ was defined from the Lagrangian, as usual), we start with the equation

$$\mathcal{S}[\Psi] = \mathcal{H}_s \Psi = \sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} \Psi \quad (44)$$

In order to construct a general analytic representation for above equation, let us set $\omega^2 = m^2$ and $-\mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right) = -\mathbb{G}$.

Because we are treating it now with a supermanifold instead of the simple flat space, it is necessary to give a consistent definition of integration on it, in order to perform successfully the construction of the general analytical representation of eq.(44). To do this we make firstly some remarks on superspace and supermanifolds.

The supermanifolds we are dealing with are modelled over flat superspace $B_L^{(D_0, D_1)}$, the cartesian product of D_0 copies of $B_{L,0}$ and D_1 copies of $B_{L,1}$, where $B_{L,0}$ and $B_{L,1}$ are the even, respectively odd, subspaces of a real Grassmann algebra B_L (with L anticommuting generators). Functions from $B_L^{(D_0, D_1)}$ to B_L will be taken to be G^∞ , i.e. infinitely differentiable with respect to all arguments which in turn implies that the function admits a finite Taylor series expansion in the odd arguments, with infinitely differentiable functions of the even arguments as coefficients [29]. A $(D_0 + D_1)$ -dimensional supermanifold $M^{(D_0, D_1)}$ is constructed from $B_L^{(D_0, D_1)}$ in the usual way by means of an atlas of charts $\bigcup_{i \in I} (U_i, \varphi_i)$ with U_i

an open cover of $M^{(D_0, D_1)}$ and a homeomorphism φ_i of U_i onto an open subset of $B_L^{(D_0, D_1)}$. If the overlap $U_i \cap U_j$ is non-empty we require the transition function $\varphi_j \circ \varphi_i^{-1}$ to be G^∞ .

A chart map φ induces coordinates $\varphi^M(m) = z^M = \{x^\mu, \theta^\alpha\}$ ($M = 1, \dots, D_0 + D_1; \mu = 1, \dots, D_0, \alpha = 1, \dots, D_1$). If we change to other coordinates we require this change to respect evenness/oddness in the following sense: if (X) denotes a grading of an Grassmann element X , i.e. $(X) = 0$ if X is even and $(X) = 1$ if X is odd, then under a coordinate change $z^M \rightarrow \bar{z}^M$ we require $(z^M) = (\bar{z}^M)$.

Equipped with the notion of differentiability in $B_L^{(D_0, D_1)}$ one can construct the tangent bundle $TM^{(D_0, D_1)}$. At a point $m \in U \subset M^{(D_0, D_1)}$ the tangent space $T_{(m)}M^{(D_0, D_1)}$ is spanned (on a coordinate basis) by $\left\{ {}_M\partial = \frac{\vec{\partial}}{\partial z^M} \right\}$ (we use the de Witt conventions [30] for index manipulation in order to avoid factors of (-1)). The dual space to $T_{(m)}M^{(D_0, D_1)}$ denoted $T_{(m)}^*M^{(D_0, D_1)}$ is spanned by $\{z^M\}$ where $\langle {}_N\partial | z^M \rangle = {}_N\delta^M$ (The Kronecker delta). This in turn gives rise to the cotangent bundle T^*M . In general, field tensors of type (p, r) are elements of $\otimes^p T^*M^{(D_0, D_1)} \otimes \otimes^r TM^{(D_0, D_1)}$, then the components of a tensor of type (p, r) are displayed on a coordinate basis as

$$dz^{N_1} \otimes \dots \otimes dz^{N_p} \left({}_{N_p \dots N_1} T^{M_1 \dots M_r} \right) \otimes {}_{M_r} \partial \otimes \dots \otimes {}_{M_1} \partial$$

We will need tensors with special symmetry properties that generalize the differential forms of the ordinary differential geometry. This space of (p, r) tensors is denoted $\Lambda_p^r(M^{(D_0, D_1)})$ and spanned by

$$dz^{N_1} \wedge \dots \wedge dz^{N_p} \otimes {}_{M_r} \partial \vee \dots \vee {}_{M_1} \partial$$

where \wedge is the graded antisymmetric wedge product

$$dz^N \wedge dz^M = dz^{[N} \wedge dz^{M]} = -(-1)^{(N)(M)} dz^M \wedge dz^N,$$

where $(N) = (dz^N)$, and \vee is the graded symmetric product

$${}_N \partial \vee {}_M \partial = ({}_N \partial \vee {}_M \partial) = +(-1)^{(N)(M)} {}_N \partial \vee {}_M \partial$$

An element of $\Lambda_p^r(M^{(D_0, D_1)})$ is denominated a hyperform [31]. For $p = 0$ a hyperform is also called a derivative r -form; for $r = 0$ a hyperform is a differential p -form.

We are now in the position to introduce the following definition that precises the relationship between the supermanifold integration and the Berezin/Riemann integration:

Definition. Let $M^{(D_0, D_1)}$ be the total space of a fibre bundle $E = (M^{(D_0, D_1)}, \pi, B_0, F_1)$, and let $p \in M^{(D_0, D_1)}$. Let $(\pi^{-1}(U), \varphi)$ be a chart on the G^∞ supermanifold, where $U \subset B_0$ and $p \in U$. Because of the local trivialization property we take $\pi^{-1}(U) \cong U \times F_1$. Let A be a (D_0^H, D_1^V) hyperform with support compact in $\varphi(U) \subset O$, where O is open in $B_{L,0}^{D_0}$. In natural coordinates $z^M = \varphi^M(p)$, and with respect to the canonical basis

$$A = a(z) \Omega_{(D_0, D_1)} \frac{1}{D_0!} \frac{1}{D_1!} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{D_0}} \underset{\mu_{D_0} \dots \mu_1}{A^{\alpha_1 \dots \alpha_{D_1}}} \underset{\alpha_{D_1}}{\partial} \vee \dots \vee \underset{\alpha_1}{\partial}$$

Then

$$\int_{\pi^{-1}(u)} A := \int_{O \times B_{L,1}^{D_1}}^{Ber} a(x, \theta) dx^1 \dots dx^{D_0} d\theta^1 \dots d\theta^{D_1} \quad (45)$$

Notice that it makes no sense to demand that A has compact support in the " θ direction", as e.g. for $B_L^{(0,1)}$ the only G^∞ function of θ which has compact support is $f(\theta) = 0$. By construction, the left-hand side of the definition (45) transform with the Berezinian under the following change of coordinates

$$\{x^\mu, \theta^\alpha\} \rightarrow \{\bar{x}^\mu(x), \bar{\theta}^\alpha(x, \theta)\} \quad (46)$$

that is nothing more than a bundle morphism from one set of natural coordinates to another. This corresponds to the choice of a different section and a change of basis in the fibres. From the practical point of view we can easily see that, following the definition (45), the procedure consists of replacing the one-forms dx^μ by the integration symbols " dx^μ ", the derivative one-forms $\alpha \partial$ by the integration symbols " $d\theta^\alpha$ ", and deleting the \wedge and \vee products. This procedure is justified by the fact that both sides transform identically under the coordinate transformations (46).

We assume that $-\mathbb{G} + \omega^2$ satisfies the conditions required to be a generator of a unitary group (self-adjoint); we can write (44) as :

$$\mathcal{S}[\Psi] = \sqrt{-\mathbb{G} + \omega^2} \Psi \quad (47)$$

Using the analytic theory of fractional powers of closed linear operators (e.g. Refs. [27,32]), it can be shown that (for generators of unitary groups) we can write (47) as

$$\mathcal{S}[\Psi] = \frac{1}{\pi} \int_0^\infty [-\mathbb{G} + (\lambda + \omega^2)]^{-1} \Psi \frac{d\lambda}{\sqrt{\lambda}} \quad (48)$$

where $[-\mathbb{G} + (\lambda + \omega^2)]^{-1}$ is the resolvent associated with the operator $(-\mathbb{G} + \omega^2)$. The resolvent can be computed directly if we can find the fundamental solution to the equation

$$\frac{\partial Q}{\partial t}(\mathbf{z}_1, \mathbf{z}_2; t) + (\mathbb{G} - \omega^2) Q(\mathbf{z}_1, \mathbf{z}_2; t) = \delta(\mathbf{z}_1 - \mathbf{z}_2) \quad (49)$$

It is shown in [27] that the equation

$$i\hbar \frac{\partial \overline{Q}}{\partial t}(\mathbf{z}_1, \mathbf{z}_2; t) + \left(\frac{1}{2M} \mathbb{G} - V \right) \overline{Q}(\mathbf{z}_1, \mathbf{z}_2; t) = \delta(\mathbf{z}_1 - \mathbf{z}_2) \quad (50)$$

has the general (infinitesimal) solution

$$\overline{Q}(\mathbf{z}_1, \mathbf{z}_2; t) = \left(\frac{M}{2\pi i \hbar t} \right)^{3/2} e^{\left\{ \frac{it}{\hbar} \left[\frac{M}{2} \left(\frac{\mathbf{z}_1 - \mathbf{z}_2}{t} \right)^2 - V(z_2) \right] + \frac{ie}{\hbar c} (\mathbf{z}_1 - \mathbf{z}_2) \cdot \mathbb{A} \left[\frac{1}{2} (\mathbf{z}_1 - \mathbf{z}_2) \right] \right\}} \quad (51)$$

provided that \mathbb{A} and M are time independent. We used the midpoint evaluation in the last part for equation (51) ($\mathbb{A} \left[\frac{1}{2} (\mathbf{z}_1 - \mathbf{z}_2) \right]$) and from the supersymmetric Hamiltonian (44) that $\mathbb{A} \equiv \left(0, -i \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}}, i \mathcal{P}_\mu (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \right)$ in superspace components.

If we set $\frac{\omega^2}{i\hbar} = V$ and $M = \frac{i\hbar}{2}$ then

$$Q(\mathbf{z}_1, t; \mathbf{z}_2, 0) = \int_{\mathbf{z}(0)=\mathbf{z}_2}^{\mathbf{z}(t)=\mathbf{z}_1} \mathcal{D}\mathcal{W}_{\mathbf{z},t}[\mathbf{z}(s)] e^{\left\{ \int_0^t V[\mathbf{z}(s)] ds + \frac{ie}{\hbar c} \int_{\mathbf{y}}^{\mathbf{x}} \mathbb{A}[\mathbf{z}(s)] d\mathbf{z}(s) \right\}}$$

solves (50), where

$$\begin{aligned} \int_{\mathbf{z}(0)=\mathbf{z}_2}^{\mathbf{z}(t)=\mathbf{z}_1} \mathcal{D}\mathcal{W}_{\mathbf{z},t}[\mathbf{z}(s)] &= \int_{\mathbf{z}(0)=\mathbf{z}_2}^{\mathbf{z}(t)=\mathbf{z}_1} \mathcal{D}[\mathbf{z}(s)] e^{\left\{ -\frac{1}{4} \int_0^t \left| \frac{d\mathbf{z}(s)}{ds} \right|^2 ds \right\}} \\ &= \lim_{N \rightarrow \infty} \left[\frac{1}{4\pi\varepsilon(N)} \right]^{n\frac{N}{2}} \int_{\mathcal{M}} \prod_{j=1}^N dz_j e^{\left\{ -\sum_{j=1}^N \left[\frac{1}{4\varepsilon(N)} (z_j - z_{j-1})^2 \right] \right\}} \end{aligned}$$

and $\varepsilon(N) = t/N$.

Now we construct the solution of equation (47) for the constant \mathbb{A} case. The solution for other examples will be discussed in a future paper applying explicitly these results to the Hamiltonian formulation in supermanifolds. First, rewrite equation (51) as

$$Q(\mathbf{z}_1, \mathbf{z}_2; t) = \left(\frac{1}{4\pi t} \right)^{3/2} \exp \left\{ \frac{\|(\mathbf{z}_1 - \mathbf{z}_2)\|^2}{4t} - \mu^2 t + \frac{ie}{\hbar c} (\mathbf{z}_1 - \mathbf{z}_2) \cdot \mathbb{A} \right\}$$

Finally, using the theory of fractional powers, we note if $\mathbf{T}[t, 0]$ is the semigroup associated with $-\mathbb{G} + \omega^2$, then the semigroup associated with $[-\mathbb{G} + \omega^2]^{1/2}$ is given by^{27,32}

$$\mathbf{T}_{1/2}[t, 0] \varphi(\mathbf{x}) = \int_0^\infty \left\{ \int_{\mathcal{M}} \left(\frac{1}{4\pi t} \right)^{3/2} \exp \left[\frac{(\mathbf{z}_1 - \mathbf{z}_2)^2}{4t} - \mu^2 t + \frac{ie}{\hbar c} (\mathbf{z}_1 - \mathbf{z}_2) \cdot \mathbb{A} \right] \varphi(\mathbf{z}_2) d\mathbf{z}_2 \right\} \times$$

$$\times \left(\frac{ct}{\sqrt{4\pi}} \right) \frac{1}{s^{3/2}} \exp \left(-\frac{(ct)^2}{4s} \right) ds$$

Laplace transforming we get

$$\mathbf{T}_{1/2}[t, 0] \varphi(\mathbf{x}) = \frac{ct}{4\pi^2} \int_{\mathcal{M}} \left(\frac{1}{4\pi t} \right)^{3/2} \exp \left[\frac{ie}{2\hbar c} (\mathbf{z}_1 - \mathbf{z}_2) \cdot \mathbb{A} \right] \frac{2\mu^2 K_2 \left[\mu \left(\|\mathbf{z}_1 - \mathbf{z}_2\|^2 + c^2 t^2 \right) \right]}{\left[\|\mathbf{z}_1 - \mathbf{z}_2\|^2 + c^2 t^2 \right]} \varphi(\mathbf{z}_2) d\mathbf{z}_2 \quad (52)$$

where the order of integration was interchanged. We can easily see from expression (52) that the solution is non-local (MacDonald's function of second order) and does not coincide with the solution to the same problem where the square root has been eliminated by reparametrization and introducing constraints. It is important to note here the following:

i) from the semi-group representation of the radical operator we see that is not the same to operate with the square root Hamiltonian as that with its square or other power of this operator: *the states under which the Hamiltonian operates are sensible to the power of such Hamiltonian*.

ii) from the practical point of view the explicit determination of the functions (states) $\varphi(\mathbf{z})$ can carry several troubles in any specific physical problems. In a future paper [28] we can give a detailed study of this problem in different physical contexts.

VII. RELATION WITH THE RELATIVISTIC SCHRÖDINGER EQUATION: COMPATIBILITY CONDITIONS AND PROBABILITY CURRENTS

Looking at formula (25) the new Hamiltonian operates as ($g_{\mu\nu} = (+ - - -)$)

$$\sqrt{m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right)} |\Psi\rangle = 0$$

for instance, the action of the radical operator is

$$\left\{ \left[m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right) \right]^\gamma \right. \left. (\Psi L_\gamma) \Psi \right\}_\beta^{1/2} = 0 \quad (53)$$

that seems as a parabosonic supersymmetric version of the relativistic Schrödinger-De Broglie equation. In the next paragraph we will see that this equation corresponds to the family of equations given firstly by E. Majorana [33] and P. A. M. Dirac [15], and in its para-bosonic version by Sudarshan, N. Mukunda and C. C. Chiang in 1981[24].

We can see from the above expression that if we put the (super)momenta together in the \square operator, we obtain a more suitable equation in order to compute the currents as in the Fock-Klein-Gordon case

$$\left[(\square + m^2)_\beta^\alpha (\Psi L_\alpha \Psi) \right]^{1/2} = \left[(\square + m^2)_\beta^\alpha \Phi_\alpha \right]^{1/2} = 0 \quad (54)$$

now eliminating the exponent 1/2 and taking the Hermitian conjugation to equation we have

$$\begin{aligned} \left[(\square + m^2)_\beta^\alpha \Psi^\dagger (L_\alpha^\dagger \Psi^\dagger) \right]^{1/2} &= 0 \\ \left[(\square + m^2)_\beta^\alpha (\Psi^\dagger L_\alpha^\dagger \Psi^\dagger) \right]^{1/2} &= \left[(\square + m^2)_\beta^\alpha \Phi_\alpha^\dagger \right]^{1/2} = 0 \end{aligned} \quad (55)$$

Following the same procedure as Dirac in Ref.[15] we multiply the square of expression (54) from the left side by Φ_α^\dagger and multiply the square of expression (55) from the left side by Φ_α , integrating and subtracting the final expressions we obtain

$$\Phi_\alpha^\dagger \square \Phi_\beta - \Phi_\alpha \square \Phi_\beta^\dagger = 0 \quad (56)$$

Using the relations: $\Phi^\dagger \square \Phi = \partial_\mu (\Phi^\dagger \partial^\mu \Phi) - \partial_\mu \Phi^\dagger \partial^\mu \Phi$; $\Phi \square \Phi^\dagger = \partial_\mu (\Phi \partial^\mu \Phi^\dagger) - \partial_\mu \Phi \partial^\mu \Phi^\dagger$ in expression (56) the current for the square states Φ_α is

$$\partial_\mu (\Phi^\alpha \partial^\mu \Phi_\alpha^\dagger - \Phi^{\alpha\dagger} \partial^\mu \Phi_\alpha) = 0 = -\partial_\mu j^\mu \quad (57)$$

with $j^\mu(x) \equiv -i [\Phi^\alpha \partial^\mu \Phi_\alpha^\dagger - \Phi^{\alpha\dagger} \partial^\mu \Phi_\alpha]$.

If we suppose that a link between the relativistic Schrödinger equation (53) and our new Hamiltonian \mathcal{H} holds, the relation with the quartionic states is the following

$$i\dot{\Psi} = E\Psi$$

Squaring the above expression and having account as \mathcal{H} operates over Ψ and Ψ^\dagger , we can easily obtain

$$\begin{aligned} -\dot{\Phi}_\beta &= E^2 \Phi_\beta \\ \dot{\Phi}_\beta^\dagger &= E^2 \Phi_\beta^\dagger \end{aligned}$$

that into the explicit expression for $j_0(x)$ permits us to analyze the positivity of this component of the current for the square states Φ_α

$$j_0(x) = 2E^2 \Phi^{\alpha\dagger} \Phi_\alpha \quad (58)$$

As we have been obtain from expression (58) $j_0(x)$ for the square states Φ_α is positively definite because the energy E appears squared.

Now in order to find the current vector for the para-Bose states Ψ we proceed analogically as above for the states Φ_α arriving to

$$\Psi^\dagger \square \Psi - \Psi \square \Psi^\dagger = 0$$

as we expected because these square root states obey the square root operator equation and, for instance, also obey the equation for the squared operator (the inverse is not true in general). In fact, in some references in the literature the authors don't take care of the fact we can pass to the equation with the square root Klein-Gordon operator to its squared traditional version operating both on the same state but not the inverse (see e.g. ref.[34]). The correct form to do this is as follows: if we start with

$$\sqrt{(-\Delta + m^2)}\Psi = i\partial_t\Psi \quad (59)$$

the relation with any pseudo-differential operator A is

$$\begin{aligned} A\Psi &= \sqrt{(-\Delta + m^2)}\Psi = i\partial_t\Psi \\ AA\Psi &= iA\partial_t\Psi = A\sqrt{(-\Delta + m^2)}\Psi \\ &= (-\Delta + m^2)\Psi \end{aligned}$$

This happens clearly because Ψ obeys (59). Finally the current for the quartionic states that we were looking for is

$$\partial_\mu [(\Psi\partial^\mu\Psi^\dagger) - (\Psi^\dagger\partial^\mu\Psi)] = 0 = -\partial_\mu j^\mu$$

$$j^\mu(x) \equiv -i [\Psi\partial^\mu\Psi^\dagger - \Psi^\dagger\partial^\mu\Psi] \quad (60)$$

Is not difficult to see that in this case from expression (60) the zero component of the current is not positively definite given explicitly by

$$j_0(x) = 2E\Psi^\dagger\Psi$$

The compatibility condition, as usual, is given by the follow expression

$$[\tau_\alpha, \tau_\beta]\Psi = 0 \quad (61)$$

where we defined $\tau_\beta \equiv \left[(\square + m^2)_\beta^\alpha (\Psi L_\alpha) \right]^{1/2}$. After a little algebra and using expression (61) we arrive to

$$\left[(\square + m^2)_\alpha^\delta (\square + m^2)_\beta^\gamma \epsilon_{\delta\gamma} \right]^{1/2} \Psi = 0 \quad (62)$$

It is good to remember here that eq.(53) describes a *free* particle in a N=1 superspace and the term of interaction appears from the supersymmetry between the bosonic and fermionic fields. The last expression shows that our equation (53) is absolutely compatible and consistent because its character fermionic coming from the supersymmetric part, and for instance not necessary to introduce any extra term in order to include spin. It is well known, that it terms (put "by hand" in equations containing a second order derivatives) destroy the compatibility condition leading to the impossibility of including interactions [35].

VIII. RELATIVISTIC WAVE EQUATION

Following the arguments given in the precedent paragraphs, it is natural to propose the following form for a square root of the second order supersymmetric wave equation

$$\left\{ \left[m^2 - \mathcal{P}_0 \mathcal{P}^0 - \left(\mathcal{P}_i \mathcal{P}^i + \frac{1}{a} \Pi^\alpha \Pi_\alpha - \frac{1}{a^*} \Pi^{\dot{\alpha}} \Pi_{\dot{\alpha}} \right) \right]_\beta^\alpha (\Psi L_\alpha) \Psi \right\}^{1/2} = 0 \quad (63)$$

In order to reduce the expression (63) to the simplest form it is necessary pass to the center of mass of the system and redefining the variables as:

$$t \rightarrow (aa^*)^{-1/2} t, \quad \theta \rightarrow a^{-1/2} \theta, \quad \bar{\theta} \rightarrow a^{-1/2} \bar{\theta}$$

we obtain the following expression

$$\left\{ \left[|a|^2 \partial_0^2 + \frac{1}{4} (\partial_\eta - \partial_\xi + i \partial_0 (\sigma^0) \xi)^2 - \frac{1}{4} (\partial_\eta + \partial_\xi + i \partial_0 (\sigma^0) \xi)^2 + m^2 \right]_\beta^\alpha \Phi_\alpha \right\}^{1/2} = 0 \quad (64)$$

where

$$\eta \equiv (\bar{\theta} + \theta), \quad \xi \equiv (\bar{\theta} - \theta), \quad \text{and} \quad \partial_0 (\sigma^0)_{\alpha\dot{\beta}} \left(\bar{\theta}^{\dot{\beta}} - \theta^\alpha \right) \equiv \partial_0 (\sigma^0) \xi \quad (65)$$

The trick that we used above [21,36,37] is based on the observation that the expression (63) has similar form that the equation for an electron in constant electromagnetic field (with

$\mathcal{P}_\mu(\sigma^\mu)_{\alpha\dot{\beta}}$ as the constant electric field). Imposing the condition $\partial_\eta \Phi_\alpha = 0 \Rightarrow \Phi_\alpha(\xi)$; the "square" of the solution eigenfunction of eq.(63) takes the form

$$\Phi_\gamma(t) = e^{A(t)+\xi\varrho(t)}\Phi_\gamma(0) \quad (66)$$

with $\varrho(t) = \phi_\alpha + \bar{\chi}_{\dot{\alpha}}$ (i.e. chiral plus anti-chiral parts). The system of equations for $A(t)$ and $\varrho(t)$ that we are looking for, is easily obtained inserting the expression (66) in the eq.(64)

$$\begin{aligned} |a|^2 \ddot{A} + m^2 &= 0 \\ \ddot{\bar{\chi}}_{\dot{\alpha}} - i\frac{\omega}{2}(\sigma^0)_{\dot{\alpha}}^{\alpha} \phi_\alpha &= 0 \\ -\ddot{\phi}_\alpha + i\frac{\omega}{2}(\sigma^0)_\alpha^{\dot{\beta}} \bar{\chi}_{\dot{\beta}} &= 0 \end{aligned}$$

The above system can be solved given us the following result

$$A = -\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2; \quad c_1, c_2 \in \mathbb{C} \quad (67)$$

and

$$\phi_\alpha = \overset{\circ}{\phi}_\alpha (\alpha e^{i\omega t/2} + \beta e^{-i\omega t/2}) + \frac{2i}{\omega} (\sigma^0)_\alpha^{\dot{\beta}} \bar{Z}_{\dot{\beta}} \quad (68)$$

$$\bar{\chi}_{\dot{\alpha}} = (\sigma^0)_{\dot{\alpha}}^{\alpha} \overset{\circ}{\phi}_\alpha (\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2}) + \frac{2i}{\omega} (\sigma^0)_{\dot{\alpha}}^{\alpha} Z_\alpha \quad (69)$$

where $\overset{\circ}{\phi}_\alpha$, Z_α and $\bar{Z}_{\dot{\beta}}$ are constant spinors. The superfield solution for the square states that we are looking for, have the following form

$$\Phi_\gamma(t) = e^{-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2} e^{\xi\varrho(t)} \Phi_\gamma(0) \quad (70)$$

with

$$\varrho(t) = \overset{\circ}{\phi}_\alpha [(\alpha e^{i\omega t/2} + \beta e^{-i\omega t/2}) - (\sigma^0)_{\dot{\alpha}}^{\alpha} (\alpha e^{i\omega t/2} - \beta e^{-i\omega t/2})] + \frac{2i}{\omega} \left[(\sigma^0)_\alpha^{\dot{\beta}} \bar{Z}_{\dot{\beta}} + (\sigma^0)_{\dot{\alpha}}^{\alpha} Z_\alpha \right] \quad (71)$$

and

$$\Phi_\gamma(0) = \langle \Psi(0) | L_\gamma | \Psi(0) \rangle \quad (72)$$

that is nothing more than the mean value of L_γ between the states $|\Psi\rangle$ in the initial time, where the subalgebra is the Heisenberg-Weyl algebra (with generators a , a^+ and $(n + \frac{1}{2})$). As we have pointed out in Section V, the states $|\Psi\rangle$ span all the Hilbert space and for instance,

we can not obtain useful information from the point of view of the topology of the group manifold then, about the spin.

The dynamics of the square root fields, in the representation that we are interested in, can be simplified considering these fields as coherent states in the sense that are eigenstates of a^2

$$\begin{aligned} |\Psi_{1/4}(0, \xi, q)\rangle &= \sum_{k=0}^{+\infty} f_{2k}(0, \xi) |2k\rangle = \sum_{k=0}^{+\infty} f_{2k}(0, \xi) \frac{(a^\dagger)^{2k}}{\sqrt{(2k)!}} |0\rangle \\ |\Psi_{3/4}(0, \xi, q)\rangle &= \sum_{k=0}^{+\infty} f_{2k+1}(0, \xi) |2k+1\rangle = \sum_{k=0}^{+\infty} f_{2k+1}(0, \xi) \frac{(a^\dagger)^{2k+1}}{\sqrt{(2k+1)!}} |0\rangle \end{aligned} \quad (73)$$

From a technical point of view these states are a one-mode squeezed states constructed by the action of the generators of the $SU(1,1)$ group over the vacuum. For simplicity, we will take all normalization and fermionic dependence or possible CS fermionic realization, into the functions $f(\xi)$. Explicitly at $t=0$

$$\begin{aligned} |\Psi_{1/4}(0, \xi, q)\rangle &= f(\xi) |\alpha_+\rangle \\ |\Psi_{3/4}(0, \xi, q)\rangle &= f(\xi) |\alpha_-\rangle \end{aligned} \quad (74)$$

where $|\alpha_\pm\rangle$ are the CS basic states in the subspaces $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ of the full Hilbert space. From expression (70) and expressions (42) we obtain

$$\Phi_\alpha(t, \lambda) = \langle \Psi_\lambda(t) | \mathbb{L}_\alpha | \Psi_\lambda(t) \rangle = e^{-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2} e^{\xi \varrho(t)} \langle \Psi_\lambda(0) | \begin{pmatrix} a^2 \\ (a^+)^2 \end{pmatrix}_\alpha | \Psi_\lambda(0) \rangle \quad (75)$$

$$\Phi_\alpha(t, \lambda) = e^{-\left(\frac{m}{|a|}\right)^2 t^2 + c_1 t + c_2} e^{\xi \varrho(t)} |f(\xi)|^2 \begin{pmatrix} \alpha_\lambda^2 \\ \alpha_\lambda^{*2} \end{pmatrix}_\alpha \quad (76)$$

where λ label the helicity or the spanned subspace (e.g. \pm). The "square root" states solution of the expression (63) take the following form

$$\Psi_\lambda = e^{-\frac{1}{2} \left[\left(\frac{m}{|a|} \right)^2 t^2 + c_1 t + c_2 \right]} e^{\frac{\xi \varrho(t)}{2}} |f(\xi)| \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\lambda \quad (77)$$

where $\lambda = 1/4, 3/4$. Notice the difference with the case in which we used the HW realization for the states Ψ

$$|\Psi\rangle = \frac{f(\xi)}{2} (|\alpha_+\rangle + |\alpha_-\rangle) = f(\xi) |\alpha\rangle \quad (78)$$

where, however, the linear combination of the states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ span now the full Hilbert space being the corresponding λ to this CS basis $\lambda = \frac{1}{2}$. The "square" state at $t=0$ are

$$\begin{aligned}\Phi_\alpha(0) &= \langle \Psi(0) | L_\alpha | \Psi(0) \rangle = \langle \Psi(0) | \begin{pmatrix} a \\ a^+ \end{pmatrix}_\alpha | \Psi(0) \rangle \\ &= f^*(\xi) f(\xi) \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\alpha\end{aligned}\quad (79)$$

The square state at time t

$$\Phi_\gamma(t) = e^{-\left(\frac{m}{|a|}\right)^2 t^2 + c'_1 t + c'_2} e^{\xi \varrho(t)} |f(\xi)|^2 \begin{pmatrix} \alpha \\ \alpha^* \end{pmatrix}_\alpha \quad (80)$$

And the "square root" solution becomes now to

$$\Psi(t) = e^{-\frac{1}{2}\left[\left(\frac{m}{|a|}\right)^2 t^2 + c'_1 t + c'_2\right]} e^{\frac{\xi \varrho(t)}{2}} |f(\xi)| \begin{pmatrix} \alpha^{1/2} \\ \alpha^{*1/2} \end{pmatrix} \quad (81)$$

We can see the change in the solutions from the choice in the representation of the Hilbert space. The algebra (topological information of the group manifold) is "mapped" over the spinors solutions through the eigenvalues α and α^* . Notice that the constants c'_1 c'_2 in the exponential functions in expressions (80) and (81) differ from the c_1 and c_2 in (76) and (77), because these exponential functions of the Gaussian type come from the action of a unitary operator over the respective CS basic states in each representation (h_3 or HW). These constants can be easily determined as functions of the frequency ω as in ref. [5] for the Schrodinger equation. A detailed analysis of this point and the other type of solutions will be given elsewhere [28].

About the possible algebras that contain an $SU(1,1)$ as subgroup that can lead or explain the fermionic factors of type $e^{\frac{\xi \varrho(t)}{2}} |f(\xi)|$ in the solutions are 2 subgroups that are strong candidates [38]: the supergroup $OSP(2,2)$ [2] and the supergroup $OSP(1/2, \mathbb{R})$ [39]. In the case of the $OSP(2,2)$ we have bosonic and fermionic realizations and the CS and SS can be constructed from the general procedure given by M. Nieto et al. in Refs.[1-3]. On the other hand, the $OSP(1/2, \mathbb{R})$ realization is more "economic", the number of generators is minor than in the $OSP(2,2)$ case and the realization is bosonic: the K_\pm and K_0 generators operate over the bose states and the HW algebra given by a and a^+ operate over the fermionic part. In this case the CS and the SS that can be constructed are eigenstates of the displacement

and squeezed operators respectively but they cannot minimize the dispersion of the quadratic Casimir operator, so that they are not minimum uncertainty states.

The important point to remark here is that when we describe from the mostly geometrical grounds any physical system through $SU(1,1)$ CS or SS, the orbits will appear as the intersections of curves that represent constant-energy surfaces, with one sheet of a two sheeted hyperboloid- the curved phase space of $SU(1,1)$ or Lobachevsky plane- in the space of averaged algebra generators. In the specific case treated in this paper, the group containing the $SU(1,1)$ as subgroup linear and bilinear functions of the algebra generators can factorize operators as the Hamiltonian or the Casimir operator (when averaged with respect to group CS or SS), defining corresponding curves in the averaged algebra space. If we notice that the validity of the Ehrenfest's theorem for CS (SS) implies that, if the exact dynamics is confined to the $SU(1,1)$ hyperboloid, it necessarily coincides with the variational motion, the variational motion that comes from the Euler-Lagrange equations for the lagrangian

$$\mathcal{L} = \langle z | i \frac{\hat{\partial}}{\partial t} - \hat{H} | z \rangle$$

will be different if $|z\rangle = |\alpha\rangle$ or $z = |\alpha_{\pm}\rangle$, as is evident to see. It is interesting to note also that similar picture holds in the context of the pseudospin $SU(1,1)$ dynamics in the frame of the mean field approximation induced by the variational principle on nonlinear Hamiltonians [40].

IX. CONCLUDING REMARKS

In this work the problem of the physical interpretation of the square root quantum operators and possible relation with the TDHO and coherent and squeezed states was analyzed considering the simple model of superparticle of Volkov and Pashnev [9]. Besides the extension and clarification of the results of our previous works [19], of which we have already made mention in the Introduction, we can summarize as follows:

- i) the Fock construction for these fractional or "square root states" was proposed, explicitly constructed and compared with the Fock construction given in the reference[9] for the superparticle model with the Hamiltonian in standard form;
- ii) we have shown that, in contrast to [9], the only states that the square root Hamiltonian can operate with correspond to the representations with the lowest weights $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$

;

iii) there are four possible (non-trivial) fractional representations for the group decomposition of the spin structure from the square root Hamiltonian, instead of $(1/2,0)$ and $(0,1/2)$ as the case when the Hamiltonian is quadratic in momentum (e.g. Ref. [9]) as a consequence of the geometrical Hamiltonian taken in its natural square root form and the Sannikov-Dirac oscillator representation for the generators of the Lorentz group $SO(3,1)$;

now we make this research complete with the following new results:

iv) The relation between the structure of the Hilbert space of the states, the spin content of the sub-Hilbert spaces and the CS and SS realization of the physical states was established for the particular model presented here.

iv) We construct explicitly from the theory of semi-group the analytical representation of the radical operator in the $N=1$ superspace and we see that it is not the same to operate with the square root Hamiltonian as that with its square or other power of this operator from the point of view of the spectrum of the physical states: *the states under which the Hamiltonian operates are sensible to the power of such Hamiltonian* .

v) If expression constructed in iv) gives a closed representation for the radical operator, from the practical point of view the explicit determination of the functions (states) $\varphi(\mathbf{z})$ can carry several troubles in any specific physical problems.

vi) The relation between the relativistic Schrödinger equation and other type of equations that involve variables with fractional spin and the model analyzed here was established and discussed.

vii) As for the Klein-Gordon equation, the conserved currents for the "square-root" states (para-fields) and for the square states were explicitly computed and analyzed. The component zero of the current is linearly dependent on the energy E in the para-field case and for the "square" state the dependence on the energy is quadratic .

viii) The compatibility conditions were analyzed and the consistency of the proposed equation was established. The explanation of this consistency and the relation with the free dynamics and the supersymmetry of the model was given.

ix) New wave equation is proposed and explicitly solved for the time-dependent case. As for the TDHO the physical states are realized on the CS and SS basis, and the link between the topology of the (super)-group manifold and the obtained solution from the algebraic and group theoretical point of view was discussed and analyzed.

It is interesting to see that the results presented here for the superparticle are in complete agreement with the results, symmetry group and discussions for non-supersymmetric examples given in references [41,42,43], where group and geometrical quantization was used. This fact gives a high degree of reliability of our method of quantization and the correct interpretation of the radical Hamiltonian operator. It is clear that the ordinary Canonical method of quantization fails when the reparametrization procedure affects the power of the starting Hamiltonian modifying inexorably the obtained spectrum of the physical states [see e.g. [42,43]]. For instance, we conclude that quantically it is not the same to operate with the square root Hamiltonian as that with its square or other power of this operator because the obtained states (mass spectrum, spin) under which the Hamiltonian operates are sensible to the power of such Hamiltonian.; and seeing that the lowest weights of the states under the square root Hamiltonian can operate, and because not concrete action is known to describe particles with fractional statistics, superparticle relativistic actions as of [9] can be good geometrical and natural candidates to describe quartionic states [13,16,17,18] (semions).

X. ACKNOWLEDGEMENTS

I am very thankful to Professors John Klauder for his advisements, E. C. G. Sudarshan for his interest demonstrated in this work in a private communication and in particular to N. Mukunda for his interest to put in this research and give me several remarks and references on the problem of consistency of the positive energy wave equations with para-fields variables. I appreciate deeply Professor Tepper Gill's efforts to clarify to me many concepts on the correct description of the quantum systems, Hamiltonian formulation and the relation with the proper energy of these systems. I am very grateful to the Directorate of JINR, in particular of the Bogoliubov Laboratory of Theoretical Physics, for their hospitality and support.

XI. REFERENCES

1. B. W. Fatyga *et al.*, *Phys. Rev. D* **43**, 1403 (1991), and references therein.
2. V. A. Kostelecky *et al.*, *Phys. Rev. A* **48**, 1045 (1993), and references therein.
3. V. A. Kostelecky *et al.*, *Phys. Rev. A* **48**, 951 (1993), and references therein.

4. V. Ermakov, *Univ. Izv. Kiev Serie III* **9**, 1 (1880).
5. K. Husimi, *Prog. Theor. Phys.* **9**, 381, (1953).
6. J. R. Klauder and B. S. Skagerstam: *Coherent States* (World Scientific, Singapore, 1985).
7. M. D. Shelby *et al.*, *Phys. Rev. Lett.* **57**, 691 (1986).
8. C. M. Caves *et al.*, *Rev. Mod. Phys.* **52**, 341 (1980).
9. A.I. Pashnev and D. V. Volkov: Supersymmetric lagrangian for particles in proper time. *Teor. Mat. Fiz.*, Tom. 44, No. **3**, 321 (1980) [in Russian].
10. R. Casalbuoni: The Classical Mechanics for Bose-Fermi Systems. *Nuovo. Cim.*, Vol. **33A**, N. 3, 389 (1976).
11. R. Casalbuoni: Relatively and supersymmetries. *Phys. Lett.* **62B**, 49 (1976).
12. C. Lanczos: *Variational Principles in Mechanics*, (Mir, 1965), pp. 408 (Russian version).
13. Yu. P. Stepanovsky: On massless fields and relativistic wave equations. *Nucl. Phys. B* (Proc. Suppl.), **102-103**, 407 (2001).
14. S. S. Sannikov: Non-compact symmetry group of a quantum oscillator. *Zh.E.T.F.* **49**, 1913 (1965), [in Russian].
15. P. A. M. Dirac: A positive-energy relativistic wave equation. *Proc. Roy. Soc. A* **322**, 435 (1971).
16. D. P. Sorokin and D. V. Volkov: (Anti) commuting spinors and supersymmetric dynamics of semions. *Nucl. Phys.* **B409**, 547 (1993).
17. D. P. Sorokin: The Heisenberg algebra and spin. *Fortschr. Phys.* **50**, 724 (2002).
18. D. V. Volkov: Quaternions in relativistic field theories. *Piz'ma Zh.E.T.F.* **49**, 473 (1989), [in Russian].
19. D. J. Cirilo-Lombardo, *Rom. Journal of Physics* **7-8**, Vol 50, 875, (2005); *Elem. Particles and Nucl. Lett.* **6**, Vol. 3, 416 (2006); *Hadronic J.* **29**, 355 (2006).
20. A. P. Akulov and D. V. Volkov: Is the neutrino a Goldstone particle? *Phys. Lett.* **46B**, 109 (1973).
21. A. S. Bakai and Yu. P. Stepanovsky: *Adiabatic Invariants* ("Naukova Dumka", Kiev, 1981), pp. 65 [in Russian].
22. J. Sucher: Relativistic invariance and the Square-Root Klein-Gordon equation. *J. Math. Phys.* **4**, 17 (1963), and references therein.

23. S. Schweber: *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson and Co., Evanston, Illinois, 1964), pp. 56.
24. E. C. G. Sudarshan *et al.*: Dirac positive energy wave equation with para-Bose internal variables, *Phys. Rev. D* **25**, 3237 (1982).
25. E. C. G. Sudarshan and N. Mukunda, *Phys. Rev. D* **1**, 571 (1970).
26. N. Mukunda *et al.*, *J. Math. Phys.* **21**, 2386 (1980).
27. T.L. Gill and W.W. Zachary, *J.Phys.A: Math. and General* **38**, 2479, (2005).
28. D. J. Cirilo-Lombardo: work in preparation.
29. A. Rogers: A global theory of supermanifolds, *J. Math. Phys.* **21**, 1352 (1980).
30. B. De Witt: *Supermanifolds* (Cambridge University Press, Cambridge, 1984).
31. R. F. Picken and K. Sundermeyer : Integration on Supermanifolds and a Generalized Cartan Calculus, *Comm. Math. Phys.* **102**, 585 (1986).
32. K. Yosida, *Functional Analysis* (Springer, New York, 2nd. edition, 1968).
33. E.Majorana, *Nuovo Cim.* **9**, 335 (1932).
34. C. Lämmerzahl, *J. Math. Phys.* **34**, 3918 (1993).
35. R. P. Feynman and M. Gellman, *Phys. Rev.* **109**, 193 (1958).
36. D. J. Cirilo-Lombardo and Yu. P. Stepanovsky, *Problems on Atomic Science and Technology* **6**, 182 (2001).
37. Akhiezer A. I. and Beretsetsky, V. B.: *Quantum Electrodynamics*, p.p. 432 (Nauka, Moscow, 1981).
38. I. Bars and M. Gunaydin, *Commun. Math. Phys.* **91**, 31 (1983).
39. Le-Man Kuang and Xing Chen, *J.Phys.A: Math. and General* **27**, L119, (1993).
40. D. M. Jezek and H. S. Hernandez, *Phys. Rev. A* **42**, 96 (1990), and references therein.
41. M. Lachieze-Rey: On three quantization methods for a particle on hyperboloid, gr-qc/0503060, (2005).
42. R. Delbourgo: A square root of the harmonic oscillator, hep-th/9503056, (1995).
43. E. Elizalde: On the concept of determinant for the differential operators of quantum physics. *JHEP* **07**, 015 (1999).