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# INTERPOLATION AND HARMONIC MAJORANTS IN BIG HARDY-ORLICZ SPACES

ANDREAS HARTMANN

ABSTRACT. Free interpolation in Hardy spaces is characterized by the well-known Carleson condition. The result extends to Hardy-Orlicz spaces contained in the scale of classical Hardy spaces  $H^p$ ,  $p > 0$ . For the Smirnov and the Nevanlinna classes, interpolating sequences have been characterized in a recent paper in terms of the existence of harmonic majorants (quasi-bounded in the case of the Smirnov class). Since the Smirnov class can be regarded as the union over all Hardy-Orlicz spaces associated with a so-called strongly convex function, it is natural to ask how the condition changes from the Carleson condition in classical Hardy spaces to harmonic majorants in the Smirnov class. The aim of this paper is to narrow down this gap from the Smirnov class to “big” Hardy-Orlicz spaces. More precisely, we characterize interpolating sequences for a class of Hardy-Orlicz spaces that carry an algebraic structure and that are strictly bigger than  $\bigcup_{p>0} H^p$ . It turns out that the interpolating sequences are again characterized by the existence of quasi-bounded majorants, but now the functions defining these quasi-bounded majorants have to be in suitable Orlicz spaces. The existence of harmonic majorants in such Orlicz spaces will also be discussed in the general situation. We finish the paper with a class of examples of separated Blaschke sequences that are interpolating for certain Hardy-Orlicz spaces without being interpolating for slightly smaller ones.

## 1. INTRODUCTION

For a sequence  $\Lambda$  in the unit disk  $\mathbb{D}$  and a space of holomorphic functions  $X \subset \text{Hol}(\mathbb{D})$ , the interpolation problem consists in describing the trace of  $X$  on  $\Lambda$ , i.e. the set of restrictions  $X|_\Lambda$  regarded as a sequence space. This problem has been considered for many spaces and also for domains different from  $\mathbb{D}$ .

In this paper, which can be regarded as a continuation of the work in [HMNT04], we will focus on spaces included in the Nevanlinna or the Smirnov class, and in particular on Hardy-like spaces. Carleson described in 1958 the interpolating sequences for the Hardy space of bounded analytic functions on the unit disk by the condition that is now called the Carleson condition [Ca58]. It turns out that this condition still characterizes interpolating sequences in a much broader situation. It was successively proved to be the right condition in  $H^p$ ,  $p \geq 1$  [ShHSh], in  $H^p$ ,  $p < 1$  [Ka63], and for Hardy-Orlicz spaces  $\mathcal{H}_\Phi$  contained in the scale  $H^p$ ,  $p > 0$ , in [Har99] (even for the weaker notion of *free* interpolation, see the definition below).

A natural question is then to ask what happens beyond  $\bigcup_{p>0} H^p$ . It is here where we enter the territory of “big” Hardy-Orlicz spaces.

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Two observations should be made at this junction. First, it is known on the one hand that the union of all Hardy-Orlicz spaces corresponds to the Smirnov class (see [RosRov85]), and this union will be the same if only big Hardy-Orlicz spaces are included. On the other hand, interpolating sequences for the Smirnov class (and the Nevanlinna class) have been characterized recently in the paper [HMNT04]. So, one could try to examine interpolating sequences for big Hardy-Orlicz spaces in the light of these results. Since the characterization for the Smirnov and Nevanlinna classes is no longer given by the Carleson condition this leads to the still open question where exactly the Carleson condition ceases to be valid (see also the Question at the end of this section).

The second observation concerns the notion of interpolating sequences itself. When one wants to characterize for example interpolating sequences for the Smirnov or the Nevanlinna class, which kind of natural trace space can one hope for a priori? The problem had been studied in the past for a priori fixed trace spaces (see [Na56] for the Nevanlinna class and [Ya74] for the Smirnov class), but those traces turned out to be too big (Naftalevič's case) or too small (Yanagihara's case), see [HMNT04] for more detailed comments on this. The notion that finally appeared to be natural is that of *free* interpolating sequences. This notion goes back to work by Vinogradov and Havin in the middle of the seventies and has been used for general interpolation problems by Nikolski and Vasyunin for Hilbert spaces (it works also in certain Banach spaces). It has been successfully used in [HMNT04] for the Smirnov class and the Nevanlinna class (the latter being even not a topological vector space).

Let us introduce the notion of free interpolation that we will use in this paper.

**Definition.** A sequence space  $l$  is called *ideal* if  $\ell^\infty l \subset l$ , i.e. whenever  $(a_n)_n \in l$  and  $(\omega_n)_n \in \ell^\infty$ , then also  $(\omega_n a_n)_n \in l$ .

**Definition.** Let  $X$  be a space of holomorphic functions in  $\mathbb{D}$ . A sequence  $\Lambda \subset \mathbb{D}$  is called *free interpolating for  $X$*  if  $X|\Lambda$  is ideal. We denote  $\Lambda \in \text{Int } X$ .

**Remark 1.1.** For any function algebra  $X$  containing the constants,  $\Lambda \in \text{Int } X$  — or equivalently  $X|\Lambda$  is ideal — if and only if

$$\ell^\infty \subset X|\Lambda.$$

This remark is quite easy to check and a proof is given in [HMNT04]. Let us repeat this proof here for completeness. The inclusion is obviously necessary. In order to see that it is sufficient notice that, by assumption, for any  $(\omega_\lambda)_\lambda \in \ell^\infty$  there exists  $g \in X$  such that  $g(\lambda) = \omega_\lambda$ . Thus, if  $(f(\lambda))_\lambda \in X|\Lambda$ , then the sequence of values  $(\omega_\lambda f(\lambda))_\lambda$  can be interpolated by  $fg \in X$ .

An almost trivial but nevertheless useful remark in our context is that if  $\Lambda \in \text{Int } X$  (i.e.  $X|\Lambda$  is ideal) then  $\Lambda' \in \text{Int } X$  (i.e.  $X|\Lambda'$  is ideal) for any subsequence  $\Lambda' \subset \Lambda$ .

In the case of the Nevanlinna and Smirnov classes the characterization of free interpolating sequences is given in terms of the existence of (quasi-bounded) harmonic majorants of a certain density associated with the sequence  $\Lambda$ . This density is expressed in terms of Blaschke products. More precisely, let  $b_\lambda(z) = \frac{|\lambda|}{\lambda} \frac{\lambda - z}{1 - \bar{\lambda}z}$  be the elementary Blaschke factor (or Möbius transform). For a sequence  $\Lambda$  satisfying the Blaschke condition  $\sum_{\lambda \in \Lambda} (1 - |\lambda|^2) < \infty$  — which will be assumed throughout this paper — we set  $B = B_\Lambda = \prod_{\lambda \in \Lambda} b_\lambda$  for the corresponding Blaschke

product, and  $B_\lambda := B_{\Lambda \setminus \{\lambda\}}$ . Define then

$$\varphi_\Lambda(z) := \begin{cases} \log |B_\lambda(\lambda)|^{-1} & \text{if } z = \lambda \in \Lambda \\ 0 & \text{if } z \notin \Lambda. \end{cases}$$

The majorants we are interested in are positive harmonic functions. Recall that any such function  $h$  is the Poisson integral of a positive finite Borel measure  $\mu_h$  on  $\mathbb{T}$ , i.e.

$$h(z) = P[\mu_h](z) = \int_{\mathbb{T}} P_z(\zeta) d\mu_h(\zeta) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu_h(\zeta), \quad z \in \mathbb{D},$$

where  $P_z(\zeta)$  is the usual Poisson kernel. We will denote by  $\text{Har}_+(\mathbb{D})$  the set of all positive harmonic functions on  $\mathbb{D}$ . A harmonic function  $h$  for which the measure  $\mu_h$  is absolutely continuous,  $d\mu_h = w dm$  for some  $w \in L^1(\mathbb{T})$ , will be called quasi-bounded. In this case we will shortly write  $P[w]$  for  $P[\mu_h]$ .

The result of [HMNT04] we are interested in is the following (we give here only an almost complete form). Recall that the Smirnov class  $N^+$  is the class of holomorphic functions  $f$  on the unit disk such that

$$\lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta.$$

(Note that the existence of the limit on the left hand side implies the existence of the boundary values of  $f$  on  $\mathbb{T}$  a.e. appearing on the right hand side).

**Theorem** ([HMNT04]). *Let  $\Lambda$  be a sequence in  $\mathbb{D}$ . The following statements are equivalent:*

- (a)  $\Lambda$  is a free interpolating sequence for the Smirnov class  $N^+$ :  $\Lambda \in \text{Int } N^+$ .
- (b)  $\varphi_\Lambda$  admits a quasi-bounded harmonic majorant, i.e. there exists a positive weight  $w \in L^1(\mathbb{T})$  such that

$$\varphi_\Lambda(\lambda) \leq P[w](\lambda).$$

- (c) The trace space is given by

$$N^+|\Lambda = l_{N^+} := \{(a_\lambda)_\lambda : \exists h \in \text{Har}_+(\mathbb{D}) \text{ quasi-bounded, } h(\lambda) \geq \log^+ |a_\lambda|, \lambda \in \Lambda\}.$$

- (d)  $\lim_{n \rightarrow \infty} \sup_{(c_\lambda) \in \mathcal{B}_\Lambda} \sum_{\lambda: \varphi_\Lambda(\lambda) \geq n} c_\lambda \varphi_\Lambda(\lambda) = 0$ , where  $\mathcal{B}_\Lambda = \{(c_\lambda) : c_\lambda \geq 0 \text{ for any } \lambda \in \Lambda \text{ and } \|\sum c_\lambda P_\lambda\|_\infty \leq 1\}$ .

The condition (d) may appear quite technical. It is in a sense a “little-o”-version of the condition that characterizes interpolation in the Nevanlinna class. In that case (see [HMNT04]) free interpolating sequences are characterized by the fact that  $\varphi_\Lambda$  admits a positive harmonic majorant (be it quasi-bounded or not). Such a majorant exists if and only if there is a constant  $C$  such that for every finite positive sequence  $(c_\lambda)$  we have

$$(1.1) \quad \sum c_\lambda \varphi_\Lambda(\lambda) \leq C \|\sum c_\lambda P_\lambda\|_\infty.$$

(In view of discussions to come we observe that  $L^\infty = (L^1)^*$ .)

In the light of the above theorem, and since the Hardy-Orlicz spaces we are interested in are in a sense close to the Smirnov class, it seems natural to seek a condition in the spirit of condition (b) in the theorem. The modification should involve a more precise hypothesis on the weight  $w$  adapted to the defining function of the Hardy-Orlicz space that we will introduce below. We will

also see that the dual condition (d) of the theorem — or rather (1.1) — has a counterpart in the Hardy-Orlicz situation (replacing  $(L^1)^*$  by the dual of a suitable Orlicz space).

Let  $\varphi : \mathbb{R} \rightarrow [0, \infty)$  be a convex, nondecreasing function satisfying

- (i)  $\lim_{t \rightarrow \infty} \varphi(t)/t = \infty$
- (ii)  $\tilde{\Delta}_2$ -condition:  $\varphi(t+2) \leq M\varphi(t) + K$ ,  $t \geq t_0$  for some constants  $M, K \geq 0$  and  $t_0 \in \mathbb{R}$ .

Such a function is called strongly convex (see [RosRov85]), and one can associate with it the corresponding *Hardy-Orlicz class*

$$\mathcal{H}_{\varphi \circ \log} = \{f \in N^+ : \int_{\mathbb{T}} \varphi(\log |f(\zeta)|) d\sigma(\zeta) < \infty\},$$

where  $f(\zeta)$  is the non-tangential boundary value of  $f$  at  $\zeta \in \mathbb{T}$ , which exists almost everywhere since  $f \in N^+$ . Throughout this paper we shall use the notation

$$\Phi = \varphi \circ \log.$$

It should be noted that the  $\tilde{\Delta}_2$ -condition is formulated in such a way that  $\Phi$  satisfies the usual  $\Delta_2$ -condition: there exist constants  $M', K' \geq 0$  and  $s_0$  such that for all  $s \geq s_0$  we have

$$(1.2) \quad \Phi(2s) \leq M'\Phi(s) + K'$$

(see also Section 2 for more on Hardy-Orlicz spaces).

In [Har99], the following result was proved.

**Theorem** ([Har99]). *Let  $\varphi$  be a strongly convex function satisfying (i), (ii) and the  $V_2$ -condition:*

$$2\varphi(t) \leq \varphi(t + \alpha), \quad t \geq t_1,$$

where  $\alpha > 0$  is a suitable constant and  $t_1 \in \mathbb{R}$ . Then  $\Lambda \subset \mathbb{D}$  is free interpolating for  $\mathcal{H}_\Phi$  if and only if  $\Lambda$  satisfies the so-called Carleson condition:

$$\inf_{\lambda \in \Lambda} |B_\lambda(\lambda)| = \delta > 0.$$

And in this case

$$\mathcal{H}_\Phi|_\Lambda = \{a = (a_\lambda)_\lambda : |a|_\varphi = \sum_{\lambda \in \Lambda} (1 - |\lambda|)\varphi(\log |a_\lambda|) < \infty\}.$$

Observe that the Carleson condition can be reformulated in terms of  $\varphi_\Lambda$ :

$$(1.3) \quad M = \sup_{\lambda} \varphi_\Lambda(\lambda) < +\infty,$$

i.e.  $\varphi_\Lambda$  is bounded and admits a fortiori a harmonic majorant. We shall occasionally call  $M$  the constant associated with a sequence verifying the Carleson condition (such sequences are usually called  $H^\infty$ -interpolating sequences).

The conditions on  $\varphi$  in the theorem imply that there exist  $p, q \in (0, \infty)$  such that  $H^p \subset \mathcal{H}_\Phi \subset H^q$ . In particular, the  $V_2$ -condition implies the inclusion  $\mathcal{H}_\Phi \subset H^p$  for some  $p > 0$ . This  $V_2$ -condition has a strong topological impact on the spaces. In fact, it guarantees that metric bounded sets are also bounded in the topology of the space (and so the usual functional analysis tools still apply in this situation; see [Har99] for more on this and for further references). It was not clear whether this was only a technical problem or if there existed a critical growth for  $\varphi$  (below exponential growth  $\varphi(t) = e^{pt}$  corresponding to  $H^p$  spaces) giving a breakpoint in the behavior of interpolating sequences for  $\mathcal{H}_\Phi$ .

We shall now turn to the case of big Hardy-Orlicz spaces. Let  $\varphi$  be a strongly convex function with associated Hardy-Orlicz space  $\mathcal{H}_\Phi$ . Our central assumption on  $\varphi$  is the quasi-triangular inequality

$$(1.4) \quad \varphi(a + b) \leq c(\varphi(a) + \varphi(b))$$

for some fixed constant  $c \geq 1$  and for all  $a, b \geq t_0$ . This condition obviously implies that  $\mathcal{H}_\Phi$  is stable with respect to multiplication so that under this condition  $\mathcal{H}_\Phi$  is an algebra. Observe that (1.4) is an equivalent formulation of the usual  $\Delta_2$ -condition (1.2) (now for  $\varphi$  instead of  $\Phi$ ), and we will henceforth denote the condition (1.4) by  $\Delta_2$ . Note that the  $\Delta_2$ -condition obviously implies  $\tilde{\Delta}_2$ .

Another condition on  $\varphi$  will be used. We say that  $\varphi$  satisfies the  $\nabla_2$ -condition (see e.g. [Leš73]) if there exist  $d > 1$  and  $t_0 > 0$  such that

$$2\varphi(t) \leq \frac{1}{d}\varphi(dt), \quad t \geq t_0,$$

(see Section 2 for more on this).

The main result of this paper then reads as follows.

**Theorem 1.2.** *Let  $\varphi : \mathbb{R} \rightarrow [0, \infty)$  be a strongly convex function satisfying the  $\Delta_2$ -condition. The following assertions are equivalent.*

- (a)  $\Lambda$  is a free interpolating sequence for  $\mathcal{H}_\Phi$ :  $\Lambda \in \text{Int } \mathcal{H}_\Phi$
- (b) There exists a positive measurable function  $w \in L^\varphi(\mathbb{T})$  such that  $\varphi_\Lambda \leq P[w]$ .
- (c) The trace space is given by

$$\mathcal{H}_\Phi|_\Lambda = l_\Phi := \{(a_\lambda) : \exists 0 \leq w \in L^\varphi(\mathbb{T}) \text{ with } \log^+ |a_\lambda| \leq P[w](\lambda)\}.$$

If moreover  $\varphi$  satisfies the  $\nabla_2$ -condition then the above three conditions are equivalent to the following.

- (d) There exists a constant  $C > 0$  such that for any sequence of non-negative numbers  $(c_\lambda)$ ,

$$\sum_{\lambda \in \Lambda} c_\lambda \log \frac{1}{|B_\lambda(\lambda)|} \leq C \left\| \sum_{\lambda \in \Lambda} c_\lambda P_\lambda(\zeta) \right\|_{(L^\varphi)^*}$$

The standard examples of functions satisfying  $\Delta_2$  and  $\nabla_2$  are  $\varphi(t) = \varphi_p(t) := t^p$  for fixed  $p > 1$  and  $t \geq t_0$ , or  $\varphi = \psi_\varepsilon(t) := t \log^\varepsilon t$  for some fixed  $\varepsilon > 0$  and  $t \geq t_0$ . Note that  $\varphi_p$  satisfies both conditions also for  $p \in (0, 1]$  but this range is excluded by the definition of strongly convex functions.

The space  $L^\varphi$  appearing in the theorem is the standard Orlicz space of measurable functions  $u$  such that  $\varphi \circ |u| \in L^1(\mathbb{T})$ . As a consequence of the  $\Delta_2$ -condition (1.4) it turns out that its dual space  $(L^\varphi)^*$  is in fact also an Orlicz space (associated with the complementary function of  $\varphi$ , see Section 2 for more comments and details).

It is interesting to note the analogy between condition (d) of this theorem and (1.1) which characterizes the interpolating sequences for the Nevanlinna class. Recall that (1.1) was not sufficient for the existence of a quasi-bounded harmonic majorant. In our situation however, any growth strictly faster than in the  $L^1$ -situation suffices to eliminate the singular part of the measure defining the harmonic majorant (this is maybe not so surprising, there is a kind of “de la Vallée Poussin effect”, see [RosRov85, Theorem 4.14]).

As in [HMNT04] we will investigate the problem of existence of harmonic majorants in the general setting. More precisely we are interested in the question when a Borel function defined on  $\mathbb{D}$  admits a harmonic majorant  $P[w]$  with  $w \in L^\varphi$ . The answer to this problem is given by the following result which involves the so-called Poisson balayage. Recall that the Poisson balayage of a finite positive measure  $\mu$  in the closed unit disk is defined as

$$B(\mu)(\zeta) = \int_{\mathbb{D}} P_z(\zeta) d\mu(z), \quad \zeta \in \mathbb{T}.$$

**Theorem 1.3.** *Let  $\varphi$  be a strongly convex function that satisfies the  $\Delta_2$ -condition and the  $\nabla_2$ -condition. If  $u$  is a non-negative Borel function on the unit disk then the following two assertions are equivalent.*

- (a) *There exists a function  $w \in L^\varphi$  such that  $u(z) \leq P[w](z)$  for all  $z \in \mathbb{D}$ .*
- (b) *There exists a constant  $C \geq 0$  such that*

$$\sup_{\mu \in \mathcal{B}_{\varphi^*}} \int u(z) d\mu(z) \leq C,$$

where  $\mathcal{B}_{\varphi^*} = \{\mu \text{ positive measure on } \mathbb{D}: \|B\mu\|_{(L^\varphi)^*} \leq 1\}$ .

It is again interesting to point out the analogy between this result and that given in [HMNT04, Theorem 1.4]. Note also that the corresponding condition (b) in that theorem does not give a quasi-bounded majorant but only a harmonic majorant, and a more subtle condition is needed to handle the case of quasi-bounded majorants (see [HMNT04, Theorem 1.6]).

The paper is organized as follows. In the next section we shall add some more comments on Orlicz and Hardy-Orlicz spaces. The sufficiency part of our main theorem has been given in [HMNT04, Theorem 9.1] and we refer the reader to that paper for a proof. The general structure of the proof of the necessity goes along the lines of the necessary part for the Smirnov class. However, the key result [HMNT04, Proposition 4.2] does not work any longer in our context. Note that it is precisely that proposition which shows that separated Blaschke sequences are interpolating for the Nevanlinna and Smirnov classes. We will actually discuss in some details in Section 6 an example showing that such a result cannot be expected in Hardy-Orlicz classes. The example shows that there are big Hardy-Orlicz spaces which are close to each other in a sense and for which there exist even separated Blaschke sequences that are interpolating for one space but not for the other one. For this reason we need a new idea which will be discussed in Section 3. The key is to factor the Blaschke product  $B_\Lambda$  into two subproducts that behave essentially in the same way as  $B_\Lambda$  (in a sense to be made precise). This will be achieved through a theorem by Hoffman. The trace space characterization is quite immediate and will be discussed in Section 4. Concerning harmonic majorants, we will discuss this problem in Section 5. We have already insisted on the analogy between our Theorem 1.3 and Theorems 1.4 and 1.6 in [HMNT04]. The techniques that apply in our situation are more classical than those used in [HMNT04]: we will use some duality arguments and a theorem by Mazur-Orlicz on positive linear functionals (which is essentially the Hahn-Banach theorem). The equivalence of (b) and (d) of Theorem 1.2 then follows from Theorem 1.3 (it suffices to consider positive measures  $\mu$  supported on  $\Lambda$ ).

**Question.** With our big Hardy-Orlicz spaces we narrow down the gap in the description of interpolating sequences from above: coming from the Smirnov class where the harmonic majorant must have an absolutely continuous measure, and so with an integrable weight  $w \in L^1$ , we obtain now that the weight has to be in  $L^\varphi$  (and we can get in a way arbitrarily close to  $L^1$ , see also

Section 6). Now, we have already indicated that the  $\Delta_2$ -condition (1.4) implies a polynomial growth on  $\varphi$ . Observe that for classical Hardy spaces  $H^p$  — where the Carleson condition characterizes the interpolating sequences — the defining functions are given by  $t \mapsto e^{pt}$ . So one could ask what happens for defining functions with subexponential growth like e.g.  $\varphi(t) = e^{\sqrt{t}}$  and whether there are Hardy-Orlicz spaces beyond  $H^p$ ,  $p > 0$ , for which the Carleson condition still characterizes the interpolating sequences.

**Remark.** To finish this introduction, we should add a comment on “small” Hardy-Orlicz spaces, more precisely, Hardy-Orlicz spaces contained between  $H^\infty$  and  $\bigcap_{p < \infty} H^p$ . An example is given for  $\Phi(t) = e^t$  (note that this function does not define the  $\Delta_2$ -condition, see (1.4)). With a reasonable definition of the associated small Hardy-Orlicz space (e.g. those holomorphic functions on  $\mathbb{D}$  for which  $\|f\|_\Phi < \infty$ , where  $\|\cdot\|_\Phi$  is defined in (2.1)),  $H^\infty$  is in the multiplier space of  $H^\Phi$ , which implies that  $H^\infty$ -interpolating sequences are interpolating for  $H^\Phi$ . On the other hand, as in [Har99, Lemma 2.1, Theorem 2.2] one shows that a free interpolating sequence for such a  $H^\Phi$  is necessarily  $H^\infty$ -interpolating.

## 2. (HARDY-) ORLICZ SPACES

For the background on the notions related to Hardy-Orlicz classes used in this section we refer to [RosRov85], [Leś73] and [KrRu61].

Let  $\varphi$  be a strongly convex function. One way of introducing the Hardy-Orlicz class  $\mathcal{H}_\Phi$  is to take all the functions  $f \in N^+$  such that the subharmonic function  $\varphi(\log^+ |f|)$  admits a harmonic majorant on  $\mathbb{D}$  (see [RosRov85, Definition 3.15]). We have already introduced the notation

$$\Phi = \varphi \circ \log$$

keeping in mind that the function  $\Phi$  is chosen in a way guaranteeing that  $\Phi(|f|)$  is subharmonic.

Since  $f$  is in the Smirnov class, the fact that  $\Phi(|f|)$  has a harmonic majorant is equivalent to (see [RosRov85, Theorem 4.18])

$$J_\Phi(f) = \int_{\mathbb{T}} \Phi(|f(e^{it})|) dt = \int_{\mathbb{T}} \varphi(\log |f(e^{it})|) dt < \infty,$$

so that  $\mathcal{H}_\Phi$  can be defined as

$$\mathcal{H}_\Phi = \{f \in N^+ : J_\Phi(f) = \int_{\mathbb{T}} \varphi(\log |f(e^{it})|) dt < \infty\}.$$

Observe that  $\mathcal{H}_\Phi$  does not depend on the behaviour of  $\varphi$  for  $t \leq t_0$  whenever a  $t_0 \in \mathbb{R}$  is fixed.

The integral expression  $J_\Phi$  is called a modular, and it does not define a metric on  $\mathcal{H}_\Phi$ . A metric can be defined by  $d(f, g) = \|f - g\|_\Phi$  where

$$(2.1) \quad \|f\|_\Phi = \inf\{t > 0 : J_\Phi(f/t) \leq t\},$$

and  $\mathcal{H}_\Phi$  equipped with this metric is a complete space.

In our situation, thanks to the  $\Delta_2$ -condition,  $J_\Phi(af) < \infty$  for any  $a > 0$  when  $J_\Phi(f) < \infty$  so that we do not need to distinguish between the Orlicz class, the Orlicz space and the space of finite elements (in the terminology of [Leś73]).

It is clear that if  $f \in \mathcal{H}_\Phi$  then by the Riesz-Smirnov factorization  $f = Ih$  where  $I$  is an inner function and  $h$  is outer in  $N^+$ . Clearly  $J_\Phi(f) = J_\Phi(h)$ . Since  $h$  is outer in  $N^+$  we

have  $h(z) = \exp(f(\zeta + z)/(\zeta - z)w(\zeta) dm(\zeta))$  for some real function  $w \in L^1(\mathbb{T})$ , so that  $|h| = \exp(P[w])$  in  $\mathbb{D}$  which has boundary values  $\exp(w)$   $m$ -almost everywhere on  $\mathbb{T}$ . Hence

$$J_\Phi(h) = \int_{\mathbb{T}} \varphi(\log(\exp w)) dm = \int_{\mathbb{T}} \varphi(w) dm \geq \int_{\mathbb{T}} \varphi(w_+) dm = J_\varphi(w_+),$$

where  $w_+ = \max(0, w)$ . In other words  $w_+$  is in the Orlicz class  $L^\varphi$  of measurable functions  $u$  such that  $J_\varphi(u) < \infty$ .

Moreover  $\lim_{t \rightarrow \infty} \varphi(t)/t = +\infty$ , and  $L^1 = L^{\varphi_1}$  (recall that  $\varphi_1(t) = t$ ,  $t \geq 1$ ), so that by standard results on Orlicz spaces we get  $L^\varphi \subset L^1$ . In particular, if we now take any real-valued  $w \in L^\varphi$ , then  $w \in L^1$ , and we can define an outer function in the Smirnov class by

$$f_w(z) = \exp\left(\int \frac{\zeta + z}{\zeta - z} w(\zeta) dm(\zeta)\right).$$

For the same reasons as above the modulus of this function has boundary limits  $\exp(w)$  a.e. on  $\mathbb{T}$  and

$$\begin{aligned} J_\Phi(f_w) &= \int_{\mathbb{T}} \varphi(w) dm = \int_{w \geq 0} \varphi(|w|) dm + \int_{w < 0} \varphi(w) dm \\ &= \int \varphi(|w|) dm + \int_{w < 0} \varphi(w) - \varphi(|w|) dm \leq \varphi(0) + J_\varphi(w) < \infty, \end{aligned}$$

so that  $w$  gives rise to an outer function in  $\mathcal{H}_\Phi$ .

Another important fact on Hardy-Orlicz classes that will be useful for us later is an estimate on point evaluations.

Indeed, the maximal radial growth that we can attain for  $f \in \mathcal{H}_\Phi$  is

$$|f(z)| \leq \Phi^{(-1)}\left(\frac{J_\Phi(f)}{1 - |z|}\right).$$

This can be deduced from the subharmonicity of  $\Phi(|f|)$  and the change of variable  $u \mapsto b_z(u)$  (see also [Leś73, II.1.2]). Hence

$$(2.2) \quad \log |f(z)| \leq \varphi^{-1}\left(\frac{c_f}{1 - |z|}\right).$$

For classical Hardy spaces  $H^p$  one recovers from this the usual estimate on the point evaluation.

We will use the above estimate to show that certain separate sequences are not interpolating for “big” Hardy-Orlicz classes (see Section 6).

Some more tools need to be introduced in the context of Orlicz spaces. In the above arguments, we did not need to appeal to topological considerations in  $L^\varphi$ . Thus the definition of  $L^\varphi$  as the set of measurable functions  $w$  for which  $\varphi \circ |w| \in L^1$  was sufficient. Later on however we will have to consider in particular the dual and the bidual of  $L^\varphi$ , and hence we need a proper norm in  $L^\varphi$ . The alerted reader might have observed that  $\varphi$  is not a so-called  $N$ -function since for instance it does not vanish at 0 and hence it is not appropriated to define a topology on  $L^\varphi$ . In the sequel, when considering  $L^\varphi$  we will think of  $\varphi$  as being suitably replaced on  $[0, t_0]$  in order that the resulting function regarded as a function on  $[0, +\infty)$  is convex and vanishing conveniently in 0. Then, in view of the convexity of  $\varphi$  the Orlicz space  $L^\varphi$  equipped with the metric (2.1) ( $\Phi$  replaced by  $\varphi$ ) is a Banach space.

With the adjusted function  $\varphi$  we can define the so-called complementary function. It is defined by  $\varphi^*(s) = \max_{t \geq 0} \{st - \varphi(t)\}$  (it is also possible to define  $\varphi^*$  using the “inverse” of the right derivative of  $\varphi$ ). Since  $\varphi$  satisfies the  $\Delta_2$ -condition, we get  $(L^\varphi)^* = L^{\varphi^*}$  (see for instance [KrRu61] for this). Note also that  $(\varphi^*)^* = \varphi$ .

We have also mentioned the  $\nabla_2$ -condition. Recall that  $\varphi$  satisfies the  $\nabla_2$ -condition if there are constants  $d > 1$  and  $t_0 \geq 0$  such that for all  $t \geq t_0$  we have  $2\varphi(t) \leq \varphi(dt)/d$ . This condition is actually equivalent to the fact that  $\varphi^*$  satisfies the  $\Delta_2$ -condition ([KrRu61, Chapter 1, Theorem 4.2]) so that  $(L^\varphi)^{**} = (L^{\varphi^*})^* = L^{\varphi^{**}} = L^\varphi$ . In particular, if the strongly convex function  $\varphi$  satisfies both the  $\Delta_2$ -condition and the  $\nabla_2$ -condition, then  $L^\varphi$  is a reflexive space (and the converse is also true, see [Leś73, p.56]).

### 3. NECESSARY CONDITION

The central Proposition 4.1 of [HMNT04] which claims that  $\log(1/|B(z)|)$  is controlled by a (quasi-bounded) positive harmonic function for those  $z$  which are uniformly bounded away in the pseudohyperbolic metric from the zeros of  $B$  cannot be applied to our situation since the function defining the quasi-bounded majorant need not be in  $L^\varphi$ . In Section 6 we will give examples of separated sequences for which the majorants are in no  $L^\varphi$  whenever  $\varphi(t) \geq t \log^\varepsilon t$  for  $t \geq t_0$  and  $\varepsilon > 0$ . To overcome this difficulty we need a more precise results which is based on a theorem by Hoffman (see [Gar81, p. 411]):

**Theorem 3.1** (Hoffman’s theorem). *For  $0 < \delta < 1$  there are constants  $a = a(\delta)$  and  $b = b(\delta)$  such that the Blaschke product  $B(z)$  with zero set  $\Lambda$  has a nontrivial factorization  $B = B_1 B_2$  such that*

$$a|B_1(z)|^{1/b} \leq |B_2(z)| \leq \frac{1}{a}|B_1(z)|^b$$

for every  $z \in \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} D(\lambda, \delta)$  where  $D(\lambda, \delta) = \{z \in \mathbb{D} : |b_\lambda(z)| < \delta\}$  is the pseudohyperbolic disk centered at  $\lambda$  with radius  $\delta$ .

**Corollary 3.2.** *Let  $\Lambda = \{\lambda_n\}_n \subset \mathbb{D}$  be a separated Blaschke sequence. Then there exists a partition*

$$\Lambda = \Lambda_1 \dot{\cup} \Lambda_2$$

and constants  $c, \eta > 0$  such that

$$(3.1) \quad \log \frac{1}{|(B_k)_\lambda(\lambda)|} \geq c \log \frac{1}{|B_\lambda(\lambda)|} - \eta,$$

where  $B_k = B_{\Lambda_k} = \prod_{\mu \in \Lambda_k} b_\mu$  and  $(B_k)_\lambda = B_{\Lambda_k \setminus \{\lambda\}}$  if  $\lambda \in \Lambda_k$ ,  $(B_k)_\lambda = B_k$  otherwise.

*Proof.* One should first note that the constants in Hoffman’s theorem depend only on  $\delta$ .

Let  $\delta$  be the separation constant of the sequence  $\Lambda$  and fix  $\Lambda = \Lambda_1 \dot{\cup} \Lambda_2$  a partition of  $\Lambda$  obtained from Hoffman’s theorem.

Pick  $\lambda \in \Lambda$ , say  $\lambda \in \Lambda_1$  (so that in the following considerations we will assume  $k = 1$ ). Then  $\Lambda \setminus \{\lambda\} = \Lambda_1 \setminus \{\lambda\} \cup \Lambda_2$ . A careful inspection of the proof of Hoffman’s theorem (see e.g. the

indicated reference) shows that we have

$$(3.2) \quad a|B_{\Lambda_1 \setminus \{\lambda\}}(z)|^{1/b} \leq |B_{\Lambda_2}(z)| \leq \frac{1}{a}|B_{\Lambda_1 \setminus \{\lambda\}}(z)|^b$$

for  $z \in \mathbb{D} \setminus \bigcup_{\mu \in \Lambda \setminus \{\lambda\}} D(\mu, \delta)$ , and so in particular for  $\lambda$ .

We should pause here to add some comments on the proof of Hoffman's theorem given in [Gar81]. The fact that we take away  $\lambda$  implies a possible shift between the odd and the even indexed points in the strip  $T_k$  (according to the terminology in [Gar81]) containing  $\lambda$ , and so the choice of  $\Lambda_1$  and  $\Lambda_2$  depends on  $\lambda$ . However, this is of no harm since these shifts mean in fact that we just add or take away at most one term in each layer  $T_k$ . Since the layers  $T_k$  are of pseudohyperbolic constant thickness, the terms added or subtracted correspond to an  $H^\infty$ -interpolating sequence the constant (in the sense of (1.3)) of which is bounded by that of an  $H^\infty$ -interpolating sequence in a radius with given separation constant. So in the estimates (3.2), the constant  $b$  is the same as in the splitting of the original sequence  $\Lambda = \Lambda_1 \dot{\cup} \Lambda_2$  whereas the constant  $a$  should be replaced by a different one, but independent on  $\lambda$ .

So

$$\begin{aligned} \log \frac{1}{|B_{\Lambda \setminus \{\lambda\}}(\lambda)|} &= \log \frac{1}{|B_{\Lambda_1 \setminus \{\lambda\}}(\lambda)|} + \log \frac{1}{|B_{\Lambda_2}(\lambda)|} \\ &\leq \log \frac{1}{|B_{\Lambda_1 \setminus \{\lambda\}}(\lambda)|} + \log \frac{1}{a|B_{\Lambda_1 \setminus \{\lambda\}}(\lambda)|^{1/b}} \\ &= \frac{b+1}{b} \log \frac{1}{|B_{\Lambda_1 \setminus \{\lambda\}}(\lambda)|} - \log a, \end{aligned}$$

and we can set  $c = (b+1)/b$  and  $\eta = \log a$ .

The cases  $\lambda \in \Lambda_2$ ,  $k = 2$  are treated in a similar way. ■

We are now in a position to state the desired result for separated sequences.

**Corollary 3.3.** *If  $\Lambda$  is a separated sequence that is interpolating for  $\mathcal{H}_\Phi$  then there is a positive measurable function  $w$  with  $\varphi \circ w \in L^1$  such that*

$$\log \frac{1}{|B_\lambda(\lambda)|} \leq P[w](\lambda), \quad \lambda \in \Lambda.$$

*Proof.* Suppose that  $\Lambda$  is a separated sequence. By Corollary 3.2, there exists a partition

$$\Lambda = \Lambda_1 \dot{\cup} \Lambda_2, \quad \lambda \in \Lambda,$$

and constants  $c, \eta > 0$  such that for all  $\lambda \in \Lambda$

$$(3.3) \quad \log \frac{1}{|(B_k)_\lambda(\lambda)|} \geq c \log \frac{1}{|B_\lambda(\lambda)|} - \eta.$$

Since  $\Lambda$  is moreover interpolating for  $\mathcal{H}_\Phi$  there exist two functions  $f_i \in \mathcal{H}_\Phi$ ,  $i = 1, 2$ , such that

$$f_i|_{\Lambda_i} = 1, \quad f_i|_{(\Lambda \setminus \Lambda_i)} = 0.$$

Now  $\mathcal{H}_\Phi \subset N^+$ , and we can factor  $f_i$  in the following way

$$f_i = B_{\Lambda \setminus \Lambda_i} I_i h_i, \quad i = 1, 2,$$

where  $I_i$  is an inner function,  $h_i$  is outer in  $\mathcal{H}_\Phi$ :

$$h_i(z) = \exp\left(\frac{1}{2\pi} \int \frac{\zeta + z}{\zeta - z} w_i(\zeta) dm(\zeta)\right),$$

and  $(w_i)_+ \in L^\varphi$ . Then for every  $\lambda \in \Lambda_i$

$$1 = f_i(\lambda) = |f_i(\lambda)| \leq |B_{\Lambda \setminus \Lambda_i}(\lambda)| \cdot |h_i(\lambda)|,$$

so that

$$\log \frac{1}{|B_{\Lambda \setminus \Lambda_i}(\lambda)|} \leq P[w_i](\lambda) \leq P[(w_i)_+](\lambda).$$

Using (3.3), we get

$$\log \frac{1}{|B_\lambda(\lambda)|} \leq \frac{1}{c} \left( \eta + \log \frac{1}{|B_{\Lambda \setminus \Lambda_i}(\lambda)|} \right) \leq P\left[\frac{1}{c}((w_i)_+ + \eta)\right](\lambda).$$

The corollary then follows by setting  $w = \frac{1}{c}((w_1)_+ + (w_2)_+ + \eta)$  which is still in  $L^\varphi$ . ■

Let us now switch to the necessary condition in the general situation. The central trick in [HMNT04] is to decompose the disk  $\mathbb{D}$  into Whitney “cubes” that split the sequence into four pieces that are uniformly separated from each other in the pseudohyperbolic metric. This allows one to reduce the situation to the separated one.

We will repeat here the proof given in [HMNT04] for completeness adding the necessary changes for the situation of big Hardy-Orlicz spaces.

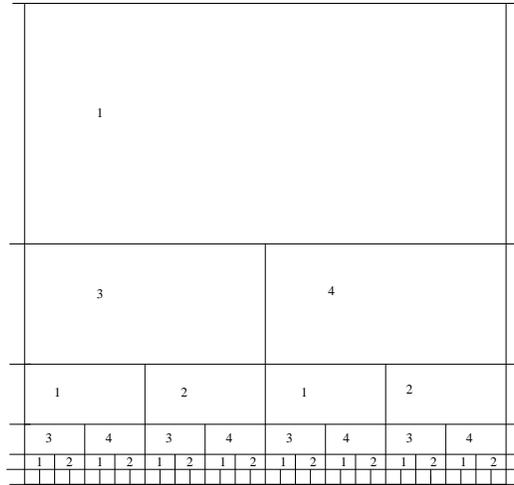


Figure 1: dyadic partition

Let  $I_{n,k} := \{e^{i\theta} : \theta \in [2\pi k 2^{-n}, 2\pi(k+1)2^{-n}]\}$ ,  $0 \leq k < 2^n$ , be the dyadic arcs and  $Q_{n,k} := \{re^{i\theta} : e^{i\theta} \in I_{n,k}, 1 - 2^{-n} \leq r < 1 - 2^{-n-1}\}$  the associated “dyadic squares”.

The splitting of the sequence into four pieces will be done in the following way:  $\Lambda = \bigcup_{i=1}^4 \Lambda_i$  such that each piece  $\Lambda_i$  lies in a union of dyadic squares that are uniformly separated from each

other (see Figure 1). More precisely, set

$$\Lambda_1 = \Lambda \cap Q^{(1)},$$

where the family  $Q^{(1)}$  is given by  $\{Q_{2n,2k}\}_{n,k}$  (for the remaining three sequences we respectively choose  $\{Q_{2n,2k+1}\}_{n,k}$ ,  $\{Q_{2n+1,2k}\}_{n,k}$  and  $\{Q_{2n+1,2k+1}\}_{n,k}$ ). In order to avoid technical difficulties we count only those  $Q$  containing points of  $\Lambda$ . In what follows we will argue on one sequence, say  $\Lambda_1$ . The arguments are the same for the other sequences.

By construction, for  $Q, L \in Q^{(1)}$ ,  $Q \neq L$ ,

$$\rho(Q, L) := \inf_{z \in Q, w \in L} \rho(z, w) \geq \delta > 0,$$

for some fixed  $\delta$ . In what follows, the letters  $j, k, \dots$  will stand for indices in  $\mathbb{N}^2$  of the form  $(n, l)$ ,  $0 \leq l < 2^n$ . The closed rectangles  $\overline{Q_j}$  are compact in  $\mathbb{D}$  so that  $\Lambda_1 \cap Q_j$  can only contain a finite number of points (they contain at least one point, by assumption). Therefore

$$0 < m_j := \min_{\lambda \in \Lambda_1 \cap Q_j} |B_\lambda(\lambda)|$$

(note that we consider the entire Blaschke product  $B_\lambda$  associated with  $\Lambda \setminus \{\lambda\}$ ). Take  $\lambda_j^1 \in Q_j$  such that  $m_j = |B_{\lambda_j^1}(\lambda_j^1)|$ .

Assume now that  $\Lambda \in \text{Int } \mathcal{H}_\Phi$ . So, since  $\ell^\infty \subset \mathcal{H}_\Phi|_\Lambda$ , there exists a function  $f_1 \in \mathcal{H}_\Phi$  such that

$$f_1(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\lambda_j^1\}_j \\ 0 & \text{if } \lambda \in \Lambda \setminus \{\lambda_j^1\}_j. \end{cases}$$

By the Riesz-Smirnov factorization we have

$$(3.4) \quad f_1 = B_{\Lambda \setminus \{\lambda_j^1\}_j} I_1 g_1,$$

where  $I_1$  is some inner function and  $g_1$  is outer in  $\mathcal{H}_\Phi$ . Hence there exists a weight  $w_1 \in L^1$  such that  $g_1 = f_{w_1}$  and  $(w_1)_+ \in L^\varphi$ . As in the proof of Corollary 3.3 we get

$$1 = |f_1(\lambda_k^1)| \leq |B_{\Lambda \setminus \{\lambda_j^1\}_j}(\lambda_k^1)| \cdot |g_1(\lambda_k^1)|,$$

so that

$$(3.5) \quad \log \frac{1}{|B_{\Lambda \setminus \{\lambda_j^1\}_j}(\lambda_k^1)|} \leq P[w_1](\lambda_k^1), \quad k \in \mathbb{N}.$$

Replacing possibly  $w_1$  by  $(w_1)_+$  we can assume  $w_1 \geq 0$ , so that  $P[w_1]$  is a positive harmonic function. By Harnack's inequality, there exists a constant  $c_H \geq 1$  such that

$$\frac{1}{c_H} P[w_1](\lambda_k^1) \leq P[w_1](z) \leq c_H P[w_1](\lambda_k^1), \quad z \in Q_k,$$

and in particular for every  $\lambda' \in \Lambda_1 \cap Q_k$ .

At this point we have to change the argument from [HMNT04]. Instead of using Proposition 4.1 of that paper we have to invoke Corollary 3.3. By construction, the sequence  $\{\lambda_j^1\}_j \subset \Lambda_1$  is separated. Moreover, as a subsequence of an interpolating sequence it is also interpolating (cf. Remark 1.1). By Corollary 3.3 there exists a function  $v_1 \in L^\varphi$  such that

$$\log \frac{1}{|B_{\{\lambda_j^1\}_j \setminus \{\lambda_k^1\}}(\lambda_k^1)|} \leq P[v_1](\lambda_k^1), \quad k \in \mathbb{N}.$$

Recall that the weight  $v_1$  can be supposed positive, so that  $P[v_1]$  is a positive harmonic function, and again by Harnack's inequality we get

$$P[v_1](\lambda_k^1) \leq c_H P[v_1](\lambda')$$

for every  $\lambda' \in \Lambda_1 \cup Q_k$ . This together with (3.5) and our definition of  $\lambda_k^1$  give

$$\begin{aligned} \log \frac{1}{|B_{\Lambda \setminus \{\lambda'\}}(\lambda')|} &\leq \log \frac{1}{|B_{\Lambda \setminus \{\lambda_k^1\}}(\lambda_k^1)|} = \log \frac{1}{|B_{\Lambda \setminus \{\lambda_j^1\}_j}(\lambda_k^1)|} + \log \frac{1}{|B_{\{\lambda_j^1\}_j \setminus \{\lambda_k^1\}}(\lambda_k^1)|} \\ &\leq P[w_1 + v_1](\lambda_k^1) \leq P[c_H(w_1 + v_1)](\lambda') \end{aligned}$$

for every  $\lambda' \in Q_k$  and  $Q_k \in Q^{(1)}$ . We set  $u_1 := c_H(w_1 + v_1)$  which is clearly in  $L^\varphi$ .

Construct in a similar way functions  $u_i$  for the sequences  $\Lambda_i$ ,  $i = 2, 3, 4$ , and set  $u = \sum_{i=1}^4 u_i$  which is still in  $L^\varphi$  by the quasi-triangular inequality (1.4). So, whenever  $\lambda \in \Lambda$ , there exists  $k \in \{1, 2, 3, 4\}$  such that  $\lambda \in \Lambda_k$ , and hence

$$(3.6) \quad \log \frac{1}{|B_\lambda(\lambda)|} \leq P[u_k](\lambda) \leq P[u](\lambda).$$

#### 4. THE TRACE SPACES

The proof of the trace space characterization is easier than that given in [HMNT04] for the Nevanlinna class.

In order to see that (c) in Theorem 1.2 implies free interpolation it suffices to observe that  $\ell^\infty \subset l_\Phi$  and use Remark 1.1.

Assume now that  $\Lambda$  is of free interpolation. Suppose that  $(a_\lambda)_\lambda \in \mathcal{H}_\Phi | \Lambda$  and  $f \in \mathcal{H}_\Phi$  is such that  $f(\lambda) = a_\lambda$ ,  $\lambda \in \Lambda$ . Let  $w \in L^1$  be the representing measure of the outer part of  $f$ , i.e.  $f = If_w$ , where  $I$  is inner and  $w_+ \in L^\varphi$ . Obviously  $\log^+ |a_\lambda| \leq \log^+(\exp(P[w](\lambda))) \leq P[w_+](\lambda)$ , and so  $(a_\lambda) \in l_\Phi$ .

Conversely, suppose that  $(a_\lambda)_\lambda$  is such that there is a positive function  $w \in L^\varphi$  with  $\log^+ |a_\lambda| \leq P[w](\lambda)$ . Since  $f_w \in \mathcal{H}_\Phi$  and  $\log^+ |f_w| = \log |f_w| = P[w]$  we have  $|a_\lambda| \leq |f_w(\lambda)|$  for every  $\lambda \in \Lambda$ . Since  $\Lambda$  is of free interpolation, i.e.  $\mathcal{H}_\Phi | \Lambda$  is ideal, there exists a function  $f_0 \in \mathcal{H}_\Phi$  interpolating  $(a_\lambda)_\lambda$ . ■

#### 5. HARMONIC MAJORANTS

We begin by recalling the definition of the Poisson balayage: for a positive finite measure  $\mu$  on the closed unit disk we set

$$B(\mu)(\zeta) = \int_{\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(z) = \int_{\mathbb{D}} P_z(\zeta) d\mu(z).$$

Let  $\text{Har}_\varphi^+ = \{P[w] : 0 \leq w \in L^\varphi\}$ . We will begin with an analog of [HMNT04, Proposition 6.1] and that does not require the  $\nabla_2$ -condition.

**Proposition 5.1.** *Let  $\varphi$  be a strongly convex function, and let  $\mu$  be a positive finite Borel measure on  $\mathbb{T}$ . Then  $\|B(\mu)\|_{(L^\varphi)^*} < \infty$  if and only if for every  $f \in \text{Har}_\varphi^+$  we have  $\int_{\mathbb{D}} h d\mu < \infty$ . Moreover*

we have the following relation:

$$\|B(\mu)\|_{(L^\varphi)^*} = \sup \left\{ \int_{\mathbb{D}} h \, d\mu : h = P[w] \in \text{Har}_\varphi^+, \|w\|_\varphi \leq 1 \right\}.$$

So, this proposition furnishes a description of those positive finite measures on  $\mathbb{D}$  that act against positive harmonic functions  $P[w]$  with  $w \in L^\varphi$ .

The proof of this result is short. It is essentially based on an application of Fubini's theorem and the definition of the norm in  $L^\varphi$  by duality. We give it for completeness.

*Proof.*

$$\begin{aligned} & \sup \left\{ \int_{\mathbb{D}} h \, d\mu : h = P[w] \in \text{Har}_\varphi^+, \|w\|_\varphi \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{D}} \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} w(\zeta) \, dm(\zeta) \, d\mu(z) : 0 \leq w \in L^\varphi, \|w\|_\varphi \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{T}} w(\zeta) \int_{\mathbb{D}} \frac{1 - |z|^2}{|\zeta - z|^2} \, d\mu(z) \, dm(\zeta) : 0 \leq w \in L^\varphi, \|w\|_\varphi \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{T}} w(\zeta) B(\mu)(\zeta) \, dm(\zeta) : w \in L^\varphi, \|w\|_\varphi \leq 1 \right\} \\ &= \|B(\mu)\|_{(L^\varphi)^*} = \|B(\mu)\|_{L^{\varphi^*}}. \end{aligned}$$

In the above identities, we have also used the fact that  $\mu$  is a positive measure so that its balayage is also positive. Hence it is enough to test against positive functions in  $L^\varphi$ .  $\blacksquare$

We are now in a position to prove Theorem 1.3.

*Proof of Theorem 1.3.* Condition (b) is clearly necessary. It suffices indeed to plug the estimate  $u \leq P[w]$  into the chain of equalities in the previous proof: if  $u \leq h = P[w]$  for some  $0 \leq w \in L^\varphi$  then  $0 \leq \int_{\mathbb{D}} u \, d\mu \leq \int_{\mathbb{D}} h \, d\mu \leq \|w\|_\varphi \|B\mu\|_{\varphi^*}$  (Hölder's inequality for Orlicz spaces, see [KrRu61, Theorem 9.3]).

Let us consider the sufficiency. So suppose that there exists a constant  $C \geq 0$  such that

$$(5.1) \quad \sup_{\mu \in \mathcal{B}_{\varphi^*}} \int u(z) \, d\mu(z) \leq C,$$

where  $\mathcal{B}_{\varphi^*} = \{\mu : \text{positive measure on } \mathbb{D} \text{ such that } \|B\mu\|_{(L^\varphi)^*} \leq 1\}$ . We want to prove that  $u$  admits a harmonic majorant  $P[w]$  with  $w \in L^\varphi$ .

We will begin as in [HMNT04] by discretizing the problem (see in particular [HMNT04, Lemma 6.3]). For this, let again  $Q_{n,k}$  be the dyadic cubes and  $z_{n,k}$  the corresponding center. Fix  $z_{n,k}^* \in Q_{n,k}$  such that  $u(z_{n,k}^*) \geq (\sup_{Q_{n,k}} u)/2$ . We will set  $u_{n,k} := u(z_{n,k}^*)$  and  $\hat{u}_{n,k} = \sup_{Q_{n,k}} u$ . Let us check that if  $u$  satisfies (5.1) then there is a constant  $C'$  such that whenever  $(c_{n,k})$  is a finite sequence of non-negative coefficients with

$$\left\| \sum c_{n,k} P_{z_{n,k}} \right\|_{\varphi^*} \leq 1,$$

we get

$$\sum c_{n,k} \hat{u}_{n,k} \leq C'.$$

Indeed, setting  $\mu := \sum c_{n,k} \delta_{z_{n,k}^*}$  we obtain a positive finite measure on  $\mathbb{D}$  such that

$$\|B(\mu)(\zeta)\|_{\varphi^*} = \left\| \int_{\mathbb{D}} P_z(\zeta) d\mu(z) \right\|_{\varphi^*} = \left\| \sum c_{n,k} P_{z_{n,k}^*} \right\|_{\varphi^*} \leq K \left\| \sum c_{n,k} P_{z_{n,k}} \right\|_{\varphi^*} \leq K.$$

In the last estimate we have used the existence of a constant  $K = K(\delta)$  such that  $|b_u(v)| \leq \delta$  implies that  $\frac{1}{K} P_v(\zeta) \leq P_u(\zeta) \leq K P_v(\zeta)$  for all  $\zeta \in \mathbb{T}$  and the fact that the Orlicz space  $L^\varphi$  has the lattice property (which can be seen by using the dual representation of the norm).

Now, using (5.1), we get

$$0 \leq \sum c_{n,k} \hat{u}_{n,k} \leq 2 \sum c_{n,k} u_{n,k} = 2 \int u d\mu \leq 2CK$$

So

$$\sup \left\{ \sum c_{n,k} \hat{u}_{n,k} : c_{n,k} \geq 0 \text{ for all } n, k \text{ and } \left\| \sum c_{n,k} P_{z_{n,k}} \right\|_{\varphi^*} \leq 1 \right\} \leq 2CK,$$

in other words, for all positive finite sequences  $(c_{n,k})$  we have

$$\sum c_{n,k} \hat{u}_{n,k} \leq 2CK \left\| \sum c_{n,k} P_{z_{n,k}} \right\|_{\varphi^*}.$$

Let  $E$  be the closure in  $L^{\varphi^*}$  of the  $\mathbb{R}$ -space generated by the Poisson kernels  $P_{z_{n,k}}$  for all  $n, k$  (which in fact corresponds to  $L^{\varphi^*}$ ). By the theorem of Mazur-Orlicz (see e.g. [Pe67, Chapter 2, Proposition 2.2]), there exists a linear continuous mapping  $T : E \rightarrow \mathbb{R}$  with same norm  $2CK$  such that

$$\hat{u}_{n,k} \leq T P_{z_{n,k}}.$$

So  $T \in E^* = (L^{\varphi^*})^* = L^{\varphi^{**}} = L^\varphi$ . Hence there exists  $w_u \in L^\varphi$  such that for all  $v = \sum c_{n,k} P_{z_{n,k}}$  we have  $Tv = \int_{\mathbb{T}} v w_u dm = \sum c_{n,k} P[w_u](z_{n,k})$ . In particular for every  $n, k$

$$0 \leq \hat{u}_{n,k} \leq T P_{z_{n,k}} = P[w_u](z_{n,k}) \leq P[(w_u)_+](z_{n,k}).$$

By Harnack's inequality the last term is bounded by  $P[c_H(w_u)_+](z)$  for and every  $z \in Q_{n,k}$  so that for every  $z \in Q_{n,k}$

$$0 \leq u(z) \leq \hat{u}_{n,k} \leq P[(w_u)_+](z_{n,k}) \leq P[c_H(w_u)_+](z).$$

Since this is true for all  $n, k$  and since clearly  $c_H(w_u)_+ \in L^\varphi$  we have achieved the proof.  $\blacksquare$

## 6. AN EXAMPLE

In this section we will consider concrete separated sequences and check whether they are interpolating for Hardy-Orlicz spaces  $\mathcal{H}_\Phi$  associated with  $\Phi = \Phi_\varepsilon = \psi_\varepsilon \circ \log$  where

$$\psi_\varepsilon = t \log^\varepsilon t$$

for some  $\varepsilon > 0$  (even if this has no special meaning for our situation one could observe that for  $\varepsilon = 1$  the space  $L^{\psi_1}$  is the Zygmund space  $L \log L$ ).

**Proposition 6.1.** *For every  $\varepsilon > 0$  there exists a separated sequence  $\Lambda$  that is interpolating for  $\mathcal{H}_{\psi_\delta}$  whenever  $0 < \delta < \varepsilon$  but not for  $\mathcal{H}_{\psi_\varepsilon}$ .*

*Proof.* Fix  $\varepsilon > 0$  and let  $\lambda_{n,k} = (1 - 1/2^n)e^{k2\pi i/2^n}$ , where  $n \in \mathbb{N}^*$ ,  $k \in \{-k_n, \dots, k_n\}$  and  $k_n := \lfloor 2^n / (n \log^{1+\varepsilon} n) \rfloor$ . Then  $\sum_{n,k} (1 - |\lambda_{n,k}|) \simeq \sum_n [1 / (n \log^{1+\varepsilon} n)] < \infty$  so that  $\Lambda = \{\lambda_{n,k}\}_{n,k}$  satisfies the Blaschke condition.

Since  $\Lambda$  is separated we have for  $\lambda, \mu \in \Lambda$  by standard estimates

$$\log \frac{1}{|b_\mu(\lambda)|} \simeq 1 - |b_\mu(\lambda)| \simeq \frac{(1 - |\mu|)(1 - |\lambda|)}{|1 - \bar{\mu}\lambda|^2}.$$

We will compute  $\log(1/|B_\lambda(\lambda)|)$  for  $\lambda = \lambda_{n,0} = 1 - 1/2^n$ ,  $n \in \mathbb{N}$ , and show that this exceeds the maximal admissible growth in  $\mathcal{H}_{\Phi_\varepsilon}$ .

$$\begin{aligned} \log \frac{1}{|B_\lambda(\lambda)|} &= \sum_{j \geq 1} \sum_{\substack{l=-k_j \\ l \neq 0 \text{ if } j=n}}^{k_j} \log \frac{1}{|b_{\lambda_{j,l}}(\lambda)|} \simeq \sum_{j \geq 1} \sum_{\substack{l=-k_j \\ l \neq 0 \text{ if } j=n}}^{k_j} \frac{(1 - |\lambda|)(1 - |\lambda_{j,l}|)}{|1 - \bar{\lambda}_{j,l}\lambda|^2} \\ &= \frac{1}{2^n} \sum_{j \geq 1} \frac{1}{2^j} \sum_{\substack{l=-k_j \\ l \neq 0 \text{ if } j=n}}^{k_j} \frac{1}{|1 - \bar{\lambda}_{j,l}\lambda|^2} \\ &\geq \frac{1}{2^n} \sum_{j \geq 2n} \frac{1}{2^j} \sum_{l=-k_j}^{k_j} \frac{1}{|1 - \bar{\lambda}_{j,l}\lambda|^2} \end{aligned}$$

We get for  $j \geq 2n$

$$(6.1) \quad \frac{1}{2^j} \sum_{l=-k_j}^{k_j} \frac{1}{|1 - \bar{\lambda}_{j,l}\lambda|^2} = \sum_{l=-k_j}^{k_j} \frac{1}{2^j} \frac{1}{|e^{l2\pi i/2^j} - r_{n,j}|^2},$$

where  $r_{n,j} = (1 - 1/2^n)(1 - 1/2^j)$ . It can be noted that for fixed  $n$ ,  $r_{n,j}$  goes rapidly and increasingly to  $\lambda = 1 - 1/2^n$  as  $j \rightarrow +\infty$  (see Figure 2). The sum in 6.1 is a Riemann sum for

$$\int_{I_j} \frac{1}{|e^{i\vartheta} - r_{n,j}|^2} d\vartheta,$$

where  $I_j = [-2\pi k_j/2^j, 2\pi k_j/2^j] = [-2\pi/(j \log^{1+\varepsilon} j), 2\pi/(j \log^{1+\varepsilon} j)]$ . Then, since the function to be integrated is continuous, we get for fixed  $n$  and  $j \geq 2n$

$$\frac{1}{2^j} \sum_{l=-k_j}^{k_j} \frac{1}{|1 - \bar{\lambda}_{j,l}\lambda|^2} \simeq \int_{I_j} \frac{1}{|e^{i\vartheta} - r_{n,j}|^2} d\vartheta$$

(for a better control on the constants it is possible to replace  $j \geq 2n$  by  $j \geq 2n + K$  for some fixed  $K > 0$ ).

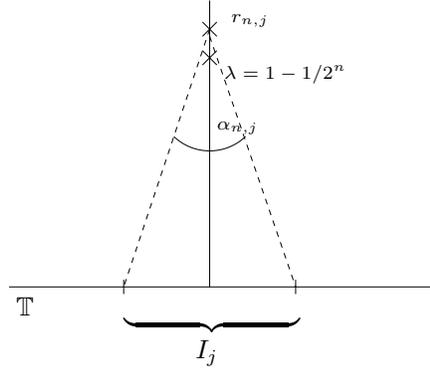


Figure 2

It is well known (see e.g. [Gar81, p.13]) that

$$\int_{I_j} \frac{1 - r_{n,j}^2}{|e^{i\vartheta} - r_{n,j}|^2} d\vartheta \simeq \frac{\alpha_{n,j}}{\pi}$$

( $\alpha_{n,j}$  is the angle indicated in Figure 2). As we have already mentioned  $1 - r_{n,j} \geq 1 - \lambda = 1/2^n$ , and so for fixed  $n$  we get  $\alpha_{n,j} \rightarrow 0$  as  $j \rightarrow +\infty$ . So  $\alpha_{n,j} = 2\alpha_{n,j}/2 \sim 2 \tan(\alpha_{n,j}/2) = 2(|I_j|/2)/(1 - r_{n,j}) \sim |I_j|/(1 - \lambda) = 2^n |I_j| = 4\pi 2^n / (j \log^{1+\varepsilon} j)$ . Hence

$$\begin{aligned} \log \frac{1}{|B_\lambda(\lambda)|} &\geq \frac{1}{2^n} \sum_{j \geq 2n} \frac{1}{1 - r_{n,j}^2} \sum_{l=-k_j}^{k_j} \frac{1}{2^j} \frac{1 - r_{n,j}^2}{|e^{l2\pi i/2^j} - r_{n,j}|^2} \\ &\simeq \frac{1}{2^n} \sum_{j \geq 2n} \frac{1}{1 - r_{n,j}^2} \int_{I_j} \frac{1 - r_{n,j}^2}{|e^{i\vartheta} - r_{n,j}|^2} d\vartheta \\ &\simeq \frac{1}{2^n} \sum_{j \geq 2n} \frac{1}{1 - r_{n,j}^2} \frac{4 \cdot 2^n}{j \log^{1+\varepsilon} j} \\ &= \sum_{j \geq 2n} \frac{1}{1 - r_{n,j}^2} \frac{4}{j \log^{1+\varepsilon} j} \end{aligned}$$

Moreover  $j \geq 2n$ , so that  $1 - r_{n,j}^2 \simeq 1 - r_{n,j} = \frac{1}{2^n} + \frac{1}{2^j} - \frac{1}{2^{n+j}} \simeq \frac{1}{2^n}$ . Using  $1 - |\lambda| = 1 - \lambda = 1/2^n$  and so  $n = \log(1/(1 - |\lambda|))/\log 2$  we get

$$\begin{aligned} \log \frac{1}{|B_\lambda(\lambda)|} &\gtrsim 2^n \sum_{j \geq 2n} \frac{4}{j \log^{1+\varepsilon} j} \sim 2^n \frac{4}{\log^\varepsilon(2n)} \\ (6.2) \quad &\sim \frac{4}{1 - |\lambda| \log^\varepsilon n} \sim \frac{4}{1 - |\lambda| \log^\varepsilon \log \frac{1}{1 - |\lambda|}}. \end{aligned}$$

Let us check that this is not compatible with free interpolation in  $\mathcal{H}_{\Phi_\varepsilon}$ . Suppose to the contrary that  $\Lambda$  is an interpolating sequence for  $\mathcal{H}_{\Phi_\varepsilon}$ . Then, by Theorem 1.2, there exists a positive function  $w \in L^{\psi_\varepsilon}(\mathbb{T})$  such that  $\log(1/|B_\lambda(\lambda)|) \leq P[w](\lambda)$  for every  $\lambda \in \Lambda$ . As we have already discussed in Section 2,  $f(z) = \exp(f(\zeta + z)/(\zeta - z)w(\zeta) dm(\zeta))$  is an outer function in  $\mathcal{H}_{\Phi_\varepsilon}$ . Hence

$$P[w](z) = \log |f(z)| \leq \psi_\varepsilon^{-1} \left( \frac{c_f}{1 - |z|} \right).$$

Note that  $\psi_\varepsilon(u/\log^\varepsilon u) \sim u$  as  $u \rightarrow +\infty$ , so that  $\psi_\varepsilon^{-1}(u) \sim u/\log^\varepsilon u$ . Hence, setting  $u = 1/(1 - |z|)$  we get for  $z$  sufficiently close to  $\mathbb{T}$ ,

$$P[w](z) \lesssim c_f \frac{1}{1 - |z|} \frac{1}{\log^\varepsilon \frac{c_f}{1 - |z|}}.$$

Since the right hand side of the last inequality is negligible with respect to the right hand side of (6.2) we have reached a contradiction. So, the sequence  $\Lambda$  is not interpolating for  $\mathcal{H}_{\Phi_\varepsilon}$  and hence for no  $\mathcal{H}_\Phi$  with  $\Phi = \varphi \circ \log$  and  $\varphi$  a strongly convex function with  $\varphi(t) \geq \psi_\varepsilon(t)$ ,  $t \geq t_\varepsilon$ .

In order to finish the proof, we check that the above constructed sequence is interpolating for  $\mathcal{H}_{\Phi_\delta}$  whenever  $0 < \delta < \varepsilon$ . Since  $\Lambda$  is separated it is of course interpolating for the Smirnov (and Nevanlinna) class, see [HMNT04, Corollary 1.9], which means that there is a function  $u \in L^1(\mathbb{T})$  such that

$$(6.3) \quad \log(1/|B_\lambda(\lambda)|) \leq P[u](\lambda).$$

It is known that the function  $u$  can be chosen explicitly by:

$$u = c_0 \sum_{\lambda \in \Lambda} \chi_{I_\lambda}$$

(see [HMNT04, Proposition 4.1] and also [NPT, p.124]), the interval  $I_\lambda = \{e^{it} \in \mathbb{T} : |t - \arg \lambda| \leq c(1 - |\lambda|)\}$  appearing in the above formula being the so-called Privalov shadow. It turns out that in the present situation this intuitive candidate for  $u$  is the right one to get a harmonic majorant. In other words, we have to check that  $u \in L^{\psi_\delta}$ ,  $0 < \delta < \varepsilon$ , and this will finish the proof. So, let us suppose that the constant  $c$  in the definition of  $I_\lambda$  is adapted in such a way that  $I_{\lambda_{n,k}}$  and  $I_{\lambda_{n,k+1}}$  touch without overlap (this is not really of importance). We then consider the shadow of the stage  $n$ :  $\bigcup_{j=1, \dots, k_n} I_{\lambda_{n,j}} = [-1/2^n - 1/(n \log^{1+\varepsilon} n), 1/(n \log^{1+\varepsilon} n) + 1/2^n]$  which is essentially the interval  $[-1/(n \log^{1+\varepsilon} n), 1/(n \log^{1+\varepsilon} n)]$ . So the function  $u$  is essentially equal to  $k$  on  $[-1/(k \log^{1+\varepsilon} k), -1/((k+1) \log^{1+\varepsilon} (k+1)) \cup [1/((k+1) \log^{1+\varepsilon} (k+1)), 1/(k \log^{1+\varepsilon} k)]$ . In order that  $\psi_\delta \circ u \in L^1$  it thus suffices (using some Fubini) that

$$\sum_{k \geq 1} \frac{\psi_\delta(k+1) - \psi_\delta(k)}{k \log^{1+\varepsilon} k} < \infty,$$

and this holds for  $0 < \delta < \varepsilon$ . ■

If one wishes to get closer to  $L^1$ , one could e.g. consider strongly convex functions  $\varphi(t) = t \log^\varepsilon \log t$  by choosing  $k_j = 2^j / (j \log j \log^{1+\varepsilon} \log j)$ .

We wanted to emphasize in this section the behaviour of *separated* Blaschke sequences since these were already interpolating for the Smirnov class. Our examples make it clear that the situation is much more delicate in big Hardy-Orlicz spaces.

Another and of course easier way of producing examples of interpolating sequences for our spaces is to take two  $H^\infty$ -interpolating sequences that approach each other in a critical way: if  $\Lambda_1 = \{\lambda_n\}_n$  is such an  $H^\infty$ -interpolating sequence, take  $\mu_n \neq \lambda_n$  close to  $\lambda_n$  and define  $\Lambda_2 = \{\mu_n\}_n$ . Then, for  $B = B_{\Lambda_1 \cup \Lambda_2}$  we get  $\log(1/|B_{\lambda_n}(\lambda_n)|) \simeq \log(1/|b_{\lambda_n}(\mu_n)|) =: \eta_n$ , so that suitable choices of  $(\eta_n)_n$  yield interpolating sequences for some big Hardy-Orlicz spaces which are not interpolating for others (see e.g. [HMNT04, Example 9.2] and [HaMa01] for such constructions).

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