The ghosts of the Ecole Normale. Life, death and legacy of René Gateaux

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Statistical Science
2015, Vol. 30, No. 3, 391–412
DOI: 10.1214/15-STS512

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Life, death and legacy of René Gateaux[1]

Abstract. The present paper deals with the life and some aspects of the scientific contributions of the mathematician René Gateaux, killed during World War I at the age of 25. Though he died very young, he left interesting results in functional analysis. In particular, he was among the first to try to construct an integral over an infinite-dimensional space. His ideas were extensively developed later by Paul Lévy. Among other things, Lévy interpreted Gateaux's integral in a probabilistic framework that later contributed to the construction of the Wiener measure. This article tries to explain this singular personal and professional destiny in pre- and postwar France.

Key words and phrases: History of mathematics, functional analysis, integration, probability, Wiener measure.

1. INTRODUCTION

In his seminal 1923 paper on Brownian motion, Norbert Wiener mentioned[2] that *integration in infinitely many dimensions* (*was*) *a relatively little-studied problem* and that *all that has been done on it* (*was*) *due to Gateaux*, *Lévy*, *Daniell* and himself. Following Wiener, the most complete investigations had been those *begun by Gateaux and carried out by Lévy*.

It was in 1922 that Lévy's book *Leçons d'Analyse Fonctionnelle* (Lévy, 1922) was published after his lectures given at the Collège de France in the aftermath of the Great War. Lévy's book, and, more specifically, Lévy himself, made a profound impression on Wiener. The American mathematician emphasized how Lévy explained personally to him how his own method of integration in infinitely many dimensions, which extended results Lévy found in Gateaux's works, was the convenient tool he needed for his construction of Brownian motion measure.

I shall comment later on the path linking Gateaux's works to Lévy's fundamental studies, but let me begin by discussing the circumstances which constituted the initial motivation behind the current paper. Gateaux was killed at the very beginning of the Great War in October of 1914. He died at the age of 25, before having obtained any academic position, even before having completed a doctorate. His publications formed a rather thin set of a few notes presented to the Academy of Sciences of Paris and to the Accademia dei Lincei of Rome. None of them dealt with infinite-dimensional integration. Neverthess, Gateaux's name is still known today, and even to (some) undergraduate students, through a basic notion of calculus known as *Gateaux*

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[1]In all the literature, there is a significant uncertainty regarding whether the name bears a circumflex accent or not (due to the confusion with the word *gâteau*—cake in French). In the present paper, I shall adopt the mathematician's own use of *NOT* writing the name with an accent (this is to conform with his birth certificate).

[2]Wiener (1923), page 132.

differentiability.[3] The notion, weaker than the (now) classical Fréchet differentiability, was mentioned in Gateaux's note (Gateaux, 1914a, page 311), under the name *variation première* of a functional, though it was probably already considered by him in 1913 as the name appears in Gateaux (1913a), page 326, but without any definition. Regardless, this notion was in fact only a technicality introduced by Gateaux among the general properties that a functional can have. Lévy was probably the first to name it after Gateaux.[4]

So, I wanted to understand how a basic notion of calculus had been given the name of an unknown mathematician, who died so young, before having obtained any academic position and even before having defended a thesis. Such a paradox deserved to be unraveled. It is this apparent contradiction that I want to address in this paper by presenting René Gateaux's life and death, some of his mathematical research and the path explaining why we still remember him though so many of his fellows killed during the war became only a *golden word on our public squares*, following Aragon's beautiful expression.[5]

Let me immediately reveal the key to our explanation. Beyond his tragic fate, Gateaux had two strokes of good fortune. The first one was related to the main mathematical theme he was interested in, *Functional Analysis* (Analyse Fonctionnelle) in the spirit of Volterra in Rome and Hadamard in Paris, often also called by them *functional calculus* (calcul fonctionnel).[6] At the beginning of the 20th Century, this subject was still little studied. In the years following World War I, it received unexpected developments, in particular, in the unpredictable direction of probability theory. Gateaux was therefore posthumously in contact with a powerful stream leading to the emergence of some central aspects of modern probability, such as Brownian motion as we have seen in Wiener's own words. It is very fortunate for the historian that important archival documents about Gateaux's beginnings in mathematics are still available. Gateaux had in particular been in correspondence with Volterra before, during and (for some weeks) after a sojourn in Rome with the Italian mathematician. His letters still exist today at the *Accademia dei Lincei* and provide precious insight into Gateaux's first steps. Letters exchanged between Borel and Volterra about the young man's projects and progress are also available. One such document is a letter from Gateaux to Volterra dated from 25 August 1914 and written on the battlefield. Moreover, some other material is accessible such as the military dossier, some of Gateaux's own drafts of reports about his work, and some scattered letters from him or about him by other people. This allows us to attempt to reconstruct the life of the young mathematician during his last seven or eight years.

But it is mainly due to the second stroke of fortune that some memory of Gateaux (or, at least, of his name) was preserved. Before he went to the war, Gateaux had left his papers in his mother's house. Among them were several half-completed manuscripts which were intended to become chapters of his thesis. After the death of her son, his

[3]Let me recall that Gateaux differentiability of a function ϕ defined on $\mathbb{R}^n$ is the directional differentiability: ϕ is said to be Gateaux differentiable at $\theta \in \mathbb{R}^n$ if for any vector h given in $\mathbb{R}^n$, the function $t \mapsto \phi(\theta + th)$ has a derivative at 0. Various notions of differentiability for a function have been considered by several French mathematicians under Volterra and Hadamard's influence during the first half of the 20th century. In the 1920s, Hadamard introduced an intermediate concept between Fréchet and Gateaux differentiability. In modern terminology, a function $\phi : E \to F$, where E and F are two normed spaces, is Hadamard differentiable at $\theta \in E$ if there is a continuous linear function $\phi'_\theta : E \to F$ such that, for any $h \in E$ and any choice of a family $(h_t)_{t>0}$ in E such that $h_t \to h$, one has

$$\lim_{t\to 0}\left\| \frac{\phi(\theta + th_t) - \phi(\theta)}{t} - \phi'_\theta(h)\right\|_F = 0.$$

The difference between Gateaux and Hadamard differentiability is that, for the latter, the direction h_t is allowed to change in the ratio. On this topic see Barbut, Locker and Mazliak (2014), Section 4.2, pages 15–17. Hadamard-differentiabilty is in particular adequate to deal with some asymptotic estimates in Statistics (see, e.g., van der Vaart, 1998, Chapter 20—especially page 296 and seq.).

[4]In Lévy (1922), page 51, under the name *différentielle au sens de Gateaux*, Sanger (1933) compared the various definitions formulated for the differential of a functional in his survey about Volterra's functions of lines. See, in particular, Chapter II on pages 240–253. Gateaux's definition is considered on pages 250–251.

[5]Déjà la pierre pense où votre nom s'inscrit
Déjà vous n'êtes plus qu'un mot d'or sur nos places
Déjà le souvenir de vos amours s'efface
Déjà vous n'êtes plus que pour avoir péri (Aragon, 1956).

[6]In the sequel, I shall use the expression *functional analysis* only in reference to the theories initiated by Volterra, though it today has a slightly different meaning.

mother sent the papers to the École Normale. Hadamard collected them and in 1919 passed them to Paul Lévy in order to prepare an edition in Gateaux's honor. Studying Gateaux's papers came at a crucial moment in Lévy's career. Not only did they inspire Lévy's book (Lévy, 1922), but they were a major source for his later achievements in probability theory.

The aim of the present paper is twofold: one aspect is to present an account of Gateaux's life by using valuable new archival material discovered in several places, the other is to give some hints of how his works were completed and—considerably—extended by Lévy. In that respect, it is clear that the mathematical ideas of Gateaux were developed in a direction he could not have expected; probability, for instance, was absolutely not in his mind. The appearance of the mathematics of randomness in this inheritance is undoubtedly entirely due to Lévy's powerful imagination. It is therefore well beyond the scope of this article, centered on Gateaux, to present a detailed study of Lévy or Wiener's studies on Brownian motion. The interested reader may refer to several historical expositions such as Kahane (1998) or Barbut, Locker and Mazliak (2014), pages 54–60. An account from direct participants in this story can be found in Lévy's autobiography (Lévy, 1970, page 96 and seq.), or Itô's comments on Wiener's papers (Wiener, 1976, pages 513–519).

Looking backward, Gateaux's role must not be overestimated in the history of mathematics. Contrary to some other examples of mathematicians who died young, such as Abel to cite a famous example, Gateaux had not made decisive progress in any important direction. So maybe some words are necessary to explain what a biographical approach of someone like Gateaux can teach us. The main point here is related to the Great War and the effect it produced on French mathematicians.

In her memoirs (Marbo, 1967), written at the end of the 1960s, the novelist Camille Marbo,[7] Emile Borel's widow, mentioned that after the end of World War I, her husband declared that he could not bear any more the atmosphere of the École Normale in mourning, and decided to resign from his position of Deputy Director. In 1910 Borel had succeeded Jules Tannery in the position, during a time of extraordinary success for Analysis in France with outstanding mathematicians such as Henri Poincaré, Emile Picard, Jacques Hadamard, Henri Lebesgue and naturally Borel himself.

A superficial, though impressive, picture of the effect of WWI on the French mathematical community is read through the personal life of the aforementioned mathematicians—with the obvious exception of Poincaré who had died in 1912. Picard lost one son in 1915, Hadamard two sons in 1916 (one in May, one in July) and Borel his adopted son in 1915. The figures concerning casualties among the students of the École Normale, and especially among those who had just finished their three year studies at the *rue d'Ulm*, are terrible.[8] Out of about 280 pupils who entered the École Normale in the years 1911 to 1914, 241 were sent to the front directly from the school and 101 died during the war. If the President of the Republic Raymond Poincaré could declare that *the École of 1914 has avenged the École of 1870*,[9] the price to pay had been so enormous that it was difficult to understand how French science could survive such a hemorrhage. Most of the vanished were brilliant young men, expected successors of the brightest scholars from the previous generation in every domain of knowledge. They were so young that almost none had time to start making a reputation of his own through professional achievement. As testimony of his assumed abnegation, Frédéric Gauthier, a young hellenist, who had entered the École Normale Supérieure in 1909 and was killed in July 1916 in the battle of Verdun, left a melancholic comment on this time of resignation: *My studies*, *it is true*, *will remain sterile*, *but my ultimate actions*, *useful for the country*, *have the same value as a whole life of action.*[10]

Gateaux, who died at the very beginning of the war, appears therefore to be a good representation of the lost generation of *normaliens* that I have just mentioned; he was at the same time an exception, as his very name, contrary to almost every one of his companions of misfortune, was retained in mathematics. The way in which it was retained and, above all, the direction in which his works received their

[7] Marbo is Marguerite Appel's nom de plume. She was the daughter of the mathematician Paul Appell.

[8] They were collected in a small brochure published by the École Normale at the end of the war (École Normale Supérieure, 1919).

[9] L'École de 1914 a vengé l'École de 1870 (École Normale Supérieure, 1919, page 3).

[10] Mes études, il est vrai, seront demeurées stériles, mais mes actions dernières, utiles au pays, vaudront toute une vie d'action (Annuaire, 1918).

most important development [Wiener's seminal paper (Wiener, 1923)] was, at least partly, related to the war. Lévy wrote to Fréchet in 1945:

> As for myself, I learned the first elements of probability during the spring of 1919 thanks to Carvallo (the director of studies at the École Polytechnique) who asked me to make three lectures on that topic to the students there. Besides, in three weeks, I succeeded in proving new results. And never will I claim for my work in probability a date before 1919. I can even add, and I told M. Borel so, that I had not really seen before 1929 how important were the new problems implied by the theory of denumerable probabilities. But I was prepared by functional calculus to the studies of functions with an infinite number of variables and many of my ideas in functional analysis became without effort ideas which could be applied in probability.[11]

The urgent need to renew the teaching of probability at the École Polytechnique was a side effect of the war, when much probabilistic technique had been used to direct artillery. And it is because Gateaux was dead that Lévy was in possession of his papers. Nobody can tell what Lévy's career would have been without the conjunction of these two disparate elements that his fertile mind surprisingly connected.

I began to be interested in Gateaux's story when we were preparing the edition of Fréchet and Lévy's correspondence with Barbut and Locker in 2003 [an English edition (Barbut, Locker and Mazliak, 2014) was recently published]. Since that date, a lot of work has been done concerning the involvement of scientists in the Great War, resulting in an increasing number of publications, and, in particular, the approach of the centennial year was met by a flow of papers and books in many countries so that it is difficult to provide an exhaustive list. Let me mention, among many others, the interesting contributions [Pepe (2011), Onghena (2011) or the books Aubin and Goldstein (2014) and Downing (2014)]. By the way, the centennial was also an occasion for economists to remember Gateaux's work (Dugger and Lambert, 2013).

[11] Barbut, Locker and Mazliak (2014), page 139.

A focus on Gateaux therefore allows us to shed some light on some specific aspects of mathematics before and after the Great War and to understand how such an event may have influenced their development, not only in technical aspects but also because of its terrible human cost.

The paper is divided into four parts. In the first I describe Gateaux's life before he went to Rome in 1913. Then I present the critical period in Rome with Volterra. The third part treats his departure to the army and his last days. Finally, there is a slightly more technical part which considers the work of Gateaux and how it was recovered by Lévy and considerably extended by him so that it became a step toward the construction of an abstract integral in infinite dimensions and then of modern probability theory.

2. A PROVINCIAL IN PARIS

We do not know much about Gateaux's life before he entered the École Normale. Gateaux did not belong to an important family and, moreover, his family unit consisted only of his parents, his younger brother Georges and himself. Neither of the brothers had direct descendants, as both boys died during WWI. I have met a distant member of his family, namely, the great-great-great-great-grandson of a great-great-great-grandfather of René Gateaux, Mr Pierre Gateaux, who still lives in Vitry-le-François and most kindly offered access to the little information he has about his relative.

René Eugène Gateaux was born on 5 May 1889 in Vitry-le-François in the département of Marne, 200 km east from Paris.[12] René's father Henri, born in 1860, was a small contractor who owned a saddlery and cooperage business in the outskirts of Vitry. His mother was Marie Roblin, born in Vitry in 1864. René's family on his father side came originally from the small town of Villers-le-Sec at 20 km from Vitry, the rural nest of Gateaux's family. René's birth certificate indicates that Eugene Gateaux (Henri's father) was a proprietor and Jules Roblin (Marie's father) was a cooper; the grandparents acted as witnesses when the birth was registered at the town

[12] Abraham de Moivre was born there 222 years earlier, before the wars of religion forced him to leave for London where he spent all his scientific career. François Jacquier was also born there 178 years earlier. A local historian from Vitry, Gilbert Maheut, has written several short papers about his three mathematician fellow-citizens. See, in particular, Maheut (2000).

hall. Eugène's birth certificate indicates that he was born in Villers-le-Sec in 1821 and that his father was a carpenter. Perhaps René's grandfather came to Vitry to create his business and employed Marie's father as a cooper. As already mentioned, the couple had two children: René is the elder; the second one, Georges, was born four years later in 1893. René's father died young, in 1905, aged 44, and the resulting precarious situation may have increased the boy's determination to succeed in his studies.

I have no details on René Gateaux's school career; he was a pupil in Vitry and then in Reims. The oldest handwritten document I have found is a letter to the Minister of Public Instruction on 24 February 1906 asking for permission to sit for the examination for admission to the École Normale Supérieure[13] (science division), although he had not reached the regular minimum age of 18.

Two things can be deduced from this document. The first is that Gateaux was a student in a Classe Préparatoire in the lycée of Reims.[14] Our second inference is that Gateaux was a brilliant student in his science classes. He probably obtained his baccalauréat in July 1904 at the age of only 15. Gateaux was a sufficiently exceptional case for an inspector (coming at the Lycée of Reims in March 1907) to mention in his report that Gateaux had obtained the extraordinary mark of 19 (out of 20) to a written test in mathematics.[15] However, he was not admitted to the École Normale on his first attempt in 1906, but only in October 1907 after a second year in the class of Mathématiques Spéciales, as was usually the case.

What was it like to be a provincial in Paris? Jean Guéhenno, born in 1890, and admitted in 1911 in the literary section, has written some fine pages on the subject in his *Journal d'un homme de 40 ans* (Guehenno, 1934—see, in particular, Chapter VI, "Intellectuel"). There he describes the École Normale Supérieure of the years before the Great War through the eyes of a young man from a poor provincial background (much poorer, in fact, than Gateaux's) and how he was dazzled by the contrast between the intellectual riches of Paris and the laborious tedium of everyday life in his little industrial town in Brittany. We also have an obituary (Annuaire, 1918, pages 136–140) written in 1919 by two of Gateaux's fellow students from the 1907 science section of the École Normale, Georges Gonthiez and Maurice Janet. They described Gateaux as a good comrade with benevolence and absolute sincerity, who soon appeared to his fellow students as one of the best mathematicians of the group.

After the entry at the École occurred an event in the young man's life of undoubted importance since Gonthiez and Janet devote many lines to it. Gateaux became a member of the Roman Catholic Church. He joined the church *with fervour*, wrote his two fellows. Such a decision in 1908 may seem surprising: the separation laws between Church and State had been passed in 1905 and the Roman Church stood accused for its behavior during the Dreyfus Affair. However, there was concurrently a revival of interest in Catholicism as a counterweight to triumphant positivism. Such a current was well represented at the École Normale (Gugelot, 1998). Among Gateaux's fellows was Pierre Poyet, who chose a religious life and died a few months before he could make his vows as a Jesuit.

René's conversion to Catholicism, which had a profound effect on his spiritual life, created difficulties for him at the École Normale. Gateaux explained in a letter to Poyet (quoted in Bessieres, 1933) that his conversion was received badly by his fellows and some professors. Several pages are devoted to Gateaux in Béssières' biography of Poyet (Bessieres, 1933). So far, all efforts to locate Poyet's personal papers have been fruitless, nevertheless, the obituary by Gonthiez and Janet in Annuaire (1918) testifies not only to the incomprehension felt by Gateaux's fellows, but also to how they were impressed by the similarity of the methods used by him to progress in his mathematical and spiritual lives.[16]

[13]After the defeat of 1870, the prestige of the École Polytechnique faded and the École Normale Supérieure became the major center of scientific life in France at the turn of the century. The École Polytechnique was to regain a real importance for scientific research only much later in the 20th Century. Paul Lévy, who chose to go to the Polytechnique instead of the École Normale to please his father, was a real exception in mathematical research at the beginning of 20th Century. He also slightly suffered from the situation by not belonging to Borel's or Hadamard's usual network of *normaliens*.

[14]The Classes Préparatoires are the special sections in the French school system that train students for the competitive examinations for entry to the "Grandes Écoles," such as the École Polytechnique or the École Normale Supérieure.

[15]Archives départementales de la Marne.

[16]Bessieres (1933) provides a surprising picture of the mystic atmosphere present at the École Normale around Poyet.

In 1910, Gateaux passed the Agrégation of mathematical sciences where he obtained the 11th rank out of 16. This was not a very good rank, so it left him no possibility of obtaining a grant to devote himself entirely to research, as had been the case for Joseph Pérès, for instance (on which I shall comment later). On 8 July 1912, a ministerial decree appointed Gateaux as Professor of Mathematics at the Lycée of Bar-le-Duc, the principal town of the département of Meuse, 250 km east from Paris, and not very distant from his native town.

Before taking up this position, Gateaux should have fulfilled his military obligations. From March 1905 (Journal Officiel de la République Française, 1905), a new law replaced the July 1889 regulation for the organization of the army. The period of active military service had been reduced to 2 years, but conscription became, in theory, absolutely universal. Gateaux was particularly affected by article 23 stipulating that the young men who entered educational institutions such as the École Normale Supérieure could, at their choice, fulfill the first of their two years of military service in the ranks before their admission to these institutions or after their exit. Gateaux had chosen the latter option (Gateaux, 1922b). In October 1910, Gateaux joined the 94th Infantry regiment where he was a private. In February 1911 he was promoted to *caporal* (corporal), and finally was declared second lieutenant in the reserve in September 1911. He had to follow some special training for officers; the comments made by his superiors on the military file indicate that the supposed military training at the École Normale had been more virtual than real. On the special pages devoted to his superior's appraisal, one reads that, though having very good spirit, René was hardly prepared for his rank, but that the second semester 1912 (which ended in fact in September 1912) seems to have been better. He had followed a period of instruction for shooting and obtained very good marks. A final comment in the military file has a strange resonance with what happened two years later. Gateaux's superior mentioned that he was able to lead a machine-gun section.

In October 1912, Gateaux, freed from the active army, began his lectures at the Lycée of Bar-le-Duc. Gateaux's (very thin) personnel file contains a personal identification form and a decree of the Minister of Public Instruction on 2 October 1913 granting him one-year's leave with an allocation of 100 francs for that year, as well as a handwritten document showing that he had obtained a David Weill grant for an amount of 3000 francs.

3. THE ROMAN STAGE

Gateaux had indeed begun to work on a thesis with themes closely related to functional analysis *à la Hadamard*. I have found no precise information about how Gateaux chose this subject for his research, but it is plausible that he was advised to do so by Hadamard himself. In 1912, Hadamard had just delivered a series of lectures on functionals at the Collège de France and had entered the Academy of Science in the same year. Paul Lévy had, moreover, defended his own brilliant thesis on similar questions in 1911. As well, a young French normalien of the year before Gateaux, Joseph Pérès, had in 1912–1913 benefited from a David Weill grant offered for a one-year stay in Rome with Volterra. Volterra himself, invited by Borel and Hadamard, had come to Paris for a series of lectures on functional analysis, edited by Pérès and published in 1913 (Volterra, 1913b). These were thus good reasons for Gateaux to be attracted by this new and little explored domain. For a young doctoral student the natural people to be in contact with were Hadamard in Paris and Volterra in Rome.[17] Pérès's example encouraged Gateaux to go to Rome. Some years later, when Hadamard wrote a report recommending Gateaux for the posthumous attribution of the Francœur prize, he mentioned that the young man had been *one of those who*, *inaugurating a tradition that could not be overestimated*, *went to Rome to become familiar with M. Volterra's methods and theories*.[18]

On the occasion of the centennial of Volterra's birth, in 1960, a volume was edited by the Accademia dei Lincei in Rome in which Giulio Krall devoted several pages to Volterra's research on the phenomenon of *hysteresis*, the "memory of materials," which describes the dependence on time of the state of deformation of certain materials. To model

[17] On Hadamard, a star of the French mathematical stage of the time, the reader can refer to the book (Mazya and Shaposhnikova, 1998). Two biographies of Vito Volterra have recently been published (Goodstein, 2007; Guerraggio and Paoloni, 2013), and the reader can also find information in the annotated edition of the correspondence between Volterra and his French colleagues during WWI (Mazliak and Tazzioli, 2009).

[18] Il fut un de ceux qui, inaugurant une tradition à laquelle nous ne saurions trop applaudir, allèrent à Rome se former aux méthodes et aux théories de M. Volterra (Hadamard, 1916). On the development of student exchanges between Paris and Rome in these years, see Mazliak (2015).

such a situation, Volterra was led to consider *functions of lines* (funzione di linea), later called *functionals* (fonctionnelle) by Hadamard and his followers, which is to say a function of a real function representing the state of the material, and to study the equations they must satisfy. These equations happen to be an infinite-dimensional generalization of partial differential equations. As Krall mentions,[19] *from mechanics to electromagnetism*, *the step was small*, and Volterra's model was applied to different physical situations, such as electromagnetism or sound produced by vibrating bars.[20] In 1904, the King made Volterra a Senator of the Kingdom, mostly honorary, but giving the recipient some influence through his proximity with the men of power.

Such a combination of science and politics appealed to Borel, who had a deep friendship with Volterra.[21] Borel had a part in Gateaux's decision to go to Rome, at least as an intermediary between the young man and Volterra. We indeed find a first indication of this Roman project in their correspondence. Borel wrote to Volterra on 18 April 1913 that he intended to support René's request for the grant, and joined a letter written by René Gateaux where he explained his research agenda. Borel then asked Volterra to write a short letter of support for this project to Liard, the Vice-Rector of the Paris Academy, and also mentioned that he lent the two books published by Volterra on the functions of lines to Gateaux [more precisely, the book Volterra (1913a) and the proofs of Volterra (1913b)]. On 30 June 1913, Borel communicated the good news to Volterra: a David Weill grant had been awarded to Gateaux for the year 1913–1914.

Gateaux's aforementioned letter to Borel[22] was in Volterra's archives, and, consequently, we know precisely what his mathematical aims were when he went to Rome. Gateaux considered two main points of interest for his future research. The first one is classified as *Fonctionnelles analytiques* (Analytical functionals) and is devoted to the extension of the classical results on analytical functions: the Weierstrass expansion, the equivalence between analyticity and holomorphy and the Cauchy formula. The second one is devoted to the problem of integration of a functional.

Gateaux started from the definition Fréchet had proposed in 1910 for an analytical functional (Fréchet, 1910) based on a generalization of a Taylor expansion. A functional[23] U is homogeneous with order n if for any $p \geq 1$ and any given continuous functions $g_1, \dots, g_p$ over $[a, b]$, the function defined on $\mathbb{R}^p$ by

$$(\lambda_1, \dots, \lambda_p) \mapsto U(\lambda_1 g_1 + \cdots + \lambda_p g_p)$$

is a homogeneous polynomial of degree less than n.[24] Now, a functional U is by definition analytical if it can be written as

$$U(f) = \sum_{n=0}^{\infty} U_n(f),$$

where U_n are homogeneous functionals of order n (Fréchet, 1910, page 214; see also Taylor, 1970).

Gateaux first proposed to obtain properties of the terms $U_n(f)$ in the previous expansion of an analytical functional. Then, he intended to obtain the equivalence between the analyticity of the functional U and its complex differentiability (holomorphy) and to deduce a definition of analyticity by a Cauchy formula. For that purpose, as he wrote, one needs a definition of the integral of a real continuous functional over a real functional field. This may be the first appearance of questions around infinite-dimensional integration. In this programmatic letter, Gateaux suggested the way he wanted to proceed, inspired by Riemann integration:

> Let us restrict ourselves to the definition of the integral of U in the field of the functions $0 \leq f \leq 1$. Let us divide the interval $(0, 1)$ into n intervals. (...) Consider next the function f in any of the partial intervals as equal to the numbers $f_1, \dots, f_n$ which are between 0 and 1. $U(f)$ is a function of the n variables

[19] Krall (1961), page 17.

[20] Volterra himself was involved in this subject through an important collaboration with Arthur Gordon Webster from Clark University in the USA. See the interesting webpage `http://physics.clarku.edu/history/history.html#webster`.

[21] On the beginning of the relationship between Borel and Volterra, see Mazliak (2015).

[22] Dated from Bar-le-Duc, 12 April 1913.

[23] Throughout the paper, the functionals considered are always defined on the set of real functions over a given interval $[a, b]$.

[24] Fréchet's definition is in fact given in a different way by means of a property inspired by a characterization he had proved for real polynomials (Fréchet, 1910, page 204); however, he proves (page 205) that the two properties are equivalent.

> $f_1 \cdots f_n : U_n(f_1, \ldots, f_n)$. Let us consider the expression
>
> $$I_n = \int_0^1 \int_0^1 \cdots \int_0^1 U_n(f_1, \ldots, f_n) \cdot df_1 \cdots df_n.$$
>
> Suppose that n increases to infinity, each interval converging to 0, and that I_n tends to a limit I independent of the chosen divisions. We shall say that I is the integral of U over the field $0 \leq f \leq 1$.

Gateaux's intention was to study whether the limit I exists for any continuous functional U or if an extra hypothesis was necessary. In the last paragraph, Gateaux mentioned the possible applications of this integration of functionals, such as the residue theorem. All the applications he mentioned belong in addition to the theory of functions of a line. There is no hint of a possible connection with potential theory. This does not appear in the papers published by Gateaux. As it is a central theme of Gateaux's posthumous texts, it is plausible that he became conscious of the connection only during his stay in Rome—perhaps under Volterra's influence.

On 28 August 1913, Gateaux wrote directly to Volterra for the first time, informing him of his arrival in October and also mentioning that he had already obtained several results for the thesis in Functional Analysis which he was working on. Gateaux may have enclosed a copy of his first note to the Comptes-Rendus (Gateaux, 1913a), published on 4 August 1913 and containing the beginning of his proposed program. The note is in fact rather limited to an exposition of results and does not contain any proof, apart from a sketch of how to approximate a continuous functional U by a sequence of functionals of order n uniformly over each compact subset of the space of continuous real functions on $[0, 1]$.[25]

About Gateaux's stay in Rome, I do not have many details. An interesting document, found in the Paris Academy, is the draft of a report written by Gateaux at the end of his stay for the David Weill foundation.[26] He mentioned there that he had arrived in Rome in the last days of October and that he followed two of Volterra's courses in Rome (one in Mathematical Physics, the other about application of functional calculus to Mechanics). Gateaux seems to have worked quite actively in Rome. A first note to the Accademia dei Lincei (Gateaux, 1913b) where he extended the results of his previous note to the Paris Academy was published in December 1913. On a postcard sent by Borel to Volterra on 1 January 1914, Borel mentioned how he was glad to learn that Volterra was satisfied with Gateaux. The young man published three more notes during his stay (Gateaux, 1914a, 1914b, 1914c), and also began to write more detailed articles—found after the war among his papers.[27]

On 14 February 1914, Gateaux made a presentation to Volterra's seminar[28] in which he mainly dealt with the notion of functional differentiation. He recalled that Volterra introduced this notion to study problems including hereditary phenomena, and also that it was used by others (Hadamard and Paul Lévy) to study some problems of mathematical physics—such as the equilibrium problem of fitted elastic plates—through the resolution of equations with functional derivatives.

Gateaux came back to France at the beginning of the summer, in June 1914. He expected to go back soon to Rome, as he was almost certain, as Borel had written to Volterra,[29] to obtain the Commercy grant he had applied for. Gateaux soon wrote that the grant had been awarded.[30] In the same letter, he mentioned that he had completed a first version of a note on functionals requested by Volterra to append it to the German translation of his lectures on functions of lines (Volterra, 1913b). During this month, he had also met the Proviseur of the Lycée in Bar-le-Duc on July 20th, as the man sadly observed in a letter after Gateaux's death.[31]

[25] We need not dwell upon this technical result here, which had already been obtained by Fréchet previously (Fréchet, 1910, page 197) in a slightly more intricate way. Let me only observe that Gateaux's elementary technique involves the replacement of the function z by a linear function over each subdivision $[\frac{i}{n}, \frac{i+1}{n}]$ of the interval $[0, 1]$. A final perfecting of Gateaux's proof is presented by Lévy (1922), pages 105–107.

[26] A very touching aspect of the report written by Gateaux for the David Weill foundation can be found in the pages where he described the nonmathematical aspects of his journey. Gateaux mentioned how he regretted that Italy and the Italian language were so little known in France, when, on the contrary, France and French were widely known within Italian society.

[27] Lévy (in Gâteaux, 1919b, page 70) mentioned that, in one case, two versions of the same paper were found, both dated March 1914.

[28] His lecture notes were found among his papers.

[29] Borel to Volterra. 3 April 1914.

[30] Gateaux to Volterra, 14 July 1914.

[31] Postcard dated from 7 December 1914.

4. IN THE STORM

A serious danger of war had in fact been revealed only very late in July 1914 in public opinion, and the French mostly received the mobilization announcement on August 2nd with stupor. Like the majority, Gateaux has been caught napping by the beginning of the war. He was mobilized in the reserve as lieutenant of the 269th Infantry regiment, member of the 70th infantry division. The diaries of the units engaged in the war[32] permit us to follow Gateaux's part in the campaign in a very precise way. He was appointed on August 6th as the head of the 2nd machine-gun section of the 6th Brigade when the unit was formed in Domgermain, a suburb of the city of Toul.[33] The regiment paused beyond Nancy the next day and was supposed to go further East, but the German army's fire power stopped it brutally a few days later near Buissoncourt, 15 kilometers east of Nancy. At the end of August, the main task of the 70th infantry division was to defend Nancy's southeast sector.

The centennial year 2014 was an occasion for many people to better realize how horrific the first few weeks of the war were on the French side. August 1914 was the worst month of the whole war in terms of casualties, and some of the figures defy belief. On 22 August 1914, for example, the most bloody day of the whole war for the French, 27,000 were killed in the French ranks (Becker, 2004). The appallingly high number of casualties was due to an alliance between the vulnerability of the French uniform [with the famous *garance* (red) trousers up to 1915...], the self-confidence of the headquarters who had little consideration for their men's lives, and the clear inadequacy of many leaders in the field. Prochasson[34] advances two hypotheses to explain why the casualties among the Grandes Écoles' students (École Normale Supérieure in particular) were so dramatic. As they were often subordinate officers, the young students were the first killed, as their rank placed them in the front of their section. But also, they were sometimes moved by a kind of stronger patriotic feeling that may have driven them to a heroism beyond their simple duty.[35] This is evident in Marbo's testimony about her adopted son Fernand, who explained to her that, as a socialist involved in the fight for the understanding between peoples and peace, he wanted to *be sent on the first line in order to prove that he was as brave as anyone else*,[36] and added that *those who would survive will have the right to speak loudly in front of the shirkers*.[37]

Gateaux's last letter to Volterra is dated August 25th. Gateaux alluded there to the ambiguous situation of Italy. Though officially allied to the Central Empires, the country had carefully proclaimed its neutrality, an interesting point described at length in Rusconi (2005). Senator Volterra immediately sided with France and Great Britain and wrote passionate letters to his French colleagues as early as the beginning of August to express the hope that Italy would join them.[38] On 24 October 1914, in a letter to Borel, he asked for news

> from Mr. Gateaux, Mr. Pérès, Mr. Boutroux and Mr. Paul Lévy and other young French friends. I have received a letter from Mr. Gateaux from the battlefield and then no other. And this is why I am very worried about his fate and that of the others.[39]

Borel answered Volterra's letter on November 4, telling him that Pérès and Boutroux were discharged and that he did not know where Gateaux was.[40] As we have seen, Gateaux was in Lorraine at the end

[32] They were put on-line by the French Ministry of Defense `http://www.memoiredeshommes.sga.defense.gouv.fr`.

[33] Gateaux used headed notepaper from the *Hotel & Café de l'Europe* in Toul for his last letter to Volterra on August 25th.

[34] Prochasson (2004), pages 672–673.

[35] Prochasson mentions the famous example of Charles Péguy and the less well-known one of the anthropologist Robert Hertz who unceasingly asked his superiors for a more exposed position and was killed in April 1915.

[36] Être envoyé en première ligne afin de prouver qu' (il était) aussi courageux que n'importe qui.

[37] Ceux qui survivront auront le droit de parler haut devant les embusqués (Marbo, 1967, page 166).

[38] See Mazliak and Tazzioli (2009) where Volterra's attitude is thoroughly studied.

[39] M. Gateaux, M. Pérès, M. Boutroux, M. Paul Lévy et d'autres jeunes amis français. (...) J'avais reçu une lettre de M. Gateaux du champ de bataille et ensuite je n'en ai reçu pas d'autre c'est pourquoi je suis très inquiet sur son compte ainsi que sur les autres.

[40] The tone of this letter was slightly less confident than the previous ones. This was the moment when the enormous losses of the first weeks began to filter through. Borel wrote that at the École Normale, several young men with a bright scientific future had already disappeared and that the responsibility of *those who wanted this war* was really terrible.

of August. The French army went steadily backward, and was closer and closer to being crushed between the two wings of the German army (one coming from the north through Belgium, the other from the east through Lorraine and Champagne). Then occurred the unexpected *miracle* of the Battle of the Marne (6–13 September 1914), which suddenly stopped the German advance, rendering the Schlieffen Plan a failure. Vitry-le-François had been occupied by the Germans during the night of the 5th of September, but they were compelled to leave and to withdraw toward the East on September 11th.[41] From September 13th, the French went again slowly toward the East, chasing after the retreating Germans.

At the end of September, the French and British and the German headquarters became aware of the impossibility of any further decisive motion on the front line running from the Aisne to Switzerland; each realized that the only hope was to bypass their enemy in the zone between the Aisne and the sea which was still free of soldiers.

General Joffre decided to withdraw from the Eastern part of the front (precisely where Gateaux was) a large number of divisions and to send them *by railway* to places in Picardie, then in Artois and finally to Flanders to try to outrun the Germans. The so-called *race for the sea* lasted two months and was very bloody.

The 70th division was transported between September 28th and October 2nd from Nancy to Lens, a distance of almost 500 km.[42] Gateaux's division received the order to defend the East of Arras. On October 3rd, Gateaux's regiment was in Rouvroy, a small village, 10 km southeast from Lens, and Gateaux was killed at one o'clock in the morning while trying to prevent the Germans from entering the village. In the confusion of the bloodshed, the corpses were not identified before being collected and hastily buried in improvised cemeteries. Gateaux's body was buried near St. Anne Chapel in Rouvroy, a simple cross without inscription marking the place.[43]

René's death was officially established only on 28 December 1915.[44] But it is only long after, on 8 December 1921, that Gateaux's corpse was exhumed and formally identified, and finally transported to the necropolis of the military cemetery of the Bietz-Neuville St Vaast.[45] The last document of the military dossier is a letter from the Minister of War, dated 22 June 1923, informing the mayor of Vitry-le-François that the Lieutenant René-Eugène Gateaux had officially been declared *Dead for France*.

The detailed chronology of how the academic world learned of Gateaux's death is not entirely clear. As already mentioned, the Principal of Bar-le-Duc Lycée wrote the postcard in December 1914, but it was clearly an answer to a letter he had received.[46]

Only on December 10th did Borel write to Volterra about Gateaux's death (Mazliak and Tazzioli, 2009, page 47), mentioning his anxious hope that of the dozens of pupils of the École Normale considered as lost, there will be at least one or two who will come back at the end of the war. Volterra sadly answered some days later (Mazliak and Tazzioli, 2009, page 48) and wrote that he was sure that René would

[41] A vivid account of this moment was written after the war by a witness (Nebout, 1922). Though Gonthiez and Janet wrote in Annuaire (1918) that they could *easily imagine all the pain he* (*Gateaux*) *would have felt when he learned that the enemy had taken the city of Vitry-le-François where his poor mother had remained*, it is not clear whether Gateaux had learnt the fact at all, due to the general confusion. I refer to Becker (2004) or to several articles of Audoin-Rouzeau and Becker (2004) for the description of this phase of the war.

[42] According to the diary of the 269th Infantry regiment, the order to board the trains, received on September 28th, was carried out the next day. With an impressive organizational efficiency, the trains followed a circuitous route to join Artois: Troyes, Versailles, Rouen before stopping at Saint-Pol sur Ternoise on October 1.

[43] According to the army file, René's mother was informed on October 4 that her son was reported missing. On March 16th 1916, her other son and only remaining child, René's brother Georges, was killed in the Mort-Homme before Verdun. Much later, René's mother passed away on 24 February 1941 in Vitry-le-François, some months after having seen her city devastated by the German invasion.

[44] This was done based on evidence given by Henri-Auguste Munier-Pugin, warrant officer, and Albert Garoche, sergeant, in the 269th Infantry regiment.

[45] Gateaux's grave is number 76 at Bietz-Neuville. Gateaux's mother was informed of this fact on 5 January 1922.

[46] This postcard is, however, a decisive link between Hadamard and the papers left by Gateaux. It was probably addressed to Hadamard or Borel, though I found it by chance in the huge archive of Fréchet material in the Paris Academy of Science. Another possibility is that the letter was addressed to Fréchet who happened to know the Proviseur as well as Gateaux well enough to have this exchange. If this hypothesis is true, it may be Fréchet who recovered Gateaux's papers and transmitted them to Hadamard. We shall see a point below that corroborates this version.

have had a great future. The same day a telegram was sent to the École Normale by Volterra in the name of the Mathematical seminar in Rome.

As early as August 1915, Hadamard took the necessary steps to obtain the award of one of the Paris Academy's prizes for Gateaux. In a letter dated 5 August 1915 (and probably addressed to Picard as Perpetual Secretary), Hadamard mentioned the following: Gateaux *has left very advanced research on functional calculus* (*his thesis was composed to a great extent*, *and partly published in notes to the Academy*), *research for which M. Volterra and myself have a great regard*.[47] At the meeting of 18 December 1916, the Francœur prize was awarded to Gateaux (Hadamard, 1916, pages 791–792). It is interesting to read in Hadamard's short report the following section:

> [Gateaux] was following a much more audacious way, which promised to be very fruitful, by extending the notion of integration to the functional domain. Nobody could predict the development and the range this new series of research would attain. This is what has been interrupted by events.[48]

It is plausible that Hadamard had only superficially looked at Gateaux's papers, since he himself was caught in the storm of events, losing his two sons during the summer of 1916. Nevertheless, he did at least notice that one major interest in the last period of Gateaux's work was integration over the space of functionals. As we shall see, this was precisely why he spoke to Lévy about Gateaux.

5. THE MATHEMATICAL DESTINY

5.1 Lévy's Interest in Infinite-Dimensional Integration

> In January 1918, I was lying on a bed in a hospital, when I suddenly thought again of functional analysis. In my early work, I had never thought of extending the notion of an integral to spaces with infinite dimensions. It suddenly appeared to me that it was possible to attack this problem starting with the notion of mean in a sphere of the space of square summable functions. Such a function can be approximated by a step function, the number n of its distinct values growing constantly. The desired mean may then be defined as the limit of the mean in a sphere of the n-dimensional space. Obviously, this limit may not exist; but in practice, it does often exist (Lévy, 1970, page 58).

Thus, Lévy described how he became interested in infinite-dimensional integration. It is not easy to decide whether this happened as suddenly as he wrote, just following the train of his thoughts. Regardless, it is sometimes forgotten today that Lévy, before becoming one of the major specialists in Probability theory of the 20th Century, had been a brilliant expert in functional analysis.[49] As we shall see, it is a remarkable fact that his studies in functional analysis led him rather naturally to probabilistic formulations of problems. At the end of 1918, the Paris Academy of Sciences, following Hadamard's proposal, decided to call upon Lévy for the *Cours Peccot* in 1919.[50] Lévy's book *Leçons d'Analyse Fonctionnelle* (Lévy, 1922), on which I shall comment later, is based on these Peccot lectures.

The first document in which the question is explicitly mentioned is a letter to Volterra written in the early days of 1919:

[47] (Gateaux) laisse sur le calcul fonctionnel des recherches fort avancées (sa thèse était en grande partie composée, et représentée par des notes présentées à l'Académie), recherches auxquelles M. Volterra, comme moi-même, attache un grand prix.

[48] (Gateaux) allait s'engager dans une voie beaucoup plus audacieuse, et qui promettait d'être des plus fécondes, en étendant au domaine fonctionnel la notion d'intégrale. Nul ne peut prévoir le développement et la portée qui auraient pu être réservés à cette nouvelle série de recherches. C'est elle qui a été interrompue par les événements.

[49] On that topic, see, in particular, Barbut, Locker and Mazliak (2014), pages 44–54.

[50] The Cours Peccot was (and still is) a series of lectures in mathematics given at the Collège de France and financed by the Peccot Foundation. It is a way to promote innovation in research by offering financial support and an audience to a young mathematician. Borel had been the first lecturer in 1900, followed by Lebesgue. In Lévy's time, the age of the lecturer was meant to be less than thirty. However, the losses of the war had been so heavy among young men that the choice of the thirty-three year old Lévy was reasonable. It is also plausible to think that Gateaux would have been a natural Peccot lecturer had he survived the war. As Lévy's appointment is almost concomitant with Hadamard asking to take care of Gateaux's papers, it is possible that there is a connection between the two events.

> As I was recently interested in the question of the extension of the integral to functional space, I spoke about the fact to Mr. Hadamard who mentioned the existence of R. Gateaux's note on the theme. But he could not give me the exact reference and I cannot find it. (...) Though I am still mobilized, I am working on lectures I hope to give at the Collège de France on the functions of lines and equations with functional derivatives, and on this occasion I would like to develop several chapters of the theory. (...) I think that the generalization of the Dirichlet problem must present greater difficulties. Up to now, I was not able to extend your results on functions of the first degree and your extension of Green's formula. This is precisely due to the fact that I do not possess a convenient expression for the integral.[51]

As can be seen from this quotation, Lévy's views on infinite-dimensional integration were related to his studies in potential theory. The central problem of the classical mathematical potential theory is to find a harmonic function U in a domain R with given values on the boundary S (Dirichlet problem) or given values of the normal derivatives on S (Neumann problem). In 1906, Hadamard (1906) proposed to make use of variational techniques from Volterra's theory of functions of lines in order to study more general forms of these problems, for instance, when the border is moving with time, and, in particular, to find Green functions used in the integral representation of the solutions. These problems would make up Lévy's thesis, defended in 1911.

As Lévy wrote to Volterra, to study these questions in infinite-dimensional functional spaces, one needs to be able to integrate over these spaces. Volterra was not the only person Lévy had contacted. He wrote to Fréchet on the same topic at the very end of the year 1918.[52] Fréchet had indeed proposed in Fréchet (1915) a theory of integration over abstract spaces in 1915, usually considered as the first attempt to define a general integral.[53]

On 6 January 1919, Lévy wrote to Fréchet

> About Gateaux's papers, I learned precisely yesterday that M. Hadamard had put them in security at the École Normale during the war and had just taken them back. Nothing is therefore yet published.[54]

From this, I infer that Fréchet mentioned Gateaux's papers to Lévy, probably because he had an idea of what they contained. This could also be a hint that the papers arrived to Hadamard during the war via Fréchet and that Fréchet was the addressee of the postcard from the Principal of Bar-le-Duc.

On January 12, Lévy sent another letter to Volterra:

> M. Hadamard has just found several of Gateaux's unpublished papers at the École Normale. I have not seen them yet but maybe I'll find what I am looking for in them.[55]

Volterra answered on January 15, writing that none of Gateaux's publications concerned integration. He nevertheless added

> Before he left Rome, we had discussed about his general ideas on the subject, but he did not publish anything. I suppose that in the manuscripts he had left, one may probably find some notes dealing with the problem. I am happy that they are not lost and that you have them in hand. The question is very interesting.[56]

[51] M'étant occupé récemment de la question de l'extension de la notion d'intégrale multiple à l'espace fonctionnel, j'en ai parlé à M. Hadamard qui m'a signalé l'existence d'une note de R. Gateaux sur ce sujet. Mais il n'a pas pu m'en donner la référence exacte et je ne puis réussir à la trouver. (...) Quoiqu'encore mobilisé, je travaille à préparer un cours que j'espère professer au Collège de France sur les fonctions de lignes et les équations aux dérivées fonctionnelles et à cette occasion, je voudrais développer davantage certains chapitres de la théorie. (...) Je crois que la généralisation du problème de Dirichlet doit présenter plus de difficultés. Je n'ai pu jusqu'ici profiter pour le cas général de vos travaux sur les fonctions du premier degré et l'extension de la formule de Green. Ceci tient précisément à ce que je n'ai pas encore mis la notion d'intégrale multiple sous une forme commode pour ce but. (Lévy to Volterra, 3 January 1919.)

[52] See Barbut, Locker and Mazliak (2014), page 69.

[53] These are Kolmogorov's terms in Kolmogoroff (1977). On this matter, see, for instance, Shafer and Vovk (2006).

[54] Barbut, Locker and Mazliak (2014), Lettre 2.

[55] M. Hadamard vient de trouver plusieurs mémoires non publiés de Gateaux à l'École Normale. Je ne les ai pas encore vus mais peut-être y trouverais-je ce que j'y recherche.

[56] Nous avons causé avant son départ de Rome des idées générales sur ce sujet mais il n'a rien publié là-dessus. Je

As already mentioned, Hadamard entrusted Lévy with the posthumous edition of Gateaux's papers. He published it in three parts as Gateaux (1919a, 1919b) and (1922a). In February 1919, Lévy began to describe the precise content of what he had found in Gateaux's papers to Fréchet.

5.2 Gateaux's Integration of Functionals

Integration over infinite-dimensional spaces was certainly the most important subject considered by Gateaux. This can be read in Hadamard's comment that follows:

> The fact that he chose functional calculus reveals a broad mind, scornful of small problems or of the easy application of known methods. But the event proved that Gateaux was able to consider such a study under its widest and most suggestive aspect. And it is what he indeed did, with integration over the functional field, to speak only about this example, the most important, that represents a path that is new and the theory.[57]

Gateaux's views on integration are the subject of the first paper edited by Lévy in 1919 (Gateaux, 1919a). Lévy completed this presentation (and considerably extended it) in Part III of Lévy (1922), Chapter II, page 274.

As said before, when I commented on Gateaux's letter to Volterra expositing his research program, Gateaux's interest in infinite-dimensional integration originated in an attempt to extend Cauchy's formula and his first idea was to use a Riemann-type approach.

Gateaux considered the ball[58] consisting of all square integrable functions over $[0,1]$ with the property $\int_0^1 x(\alpha)^2\,d\alpha \leq R^2$.[59] He defined a function x to be *simple of order* n if it assumes constant values $x_1, x_2, \ldots, x_n$ over each subinterval $[0, \frac{1}{n}[, \ldots, [\frac{n-1}{n}, 1]$. In order that a simple function x belongs to the ball, one must therefore have $x_1^2 + x_2^2 + \cdots + x_n^2 \leq nR^2$. The set of simple functions of order n belonging to the ball is called the nth section of the ball. This set corresponds to a ball in $\mathbb{R}^n$ centered at 0 with radius $\sqrt{n}R$.

As the volume V_n of a ball with radius $\sqrt{n}R$ in dimension n is asymptotically equivalent to $\frac{(2\pi\mathrm{e})^{n/2}}{\sqrt{n\pi}}R^n$ (Lévy, 1922, page 265), it tends to zero or infinity for $n \to \infty$, depending on the value of R. This fact constitutes the central problem for the definition of the integral: in functional space, a subset has generally a volume equal to zero or infinity, and this forbids the direct extension of the Riemann integral through an approximating step-function sequence.

Gateaux seems to have been the first to propose a natural way to bypass the problem by defining the integral as a limit of mean values. Consider a functional U defined and continuous on the ball $\int_0^1 x(\alpha)^2\,d\alpha \leq R^2$. Its restriction U_n to the nth section can be considered as a continuous function of the n variables $x_1, x_2, \ldots, x_n$ and, therefore, it admits a mean value

$$\mu_n = \frac{\int_{x_1^2+x_2^2+\cdots+x_n^2 \leq nR^2} U_n(x_1, \ldots, x_n)\,dx_1 \cdots dx_n}{V_n}.$$

Under some circumstances, the sequence (μ_n) admits a limit which is called the mean value of U over the ball of the functional space. Gateaux's main achievement in Gateaux (1919a) was to obtain the value of the mean for important types of functionals.

He began by considering functionals of the type $U : x \mapsto f[x(\alpha_1)]$ where x is a point of the functional space, f a continuous real function and α_1 a fixed point in $[0,1]$. As α_1 is fixed, $x(\alpha_1)$ is one of the coordinates when x is taken in the nth section.[60]

pense que dans les notes manuscrites qu'il a laissées, on pourra bien probablement trouver quelques notes sur ce sujet. Je suis heureux qu'elles ne soient pas perdues et qu'elles se trouvent dans vos mains. La question est très intéressante.

[57] Le fait qu'il ait choisi le calcul fonctionnel révélait un esprit aux vues larges, dédaigneux du petit problème ou de l'application facile de méthodes connues. Mais le fait prouva que Gateaux était capable de considérer une telle étude sous son aspect le plus large et le plus suggestif. Et c'est effectivement ce qu'il fit, avec l'intégration sur le champ fonctionnel, pour ne mentionner que cet exemple, le plus important, qui représente une voie entièrement nouvelles et de très grandes perspectives pour la théorie (Annuaire, 1918, page 138).

[58] To fit better with modern terminology, I use the word *ball*, though Gateaux and Lévy systematically use *sphere*.

[59] In fact, Gateaux started from a *continuous* function x. However, as Lévy explained to Fréchet in a long letter dated 16 February 1919 (Letter 5 in Barbut, Locker and Mazliak, 2014), it is more natural to consider measurable functions, that is, to work with the (now) usual space L^2. This is what he does in Lévy (1922).

[60] Gateaux considers this functional although it is clearly not continuous. Gateaux had not sorted out the role of continuity in his work on the infinite-dimensional. It is likely that he would have improved the apparent incoherence in a subsequent rewriting of the paper. We shall see that Lévy fixed the question in Lévy (1922).

Therefore, the [$(n-1)$-dimensional] volume of the intersection of the ball of radius R with the plane $x(\alpha_1) = z$ (with $0 \le z^2 \le nR^2$ or, equivalently, $-\sqrt{n}R \le z \le \sqrt{n}R$) is given by

$$(\sqrt{nR^2 - z^2})^{n-1} \cdot V_{n-1},$$

where V_k is the volume of the unit ball in dimension k. A classical result is that for any $k \ge 2$, V_k satisfies the induction formula $V_k = 2V_{k-1} \int_0^{\pi/2} \cos^k \theta \, d\theta$.

Now, the mean of the functional U over the nth section is given by

$$\frac{1}{(\sqrt{n}R)^n \cdot V_n} \cdot \int_{-\sqrt{n}R}^{+\sqrt{n}R} f(z)((\sqrt{nR^2 - z^2})^{n-1} \cdot V_{n-1}) \, dz.$$

Performing the change of variables $z = R\sqrt{n}\theta$ transforms the previous expression into

$$\frac{1}{\int_{-\pi/2}^{\pi/2} \cos^n \theta \, d\theta} \int_{-\pi/2}^{\pi/2} f(R\sqrt{n} \sin \theta) \cos^n \theta \, d\theta.$$

It is seen that the preponderant values for θ in the last integral are those around 0, and $\int_{-\pi/2}^{\pi/2} \cos^n \theta \, d\theta$ is known to be asymptotically equivalent to $\sqrt{\frac{2\pi}{n}}$. Under "some regularity conditions" for f, the previous expression is therefore approximately equal to

$$\frac{1}{\sqrt{(2\pi)/n}} \int_{-\alpha\sqrt{n}}^{\alpha\sqrt{n}} f\left(R\sqrt{n} \sin \frac{\psi}{\sqrt{n}}\right) \cos^n \frac{\psi}{\sqrt{n}} \frac{d\psi}{\sqrt{n}}$$

for any $\alpha > 0$ and sufficiently large n.

Using a Taylor expansion, and letting n go to infinity, the latter expression converges to

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(R\psi) \mathrm{e}^{-\psi^2/2} \, d\psi, \tag{1}$$

defined by Gateaux as the mean of U over the ball of all square integrable functions over $[0,1]$ such that $\int_0^1 x(\alpha)^2 \, d\alpha \le R^2$. He asserted that this result can be generalized for functionals of the type

$$U(x) = \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_p \cdot f[x(\alpha_1), \dots, x(\alpha_p), \alpha_1, \dots, \alpha_p]$$

for which the mean value is given by

$$\frac{1}{(2\pi)^{p/2}} \cdot \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_p \int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_p \cdot f(Rx_1, \dots, Rx_p, \alpha_1, \dots, \alpha_p) \cdot \mathbf{e}^{-(x_1^2 + \cdots + x_p^2)/2}. \tag{2}$$

The rigorous existence of the limit was not explained by Gateaux, as Lévy wrote to Fréchet in his letter of 12 February 1919. Obviously, for Gateaux, as Lévy himself wrote in the foreword of Gateaux (1919a), the present state of his papers was certainly not a final one.[61] And in the long note Lévy added at the end of the article (Gateaux, 1919a, page 67), he described the attempts made by Gateaux to obtain the limit in several situations. For Lévy, the priority was to fill the gap left by Gateaux and to try to obtain the existence of the mean value for the most general functionals.

Gateaux (1919a, page 52) also considered continuous (with respect to uniform norm) functionals U satisfying the following property: for any $\varepsilon > 0$, there is an n_0 such that, for $n \ge n_0$ and for any two functions x and y satisfying $\int_0^1 x(\alpha)^2 \, d\alpha \le R^2$ and assuming the same mean value over each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$,[62] one has $|U(x) - U(y)| < \varepsilon$. Following Gateaux, for such a functional, the mean value is given by the value at the center 0 of the ball (the function constantly equal to 0), and it can therefore be considered as a *harmonic* functional. The previously mentioned property of U was natural to Gateaux: he had proved in Gateaux (1913b) that, under such a condition, a continuous functional U can be well approximated over the ball $\int_0^1 x(\alpha)^2 \, d\alpha \le R^2$ by $U(y_n)$, where y_n belongs to the nth section of the sphere and takes on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ the value $\int_{(i-1)/n}^{i/n} x(t) \, dt$.

After he began to scrutinize Gateaux's paper, Lévy became convinced that Gateaux's requirement of continuity with respect to the uniform norm for a functional U was in fact much too restrictive. As early as 16 February 1919,[63] he mentioned the fact

[61] As can be seen, Gateaux used a technique close to Laplace's method for the estimation of the limit. This method for asymptotic estimation of integrals was currently taught to students in Paris, but usually without much care for the convergence conditions. This may also explain that Gateaux did not pay much attention to this aspect of the question in his manuscript.

[62] Which is to say that $\int_{(i-1)/n}^{i/n} x(t) \, dt = \int_{(i-1)/n}^{i/n} y(t) \, dt$ for every i such that $1 \le i \le n$.

[63] Barbut, Locker and Mazliak (2014), page 115.

to Fréchet. And in the final version of his ideas on the question, in Lévy (1922), page 277, he arrived at a striking conclusion: under very general assumptions, such a continuous functional takes *almost everywhere* the same constant value b, meaning that for any $\varepsilon > 0$, the volume of the subset of these functions x in the nth section of the ball satisfying $|U(x) - b| > \varepsilon$ tends to 0 with $n \to \infty$. The mean of such a functional is therefore obviously equal to this value b. Lévy gives (Lévy, 1922, page 275) a simple example illustrating this situation. Consider the functional defined on the ball $\int_0^1 x(\alpha)^2\, d\alpha \le R^2$ by $U(x) = \varphi(r)$, where φ is a given continuous function on $\mathbb{R}_+$ and $r^2 = \int_0^1 x^2(\alpha)\, d\alpha$. The volume of the ball $B_n(\sqrt{n}R)$ with radius $\sqrt{n}R$ centered in 0 in $\mathbb{R}^n$ is proportional to $(\sqrt{n}R)^n$; hence, for any given $0 < \varepsilon < 1$, the quotient of the volumes of $B_n((1-\varepsilon)\sqrt{n}R))$ and $B_n(\sqrt{n}R)$ tends to 0, which means that when n grows, the volume is more and more concentrated close to the surface. Therefore, $\varphi(R)$ is essentially the only value assumed by φ in the ball counting for the calculation of the mean.[64]

5.3 Lévy's Probabilistic Interpretation

I have already mentioned that in 1919, Lévy had his first contact with probability theory when he was asked to teach probability at the École Polytechnique.[65] This was exactly the same period he was studying Gateaux's papers and preparing their publication. One may observe that probability theory takes no part in the various notes presented by Lévy to the Paris Academy of Sciences as he progressed in his work on Gateaux (Lévy, 1919a, 1919b, 1919c, 1921).[66] But when he wrote his book Lévy (1922) he often adopted probabilistic reasonings as relevant for his considerations about the mean in a functional space and it seems that a kind of extraordinary junction occurred during these years in Lévy's mind, resulting in unifying his mathematical interests in functional calculus and probability theory.[67]

Let us try to understand how probability entered Lévy's considerations about the mean in functional spaces [third part of Lévy (1922)]. Consider (Lévy, 1922, page 266) a given hyperplane H containing 0 in $\mathbb{R}^n$ and define the coordinate z as the distance to H. Let us consider the fraction of the ball centered at 0 with radius $R\sqrt{n}$, comprised between the hyperplanes $z = R\xi_1$ and $z = R\xi_2$. The ratio of the volume of this fraction to the total volume of the ball is equal to

$$\frac{\int_{\xi_1/\sqrt{n}}^{\xi_2/\sqrt{n}} \cos^n\theta\, d\theta}{\int_{-\pi/2}^{+\pi/2} \cos^n\theta\, d\theta}$$

which tends to

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_1}^{\xi_2} \mathbf{e}^{-x^2/2}\, dx. \tag{3}$$

More generally, consider p hyperplanes containing 0 and call $z_1, z_2, \dots, z_p$ the distances to these hyperplanes. The volume of the intersection of p regions $R\xi_i' < z_i < R\xi_i''$ $(i = 1, 2, \dots, p)$ is a fraction of the total volume equal to

$$\frac{1}{(2\pi)^{p/2}} \int_{\xi_1'}^{\xi_1''} dx_1 \int_{\xi_2'}^{\xi_2''} dx_2 \cdots \int_{\xi_p'}^{\xi_p''} dx_p \cdot \mathbf{e}^{-(x_1^2+x_2^2+\cdots+x_p^2)/2}.$$

This is, writes Lévy, a direct consequence of the independence of the random variables z_i, each following a Gaussian distribution according to the previous result. In order to prove the desired independence, writes Lévy, it is sufficient to prove that the conditions $z_i = R\xi_i, i = 1, 2, \dots, p-1$ do not influence the distribution of z_p. The intersection of these conditions is a hyperspace H with dimension $n - p + 1$, included in a hyperplane $r = kR$ (r being the distance between 0 and H). Now, the intersection of H and the ball of radius $R\sqrt{n}$ is a ball with dimension $n - p + 1$ and radius $R\sqrt{n - k^2}$, asymptotically equivalent to $R\sqrt{n - p + 1}$ when n tends to

[64] The concentration of measure phenomenon became an important field of research following Milman's systematic study of asymptotic geometry in Banach spaces during the 1970s. It has many important applications, especially in probability theory by providing exponentional inequalities of Gaussian type. See Ledoux (2001) for a panoramic view of this question.

[65] For more details about this story, I refer the reader to Barbut and Mazliak (2008a).

[66] Lévy began, however, to work on independent probabilistic questions at the same time. See, in particular, Fischer (2011), page 218 *and seq.* for Lévy's investigations on characteristic functions and the central limit theorem, and Barbut, Locker and Mazliak (2014), pages 40–44, more specifically about Lévy's investigations on stable distributions.

[67] Recall here his own mention that he *was prepared by functional calculus for the study of functions with an infinite number of variables and* (*that*) *many of* (*his*) *ideas in functional analysis became without effort ideas which could be applied in probability* (Barbut, Locker and Mazliak, 2014, page 156).

infinity. Moreover, $n-p+1$ tends to infinity with n. Therefore, concludes Lévy, the distribution of z_p is given by the formula (3), hence the desired independence.

As the reader can see, Lévy's proof is based on a kind of intuitive approach which would become his typical trademark in numerous later works in probability. In particular, the sketchy use of conditional densities seems almost sloppy for a modern mathematician's eye, but Lévy was never embarrassed with such technicalities in his proofs. For Lévy, the essential task was to understand the deep nature of the mathematical situation. In so doing, he had a lot in common with Poincaré's conception of what is a rigorous proof in mathematics. Not only beyond the purely logical proofs, mathematically insignificant, but also beyond the analytical proofs which logically deduce theorems from definitions and axioms, Poincaré defended the necessity of a specific intuition for a mathematician, a *geometrical* spirit using his senses and his imagination in order to perceive this *touch of something which realizes the unity of the proof*.[68]

The probabilistic framework allowed Lévy to explain Gateaux's formula (1) for the mean of the functional $U(x)=f[x(\tau)]$ in what seems to him a more convincing way (Lévy, 1922, page 278). If x is in the ball with radius $R\sqrt{n}$, the probability of the event $R\xi_1 \le x(\tau) \le R\xi_2$ tends to $\frac{1}{\sqrt{2\pi}}\int_{\xi_1}^{\xi_2} \mathrm{e}^{-\xi^2/2}\,d\xi$ when $n\to\infty$, so that the mean of U is given by (1). Moreover, the mean of $U(x)=\varphi(x(t_1),x(t_2),\dots,x(t_p))$ is immediately obtained using the fact that $x(t_1),x(t_2),\dots,x(t_p)$ are i.i.d. variables having a centered Gaussian distribution with variance R^2 (Lévy, 1922, page 281). Probabilistic reasoning also enables us to explain the concentration of the mass at the surface of a ball in the functional space (Lévy, 1922, page 283). By the law of large numbers, $\frac{x_1^2+\cdots+x_n^2}{n}$ tends to R^2 and, therefore, for any $\varepsilon>0$, the probability that $\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}$ does not belong to $[R-\varepsilon,R+\varepsilon]$ tends to 0 when $n\to\infty$. Therefore, concludes Lévy, the part of the nth section one must take into account for the computation of the mean of a functional is in the neighborhood of the surface of the sphere with radius $R\sqrt{n}$.

In Chapter VI (Lévy, 1922, Part Three, page 421), Lévy studies the general question of the existence of the mean for a functional. As we have seen in the previous subsection, Lévy considered continuity with respect to the uniform norm as too strong a condition because it implied that the functional is almost surely constant. In this chapter, he highlights that in order to obtain a convenient condition for the existence of the mean, it is necessary to look at the probability distribution of the values of the function x rather than at the values themselves.

As a basic example he considers the mean of the functional $U(x)=F(f)$ in the ball with radius R, where f is the probability distribution function (called by Lévy *fonction sommatoire*) of x over the space $[0,1]$ equipped with Lebesgue measure λ.[69] Lévy's reasoning is as follows. If x belongs to the nth section of the ball, it is a function constant in each interval $[\frac{i-1}{n},\frac{i}{n}]$ with value x_i, such that $x_1^2+x_2^2+\cdots+x_n^2\le nR^2$. In the limit $n\to\infty$, the x_i are independent Gaussian random variables with variance R^2, and the probability distribution function associated with this x is the Gaussian distribution function with variance R^2 denoted by φ.[70] This allows Lévy to conclude (Lévy, 1922, page 424) that the mean of U is equal to $F(\varphi)$.

As a generalization of the previous result, Lévy studies functionals U satisfying a condition which, though weaker than continuity with respect to uniform topology, guarantees a good approximation of the functional by its values on the nth section. The most general property [called $\mathcal{H}$ by Lévy (1922),

[68] *Ce je ne sais quoi qui fait l'unité de la démonstration.* On Poincaré's conceptions, see the recent paper (Kebaïli, 2014). Lévy's intuitive approach is the precise aspect that explains what Itô wrote later, about his difficult work to *translate* Lévy. *At that time*, writes Itô, *it was commonly believed that Lévy's works were extremely difficult, since Lévy, a pioneer in the new mathematical field, explained probability theory based on his intuition. I attempted to describe Lévy's ideas using precise logic that Kolmogorov might use* (Itô, 1998).

[69] This is to say that x is considered as a random variable on the probability space $[0,1]$ with Lebesgue measure λ. Hence, $f(\xi)=\lambda\{t\in[0,1], x(t)\le\xi\}$.

[70] To explain this in modern terms, consider a sequence of independent random variables $(X_n)_{n\ge1}$, each with the standard normal distribution. By the law of large numbers, the sequence $\frac{1}{n}\sum_{k=1}^n \mathbb{1}_{X_k\le x}$ tends almost surely to $P(X_1\le x)$. Choose a ω for which the convergence occurs and, for each n, define a random variable Z_n on the probability space $([0,1],\lambda)$ by $Z_n(t)=X_i(\omega)$ if $\frac{i-1}{n}\le t<\frac{i}{n}$. Then $(Z_n)_{n\ge1}$ converges in distribution to the standard normal distribution. Lévy is extremely elliptic in his proof (he only mentions *des raisonnements connus de calcul des probabilités*). He may have had the intuition that the dependence on ω in the previous construction would not create real difficulties as results from the Glivenko–Cantelli theorem.

page 424] he considers is the following: for each given $\varepsilon > 0$, there is a n such that, if x and y are two functions in the ball such that in every interval $[\frac{i-1}{n}, \frac{i}{n}]$ the probability distribution function of x and y is the same,[71] $|U(y) - U(x)| < \varepsilon$. However, Lévy was not able to prove the approximation result he was looking for in all the desired generality, but he asserted that the result was *reliable* for the functional satisfying the property $\mathcal{H}$ (Lévy, 1922, page 427).

As it is seen, probability reasoning is omnipresent in the Third Part of Lévy (1922). Lévy was certainly conscious of the profound originality of his approach and desired to convince everyone of its interest. The complicated relations between the prominent French mathematicians (Borel and Hadamard in the first place) and probability theory was considered in several studies (see Bru, 2003 and Durand and Mazliak, 2011 and the references included for more details). It was observed that from the very beginning of his interest in probability, Lévy felt himself unjustly despised for his choice,[72] though he was comforted by Wiener's reaction to his approach (I shall come back on that point in the next subsection).

This lack of interest of the leading French mathematicians in probability (Borel was the exception) may be an explanation why absolutely no reference to probability can be located in Gateaux's papers, even when he observed the remarkable appearance of the Gaussian distribution in the limit expression (1). In Borel (1906),[73] Borel had proved that if B_n is the ball of $\mathbb{R}^n$ centered in 0 with radius $R\sqrt{n}$, and $V_n(u)$ the volume of the portion $u \leq x_1 \leq u + du$ of B_n, the ratio of $V_n(u)$ to the total volume of B_n tends to $\frac{1}{\sqrt{2\pi}R}\mathrm{e}^{-u^2/(2R^2)}\,du$.[74] Borel's interest was statistical mechanics, more precisely, for Maxwell and Boltzmann's kinetic theory of gases. In his presentation, the spheres represent surfaces in the phase space of equal total kinetic energy. In a complement to his translation of Ehrenfests' paper on statistical mechanics in *Encyclopédie des Sciences Mathématiques* (Borel, 1914b, page 273), Borel mentions studies about the n-dimensional sphere as the first example of mathematical research inspired by statistical mechanics. He even audaciously asserts that one should consider the results about surfaces and volumes in high dimensions as connected to statistical mechanics. However, in contrast to Maxwell, who, in his fundamental paper in 1860, had emphasized the coincidence between the distribution law for the speeds of the particles and the distribution *governing the distribution of errors among observations by use of the so-called* least-squares method,[75] Borel did not mention any possible connection with the law of errors in Borel (1906). The only reference is in Borel (1914a), page 66, without any probabilistic interpretation, just mentioning that the Gaussian distribution function was a well-tabulated distribution function which allows it to be used for computations.

It is probably the desire to explain to a large audience why probabilistic tools were useful that prompted Lévy to write a nontechnical paper for the *Revue de Métaphysique et de Morale* (Lévy, 1924). Lévy explains there the general ideas leading to his conception of the mean value, based on probability considerations over general sets.[76] As an elementary example, he considers the situation of non-negative integers as today in probabilistic number theory. If f is a function defined on $\mathbb{N}$ (f could typically be the indicator of a subset $A \subset \mathbb{N}$), the mean of f is defined as the limit of $\frac{1}{N}\sum_{k=1}^{N} f(k)$ when N tends to infinity. In particular, $P(A) = \lim_{N\to+\infty} \frac{1}{N}\,\mathrm{Card}\{n \in \mathbb{N}, n \in A\}$.[77] The paper includes a presentation of

[71] This means that considered as random variables on the probability space $[\frac{i-1}{n}, \frac{i}{n}]$ with probability measure $n \cdot \lambda$, the two functions x and y restricted to this interval have the same distribution.

[72] On that topic, see, in particular, Barbut and Mazliak (2008b).

[73] Reprinted as *Note I* in his book (Borel, 1914a).

[74] The result, usually known today under the name *Poincaré's lemma*, has in fact nothing to do with Poincaré, according to Diaconis and Freedman (1987). Moreover, Stroock (2010) discovered that Mehler had already obtained the result in 1866 in a purely analytical context [see Stroock (2010), page 68, footnote 3 for an exact reference and comments].

[75] Maxwell (1860), Prop. IV and following comments.

[76] Interestingly, Lévy asserts (Lévy, 1924, page 149) that the article is the development of his last lecture of the Cours Peccot of 1919, meaning that the aforementioned junction between probability and his studies in functional calculus appeared quite early in his mind. This is corroborated by his first letters to Fréchet (Barbut, Locker and Mazliak, 2014, Letters 1–5, before February 1919). If probability is never mentioned explicitly there, one may observe how gradually Lévy is closer to probabilistic reasoning. A good example is found in Letter 3 (Barbut, Locker and Mazliak, 2014, page 55) where Lévy writes about his desire to find a way of expressing that functions u such that $\int u'^2$ is large are *less probable*.

[77] Therefore, if one randomly draws a point from $\mathbb{N}$, there is, for instance, one chance over two that it is an even integer, a rather comforting result for the mind…

Gateaux's work on infinite-dimensional integration and the idea behind the extension to more general functionals. Lévy was probably rather satisfied with the picture he had provided in his paper, as he decided to reprint it as an appendix in his treatise of probability published the next year (Lévy, 1925a). Another attempt to disseminate his considerations on functional analysis was also done in 1924. Henri Villat asked Lévy to write a small booklet for his new series *Mémorial des Sciences Mathématiques*. Lévy (1925b) contains 56 pages and appears in fact as a survey of the book (Lévy, 1922). Lévy updated his bibliography and Daniell's and Wiener's works were now quoted.

5.4 Wiener Measure: Daniell Versus Gateaux's Integrals

As mentioned in the Introduction, it is well beyond the scope of this article to provide a complete description of the fundamental works where Wiener built the first mathematical model of Brownian motion; on that topic, I refer the reader to Itô's comments in Wiener (1976) and to Chatterji (1993), Kahane (1998) and Barbut, Locker and Mazliak (2014), pages 54–60. The aim of this section is more modest: to try to explain how Wiener became acquainted with Gateaux's approach to integration and how he eventually used it in his epoch-making paper (Wiener, 1923).

In the second half of the 1910s, the British mathematician Percy J. Daniell (1889–1946), then holding a position at the Rice Institute in Houston, Texas, was interested in extending Lebesgue integration to infinite-dimensional spaces.[78] Daniell wrote two important papers (Daniell, 1917, 1918) on the subject. His approach was to consider the integral as an operator on functions satisfying certain properties, such as linearity and a monotone convergence theorem on a restricted class of functions T_0, and to prove that these properties allow one to extend integration to the class T_1 of limits of sequences in T_0. It can be seen that such a construction is directly inspired by Lebesgue.[79]

Wiener's first work on functionals (Wiener, 1920) appeared in 1920. Wiener proved there that Daniell's method can be applied to define the integral of a functional, taking as basis T_0 a set of step functions for which the integral is defined as a mean. Probably shortly before publication, Wiener added the following footnote (Wiener, 1920, page 67):

> The use of mean instead of integral is found in the posthumous papers of Gateaux (Bulletin de la Société Mathématique de France, 1919). This was however unknown to me at the time I wrote this article.

We do not know exactly when Wiener was informed of the existence of Gateaux's works. A possible hypothesis is that he became aware of them during his journey in France in 1920 when he came to the Strasbourg International Congress and met Fréchet and Volterra.

The next year, Wiener published his first papers on Brownian motion. In the first one (Wiener, 1921a), he starts from Einstein's result: at time t the probability that the position $f(t)$ of a particle on a line belongs to the interval $[x_0, x_1]$ has the form $\frac{1}{\sqrt{\pi c t}} \int_{x_0}^{x_1} \mathrm{e}^{-x^2/ct}\,dx$ where c is a constant (taken equal to 1 by Wiener, corresponding to a good choice of units). The path $x = (f(t), 0 \le t \le 1)$ of the particle is a real-valued continuous function on $[0,1]$. Thus, if we consider as functional a function of this path, a natural question arises of defining its average value. Due to the independence of increments in the Brownian motion, asserts Wiener, it is reasonable to associate to a functional of the form $F = \Phi(f(t_1), \dots, f(t_n))$ depending only on the values of f at some finite number of values of t, a mean, denoted $A[F]$ by Wiener, defined by

$$A[F] = \frac{1}{\sqrt{\pi^n t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \cdots \cdot \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \Phi(x_1, \dots, x_n) \cdot \mathrm{e}^{-x_1^2/t_1 - (x_2 - x_1)^2/(t_2 - t_1) - \cdots - (x_n - x_{n-1})^2/(t_n - t_{n-1})} \cdot dx_1 \cdots dx_n.$$

[78]A very complete description of Daniell's work and personality can be found in the paper (Aldrich, 2007).

[79]Lévy always coolly accepted nonconstructive approaches, which, for him, probably did not sufficiently reveal the *touch of something* (in the words of Poincaré, see note 68 above) at the heart of a mathematical concept. Thus, he did not hide his moderate appreciation of Daniell's work on integration to Fréchet. He wrote to him *if nothing important has escaped me, Daniell has given not a definition of the integral but an extension of the notion of integral from a restricted domain to a larger one. That is a Lebesgue-kind work* (*Barbut, Locker and Mazliak, 2014*, *page* 86).

In particular, observes Wiener, if $F(f) = f(t_1)^{m_1} \cdot f(t_n)^{m_n}$, one may compute an explicit value for $A[F]$. Therefore, if a functional F is analytical in the sense of Volterra, which means that it can be expanded as a sum of functionals of the type

$$\int_0^1 \cdots \int_0^1 f(x_1) \cdots f(x_n) \cdot \varphi_n(x_1, \ldots, x_n)\, dx_1 \cdots dx_n,$$ [80]

the mean of F is defined as the sum of the corresponding terms

$$\int_0^1 \cdots \int_0^1 A[F_n] \varphi_n(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

[where F_n is the functional $f \mapsto f(x_1) \cdots f(x_n)$] when this series is convergent. Wiener's paper proves that, with this definition, the mean satisfies the classical properties of integrals such as linearity or the possibility of exchanging infinite summation and integration. Wiener quotes Gateaux (Wiener, 1921a, Note 1, page 260) for having proposed using analytical functionals in the definition of the mean of a functional. As we have seen, it is true that Gateaux had such an idea in mind from the very beginning (see his programmatic letter to Borel), but, contrary to Wiener's assertion, the idea does not seem to be explicit in Gateaux (1919a). Wiener adds that Gateaux's definition is, however, not well adapted to the treatment of Brownian motion.

Wiener published his second study (Wiener, 1921b) in the next issue of the Proceedings of the National Academy of Sciences. The aim of this new paper was to show that the use of the definition of the mean provided in Wiener (1921b) allowed one to obtain a direct proof (moreover, under a somehow lighter hypotheses) of Einstein's formula for the mean quadratic displacement of the Brownian particle in a viscous medium. Once again, Gateaux is mentioned as having proposed another construction of the mean:

> To determine the average value of a functional, then seems a reasonable problem, provided that we have some convention as to what constitutes a normal distribution of the functions that form its arguments. Two essentially different discussions have been given on this matter: one, by Gateaux, being a direct generalization of the ordinary mean in n-space; the other, by the author of this paper, involving considerations from the theory of probabilities (Wiener, 1921b, page 295).

During the Summer of 1922, Wiener came again to France and met Lévy for the first time during his vacation in Pougues les Eaux, a spa in central France, and discussed Lévy's book on functional analysis. Lévy narrates the meeting in his autobiography, where he emphasizes that Wiener was almost the only one who immediately recognized the depth of Part III of his book [Lévy (1922, 1970, page 86—and also on page 65)]. He adds he had reasons to think that this third part was the origin of Wiener's memoir (Wiener, 1923) on Brownian motion.

Indeed, in the introduction of Wiener (1923), Wiener pays full tribute to Lévy:

> The present paper owes its inception to a conversation which the author had with Professor Lévy in regard to the relation which the two systems of integration in infinitely many dimensions—that of Lévy and that of the author—bear to one another. For this indebtedness the author wishes to give full credit (Wiener, 1923, page 132).

Gateaux is now clearly treated by Wiener only as a precursor, and Lévy has become the major source of inspiration. Besides, Wiener wrote (Wiener, 1923, page 132) that Gateaux had begun investigations on integration in infinitely many dimensions which had been *carried out by Lévy* in Lévy (1922).[81]

In Wiener (1923), Wiener reconsidered the results of his previous papers on Brownian motion. Contrary to what he had done in Wiener (1921a) where the mean of a functional $F = \Phi(f(t_1), \ldots, f(t_n))$ of the trajectory was given *a priori*, he now used Lévy's studies of the n-dimensional sphere and the Gateaux–Lévy definition of the mean as a limit of the means over the nth sections in order to:

[80] Wiener considers in fact a generalization of this situation where the functionals are defined by means of Stieltjes integrals.

[81] It took some time for Gateaux–Lévy or Daniell considerations on infinite-dimensional integration to be widely known. For instance, in 1930, the Danish mathematician Børge Jessen (1907–1993) defended a doctoral thesis with the title *Contribution to the theory of the integration of the functions of an infinity of variables* and was totally unaware of the previous works on the topic. See Bru and Eid (2009).

(1) *deduce* that at time t, the probability distribution of the position is Gaussian,[82]

(2) *define* the related measure on the space of continuous functions (Wiener measure),

(3) *prove* the value of the mean of the aforementioned functional he had postulated in his previous works,[83]

(4) *derive* the expression of the mean of an analytic functional with a new proof.[84]

Section 10 of Wiener (1923) is devoted to proving that, for the functionals previously considered, Daniell's extension of the mean Wiener had introduced in Wiener (1920) gives the same value to the integral.[85]

Finally, observe that Wiener's paper is not absolutely conclusive about the use of Daniell's versus Gateaux–Lévy's approach, though I can certainly interpret Wiener's choice to write the paper starting from the latter as recognition of its more intuitive character. Besides, it is well known that Lévy was never a great supporter of abstract constructions of Brownian motion. In his autobiography (Lévy, 1970, page 98), Lévy, who was not shy about emphasizing his missed opportunities, regretted how he let Wiener get ahead of him in the construction of Brownian motion though all the necessary material was in Lévy (1922). Lévy did sometimes slightly exaggerate his own role [as, e.g., when he wrote about Kolmogoroff's *Grundbegriffe* (Lévy, 1970, page 68)]. In the case of Brownian motion, however, one can understand his regrets.

The geometric approach to Brownian motion was quite fertile in the 20th century. McKean (1973) has explained how thinking of the Wiener measure as a uniform distribution over the *infinite-dimensional sphere of radius* $\sqrt{\infty}$, a direct consequence of Lévy's considerations in Lévy (1922), was successfully used by Japanese mathematicians in the 1960s to describe the geometry of Brownian motion. In another direction, in 1969, Gallardo (1969) made the observation that *Poincaré's lemma* could be connected with the fact that if $X^n(t) = (X_1(t), \ldots, X_n(t))$ is an n-dimensional Brownian motion starting at 0, if one denotes by T_n the first passage time of X^n on the sphere centered at 0 and with radius $\sqrt{n}$, then $T_n \to 1$ in probability and $X^n(T_n)$ follows the uniform distribution on the n-dimensional sphere of radius $\sqrt{n}$. Yor later developed these considerations (see Yor, 1997).

[82]Wiener (1923), pages 136–137. This was a decisive step forward with respect to Wiener (1921a) where Wiener took Einstein's Gaussian form as a starting point

[83]Wiener (1923), page 153.

[84]Wiener (1923), page 165.

[85]The construction of the Wiener measure via Daniell's extension is tightly related to the theorem of extension Kolmogorov would provide 10 years later in his *Grundbegriffe* (Kolmogoroff, 1977). On that topic, consult Shafer and Vovk (2006), in particular, Section 5.1, page 87.

6. CONCLUSION

It has often been said that after World War I, the French Grandes Écoles, the École Normale especially, were crowded with the ghosts of the students from the 1910s who disappeared during the conflict. Of course, these dead of the Great War were essentially very young men who had scarcely finished their graduate studies and whose names are hardly known to us today. René Gateaux, who died at the age of 25 in October 1914, is an example both representative and exceptional of the student victims of the war—exceptional because, despite being very young, he left scientific work that could be carried on by others.

Bourbaki, when he eventually added some words about probability theory in the chapter devoted to integration in nonlocally compact spaces of a late edition of his *Eléments d'histoire des mathématiques* (Bourbaki, 1984, pages 299–302),[86] mentioned the path linking Borel's consideration on kinetic theory of gases to the Wiener measure with Gateaux's and Lévy's works as fundamental steps.

Though uncompleted, Gateaux's mathematical studies were recovered and extended by Paul Lévy for whom they became a catalyst for a renewal of his scientific interests in probability. It is due to Lévy's work of editing and extension that today we remember Gateaux.

ACKNOWLEDGMENTS

I want to express my deep thanks to the people who kindly contributed to the preparation of this paper. First, to M. Pierre Gateaux, a distant relative of René Gateaux, for his warm welcome in

[86]The complicated story of Bourbaki's attitude to integration and, in particular, of Dieudonné's resistance to abstract integration is well known and presented in detail in Schwartz (1997). I shall not make further comments on it here.

Vitry and his help in my research for René's familial background, and to Gilbert Maheux, a local historian in Vitry le François, for his enthusiastic help. In the course of this research, I have visited several archives. I would like to thank Giorgio Letta of the mathematical department in Pisa and member of the Accademia dei Lincei for his efficient help, and also those responsible for the archives in Rome for their help and kindness. Florence Greffe and the staff of the archives of the Academie des Sciences in Paris helped me in obtaining permission to consult Gateaux's manuscripts deposited at the Académie. Valuable assistance was provided by Francoise Dauphragne at the archives of the École Normale Supérieure from whom I obtained a photograph of the 1907 pupils. And, last but not least, through a visit to the archives of the Paris Archdiocese and the help of Father Ph. Ploix, I discovered the only picture I know of where Gateaux is explicitly identified. I thank also my colleagues Olivier Guédon, Bernard Locker, Stéphane Menozzi and Marc Yor for several interesting discussions relating to Gateaux's life and work. I owe to John Aldrich and Daniel Denis the transformation of the original version of the text into what can be called real English. Finally, I am glad to mention the anonymous referees and to thank them for having generously provided numerous suggestions which, I hope, have eventually largely improved the paper.

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