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The ghosts of the Ecole Normale

Life, death and destiny of René Gateaux

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17th January 2007

Abstract

The present paper deals with the life and some aspects of the scientific contribution of the mathematician René Gateaux, killed during World War 1 at the age of 25. Though he died very young, he left interesting results in functional analysis. In particular, he was among the first to try to construct an integral over an infinite dimensional space. His ideas were extensively developed later by Lévy. Among other things, he interpreted Gateaux's integral in a probabilistic framework that later led to the construction of Wiener measure. This article tries to explain this singular personal and professional destiny in pre and postwar France. It also recalls the slaughter inflicted on French students during the conflict.

Keywords and phrases : History of mathematics, functional analysis, integration, Brownian motion

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INTRODUCTION

*E quando che meno
ti pensi nel seno
ti vien a finire
bisogna morire
Se tu non vi pensi
hai persi li sensi
sei morto e puoi dire
bisogna morire²*

In her memoirs [40] written at the end of the 1960's, the novelist Camille Marbo, daughter of the mathematician Paul Appell and Emile Borel's widow, mentions that, after the end of World War 1, her husband declared that he could not bear any more the atmosphere of the Ecole Normale in mourning, and decided to resign from this position. Since 1910 (when he succeeded to Jules Tannery), Emile Borel had been the vice-director of the Ecole Normale Supérieure during a time of extraordinary success for this institution. It was in particular the case for Borel's favorite discipline. Mathematical analysis experienced at this moment a complete revolution. The epicentre was in Paris, due to the presence of such outstanding personalities as Henri Poincaré, Emile Picard, Jacques Hadamard, Henri Lebesgue, and naturally Borel himself with the new measure of sets he introduced at the end of 19th century.

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²And when you are least/ thinking of it, in your breast / all comes to an end / We must die / If you do not think of this / you have lost your senses / you are dead and you can say / We must die. [33]

To have a superficial, though impressive reading of the effect of war on the French mathematical community, one may follow the personal life of the aforementioned mathematicians - except of course Poincaré who died in 1912 . Picard lost one son in 1915, Hadamard two sons in 1916 (one in May, one in July !), Borel lost his adopted son in 1915. The figures concerning casualties among the students of the Ecole Normale or among those who had just finished their 3 years studies at the *rue d'Ulm* are quite unbelievable. They are collected in a small brochure edited by the Ecole Normale at the end of the war [12]. To mention only the most dramatic ones, let us say that out of the about 280 pupils who entered the Ecole Normale in the years 1911 to 1914, 241 were mobilized in the Army directly from the school and 101 died during the war. If the President of the Republic Raymond Poincaré could declare that *the Ecole of 1914 had taken revenge over the Ecole of 1870*, the price to be paid had been so enormous that it was difficult to understand how French science would survive such a hemorrhage. Most of the dead were brilliant young men, the expected successors of the brightest scientists and scholars from the previous generation in every domain of knowledge. Obviously, they were so young that practically not any of them had had time to prove themselves and start making a name of his own through a professional achievement. As a testimony of melancholy and assumed abnegation, Frédéric Gauthier, a young hellenist, who entered the Ecole Normale Supérieure in 1909 and was killed in July 1916 during the Verdun battle, wrote : *My studies, it is true, will stay sterile, but my ultimate actions, useful for the country, have the same value as a whole life of action* ([1]).

In this paper, we shall focus on René Gateaux's case who died at the very beginning of the war. Gateaux appears at the same time as a good representative of the lost generation of *normaliens* that we have just mentioned, and also as an outstanding case, as his name, contrary to practically every one of his companions of misfortune, has been retained in mathematics that were his professional field. A superficial proof for this assertion is that Gateaux's name is still nowadays quite well known by undergraduate students of differential calculus, through the *Gateaux differential* which is the directional derivative for a function over a linear space and is therefore more general than the usual (Fréchet) global differential. The notion of differential was only a small (though important) technicality in Gateaux's mathematical researches . But that the name of an unknown mathematician, who died so young, before having occupied any academic position and even before having defended a thesis, had been given to a general notion of calculus appears as an enigma and deserves a study. It is this apparent contradiction that we want to approach in this paper through the presentation of the life and death, the mathematical studies and the mathematical destiny of René Gateaux.

Let us immediately reveal what seems to be the key explanation for the paradox. In fact, beyond his unhappy fate, it appears that Gateaux met two lucky chances. The first one concerns the subject he was interested in, the Functional Analysis. At the beginning of the 20th Century, this subject was still a rather confidential domain, directed by two huge mathematicians : one in Paris, Jacques Hadamard, the other in Rome, Vito Volterra, who had been the true creator of the subject in the years 1890 (see below). In the years following the end of World War 1, Functional Analysis received an unexpected development, in particular in the unpredictable direction of probability theory. Gateaux was therefore posthumously in the center of a powerful stream that led to the creation of some aspects of modern probability theory. And it is a wonderful bit of luck for the historian that a rather important archival documentation about Gateaux's beginnings in mathematics is accessible up to now. Gateaux had in particular been in correspondence with Vito Volterra before, during and (for some weeks) after the time spent in Rome with the Italian

mathematician. His letters still exist today at the Academia dei Lincei and offer a unique access to Gateaux's first steps, as well as are kept letters between Borel and Volterra about the young man's projects and progresses. To illustrate how amazing appear these exchanges, let us just mention that the last letter from Gateaux to Volterra is dated of August 25th, 1914, sent from the front in the middle of the raging battle. Moreover, some other material is accessible such as the military dossier, some of Gateaux's own drafts of reports about his work, and some scattered letters from or about him by other people. This allows to reasonably reconstruct the life of the young mathematician during his 7 or 8 last years.

But it is mainly the second chance to which Gateaux's memory pays a tribute for not having become only *a golden word on the squares of our towns*³ as it is dramatically evoked in Aragon's verses. Just after leaving for the Army, Gateaux had left a heap of papers in his mother's house, among which several semi-achieved papers that were supposed to become chapters of his thesis. These papers were sent after his death by his mother to the Ecole Normale. Hadamard collected them and, in 1919, transmitted them to Paul Lévy for the preparation of their edition in honour of Gateaux's name. This work over Gateaux's papers was an extremely important moment in Lévy's career. Not only from it was issued Lévy's wonderful book *Leçons d'Analyse Fonctionnelle* of 1922, but it is also possible to find there a major source of his future achievements in probability theory.

The paper is divided in four parts. In the first one, we try to present a sketch of Gateaux's life up to 1913. Then we present the decisive event in his short professional life, his stay in Rome with Volterra. In the third part, we give another biographical sketch about his depart for the army and his last days. Finally, in a slightly more technical part, we shall turn to the mathematical work of the young mathematician, how it was recovered and considerably extended by Lévy and appeared afterwards as a step in the construction of abstract integration and modern probability theory.

Acknowledgement: I want to express my deep thanks to many people who kindly contributed to the preparation of this paper. Firstly, I want to thank M.Pierre Gateaux, distant relative of René Gateaux, for his warm welcome in Vitry and his dynamic help in my research for René's familial background. Next, I had the occasion to visit several archives services, obtaining from them priceless documents. In first place, I want to thank Giorgio Letta, from the mathematical department in Pisa, Italy, member of the Academia dei Lincei, for his extremely efficient help. He obtained for me exceptional conditions for consulting Volterra's fund at the Accademia dei Lincei. Of course, those in charge of these archives in Rome deserve also all my gratitude for their help and kindness, as well as Florence Greffe and all the personal of the archives in Académie des Sciences in Paris, in particular for having obtained for me the authorization for consulting Gateaux's manuscripts deposited in the Academy. Another valuable source was provided by Françoise Dauphagne, from the archives of the Ecole Normale Supérieure from whom I obtain in particular a picture of the 1907 pupils, which may be the only known portrait of Gateaux.

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*Déjà la pierre pense où votre nom s'inscrit
Déjà vous n'êtes plus qu'un mot d'or sur nos places
Déjà le souvenir de vos amours s'efface
Déjà vous n'êtes plus que pour avoir péri ([2])*

Finally, thank you also to Olivier Guédon, Bernard Locker, Stéphane Menozzi and Marc Yor for several interesting discussions we have had on subjects connected to Gateaux's life and work that definitely improved the content of the paper, and to Jeannine Guy for the patience she needed to correct many places of the English text.

1. A PROVINCIAL IN PARIS

We do not know much about Gateaux's life before his entrance at the Ecole Normale. Gateaux was not member of a famous family, and moreover he belonged to a very restricted family unit composed only of his parents, his young brother Georges and himself. None of the brothers had any direct descendants, as both boys died during the war. We have met a distant member of his family, namely the great-great-great-great-grandson of a great-great-great-grandfather (!) of René Gateaux, M.Pierre Gateaux, who still lives in Vitry-le-François and very friendly gave us access to the little information he gets about his relative.

René Eugène Gateaux was born on May 5th, 1889 in Vitry-le-François in the département of Marne, 200 km east to Paris. Vitry-le-François is the sous-préfecture of this département, a town of some industrial and military importance, as its history testifies. A smaller town, named Vitry-en-Perthois was burnt during the Saint-Dizier siege by Charles-Quint in 1544 and in 1545 the king François the 1st decided to rebuild a new fortified town almost at the same place, to which he would give his own name François and his personal symbol, the salamander. He called the Italian architect Girolamo Marini for that purpose. The town takes the shape of a draughtboard, a square of 612 meters of side, with perpendicular street which can be still observed today as the original plan was conserved for the reconstruction of the city destroyed at 90 % by German bombings in May and June 1940. A tourist guide, the famous Guide Michelin in its first edition, indicates a population of 7700 inhabitants for Vitry-le-François in 1900 ([42]). Observe also that another very famous mathematician was born 222 years before Gateaux in the same small town, Abraham de Moivre, who had to leave for London where he spent all his outstanding scientific career. A local historian from Vitry, Gilbert Maheux, has written several short papers concerning his brilliant mathematicians fellowcitizens. See in particular [39].

Not much is known about René's parents. His father Henri Eugène Gerasime was born in 1860. He was a small contractor who owned an enterprise of saddlery and cooperage located in the suburbs of Vitry. The mother was Marie Alexandrine Roblin; she was born in Vitry on September 26th, 1864. As Pierre Gateaux told us, René's family on his father side was originated from the small town of Villers-le-Sec at 20 km from Vitry, that seems to be the rural nest of Gateaux family. On the birth declaration of René, it is indicated that Eugène Gateaux (Henri's father) is owner and Jules Roblin (Marie's father) is cooper : the grandparents acted as witnesses for the child's birth at the cityhall. As Eugène's birth declaration stipulates that he was born in Villers-le-Sec in 1821 and his father was a carpenter, maybe René's grandfather came to Vitry to create his business where Marie's father was working as cooper . As already mentioned, the coupled had two children : René is the elder, the second one, Georges was born four years later on August 27th, 1893. René's father died young, on July 28th, 1905, aged 44. The precarious situation that may have resulted from this event may have increased the young boy's motivation in his studies. We do not know precisely what was the school career of René Gateaux. He was pupil in Vitry and then in Reims. The only sure information comes from the first of his own written testimonies. On February 24th, 1906, Gateaux signed a letter to the Ministry of Education to get permission to sit for the examination for the admission to the Ecole Normale Supérieure (science division)

in 1906, though he had not attained the regular minimum age of 18 (he was to be 17 in May 1906). We can gather two informations from this document. One is that Gateaux was a student of Classe Préparatoire at Reims lycée, an institution of renown founded by Bonaparte in 1804 that continued an old medieval tradition of education in the Collegium bonorum puerorum created in the 13th Century by the archdiocese of Reims.

The Classes Préparatoires are the special sections in the French schooling system devoted to train students for the entrance examination to the 'Grandes Ecoles'. It is well known that these Grandes Ecoles, beginning by the most prestigious one the Ecole Polytechnique, had been created at the end of the 18th Century to provide the officers, engineers and managers for the economic and technical development of France, and also teachers for secondary schools with the creation of the Ecole Normale Supérieure. Though the creation of several schools had anticipated the Revolution, it is mostly during this period that most of them were created, as the pediment of the building on 45 rue d'Ulm still proudly reminds today, and above all received the mission to create a caste of civil and military servants, a kind of republican aristocracy, dedicated to the new Republic, that would help build a new France.

The second information is, without surprise, that Gateaux should have been a brilliant student in these scientific classes as his demand for an age dispense reveals how young he was. He should have obtained his baccalauréat in July 1904 at the age of only 15. It is however worth noticing that he did not pass the examination for the Ecole Normale on his first attempt in 1906, but entered the institution only with the 1907 session after a second year in the class of Mathématiques Spéciales as was usually the case (and is still today).

In October 1907, Gateaux is therefore in Paris and enters the venerable institution. It is hard today to have an exact feeling of what can have been the atmosphere of rich intellectual life of the Ecole Normale of these years, especially in mathematics. In the first half of the 19th Century, the scientific prestige had belonged unmitigatedly to the Ecole Polytechnique, illustrated by such mathematicians as Laplace, Poisson or Cauchy. Let us for example remind Galois' failure to enter the Ecole Polytechnique and his reluctant application to the Ecole Normale, a *pale shadow* for the former as described by Stewart in his Galois' biographical sketch ([46]). But since 1870, the prestige of the Ecole Polytechnique has clearly faded and the Ecole Normale Supérieure became the real center of intellectual life in France at the turn of the century.

Jean Guéhenno, born in 1890, has written admirable pages in his *Journal of a 40 years old man* [24] (see in particular the Chapter VI, called 'Intellectuel') where he described what was in these years preceding WWI life like at the Ecole Normale Supérieure (to which he was admitted in 1911 - in the literary section) as seen by a young man from a poor provincial extraction (certainly much poorer in fact for Guéhenno than for Gateaux), who was dazzled by the contrast with the laborious tedious everyday's life of a little industrial town in Brittany. *I wondered, writes Guéhenno, what would have become of me, if I had stayed at the factory, if I had refused to upstart, to stand out, to become a member of the upper crust of the society. I would have become a good worker, an 'operator'. [...] I would have fought in real battles, with real people, for real causes: some more leisure and some more bread. My friend severely answered that I was lying to myself, that all this was false remorse and false regrets, that I must not repent to be now better armed for the service of those I loved if I really wanted to serve them. [...] However, the thing was more delicate that he could have thought. [...] Without even noticing, I ceased to be the young tough and decided plebeian I was at the factory.* This was certainly also the kind

of thoughts Gateaux may have had, after his admission to the Ecole Normale in Paris after his provincial years.

We have a testimony, an obituary written in 1919 by two companions of Gateaux in the 1907 science section of the Ecole Normale ([1], pp.136 to 140), Georges Gonthiez and Maurice Janet. They write : *He was one of these good comrades with whom one likes to chat. His benevolence and absolute sincerity were immediately felt; he was among those who knew how to listen and to empathize with the other's thoughts. Maybe others were more assertive of their personality, more inclined to prove the originality of their spirit and character. It is without noise that Gateaux's personality blossomed, following the way he judged to be the best possible, and his personality unceasingly and smoothly strengthened. He had this freshness of spirit of the right natures not yet offended by life. When he arrived at the Ecole, he quietly opened his spirit to new subjects with the natural easiness and the calm of a modest, self-confident and beautiful intelligence. [...] He soon appeared to us as one of the best mathematicians in our group, serious-minded, and quick to focus on the essential. He liked to deal with all kind of philosophical or general questions. .*

At that moment occurred an unexpected event in the life of the young man, certainly important as Gonthiez and Janet devote many lines to it. During his second year at the Ecole Normale, Gateaux became a member of the Roman Catholic Church. He joined the church *with fervour* write his two fellow companions of the Ecole Normale. In this year 1908, so close to the events of the beginning of the 20th century, where the Roman Catholic Church was under accusation for its dubious behaviour during the Dreyfus Affair, and after the vote in 1905 of the separation laws, such a decision may surprise. However, a strong catholic group attended the Ecole Normale on these days (let us only mention Pierre Poyet, one of Gateaux's comrades among the 1907 pupils, who took orders and became a Jesuit) and Gateaux had certainly the occasion to move in these circles.

This conversion, about which we did not find any trace apart from this account in the obituary [1], seems to have played a considerable role in Gateaux's spiritual life. *This beautiful childish ingenuousness is what was most impressive in Gateaux and this is probably why he succeeded so easily to a religious life.*

Maybe his very quick and drastic conversion had not been taken very seriously at the beginning. He did not raise questions on the subject. In his presence, knowing his first steps, we actually were reluctant to do so, afraid of shocking him. We did not doubt he would have the same beautiful nonchalance in his spirituality that the one he had for examinations, competitions and all the honors of the life .

*It is indeed striking to see how he dealt with both his religious and secular lives. His great abilities for the intellectual work, a delicate health and the absence of any self-love may have passed for indolence. He was rather quickly tired, and, if he did not feel himself in a sufficiently good mood for working, he stopped immediately and went away for a walk. He rarely imposed a task to himself in advance and somehow left circumstances lead him. But when he was in the right mood, he amazed us by his quickness to focus on the heart of a problem, to organize an Agrégation lecture⁴ around a central, though sometimes a little too elevated, idea. One may hear him jump with exclamations as he went along his discoveries. And later, Gonthiez and Janet recall an amazing sentence Gateaux had written in his diary (which may still exist somewhere ?) that an ecclesiastic who followed him closely communicated to them : *I asked God to make a**

⁴that is to say a mock lecture of the type that candidates to the Agrégation competition - the degree to obtain a position as a secondary school teacher - has to present to the board of examiners.

saint out of me. . . Maybe I shall need to resign from my profession and to follow Jesus and preach. I do not know how this would be done. Maybe also God would ask me to stay in the University. He will later let me know. .

In 1910, Gateaux passed the Agrégation of mathematical sciences. This was the normal way for a young normalien who wanted to enter an academic career, to become a secondary school teacher after the Ecole Normale study while he prepared his thesis. Most of the French mathematicians of the time, Hadamard, Lebesgue, Fréchet. . . had followed this path. On July 8th, 1912 a ministerial decree appointed Gateaux as Professor of Mathematics at the Lycée of Bar-le-Duc, the maintown of the département of Meuse, 250 km east from Paris, and not very distant from his native Vitry-le-François.

Before taking up this post, Gateaux had the boring but necessary task to fulfill his military obligations. Since March 23rd, 1905 ([28]), a new law replacing the law of July 16th, 1889 for the organization of the army had been voted by the Parliament, where the length of the active military service had been reduced to 2 years, but there were much more candidates. The first article of both laws quite drastically stipulates that *every French man owes a military service to the nation*, but the 1889 law left aside a lot of people from the military service. There was in particular a quite intricate system of random draft described in the text of the law ([29]). The law was also quite lax for prospective civil servants such as future teachers or future priests. The presence of this last category in the law was certainly an unbearable detail for the anticlerical Government that assumed power in France in 1902 after the swirls of the Dreyfus Affair. In fact, the latter (teachers, priests. . .) were only asked to fulfill one year of military service, whilst the normal length was 3 years. In 1905, as a continuation of the reforms inspired by the radical majority at the Parliament, the law of conscription was changed to what seemed to be a balanced compromise. On one hand, the length of the military service was shortened to 2 years, but on the other hand the conscription became theoretically absolutely universal.

Gateaux was particularly concerned by the article 23 of the 1905 law. It reads as follows : *[The young men] who passed the entrance examination to the Ecole Normale Supérieure, to the Ecole Forestière, to the Ecole Centrale des Arts et Manufactures, to the Ecole Nationale des Mines, to the Ecole des Ponts-et-Chaussées or to the Ecole des Mines de Saint-Etienne can, at their choice, fulfill the first of their two years of military service in the ranks before their admission in these schools or after their exit. [. . .] The students of the aforementioned schools receive a military instruction preparing them to the grade of second lieutenant of the reserve in these institutions. [. . .] The young men who [. . .] have not accomplished one year of military service before their admission in the schools, accomplish one year in the ranks at the exit and then serve for one year as second lieutenant in the reserve [. . .] or as second lieutenant in the active army.* Students from the Ecole Normale who did not choose to go immediately into the army for their first year were asked to sign a voluntary 5 years commitment before entering the school (that is to say their 3 years of Ecole and two years of active military service).

Gateaux military dossier ([15]) stipulates that Gateaux signed his voluntary engagement in Vitry on October 12th, 1907 and asked to be incorporated only at the end of his study time. On October 10th, 1910 Gateaux joined the 94th Infantry regiment where he is a basic 'second class' soldier. On February 18th, 1911 he became *caporal* (a kind of private first class), and finally is declared second lieutenant in the reserve by the President of the Republic on September 17th, 1911. Then Gateaux had to follow some special training for officers, and the comments made by his superiors on the military dossier indicate that the supposed military training at the Ecole Normale had (not

surprisingly) been more virtual than real. On the special pages devoted to his superior's appraisal, one reads : *1st Semester 1912 : M. Gateaux strives to acquire the aptitude to command and to his functions of second lieutenant. He has very good spirit, is very intelligent, zealous, and conscientious, but he was quite badly prepared for his rank.* The second semester 1912 (which ended in fact in September) seems however to have been more convincing. It is written that *Gateaux has made many progresses. Very intelligent, very conscientious, and very good feelings, willing to do for the best. He has become a good section leader, capable of useful tasks in case of mobilization. He has followed a period of instruction for shooting and obtained very good marks during it. He is able to lead a machine-gun section.* This last sentence has a strange resonance with Gateaux's death exactly two years later, while operating on the machine-guns of his section. But let us come back to 1912. On October 4th, Gateaux, freed from the active army, begins his lectures at the Lycée of Bar-le-Duc. Quite unfortunately, the archives of the Lycée from that time seem to have rotten in the cellar of the institution. The few remaining documents were recently transferred to the Archives of the département of Meuse. Among them, the register of salaries on which one can follow the evolution of Gateaux salary from his nomination in October 1912 in replacement of Georges Reynaud, who obtained a position at the college of Embrun and was paid until September 30th included. Paid 285 francs on October, Gateaux received 237 francs from November to February due to a special tax, and his salary increased to 316 francs between March and August, and to 379 francs in September. In the archives in Bar-le-Duc, a very thin personal dossier for Gateaux as professor of mathematics is also kept. Apart from a rather brief description, the dossier contains two letters asking the Minister of Education for a permission to go on leave for one year study in Rome.

2. THE ROMAN STAGE

Gateaux indeed had begun to prepare a thesis in mathematics, on themes closely related to functional analysis *à la Hadamard* and its application to the theory of potential. We did not find precise details about how Gateaux did chose this subject for his researches, but it is plausible that he was advised to do so by Hadamard himself who had just read lectures at the Collège de France and entered the Academy of Science in 1912. There were good reasons for Gateaux to be attracted by these new and hardly explored domain. Paul Lévy had defended his own brilliant thesis about similar questions in 1911. And also, a young French normalien of the year before Gateaux, Joseph Pérès, had in 1912-1913 benefited from a grant offered by the David Weill fundation for a working one-year period in Rome with Volterra. Volterra himself, invited by Borel and Hadamard, came to Paris for a series of lectures on functional analysis, published in 1913 ([49]) and whose redaction was precisely made by Pérès.

For a young doctorate student the natural persons to be in contact with in these years were obviously Hadamard in Paris and Volterra in Rome. Vito Volterra is such a huge personality that it is clearly not the proper place to make a description of his various activities, and we shall only give a brief description of him. At the occasion of the centennial of his birth, in 1960, a volume was edited by the Accademia dei Lincei in Rome, in which Giulio Krall tried to draw a short biography of Volterra ([32]). Krall devotes several pages to the first striking Volterra discoveries in the field of partial derivatives equations of mathematical physics, when he found out a mistake in a Sonya Kowalevska's paper about the light in crystalline middle. It is however the questions studied later by Volterra, since 1896, that are more of interest for us. Volterra was involved in researches about the phenomenon of *hysteresis*, that is to say of the 'memory of materials', the

fact that the state of deformation of some materials depends on the deformation at previous times : it is therefore necessary to take into account a whole trajectory of the successive states for a complete description. To modelize such a situation, Volterra was led to consider *functions of lines* (funzione di linea), called later *functionals* (fonctionnelle) by Hadamard and the French school, which is to say a function of a real function representing the state of the material, and to study the equations they must satisfy. These equations happen to be an infinite-dimensional generalization of partial derivatives equations. As Krall mentions ([32], p.17), *from mechanics to electromagnetism, the step was small*, and Volterra's model was application to different physical situations, such as electromagnetism or sound produced by vibrating bars. Volterra himself was involved in this last subject at the occasion of an important collaboration with Arthur Gordon Webster from Clark University in USA⁵. This question was later a source for Hostinský's attention to Volterra's equations, and had some influence on his further investigations on Markov processes - see [27]). Volterra was also interested in applications of functional analysis to biology, in particular for providing models of competition between species. As it is seen, the italian mathematician appears as a very rich scientific personality, to which should be added another side of his career, the statesman : in 1904, Volterra was appointed by the king as Senator of the Kingdom, a function mostly honorary, but giving to the recipient an official status and a real influence.

This mixture of interests for science and politics obviously created a proximity of mind between Volterra and Borel. The two men began a friendly relationship since the end of the 19th Century, and exchanged a huge correspondence almost entirely conserved at the Accademia dei Lincei in Rome and the Académie des Sciences in Paris. Borel has certainly played a role in Gateaux's decision to go to Rome, at least as a medium between the young man and Volterra. We have indeed a first trace of this Roman project in their correspondence. On April 18th, 1913, Borel wrote to Volterra that *another young man, who is also my former student, M.Gateaux and presently teacher at Bar-le-Duc lycée, advised about his intention to solicit a study grant to continue his research. I recommended to him to apply for a David Weill grant and to go to Rome, if you may welcome him. I join his letter where he mentions that he asked for the grant, and presents what he intends to do. He is less advanced than Pérès was, as part of his time is devoted to teaching, while Pérès had a study grant. I am also inclined to think that he may be less distinguished than Pérès, but I think nevertheless that he will derive benefit from the grant he would obtain and I have the intention to support his request.* Borel then asked Volterra to write a small letter of agreement for this project to Liard, the Vice-Rector of the Paris Academy, and also mentioned that he lended the two books published by Volterra on the functions of lines to Gateaux (more precisely, the book [48] and the proofs of [49], this last book being in the press at the time). Volterra indeed wrote the letter in favor of Gateaux (Borel thanked him for that on April 28th), and finally, on June 30th, 1913, Borel communicated the good new to Volterra : a David Weill grant was attributed to Gateaux for the year 1913-1914. In the following letter, on July 3rd, Borel regrets that it had not been possible to attribute a second grant for Soula and evokes that the latter may be Gateaux's successor the next year (that is to say, 1914-1915). The exchanges of students between Paris and Rome were probably appreciated during these years preceding the war : in August 1913, Volterra asked Borel to welcome an Italian student, Armellini, at the Ecole Normale. Some years later, when Hadamard wrote a report to comment the posthumous attribution of the Francœur prize to Gateaux, he mentioned that the young man had been *one of those who, inaugurating a*

⁵See the interesting webpage <http://physics.clarku.edu/history/history.html#webster>

tradition we could not overestimate, went to Rome to become familiar with M. Volterra's methods and theories ([26]).

Gateaux's aforementioned letter to Borel (dated from Bar-le-Duc, April 12th, 1913) was indeed in Volterra's archives, and we have therefore the extreme luck to know precisely what were the mathematical aims of Gateaux, when he went to Rome. Gateaux considered two main points of interest for his future researches. The first one is classified as Analytical functionals, and is devoted to extension of the classical results on analytical functions to functionals : the Weierstrass expansion, the equivalence between analyticity and holomorphy and the Cauchy formula.

Volterra had in the first place called analytical a functional of the form

$$U(f) = \sum_{n=0}^{\infty} \int_a^b K_n(x_1, \dots, x_n) f(x_1) \dots f(x_n) dx_1 \dots dx_n,$$

but Fréchet had proposed in 1910 ([13]) a more general definition through a kind of generalized Taylor development

$$U(f) = \sum_{n=0}^{\infty} U_n(f)$$

where U_n is a homogeneous functional of order n (i.e. $U_n(\lambda f + \lambda' f')$ is an homogeneous polynomial in λ and λ' , with degree n). Gateaux proposed to study the expression of the $U_n(f)$ more precisely. Then, he intended to obtain the equivalence between the analyticity of the functional U and its complex differentiability (holomorphy). And finally, he wanted to consider a definition through a kind of Cauchy formula. For that purpose, as Gateaux writes, *one needs a definition of the integral of a functional over a real functional field.* This may be the first appearance of questions around infinite dimensional integration, that will become the most important mathematical trace left by Gateaux (see below, section 4). In this programmatic letter, Gateaux suggests the way he wants to follow, inspired by the Riemann integration : he did not seem to have immediately understood the necessity of considering mean values. Gateaux writes : *We restrict ourselves to the definition of the integral of U on the field of the functions $0 \leq f \leq 1$. Let us divide the interval $(0,1)$ into n intervals. [...] Consider next the function f in any of the partial interval as equal to the numbers f_1, \dots, f_n which are between 0 and 1. $U(f)$ is a function of the n variables $f_1 \dots f_n : U_n(f_1, \dots, f_n)$. Let us consider the expression*

$$I_n = \int_0^1 \int_0^1 \dots \int_0^1 U_n(f_1, \dots, f_n) df_1 \dots df_n.$$

*Suppose that n increases to infinity, each interval converging to 0, and that I_n tends to a limit I independent of the chosen divisions. We shall say that I is the integral of U over the field $0 \leq f \leq 1$. And Gateaux concludes : *I propose to study whether the limit I exists for U a continuous functional, or whether it is necessary to add another hypothesis.* In a last paragraph, Gateaux mentions the possible applications of such an integration of functionals, such as the residue theorem... It is remarkable that all these applications are within mathematics : for instance, there is not any comment about the theory of potential, which is present in good place in the texts written after Gateaux's stay in Rome. We could deduce in Rome he became conscious of the connection - perhaps under advices by Volterra. Anyway, potential theory is absent from all the works published by Gateaux during his life. On August 28th, 1913, Gateaux wrote directly to Volterra for the first time, warning him of his planned arrival in October, and also mentioning that he had already obtained several results for the thesis in Functional analysis he was preparing.*

Gateaux may have joined an exemplary of his first publication to the letter, a note to the Comptes-Rendus ([16]), published on August 4th, 1913 and containing the beginning of fulfilment of the proposed program.

About Gateaux's stay in Rome, we do not have many details. An interesting document, found in the fund of the Académie des Sciences in Paris, is the draft of a report Gateaux had to write at the end of his journey, probably for the David Weill foundation. He mentioned there that he arrived in Rome *in the last days of October*, at the precise moment of elections for the Parliament, so that *lectures were delayed until the end of November*. As Gateaux only mentions that he followed two Volterra's lectures in Rome (one of Mathematical Physics, the other about application of functional calculus to Mechanics), it is probable that the delay refers to Volterra political involvements. But as Gateaux wrote, this enabled him to *learn the few Italian he needed to perfectly understand the lectures*. He added: *If long studies are necessary to know Italian perfectly, in return a few weeks are sufficient to a French to learn the rudiments of it*. He lived on the Corso Vittorio-Emmanuele at the number 72, near the Torre Argentina square, as he indicated in the heading of the letter he sent to Volterra on November 21st, 1913 asking him for a meeting in order to discuss his research program. Gateaux seems to have worked quite actively in Rome. A first note in the Rendiconti dell'Accademia dei Lincei is published in December 1913 ([17]) where he extended the results of the previous paper. Moreover, on the postcard sent by Borel to Volterra on January 1st, 1914 with his best wishes for the new 1914 year (a sentence sounding strangely to the ears of one knowing what was going to happen soon. . .), Borel mentioned how he was glad to learn that Volterra was absolutely satisfied with Gateaux in Rome. And indeed, the young man published three more notes during his stay ([18],[19],[20]), but also began the redaction of more detailed articles - found afterwards among his papers after the war. Lévy (in [22], p.70) mentioned that two versions of this paper were found, both dated from March 1914. On February 14th, 1914, Gateaux read a lecture at Volterra seminary : his lecture notes were kept among his papers. Gateaux mainly dealt with the notion of the functional differentiation. He recalled that Volterra introduced this notion to study problem including an hereditary phenomenon, but also that it was used by others (MM.Hadamard and Paul Lévy) to study some problems of mathematical physics - such as the equilibrium problem of fitted elastic plates - finding a solution to equations with functional derivatives, or in other words by calculating a relation between this functional and its derivative.

A very touching aspect of the report written by Gateaux for the Weill's foundation concerns the pages where he described the non-mathematical aspects of his journey. He explained that he had at heart to travel in this new country he wanted to discover, and his words certainly sound as those of a man travelling for the first time. Before arriving in Rome, he explained, he went through Switzerland. He described at length the beauty of nature as well as the magnificent ingeneering exploits of the Gothard way. Then he passed to Milan where, he wrote, he *was glad to see the Lord's supper by Leonardo da Vinci [. . .] I had a particular pleasure to see a masterpiece now so popular that a great number of families in our country have a printed reproduction of it*. He went to Venice, to Florence, mentioning that he preferred the first as *I was, and still am, more sensitive to natural beauties than to artistic ones. For the first, my education had begun in childhood, when for the other it was quite lately*. But, he concluded, *in front of beauty, the mind becomes educated and quickly learns the taste of what it hardly appreciated in the first place*. He also went to Tunisia and Algeria. He mentioned that he was displeased by the lack of respect dispalyed by the French towards the arab cities of Algiers or Constantine - contrary to that of Tunis that has

been preserved. After, eventually, a description of his student's life in Rome, Gateaux concluded this moving document by mentioning how he regretted that Italy and Italian language were so badly known in France, when, on the contrary, France and French language are widely known in Italy.

Gateaux probably came back to France at the beginning of the summer, in June 1914. He certainly expected to go back soon to Rome as he was almost certain, as Borel had written to Volterra on April 3rd, 1914, to obtain the Commercy grant he had applied for. And indeed, in a letter to Volterra dated from Vitry-le-François on July 14th, 1914, Gateaux wrote that Borel informed him that the grant was accepted. In the same letter that he wrote a first version of a note on functionals that Volterra had asked him for joining it to the German translation (!) of his lectures on line functions ([49]), an impressive detail two weeks before the conflagration of Europe. During this hot month, he had also met the Principal of the Lycée of Bar-le-Duc on July 20th, as the man sadly observed in a letter dated from December 7th, 1914. The Principal wrote : *I saw M.Gateaux, for the last time, on July 20th. Neither me nor him did think about war. He spoke at length about his stay in Rome and told me that his thesis was almost finished. He therefore left some works deserving publication and it seems to me that you could tell his mother about the fact.*

This small postcard has been a decisive link between Hadamard and the papers left by Gateaux. Without it, the name Gateaux may indeed have only become a golden word on a monument. It was probably addressed to Hadamard or Borel, but we found it by an unexplainable chance in Fréchet's huge archival material in the Academy of Science in Paris. Another hypothesis is nevertheless that Fréchet was precisely the destinator of the letter, and happened to know the Principal as well as Gateaux in a sufficiently intimate way to have this exchange : if this hypothesis is true, it is maybe Fréchet who recovered the papers of the young mathematician and transmitted them to Hadamard. We shall see below a point that corroborates this possibility.

3. IN THE STORM

Gateaux seems to have been caught napping by the beginning of the war. The danger of war had in fact been considered only very lately in July 1914, and most people received the mobilization announcement on August 2nd with stupor. In her memoirs, Camille Marbo ([40], pp.158 et seq.) describes how, during the first days of July, the examiners for the admission to the Ecole Normale, including Lebesgue and her husband Borel, joked and chatted. She writes that it is only after the Austrian ultimatum to Serbia and the Serbian answer on July 25th that *anguish invaded Paris*, though Borel did not want to believe that *men can be so insane*. The couple left Paris for Brittany during the last week of July for their summer vacations as it had been organized earlier, but they had to return to Paris hurriedly on August 1st.

Gateaux was mobilized in the reserve as lieutenant of the 269th Infantry regiment, member of the 70th infantry division. An impressive website (www.chtimiste.com) draws the list of all French regiments involved in World War I, and gives a lot of accessible informations on them. The quartering of Gateaux's regiment took place in Nancy or in Toul. It was probably Toul in Gateaux's case, as he used headed notepaper from the *Hotel & café de l'Europe* in Toul for his last letter to Volterra on August 25th. The first battles were successful for the French, but after the euphoria of the very beginning of August, the hard reality of the force of German army obliged the French troops to withdraw day after day. At the end of August, the task of the 70th Infantry division was in first place to defend the south-east sector of Nancy.

It is hardly realized nowadays how the first weeks of the war counted incredibly heavy losses on the French side. August 1914 is the worst month of the whole war in terms of casualties, and some of the figures are beyond belief. On August 22nd, 1914, for example, the most bloody day of the war for the French, there had been 27.000 killed in the French ranks ([4]). These atrociously severe figures are due to the alliance of the vulnerability of the French uniform (with the famous *garance* (red) trousers up to 1915...), the self-confidence of the generals who clearly had little consideration for the lives of their men, and also the clear inadequacy of many leaders on the field. Prochasson ([44], pp.672-673) proposes two hypotheses to explain why the casualties in the Grandes Ecoles (Polytechnique and Ecole Normale Supérieure in particular) were so dramatic. Firstly, it was due to their grade. As they were often subordinate officers, the young students were the first killed as their rank placed them in front of their section. But in the second place, they were also sometimes moved by a kind of stronger patriotic feeling that may have driven them to an heroism beyond their simple duty. Prochasson mentions the famous example of Charles Péguy, and the less well-known one of the anthropologist Robert Hertz who unceasingly asked his superiors for a more exposed position, and has been killed in April 1915. We may also mention Marbo's testimony about her adopted son Fernand. He explained to her that, as a socialist involved in the fight for the understanding between peoples and peace, he wanted to *be sent on first line in order to prove that [he was] as brave as anyone else*. He added :*Those who would survive will have the right for speaking loudly in front of the shirkers* ([40], p.166).

As mentioned before, Gateaux's last letter to Volterra is dated on August 25th. It is addressed to *Monsieur Volterra, Sénateur du Royaume*. It is worth quoting this extraordinary document entirely.

Buissoncourt (Meurthe-et Moselle), August 29th, 1914

Mr Senator,

I deeply thank you for your letter I have just received, in the fields of Lorraine where we live days and nights under the sound of guns.

I think that the translation of the *lectures on functions of lines* will be, as you mention, delayed for a long time. And as for your *Memoir on permutable functions*, that I would have read with so much interest, God knows when I will have the chance to study it !

I want to tell you how I had been happy to learn that Italy, not only remains neutral, but also comes somewhat closely towards France. All the French people has been sensitive to that point, and greatly appreciated the attitude of Italy. May that act encourage our two countries to know better each other and to become more intimate!

The mail service works with much irregularity. I think I shall write to you soon again and I hope that at least a part of my letters will join you. I hope it will find you in good health as well as your family. As for me, I perfectly bear the stress of the battle.

Please recall me to the professors I had the honor to meet in Rome. Will you transmit my most respectfully regards to Mrs Volterra and accept my most respectful feelings.

R.Gateaux, Lieutenant at the 269th IR, 139th brigade, 70th division of reserve, by Troyes (Aube)

The fact that Gateaux alludes to the ambiguous situation of Italy is an interesting point. Italy was indeed in a strange position. As it had accepted to sign in the 1880's the military partnership treaty with Germany and Austro-Hungary proposed by Bismarck, Italy was officially member of the Triplice alliance. However, in 1914, the country was very hostile to any participation to the war. Contrary to what Gateaux writes, it was not clearly inclined towards France. France was an irritating neighbour: its support in the battle for Independence during the 1860's had been reluctant and pricey (France had then annexed Nice and the Savoy) that Italy felt obliged to enter the Triplice to counterbalance French policy. Of course, Italy had not more friendly feelings towards Austrians : it had been so hard to get rid of them 40 years earlier! Therefore, claiming with some hypocrisy that the treaty only obliged Italy to engage in war on the side of the Central Empires in case they had been attacked (this time, Austria opened the ball by declaring war to Serbia), the Italian government declared that the country would remain neutral. It lasted so until May 23rd, 1915, when Italy officially engaged in the war on the British and French side. In the meantime, there had been an unbelievable period of bargaining, both sides trying to attract Italy to its cause. Basically, the deal was clear : Italy, if it were ever engaged in a war, would choose the side that could, as reward, offer the *terre irredente* which had remained in the lap of Austria after 1870 : the Trentino, the east coast of Adriatic sea including Trieste and Fiume. At first glance, it could seem strange that such a reward may be expected from Austria, but Germany could have made pressure (and it made indeed) on its weaker ally to accept the bargain. And as mentioned before, the Allies could have promised the Trentino or the Dalmatia coast, but what would have been the price. There are testimonies from soldiers in Piemont from the beginning of 1915 where they explain that until the last moment they did not know which direction they would have to follow if the war outbreaks, west or east. The game played by the main Italian leaders, San Giuliano, Giolitti, Salandra and Sonnino during this incredible period is described in detail in [45], as is described in detail in [38] the other bargain in Versailles where the disappointed Italians could not receive the desired lands after the war. The Italian negociator Orlando left the Peace Conference, slamming the door in front of what d'Annunzio soon angrily described as the *vittoria mutilata*.

The senator Volterra immediately stood up for France and Britain. The intensive exchanges he had with his French colleagues in Paris, Appel, Borel, Painlevé, Picard and others just after the beginning of the war do not leave any doubt on that point. Volterra even, *volens nolens*, exaggerated the positive state of opinion towards the Entente. Much more plausible is San Giuliano's opinion in his letter to the Italian ambassador in Paris on August 12th, 1914: *In the Italian public opinion, there are three trends: the strongest is for neutrality; a very weak trend would like us to assist our present allies; another stronger one would be in favor of an attack against Austria despite of the treaty with the Triplice, but this trend is cooled and made suspicious by the fact that, despite of their naval superiority, France and England have still not attacked the Austrian fleet. I do not think that there are very active sympathies for France but the moderation of France before and since the beginning of the war has made an optimal impression. On the other hand, there are no antipathies against Germany but there is a big reprobation against her behaviour. There exists very deep aversion against Austria. Our information do not show that if we had marched with Germany and Austria there would have been a revolution. Surely the Italian people would have fulfilled its duty with patriotism but also quite unwillingly.*

On September 3rd, 1914, Borel writes to Volterra : *I have been most touched by your salutation transmitted by M.Appel. The attitude of Italy is one of the objective reasons we have to think that*

we defend the cause of civilization, liberty and right and this will give us the strength to fight until the end without being demoralized by any hard moment, as time will obviously guarantee our success. Volterra answered on September 14th : *I am not doubtful, and I have never been, about the triumph of your great and noble country that I love with all my heart. France fights for justice and liberty and for the cause of civilization against the violence of the most brutal and horrible imperialism whose aim was to put Europe into slavery. [...] In my opinion, the role and the mission of Italy is to leave neutrality and to take the side of France and its allies against Austria and Germany.* In October 16th, 1914, Borel asks Volterra to send him articles from Italian intellectuals in Italian newspapers, commenting on the *German intellectuals' call to the civilized world*, in order to translate them and publish them in his journal, the *Revue du Mois*. The journal came out until Borel's own departure to the front in 1915. Volterra answers on October 24th, indicating some possible interesting articles, but also asking for some news from *M.Gateaux, M.Pérès, M.Boutroux and M.Paul Lévy and other young French friends. I have received a M.Gateaux letter from the battlefield and afterwards I did not receive any other. And this is why I am very worried about his fate and the other's one.* Volterra, as Borel in fact, then ignored Gateaux's death. Borel answers to Volterra's letter on November 4th, informing him that Pérès and Boutroux were reformed and that he did not know where was Gateaux. Quite interestingly, the tone of this letter is slightly less confident than the previous ones. This was the moment when the enormous losses of the first weeks began to filter. Borel writes: *At the Ecole Normale, several young men with a bright scientific future have already disappeared. The responsibility of those who wanted this war is really terrible. The entrance of Turkey on stage may create an extension whose consequence will be to shorten the war and reduce the sum of the sufferings. But I understand very well that the statesmen who have the responsibility for action do hesitate to attract on their country a part of these sufferings as long as they can avoid them. I think that everyone realizes that in our country ; the greater will be the recognition and the sympathy towards the nations who will join us to permanently annihilate the prussian dream of domination over the whole world.*

Let us come back to Gateaux. We have left him at the end of August in Lorraine. The Schlieffen's plan of the German Headquarters seemed to work perfectly as the French went back and back and were more and more close to be crunched between the two wings of the German army (one coming from the north through Belgium, the other from the east through Lorraine and Champagne). Then happened the unexpected *miracle* of the Battle of the Marne (September 6th-13th, 1914) that suddenly stopped the German progression, concluding the Schlieffen's plan by a failure (almost the same plan, 26 years later, ended with a real success...). During these days, Vitry-le-François, where Gateaux's mother stayed, had been occupied by the Germans during the night between 5th to 6th of September, but they were compelled to leave the place and to withdraw towards east on September 11th. A vivid tale of this moment was written after the war by a witness ([43]). Though Gonthiez and Janet wrote in [1] that they *easily imagine all the pain he [Gateaux] should have felt when he learnt that the enemy had taken the city of Vitry-le-François where his poor mother had stayed*, it is not clear whether Gateaux has learnt the fact at all, due to the general confusion. We refer to [4] or to several articles of [11] for the description of this phase of the war. From the plan of campaign of the 70th Infantry division, one may infer the movements of Gateaux's regiment. From September 13th, the French chased after the withdrawing German troops beyond Nancy on September 24th, 1914. This was the precise moment when the French and German headquarters became aware of the impossibility of any further movement on a front

line running from the Aisne to Switzerland and so that the only hope was to bypass the enemy in the still almost free of soldiers zone between the Aisne and the sea. The general Joffre decided to withdraw from the eastern part of the front (precisely where Gateaux was) a large number of divisions and to send them by rail to places in Picardie, then to Artois and finally to Flandres to try to outrun the Germans. The so-called *race for the sea* lasted two months and was incredibly bloody.

The 70th division was transported between September 28th and October 2nd from Nancy to Lens, on a distance of almost 300 km ! It received the order to defend the east of Lens and Arras. On October 3rd, Gateaux's regiment was in Rouvroy, a small village, 10km south-east from Lens and Gateaux was killed at 1 o'clock in the morning, while trying to prevent the Germans to enter the village. In the general confusion of the bloodshed, the corpses were not identified before being collected and hastily buried in improvised cemeteries. Gateaux's body was buried near St Anne Chapel in Rouvroy, a simple cross without inscriptions marking the place. Following the military dossier, René's mother was informed on October 4th that her son was reported missing. On March 16th, 1916 her other son and only child, René's brother Georges was killed in the Mort-Homme in front of Verdun. Much later, at the end of this life of pains, René's mother passed away on February 24th, 1941 in Vitry-le-François, some months after having seen her city annihilated by the German invasion. . .

The official act of René's death was established only on December 28th, 1915 under the evidence given by Henri-Auguste Munier-Pugin, warrant officer and Albert Garoche, sergeant in the 269th Infantry regiment. But it is only long after, on December 8th, 1921, that Gateaux's corpse was exhumated and formally identified, and finally transported to the necropole of the military cemetery of the Bietz-Neuville St Vaast where Gateaux's grave is number 76. Gateaux's mother was informed on January 5th, 1922 of the fact. The last document of the military dossier is a letter from the Ministry of war, dated June 22nd, 1923, informing the mayor of Vitry-le-François that the Lieutenant René-Eugène Gateaux had officially been declared *Dead for France*.

The chronology of how Gateaux's death was communicated to the academic sphere is not completely clear. As already mentioned, the Principal of Bar-le-Duc Lycée writes on December 7th, 1914 to someone who may be Borel or Hadamard, but it is clearly an answer to a letter he has received, asking him about Gateaux's relatives. Answering that René had only his mother, he adds that *in the so cruel mourning she lives, she would be very touched by your condolences and it would be a painful joy for her to read your appreciation of the great intellectual value of her dear son*. To conclude his letter, the Principal of Bar-le-Duc writes : *The death of our young colleague has sadly moved all his colleagues of the Lycée in Bar who had mostly appreciated his bright intelligence, the frankness of his character and the charm of his modesty. He has bravely done all his duty until the end, but it is a great pity that he could not have lived his whole life*.

It is on December 10th, 1914 that Borel writes to Volterra about Gateaux's death. *The success will unfortunately cost irrepleceable losses ; among the sad news I have recently heard, one that caused me most grief is Gateaux's death. The conditions in which it was announced to us leave unfortunately the tiniest hope of a mistake. I want though to hope that on the dozens of pupils of the Ecole Normale considered as lost, there will be at least one or two who will come back at the end of the war*. Volterra sadly answered some days later : *Gateaux was very talented and I am sure that he had a great future. He was developing his ideas in a somehow slow but always precise way. Last year, he had worked a lot and I did not doubt that all the material for his thesis was ready. How many young lives have been the victims of this war ! It is horrible to think. . .* The

same day a telegram was sent to Borel by Volterra in the name of the Mathematical seminary in Rome :*Mathematical seminary Rome expresses deep grief received new about his former member Gateaux and all his great sympathy to the Ecole Normale.*

As soon as August 1915, Hadamard began the necessary steps to obtain the attribution of one of the prizes of the Academy of Science to Gateaux. In a letter dated August 5th, 1915 and maybe addressed to Picard as Perpetual Secretary, Hadamard mentions that Gateaux *has left very advanced researches on functional calculus (his thesis was composed to a great extent, and partly exposed in notes to the Academy), researches for which M.Volterra and myself have a big consideration.* At the seance of December 18th, 1916 the prix Francœur is attributed to Gateaux ([26], pp.791-792). It is interesting to read in Hadamard's short report the following section : *[Gateaux] was on the point to follow a much more audacious way, promising to be most fruitful, by extending the notion of integration to the functional domain. Nobody could predict the development and the range this new series of researches could have attained. This is what the events have interrupted.* It seems plausible that Hadamard has not looked in detail to the papers that Gateaux left. Hadamard himself was caught in the storm of events, loosing his two sons during the summer of 1916. However, he noticed that one major interest in the last period of the young mathematician's work was the integration over the functional space. As we shall see in the next section, this was precisely for this reason that he spoke to Lévy about Gateaux.

4. THE MATHEMATICAL DESTINY

In 1918, the Paris Académie des Sciences, following Hadamard's proposition, decides to call Lévy for the *Cours Peccot* in 1919. The *Cours Peccot* was (and still is) a series of lectures in mathematics given at the Collège de France financed by the Peccot foundation. It is a way to promote new ways in research by offering a financial support and an audience to a young mathematician. Borel had been the first lecturer in 1900, followed by Lebesgue. Normally, the age of the lecturer should have been less than 30. However, the losses of the war had been so heavy among young men that the choice of the 33 years old Lévy was reasonable. It also plausible to think that Gateaux would have been a natural Peccot lecturer, had he survived the war. As Lévy's designation is almost concomittant with Hadamard's asking of taking care of Gateaux's papers, maybe there is a connection between the two events. Anyway, on January 3rd, 1919, Lévy wrote to Volterra :*As I was recently interested in the question of the extension of the integral to the functional space, I spoke about the fact to M.Hadamard who mentioned the existence of a R.Gateaux's note on the theme. But he could not give me the exact reference and I cannot find it. [...] Though I am still mobilized, I work on lectures I should read at the Collège de France on the functions of lines and the equations with functional derivatives and at this occasion I would like to develop several chapters of the theory. [...] I think that the generalization of the Dirichlet problem should be more difficult. Up to now, I was not able to extend your results on functions of the first degree and your extension of Green's formula. This is precisely due to the fact that I do not possess a convenient expression for the integral.* On January 12th, Lévy sends a new letter to Volterra : *M.Hadamard has just found several unpublished Gateaux's papers at the Ecole Normale. I have not seen them yet but maybe I'll find what I am looking for in them.* Volterra answers on January 15th to both letters, telling Lévy that none of Gateaux's publications concerned the integration. He nevertheless adds : *We had chatted before he left Rome about his general ideas on the subject, but he did not publish anything. I suppose that in the manuscripts*

he had left, one may probably find some notes dealing with the problem. I am happy that they are not lost and that you have them in hands. The question is very interesting.

Integration over infinite dimensional spaces was certainly the most important subject considered by Gateaux. This may be read in Hadamard's following lyrical comment ([1], p.138) : *The fact that he chose the functional calculus revealed a wide-spreading mind, scornful of the small problem or of the easy application of known methods. But the event proved that Gateaux was able to consider such a study under its wider and more suggestive aspect. And it is what he indeed did, with the integration over the functional field, to speak only about this example, the most important, that represents an entirely new path and very large prospects offered to the theory.*

Volterra was in fact not the only person Lévy had contacted these days about general integration. As can be seen in the first letters of their correspondence, it was also on that theme that he entered in contact with Fréchet (see [3]) at the very end of the year 1918. Fréchet had in 1915 proposed a first attempt to define an integration theory over abstract spaces. In 1933, Kolmogorov ([31]) considered this construction by Fréchet as the first attempt of the kind.

On January 6th, 1919 Lévy writes to Fréchet : *About Gateaux's papers, I learnt precisely yesterday that M.Hadamard had put them in security at the Ecole Normale during the war and had just taken them back. Nothing is therefore yet published.* From this sentence, we may infer that it is Fréchet who firstly wrote to Lévy about Gateaux papers, and certainly because he had an idea of what they contained. Maybe this constitutes a proof that it was indeed through Fréchet that the papers arrived to Hadamard during the war, and that Fréchet was the destinator of the aforementioned letter from the Principal of the Lycée in Bar-le-Duc? On February 12th, Lévy begins to describe to Fréchet the content of what he has precisely found in Gateaux papers, namely a first theory of harmonic functionals.

As said before when we commented on Gateaux's letter to Volterra exposing his research program, Gateaux's interest for the infinite dimensional integration originated in the extension of Cauchy formula. For Lévy, the situation was somewhat different, in connection with research devoted to the theory of potential. This theory, directly derived from the electromagnetism has the determination of the electric potential created in one point P by a repartition of electric charges in a region of the space. We shall only give a brief account of it, in order to explain how it is connected to integration. The interested reader may report to the extraordinary book by Kellogg ([30]) to obtain better information on the theory and its history.

Following Coulomb's researches about the attraction of electric charges in the middle of 18th Century, it had become clear that the structure proposed by Newton for the universal gravitation was transposable to the case of electric attraction. The study of the representation of these fields of power had been an important chapter of mathematical physics since the beginning of 19th Century.

Since Green, it had been identified that any harmonic function U (that is to say satisfying $\Delta U = 0$) in a *regular* region R of the space, is a newtonian potential decomposed in three terms

$$U(P) = -\frac{1}{4\pi} \int \int \int_R \frac{\nabla^2 U}{r} dV + \frac{1}{4\pi} \int \int_S \frac{\partial U}{\partial \nu} \frac{1}{r} dS - \frac{1}{4\pi} \int \int_S U \frac{\partial}{\partial \nu} \frac{1}{r} dS$$

where r is the distance between P and the current point Q , ν the normal vector to S at Q directed towards the outside, where S is the border of R . In this formula, the first term represents a volume distribution with density $-\frac{\nabla^2 U}{4\pi}$, the second one a regular surface distribution with density $\frac{1}{4\pi} \frac{\partial U}{\partial \nu}$

on S , and the last one is what is called a double-distribution (*potentiel double-couche* in French) with moment $-\frac{U}{4\pi}$.

The presence of this last term has an interesting physical explanation : when two electric charges are very close, they act as an infinitesimal magnet and induce a magnetic field representable as a moment. The presence of the double-distribution is therefore explained as the result of two infinitely close surface distributions.

From these considerations is issued the central problem of the mathematical theory of potential: to find a harmonic function U in a domain R giving the values of the function on the border S (Dirichlet problem) or the values of the normal derivatives on S (Neumann problem). In 1906, Hadamard ([25]) proposed to make use of variational techniques from Volterra's theory of functions of lines in order to study more general forms of these problems. They form the content of Lévy's thesis, defended in 1911. It is indeed somehow forgotten today that Lévy, before becoming one of the major specialist in probability theory in the 20th Century, had been a very good specialist of functional analysis (see in particular [3], •6, •7 and •8). It is besides quite remarkable (and the present paper gives an evidence of the fact) that the works in functional analysis from the beginning of 20th Century have led rather naturally to probabilistic formulations and problems in Lévy's mind.

As it is seen, the study of these questions in the infinite dimensional classical functional spaces puts the question of integration over these spaces. A central problem arises from the fact that generally, in infinite dimension, a subset has a volume equal to zero or infinity, and this prevents the direct extension of the Riemann integral defined through an approximating step-functions sequence.

Gateaux seems to have been the first to propose a natural way to bypass the problem by considering the integral as an asymptotic mean value. In a paper he wrote in 1924 at the end of his intense period of activity around these questions ([35]), Lévy takes stock of this definition of integral as mean value as a natural definition of the uniform probability in an infinite set. As a basic example, he considers the elementary situation of the nonnegative integers that is the foundation of what we call today the probabilistic number theory. If f is a function defined on \mathbb{N} (f could typically be the indicator of a subset $A \subset \mathbb{N}$), the expectation of f is defined as the limit of $\frac{1}{N} \sum_{k=1}^N f(k)$ when N tends to infinity. In particular,

$$P(A) = \lim_{N \rightarrow +\infty} \frac{1}{N} \text{Card}\{n \in \mathbb{N}, n \in A\}$$

and therefore, for example, if one randomly draws a point from \mathbb{N} , there is one chance over two that it is an even integer, a rather comforting result for the mind. . .

In his paper, Gateaux is mostly interested by the case of the sphere of the Hilbert space of square integrable functions over $[0,1]$. In fact, Gateaux restricts himself to the subset of continuous functions. As Lévy explains to Fréchet in a prolix letter dated from February 16th, 1919 (Letter 5 in [3]), it is more natural to consider measurable functions, that is to say to work with the (now) usual space L^2 .

Gateaux begins by considering functionals of the type $U : x \mapsto f[x(\alpha_1)]$ where x is a point of the functional space (a 'line'), f a continuous real function and α_1 a fixed point in $[0,1]$. He introduces the n -th section of the sphere $\int_0^1 x(\alpha)^2 d\alpha = R^2$ as the set of functions x assuming constant values x_1, x_2, \dots, x_n over each subinterval $[0, \frac{1}{n}], \dots, [\frac{n-1}{n}, 1]$ and therefore such that $x_1^2 + \dots + x_n^2 = nR^2$. As α_1 is fixed, $x(\alpha_1)$ is one of the coordinates when x is taken in the n -th section and therefore the $(n - 2)$ dimensional volume of the intersection of the section with the

set $x(\alpha_1) = z$ (with $0 \leq z^2 \leq nR^2$ or equivalently $-\sqrt{n}R \leq z \leq \sqrt{n}R$) is given by

$$(\sqrt{nR^2 - z^2})^{n-1} \cdot V_{n-1}$$

where V_n is the volume of the unit sphere in dimension n . It is easily seen that V_n satisfies the induction formula $V_n = 2V_{n-1} \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta$. Finally the mean of the functional U over the n -th section is given as

$$\begin{aligned} \frac{1}{V_n(R\sqrt{n})^n} \int_{-\sqrt{n}R}^{\sqrt{n}R} f(x) V_{n-1}(\sqrt{nR^2 - x^2})^{n-1} dx &= \frac{V_{n-1}}{V_n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(R\sqrt{n} \sin \theta) \cos^n \theta d\theta = \\ &= \frac{1}{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n \theta d\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(R\sqrt{n} \sin \theta) \cos^n \theta d\theta. \end{aligned}$$

It is seen that the preponderant values for θ in the integral on the right are those around 0, and the other one is known to be equivalent to $\sqrt{\frac{2\pi}{n}}$. Under ‘some regularity conditions’ for f , the previous expression is therefore approximately equal to

$$\frac{1}{\sqrt{\frac{2\pi}{n}}} \int_{-\frac{\alpha}{\sqrt{n}}}^{\frac{\alpha}{\sqrt{n}}} f(R\sqrt{n} \sin \theta) \cos^n \theta d\theta$$

or

$$\frac{1}{\sqrt{\frac{2\pi}{n}}} \int_{-\alpha}^{\alpha} f\left(R\sqrt{n} \sin \frac{\psi}{\sqrt{n}}\right) \cos^n \frac{\psi}{\sqrt{n}} \frac{d\psi}{\sqrt{n}}$$

for any $\alpha > 0$ and n large. Using a Taylor expansion, the previous expression becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f\left(R\sqrt{n}\left[\frac{\psi}{\sqrt{n}} + O\left(\frac{\psi^2}{n}\right)\right]\right) \left(1 - \frac{\psi^2}{2n} + O\left(\frac{1}{n^2}\right)\right)^n d\psi.$$

Letting n goes to infinity, this tends to $\frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} f(R\psi) e^{-\frac{\psi^2}{2}} d\psi$, and finally letting α going to infinity, one obtains the limit

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(R\psi) e^{-\frac{\psi^2}{2}} d\psi \quad (1)$$

which is defined by Gateaux as the integral of U over the sphere of the functional space. Then Gateaux asserts that this can be generalized to more general functionals. The most complicated he considers is

$$U(x) = \int_0^1 d\alpha_1 \dots \int_0^1 d\alpha_p f[x(\alpha_1), \dots, x(\alpha_p), \alpha_1, \dots, \alpha_p]$$

for which the mean value is given by

$$\frac{1}{(2\pi)^{\frac{p}{2}}} \int_0^1 d\alpha_1 \dots \int_0^1 d\alpha_p \int_{-\infty}^{+\infty} dx_1 \dots \int_{-\infty}^{+\infty} dx_p f(Rx_1, \dots, Rx_p, \alpha_1, \dots, \alpha_p) e^{-\frac{x_1^2 + \dots + x_p^2}{2}}. \quad (2)$$

The rigorous existence of the limit is not justified by Gateaux, as Lévy comments to Fréchet in the letter he sends to him on February 12th, 1919. Obviously, for Gateaux, as Lévy himself writes in the preamble of [21], the present state of his papers was certainly not a final one. And in the long note Lévy adds at the end of the article ([21], p.67), he describes several attempts made by Gateaux to obtain the limit in several situations. Anyway, it is clear for Lévy that the

priority is to fill the gap left by Gateaux and to try to obtain the existence of the mean value for the most general functionals he can find. Gateaux had suggested that the important property was the continuity, in the meaning that the functional is well approximated by its value on the n -th section. Lévy writes in the same letter to Fréchet that with such an hypothesis the definition of the integral is not difficult to legitimate but is much too restrictive. In the following very long letter to Fréchet (on February 16th, 1919), he explains why ([3], p.115). In fact, if the functional is approximated uniformly by its projection on the n -th section, it is only a very particular type of harmonic functional. In the final redaction given by Lévy to these considerations (Chapter VI of [34]), he will even be more drastic in his conclusions : Gateaux's functionals are in fact almost constant in the infinite dimensional space and therefore it is not amazing that the mean value exists for them or that they are harmonic ([34], section 138). Lévy replaces the spatial continuity required by Gateaux hypothesis by a continuity in distribution. This is already discernible in the Letter 5 to Fréchet ([3], pp.116-117) when he writes that *I see that if the size of the different values of x that contribute in the considered interval to produce the mean value is very important, on the other hand it is indifferent to know in which points of the interval these values are assumed.* In [34], Lévy has finally admitted that the right formulation for these problems is in a probabilistic framework, and it is impressive to see how in the book (and in particular in Chapter VI), Lévy makes use of probability theory to justify the passages to the limit by means of the law of large numbers.

In a way, it is surprising that not any reference to probability can be located in Gateaux's papers, and in particular some comments about the appearance of the gaussian distribution in the limit expression (1). This result, usually known today under the name *Poincaré's lemma* has in fact nothing to do with Poincaré, as already observed Diaconis and Freedman [10] and Stroock [47]. To the unnoticed exception of Mehler, who obtained the result in 1866 in a purely analytical context (see [47] for an exact reference), it was Borel who firstly obtained this convergence to the Gaussian measure when the dimension tends to infinity in 1906. In [5], a paper reprinted as "Note I" in his book [6], Borel's considerations were about statistical mechanics, more precisely about Maxwell and Boltzmann's kinetic theory of gases. Borel desired, as he himself mentions in the introduction, to convince those who look at Maxwell-Boltzmann's theory with suspicion. *Their scruples are legitimate to a certain extent : one cannot reproach a mathematician with his love for rigor ; but it seems to me possible to allay them.* And it is when constructing a model from which it is possible to infer the Maxwell-Boltzmann law for the distribution of speeds of particles that Borel was led to considerations on spheres of large dimensions. These spheres represent the surfaces in the phase space of equal total kinetic energy. In the complement to the translation of the Ehrenfests' paper in the *Encyclopédie des Sciences Mathématiques* about statistical mechanics ([7] p.273), Borel mentions the studies about n -dimensional geometry as the first example of mathematical researches inspired by statistical mechanics. He even audaciously asserts that one should consider the results about surfaces and volumes in high dimensions as connected to statistical mechanics.

From his founding paper in 1860, Maxwell had observed the coincidence between the distribution law for the speeds of the particles and the distribution *governing the distribution of errors among observations by use of the so-called least-squares method* ([41], Prop.IV and following comments). In [5], Borel does not mention the point. On the contrary, he does in [6] (p.66) but without any probabilistic interpretation : for him, the interesting fact in this appearance of

the Gaussian distribution function is only that this distribution function is well tabulated, and this allows to use it for computations.

Lévy seems to be the first to have interpreted the previous considerations in a probabilistic way. He in particular interprets the relation (2) as a convergence in distribution of the coordinates x_1, \dots, x_p conceived as random variables uniformly distributed on the sphere $S_n(\sqrt{n})$ to independent Gaussian variables when n goes to infinity ([34], pp.267-268). In Remark 20, p.282, Lévy observes that the result may be directly obtained through the law of large numbers, the uniform distribution over a n -dimensional sphere of radius \sqrt{n} being connected to the Gaussian distribution when n goes to infinity. Interestingly, probability theory does not appear in the notes Lévy had published just after the war about his works on the function of lines. It is only when he wrote his book [34] that he understood this natural framework, at a precise moment when he began to be interested in probability theory for different reasons : we refer the interested reader to [3] where this story is discussed at length. At this moment, a kind of extraordinary junction seems to have happened in Lévy's mind resulting in unifying his mathematical considerations under probabilistic reasoning. At the end of his 1906 paper ([5]), Borel had only briefly presented some hints vaguely connected with the fact. As soon as in his letter to Fréchet on February 16th, 1919, it is possible to feel how the Wiener measure shows through when Lévy considers a series of L^2 orthogonal functions for generalizing Gateaux's sections. In his autobiography ([36]), Lévy, always ready to moan about missed opportunities, had observed how in [34], *he was close to Wiener measure*. This is not a rhetoric formulation. He was indeed so close that Wiener, when speaking with him in 1922 will immediately see that Lévy's considerations to define the integral over the infinite dimensional sphere are precisely what he can use to define his Differential-space and construct the Wiener measure of the Brownian motion. Two years earlier, he has had the intention to use the results provided by Daniell, who, independently from Gateaux has also defined an integral through a limit of means ([9]). In [50] (footnote *, p.67), Wiener mentions that he had just discovered Gateaux's earlier discoveries in [20]. In 1923, at the beginning of his epoch-making paper [51] (p.56), Wiener pays tribute to Gateaux and Lévy for having provided the most complete investigations about integration in infinitely many dimensions. He also credits Lévy (pp.56-57) for having explained to him how these results could be exploited. Bourbaki, when he deigned to speak about probability theory in the chapter devoted to integration in non-locally compacts spaces of a late reedition of his *Eléments d'histoire des mathématiques* ([8]), mentions this path linking Borel's consideration on kinetic theory of gases to the Wiener measure with Gateaux and Lévy as fundamental steps. We refer the reader to •7 in [3] for more details on the beginning of the mathematical Brownian motion. In his paper [37], McKean has exposed how thinking of the Wiener measure as a uniform distribution over the *infinite dimensional sphere of radius $\sqrt{\infty}$* , a direct consequence of Lévy's considerations in [34], has been fruitfully used by a Japanese school in the 1960's to describe the geometry of Brownian motion. In another direction, in 1969, Gallardo ([14]) made the interesting observation that *Poincaré's lemma* could be connected with the fact that if $X^n(t) = (X_1(t), \dots, X_n(t))$ is a n -dimensional Brownian motion issued from 0, if one denotes by $\rho_n(t) = \|X(t)\|$ and by T_n the first passage time of X^n on the sphere \sqrt{n} , then $T_n \rightarrow 1$ in probability and $X^n(T_n)$ follows the uniform distribution on the n -dimensional sphere of radius \sqrt{n} . Yor has later developed these considerations (see [52]).

CONCLUSION

It has often been mentioned that after World War 1, the French Grandes Ecoles, the Ecole Normale especially, were crowded with the ghosts of the students from the years 1910 who largely disappeared during the conflict. Obviously, as was the case in the aforementioned examples, the dead of the Great War were essentially very young men who had scarcely finished their graduation studies and whose names are hardly known to us today.

On that prospect, the example of René Eugène Gateaux, who died at the age of 25 in October 1914 during the first battle of Artois is an example both representative and exceptional of the students victims of the war, in so far as, though he was very young, a trail of his scientific work remained after himself had disappeared. It is remarkable that Gateaux's name is still nowadays usually known by those who study mathematics : the directional Gateaux differential, an alternative to the global Fréchet differential is present in most textbooks of differential calculus.

As we have seen, in his tragic fate, Gateaux met two posthumous fortunes. Firstly, he happened to deal at the right moment with subjects of Functional Analysis. These themes shaped and developed in the immediate after-war in an unexpected way. Moreover, just before he went to the army at the mobilization, he left in a semi-achieved state a series of papers that he had begun to write while he was in Rome with Volterra. When Hadamard in 1918 entrusted Paul Lévy with the preparation of their publication in the *Bulletin des Sciences Mathématiques*, it was for Lévy the occasion of a tremendous restarting of his career, leading to numerous mathematical consequences. It was also the reason why our time still has some remembrance of a young mathematician called René Gateaux.

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