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Belkacem Bendiffalah, Claude Cibils

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B. Bendiffalah and C.Cibils

Abstract

In this note we show that a finite full subcategory of an hereditary small \mathbb{Z} -category is hereditary. As a specialization we obtain the result proved in [6, 4], namely that the endomorphism ring of a finitely generated projective module over an hereditary ring is also hereditary, see also [5]. In particular if A is an hereditary ring and e is an idempotent, the ring eAe is also hereditary.

1 Introduction

A small \mathbb{Z} -category \mathcal{C} is a category with set of objects \mathcal{C}_0 , abelian groups of morphisms and \mathbb{Z} -bilinear composition. Note that the endomorphism set of each object of \mathcal{C} is a ring (i.e. an associative unitary ring which is non necessarily commutative). Morphisms of \mathcal{C} are naturally endowed with bimodules structures over those rings.

A ring A is hereditary if submodules of projective modules are still projective. In other words an A -module can be resolved using projective modules in at most one step, that is hereditary rings are those of projective dimension at most one. Similarly a \mathbb{Z} -category is hereditary if subfunctors of projective functors from \mathcal{C} to the category of abelian groups are still projective. Recall that such functors are often called modules over the category, since for a category with a single object, such functors over the category and modules over the endomorphism ring of the object coincide.

The main result of this paper is that for an hereditary \mathbb{Z} -category, any finite full subcategory is still hereditary. The proof relies on the study of functors defined on a vertex of the category and their induction to the entire category.

As a consequence we recover the result proven by H. Lenzing [6] and R.R. Colby – E.A. Rutter [4] (see also the work of J.L. García Hernandez and J.L. Gómez Pardo [5]) : let A be an hereditary ring and let P be a finitely generated module, then $\text{End}_A P$ is also hereditary. In particular, let e be an idempotent of an hereditary ring. Since eAe is the endomorphism ring of the projective module Ae , the ring eAe is hereditary. Note that a trivial situation occurs when the idempotent e is central. In this case $eAe = Ae$ and $A = Ae \times A(1 - e)$. Then

$$\text{proj dim } A = \max\{\text{proj dim } Ae, \text{proj dim } A(1 - e)\}.$$

Recall that for rings A with projective dimension strictly larger than 1 the projective dimension of eAe can be arbitrary, examples are provided by the Auslander-Reiten algebra of a finite representation type algebra over a field. In this respect the result for hereditary rings and categories is an exception.

The link between the main result and it's specialization is as follows. Let A be an hereditary ring and let e be an idempotent. Consider the \mathbb{Z} -category with two objects denoted e and $1 - e$, having endomorphisms rings eAe and $(1 - e)A(1 - e)$; the bimodules of morphisms between different objects are given by $(1 - e)Ae$ and $eA(1 - e)$. Composition is provided by the product of A .

The preceding construction is a special instance of the expansion process described in [3]. It follows for instance from this paper that modules over the expanded category coincide with A -modules, hence the category is hereditary in case A is hereditary. Consequently, eAe is an hereditary ring using Theorem 3.1.

2 Categories and modules

A \mathbb{Z} -category is a small category \mathcal{C} such that each set of morphisms ${}_y\mathcal{C}_x$ from an object x to an object y is a \mathbb{Z} -module, and such that the composition of morphisms is \mathbb{Z} -bilinear. Of course, any ring A provides a single object \mathbb{Z} -category, this way a \mathbb{Z} -category can be considered as a ring with "several objects", see for instance [7].

A \mathcal{C} -module is a \mathbb{Z} -functor \mathcal{M} from \mathcal{C} to the category of \mathbb{Z} -modules. In other words \mathcal{M} is given by a family of \mathbb{Z} -modules ${}_x\mathcal{M}$ and a family of "actions" ${}_y\mathcal{C}_x \otimes_{\mathbb{Z}} {}_x\mathcal{M} \longrightarrow {}_y\mathcal{M}$ verifying the usual module axioms. Morphisms between modules are natural functor transformations, providing an abelian category. In

case the \mathbb{Z} -category is a single object category, modules coincide with usual modules over the endomorphism ring of the object.

By definition a \mathcal{C} -module \mathcal{M} is projective if the covariant functor $(\mathcal{M}, -)$ is exact. We record the following well known lemma, see for instance [8].

LEMMA 2.1 *Let \mathcal{C} be a \mathbb{Z} -category and let x be an object of \mathcal{C} . The \mathcal{C} -module ${}_x\mathcal{C}_x$ is projective.*

Proof. Let $\mathcal{M} \xrightarrow{h} \mathcal{N}$ be a surjective morphism and let ${}_x\mathcal{C}_x \xrightarrow{\varphi} \mathcal{N}$ be any morphism. Considering the object x , we lift φ by defining $\overline{\varphi}({}_x1_x) = {}_xm$ where ${}_xm$ verifies $h({}_xm) = \varphi(\pi_x)$. Then for any object y define $\overline{\varphi}({}_yf_x) = {}_yf_x {}_xm$.

We will need to induce modules as follows.

DEFINITION 2.2 *Let \mathcal{C} be a \mathbb{Z} -category and let M be an usual module over the ring ${}_xC_x$ for a fixed object x . The induced \mathcal{C} -module $M \uparrow$ is*

$${}_y(M \uparrow) = {}_y\mathcal{C}_x \otimes_{{}_xC_x} M$$

together with the natural actions provided by composition of morphisms of \mathcal{C} .

PROPOSITION 2.3 *Let \mathcal{C} be a \mathbb{Z} -category. Let x be an object of \mathcal{C} and let \mathcal{M} be a ${}_xC_x$ -module. Then \mathcal{M} is projective if and only if $\mathcal{M} \uparrow$ is projective.*

Proof. Assume M is free of rank one. Then $M \uparrow$ is precisely ${}_x\mathcal{C}_x$. Replacing M by a direct sum of free rank one modules, or a direct summand of it, provides that $M \uparrow$ is projective.

Conversely, assume M is not projective and let $h : X \rightarrow Y$ be a surjective ${}_xC_x$ -module together with a map $\varphi : M \rightarrow Y$ which cannot be lifted. Inducing this situation to the entire category shows that $M \uparrow$ is not projective, since a lifting of \mathcal{C} -modules would produce a lifting at each object, in particular a lifting at x .

Remark 2.4 *In case \mathcal{C} is amenable, meaning that finite coproducts exists and idempotent splits, the representable functors ${}_x\mathcal{C}_x$ are precisely the finitely generated projectives, see [8]. We do not need formally this result for the preceding considerations. Note however that any category is Morita equivalent to its amenable completion, which is obtained through additivisation and karoubianisation.*

3 Hereditary is hereditary

THEOREM 3.1 *Let \mathcal{C} be a \mathbb{Z} -category which is hereditary (any submodule of a projective module is projective). Let x be an object of \mathcal{C} . Then the ring ${}_x\mathcal{C}_x$ is hereditary.*

Proof. Let M be a projective ${}_x\mathcal{C}_x$ -module and let N be a submodule. Clearly $N\uparrow$ is a submodule of $M\uparrow$. Since $M\uparrow$ is projective we infer that $N\uparrow$ is projective. Using the Proposition above we conclude that N is projective.

Next we recall briefly the expansion process of a \mathbb{Z} -category as described in [3]. Let \mathcal{C} be a \mathbb{Z} -category provided with complete finite sets of orthogonal idempotents of endomorphisms algebras at each object. Recall that *complete* means that the sum of the idempotents at each vertex provides the unit of the endomorphism ring. We call this situation a category with an *idempotent data*. Equivalently each endomorphism ring is provided with a finite direct sum decomposition by means of left ideals.

The expanded category (with respect to this idempotent data) has each object x of \mathcal{C} replaced by the finite set of idempotents of its endomorphism ring. The morphisms from an idempotent e at x to an idempotent f at y are given by the corresponding piece of the decomposition of ${}_y\mathcal{C}_x$, namely $f({}_y\mathcal{C}_x)e$. Clearly the composition of the original category provides a well defined composition of the expanded category.

PROPOSITION 3.2 *The module categories of a \mathbb{Z} -category \mathcal{C} and of an expansion of it (given by an idempotent data) are isomorphic.*

Proof. At each object x , the abelian group ${}_x\mathcal{M}$ of a \mathcal{C} -module \mathcal{M} can be decomposed in direct summands using the action of the idempotents of the endomorphism ring given by the data. The module over the expanded category associates to the object e the piece $e({}_x\mathcal{M})$. The action of $f({}_y\mathcal{C}_x)e$ is the restriction to this direct summand of the original action, which indeed provides elements in $f({}_y\mathcal{M})$.

Conversely a module over the expanded category can be assembled as follows. Provide each object x of \mathcal{C} with the direct sum of the abelian groups associated to the idempotents at this object. The action of the entire morphisms from x to y is obtained by matrix multiplication combined with the given action.

Clearly those functors (which we have only described on objects for simplicity) are inverses each other.

EXAMPLE 3.3 Let A be a ring and \mathcal{C}_A be the corresponding single object \mathbb{Z} category. Let e be an idempotent of A , note that $\{e, 1-e\}$ is an idempotent data of \mathcal{C}_A and that the corresponding expansion is precisely the category described in the introduction, namely it has two objects called e and $1-e$ with endomorphisms rings respectively eAe and $(1-e)A(1-e)$; the bimodules of morphisms between different objects are given by $(1-e)Ae$ and $eA(1-e)$. Composition is provided by the product of A .

THEOREM 3.4 *Let A be an hereditary ring and let e be an idempotent of A . The ring eAe is also hereditary.*

Proof. Since A is hereditary the single object \mathbb{Z} -category \mathcal{C}_A is hereditary. The expanded category considered in the example above is hereditary since the categories of modules are isomorphic. Hence the endomorphisms rings at each object are hereditary by Theorem 3.1, in particular eAe is hereditary.

As quoted in the Introduction the preceding result is exceptional. We record the following :

PROPOSITION 3.5 *For each integer $n \geq 1$ there exists an algebra of projective dimension 2 with an idempotent e such that*

$$\text{proj dim } eAe = n.$$

Proof. Consider the Auslander algebra $\mathcal{A}(A)$ of a finite dimensional algebra A of finite representation type, meaning that up to isomorphism the algebra A admits only a finite number of isomorphism classes of indecomposable modules. Recall that the Auslander algebra is obtained as the opposite of the endomorphism algebra of the direct sum of one copy of each indecomposable module (no isomorphic copies are allowed).

It is well known and easy to prove – see for instance [1] – that the original algebra A is reobtained from $\mathcal{A}(A)$ using the direct sum of all the projective indecomposable summands. More precisely, let e be the sum in $\mathcal{A}(A)$ of the identities endomorphisms of each indecomposable projective module used for the construction of $\mathcal{A}(A)$. Then $A = e[\mathcal{A}(A)]e$.

Note that the Auslander algebra is well known to be of projective dimension 2, see for instance [1]. It remains to construct algebras of finite representation type with arbitrary projective dimension, for instance consider the path algebra of the quiver A_n with $n+1$ vertices all arrows from 1 to 2, from 2 to 3, ... , from $n-1$ to n , modulo the two-sided ideal generated by all the paths of length two. It is easy to prove that this algebra of finite representation type is of projective dimension n .

Next we infer the result obtained by H. Lenzing [6] and by R.R. Colby – E.A. Rutter [4].

THEOREM 3.6 *Let P be a finitely generated projective module over an hereditary ring A . Then it's endomorphism ring is hereditary.*

Proof. Consider P as a direct summand of A^n , a free A -module of rank n , which has endomorphism ring $M_n(A)$. Note that this matrix ring is hereditary since it is Morita equivalent to A . Let e be the idempotent of $M_n(A)$ corresponding to P in the decomposition $A^n = P \oplus Q$. Clearly the endomorphism ring of P is $eM_n(A)e$, hence hereditary.

We will prove now a more general version of Theorem 3.1.

THEOREM 3.7 *Let \mathcal{C} be a \mathbb{Z} -category, let F be a finite set of objects of \mathcal{C} and let \mathcal{C}_E be the full subcategory of \mathcal{C} having E as set of objects. If \mathcal{C} is hereditary then \mathcal{C}_E is also hereditary.*

In order to prove this Theorem we first recall the contraction process from [3].

DEFINITION 3.8 Let \mathcal{C} be a \mathbb{Z} category and let E be a partition of the set of objects by means of finite sets E_i indexed by a set I . The contracted category is a new \mathbb{Z} -category which objects are the sets of the partition, and morphisms from the set E_i to the set E_j are given by the direct sum of the morphisms from each element of E_i to each element of E_j . Composition is given by the matrix product combined with composition of the original category.

LEMMA 3.9 (see [3]) *Consider the contraction of a \mathbb{Z} -category along a partition of its objects by means of finite sets. Then the categories of modules of the original category and the contracted one are isomorphic.*

Proof. It is clear that the contraction of a category is provided with an extra data, namely the idempotents at each object reminiscent from the elements of the partition. More precisely, at each object of the contracted category (i.e. at a set of the partition), consider the endomorphism given by the matrix having 0 entries except for the endomorphisms of one of the objects of the set of the partition, where we consider it's identity. In other words we can expand the contracted category along this idempotent data, getting back this way to the original category. Since the modules categories of a category and of an expansion of it are isomorphic, the result is proven.

Proof of Theorem 3.7. Consider the partition of the objects of the original category given by the finite set E and each singleton corresponding to each element not belonging to E . The Lemma above shows that this contracted category is hereditary, hence the endomorphism ring at the object corresponding to E is hereditary by Theorem 3.1. Finally the full subcategory of \mathcal{C} given by E is the expansion of the single object category given by the endomorphism ring of the object E in the contracted category.

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B.B.:

Institut de Mathématiques et de Modélisation de Montpellier - I3M
Université Montpellier 2,
F-34095 Montpellier cedex 5, France.
ben@math.univ-montp2.fr

C.C.:

Institut de Mathématiques et de Modélisation de Montpellier - I3M
Université Montpellier 2,
F-34095 Montpellier cedex 5, France.
Claude.Cibils@math.univ-montp2.fr