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Satyanad Kichenassamy

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# Schauder-type estimates and applications

Satyanad Kichenassamy\*

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# 1 Introduction

The Schauder estimates are among the oldest and most useful tools in the modern theory of elliptic partial differential equations (PDEs). Their influence may be felt in practically all applications of the theory of elliptic boundary-value problems, that is, in fields such as nonlinear diffusion (in biology or environmental sciences), potential theory, field theory, or differential geometry and its applications.

Generally speaking, Schauder estimates give Hölder regularity estimates for solutions of elliptic problems with Hölder continuous data; they may be thought of as wide-ranging generalizations of estimates of derivatives of an analytic function in the interior of its domain of analyticity (Cauchy's inequalities) and play a role comparable to that of Cauchy's theory in function theory. They may be viewed as converses to the mean-value theorem: a bound on the solution gives a bound on its derivatives. The estimates generally become false if Hölder continuity is replaced by mere continuity.

Schauder estimates have three aspects, corresponding to three different ways of applying them:

- (i) they are regularity results: solutions with minimal regularity must be as regular as data permit;
- (ii) they give the boundedness of the inverse of certain elliptic operators;
- (iii) they give the compactness of these inverses.

Schauder theory has strongly contributed to the modern idea that solving a PDE is equivalent to obtaining an *a priori* bound, that is, trying to estimate a solution *before* one has constructed any solution.

We aim in the following pages to give the reader the means to make use of the recent literature on the subject. We assume the reader is familiar with the basic facts of Functional Analysis and elliptic theory (see [11]). For this reason, we give complete proofs of the most commonly used theorems used in actual applications of the estimates; we then survey the main generalizations, with emphasis on recent work. General references on Schauder estimates and their applications include [2, 26, 35, 40, 50, 53, 54, 61, 64, 72, 66, 67, 74]

## 1.1 What are Schauder-type estimates?

It is convenient to distinguish four kinds of estimates: interior, weighted interior, boundary, and Fuchsian estimates. Each of them is further divided into second-order and first-order estimates. We begin with the second-order estimates.

The interior Schauder estimate expresses that, if  $L$  is a second-order elliptic operator  $L$  with Hölder-continuous coefficients,<sup>1</sup> the  $C^\alpha$  norm of any second-order derivative of  $u$  on the ball of radius  $r$  is estimated by the sum of the  $C^\alpha$  norm of  $Lu$  on the ball of radius  $2r$ , and the supremum of  $u$  on the same ball. It therefore contains the following information:

- (i)  $u$  is as smooth as the data allow: even though  $Lu$  is just one particular combination of  $u$  and its derivatives of order two or less, the Hölder continuity of  $Lu$  implies the same regularity for all second-order derivatives;
- (ii) the regularity of  $u$  is *local*, in the sense that we require no smoothness assumption on the value of  $u$  on the boundary of the ball of radius  $2r$ ;
- (iii) the set of all functions  $u$  such that  $\sup |u|$  and  $\|Lu\|_{C^\alpha(|x|<2r)}$  are bounded by a fixed constant  $M$  is *relatively compact* in the  $C^2$  topology of the ball of radius  $r$ .

The boundary Schauder estimate expresses that if  $Lu = f$  on a bounded domain of class  $C^{2+\alpha}$  and if  $u$  is equal on the boundary to a function of class  $C^{2+\alpha}$ , then  $u$  is of class  $C^{2+\alpha}$  up to the boundary.

The scaled, or weighted interior estimates, in their simplest form, express that, if  $Lu = f$  is  $C^\alpha$ , and  $f$  is bounded, then, as one tends to the boundary, (i)  $\nabla u$  blows up at most like  $d^{-1}$ , where  $d$  is the distance to the boundary, and  $\nabla^2 u$  like  $d^{-2}$ ; (ii) the expression  $|\nabla^2 u(P) - \nabla^2 u(Q)|/|P - Q|^\alpha$  is estimated by  $C(\min(d(P), d(Q))^{-2-\alpha}$ .<sup>2</sup> In particular, this estimate does not imply that

---

<sup>1</sup>See section 2 below for the definition of the regularity classes used in this paper. Recall that an operator  $L = \sum_{ij} a^{ij} \partial_{ij} + b^i \partial_i + c$  is elliptic if the quadratic form  $a^{ij} \xi_i \xi_j$  is positive-definite.

<sup>2</sup>Following common practice, we use the "variable constant convention" according to which the same letter  $C$  is used to denote constants which may change from line to line. It is consistent as long as (i) the context makes clear on what quantities the constants depend and (ii) one is not interested in the value of the constant, but only in its existence. The convention may have been influenced by an observation by Schauder to the effect that the best constants in Schauder estimates are not well understood.

$d^2u$  is of class  $C^{2+\alpha}$  up to the boundary.

The Fuchsian estimates express that, in the above situation,  $d^2u$  is of class  $C^{2+\alpha}$  up to the boundary provided that (i) the  $a^{ij}/d^2$  and  $b^i/d$  satisfy a Hölder condition near the boundary and (ii) *either*  $L$  satisfies additional sign conditions on the lower-order terms and  $u$  is bounded, *or* both  $u$  and  $f$  satisfy a flatness condition at the boundary. Condition (i) is reminiscent of the scaling behavior of ODE of Fuchs-Frobenius type, hence the terminology.

First-order estimates are similar, with the difference that they give  $C^{1+\alpha}$  regularity of the solution if  $Lu$  is merely bounded; the conditions on the coefficients of  $L$  are also slightly weaker than in the  $C^{2+\alpha}$  case. First-order estimates are often as useful as the second-order estimates, and may generalize to nonlinear operators such as the  $p$ -Laplacian, for which the  $C^{2+\alpha}$  estimates are false.

## 1.2 Why do we need Schauder estimates?

Schauder estimates form the basis of very general existence theorems, because the compactness information they contain makes it possible to apply fixed-point theorems for compact operators (see [11, 54, 64, 66, 74]).

Schauder theory has many applications beyond existence theorems; we mention: asymptotic behavior, at infinity or near singularities; properties of eigenfunctions (Riesz-Fredholm theory [11], Krein-Rutman theorem [52]); the method of sub- and super-solutions for nonlinear problems, and bifurcation theory (see [64, 67, 74]).

Schauder theory has not been rendered obsolete by the more recent developments of Sobolev theory and variational methods, for the following reasons:

- (i) Schauder theory applies to problems without variational structure;
- (ii) it produces existence results without assuming uniqueness;
- (iii) it is more convenient than Sobolev theory in the sense that functions in  $H^k$  are Hölder continuous only for  $k$  greater than the number of space dimensions.

Of course, modern studies of nonlinear problems often use Sobolev or de Giorgi-Nash theory to obtain a modicum of regularity, and improve it using the Schauder estimates.

This survey is by no means an exhaustive report on regularity theory; in particular, the de Giorgi-Nash theorem on Hölder estimates for operators

with bounded measurable coefficients, and the literature it gave rise to, is not discussed. Special results on particular equations such as the Monge-Ampère equation, or the Laplace equation on polyhedral domains, are only briefly discussed. Regularity estimates for parabolic problems, including probabilistic methods for diffusion processes [76], fall outside the scope of this volume devoted to stationary problems, although many of the techniques are similar to those in the stationary case.

### 1.3 Why so many methods of proof?

The wide variety of methods for the derivation of Schauder estimates may be understood as follows: all proofs require the following ingredients:

- An estimate for a model problem (the Laplacian on the unit ball of  $\mathbb{R}^n$ , a half-space, or a half-ball).
- A scaling argument which transfers estimates to balls of radius  $R$ .
- A linear change of coordinates which yields the result for constant-coefficient operators.
- A passage to continuous coefficients.

The second step is often formulated in terms of weighted norms involving the distance to the boundary; the third is achieved using Korn's device, which consists in comparing the given operator with the operator with coefficients "frozen" at one point; the fourth is streamlined by the use of interpolation inequalities for (weighted) Hölder norms. In fact, the first three steps follow from the invariance properties of the Laplace operator; the variety of proofs essentially comes from the different ways to exploit these properties of the Laplace operator.

The period 1882-1934 has seen the emergence of *derivations* of estimates of which the definitive form was only gradually discovered. In this first stage, one notices a tendency to try and replace the estimation of Green's function by a direct estimation of the solution. As a result, solution methods based on the construction of integral operators which provide (approximate) inverses developed separately from regularity theory for nonlinear problems, and eventually gave rise to pseudo-differential analysis.

The subsequent period, say from 1934 to 1964, was devoted to a streamlining and elucidation of the methods, and culminated in the generalization

to systems [1] of Schauder’s estimates. At the end of this period, estimates on Green’s function had been evacuated from the variable-coefficient case. They would re-appear indirectly with the introduction of pseudo-differential inverses of elliptic operators, but pseudo-differential techniques with symbols of limited regularity are still not very well understood [77].

Once the estimates had been discovered, it became possible to look for *verifications*: efficient ways to prove that the estimates hold once they have been proved by other methods. This search has brought about a change in perspective, triggered by the needs of new applications: once the passage from constant to variable coefficients had been streamlined, it became clear that potential theory was still needed to prove the estimates in the case of the Laplacian. In other words, all of the refinements of Schauder theory were ultimately based on the direct proof of the estimates for the Laplacian. And the various proofs in this case ultimately make use of the invariance of the Laplacian under translation, rotation and scaling.

Now, a problem in which a singularity occurs at a specific point in space cannot be translation-invariant. The first step in handling such problems would be to consider scaled Schauder estimates in balls which become smaller as one approaches the singularity; these “blow-up” methods lead naturally to weighted Schauder estimates.<sup>3</sup> This approach does not yield optimal regularity. From 1990 onwards, the author showed that the correct regularity, first for hyperbolic problems, and more recently, for elliptic problems, may be understood by reducing the problem to a local model of the typical form

$$d^2\Delta u + \lambda d\nabla d \cdot \nabla u + \mu u = f(d, P), \tag{1}$$

where  $d$  is the distance to the singularity locus;  $\lambda$  and  $\mu$  are usually constants.<sup>4</sup> This leads to the Fuchsian estimates mentioned above, which form part of a systematic technique for finding the asymptotic behavior of solutions (see [50, 47, 48] and sections 5 and 6.9).

Note that Fuchsian operators had been studied for their own sake from the 1970s onwards, but the results obtained at this time were slightly weaker than those required for application to nonlinear problems. Of course, the idea of Fuchsian reduction is not to be found in earlier work.

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<sup>3</sup>The question of behavior at infinity is of a similar nature, because infinity may be replaced by an isolated singularity by inversion.

<sup>4</sup>In some cases, it is convenient to allow them to be operators.

## 1.4 Classification of proofs

The various approaches to the estimates differ in their treatment of the model case, and the characterization of Hölder continuity they use. The modern theory is dominated by the fact that interior  $H^k$  estimates for harmonic functions are now considered more or less obvious (see the beginning of the proof of theorem 16).

A first proof is based on the direct estimation of the Newtonian potential [41, 42, 69, 70, 71, 27, 63, 35]. A second proof is based on the search for comparison functions, and therefore uses only the maximum principle [9, 10]. A third proof rests on the dyadic decomposition of the Fourier transform of  $u$  [75]. A variant may be based on a characterization of Hölder continuity by mollification generalizing a property of the Poisson kernel [79]. A fourth proof rests on an integral characterization of Hölder continuity [19, 20, 32, 58, 60]. The regularity problem for minimal surfaces has led to a fifth approach: consider scaled versions of the graph of  $u$  corresponding to smaller and smaller scales and characterize their limit by a Liouville theorem [73]. A sixth proof consists in rescaling  $u - P$  where  $P$  is a second-degree polynomial [17, 15, 39].

## 1.5 Generalizations and variants

The most important cases to which the second-order estimates on bounded smooth domains may be generalized are: higher-order equations of Agmon-Douglis-Nirenberg type, for which it is possible to find a fundamental solution for a model problem with constant coefficients [1] and equations on unbounded domains [22, 65]. Scaling interior estimates yields several, non-equivalent results [22, 47, 34, 78]. The first-order estimates may hold under weaker conditions on the coefficients [24, 25, 16]. It is also possible to obtain estimates in cases when the r.h.s. is only Hölder with respect to some of the variables [30]. Slightly stronger results hold in two dimensions [35, Ch. 12]. A simple example in which the model problem is quite difficult is the case of the Laplace equation on a polyhedral domain.<sup>5</sup>

Higher-order estimates may be obtained in the obvious manner, by differentiating the equation, provided the nonlinearities are smooth. The Schauder estimates are actually true for certain fully nonlinear equations with non-smooth nonlinearities [4, 15, 35].

---

<sup>5</sup>Separation of variables shows that the smoothness of harmonic functions on a wedge-like domain, with Dirichlet conditions, depends on the opening angle of the wedge.

There are cases in which the model case is not a linear, constant-coefficient problem: for instance,

- (i) the  $p$ -Laplacian—also invariant under a similar group—has the property that solutions with right-hand side zero are not necessarily of class  $C^2$  (see [29, 44]);
- (ii) Fuchsian operators also admit non-smooth solutions with smooth data [38, 47, 48];
- (iii) sub-elliptic operators, such as those related to Carnot groups, are not close to the Laplacian either [21];
- (iv) even the Laplacian on polyhedral domains presents new features not found in the regularity theory in smooth domains. All this led to a very recent surge of activity on very simple models. Since the simplest non-trivial model beyond the Laplacian is the Fuchsian case, we briefly explain how such problems arise naturally.

When trying to generalize Schauder estimates to problems with boundary degeneracy, we saw that the local model is not the Laplacian any more: it is a problem with quadratic degeneracy of special form; it is *scale-invariant* but not *translation-invariant*. Let us mention a few further contexts where such PDEs arise: Axisymmetric potential theory leads to problems with singular coefficients such as the (elliptic) Euler-Poisson-Darboux equation

$$u_{rr} + \frac{\lambda}{r}u_r + u_{zz} = -4\pi\rho, \quad (2)$$

where  $\lambda$  is a constant. Many authors, especially Alexander Weinstein (and Hadamard) stressed long ago the usefulness of this equation and noted its remarkable behavior under transformations of the form  $u \mapsto r^\gamma u(r, z)$ . It may be treated within the framework of the general theory of degenerate elliptic PDEs (Fichera), writing it in the form

$$ru_{rr} + \lambda u_r + ru_{zz} = -4\pi r\rho.$$

In this form, the problem is reminiscent of Legendre's equation, which also admits a linear degeneracy (at  $\pm 1$ ). Motivated by the search for a higher-dimensional generalization of the expansion into Legendre functions to several variables, a general theory of the Dirichlet problem for elliptic equations

with linear degeneracy on the boundary was developed in the 1960s and 1970s. The prototype of such problems is

$$d\Delta u + \lambda \nabla d \cdot \nabla u = f(P), \quad \text{in } \Omega \quad (3)$$

where  $d$  is a smooth function of  $P \in \Omega$ , equivalent to the distance to  $\partial\Omega$  near the boundary. An analogue of Schauder estimates may be derived by an explicit computation of Green's function if  $\lambda > 0$ [38]; the essential step is the analysis of a model problem on a half-plane, by Laplace transform in the normal variable. This method does not seem to generalize to the case of quadratic degeneracy.

These considerations took a new meaning when, in the 1990s, one realized that nonlinear problems give rise, by a systematic process of reduction (see [49, 50]), to problems modeled upon the general Fuchsian-type problem

$$d^2 \Delta u + \lambda d \operatorname{grad} d \cdot \operatorname{grad} u + \mu u = f(P), \quad \text{in } \Omega. \quad (4)$$

Because of the quadratic degeneracy, the Laplace transform is not helpful. Nevertheless, it is possible to analyze indirectly this model problem (see [50, 47, 48] and sections 5 and 6.7).

## 1.6 What process were the Schauder estimates discovered by?

Many steps in the derivation of the Schauder estimates become clearer if one recalls the historical development which led from potential theory to the Schauder estimates. For this reason, we give a historical sketch, starting from Poisson (1813).

### 1.6.1 Does Poisson's equation hold?

Consider the Newtonian potential in three dimensions:

$$V(P) = \int_{\mathbb{R}^3} \frac{\rho(Q)}{|P - Q|} dQ \quad (5)$$

where  $P \in \mathbb{R}^3$  and  $|P - Q|$  is the distance from  $P$  to  $Q$  and integrals are extended over  $\mathbb{R}^3$ . This integral represents, up to a constant factor, the gravitational potential generated by the mass density  $\rho(Q)$ , if  $\rho \geq 0$ . If

$\rho$  takes positive and negative values, it may be interpreted in terms of an electrostatic potential. If the density is bounded and has limited support,  $V$  is defined by a convergent integral, and so is the corresponding force field proportional to the gradient of  $-V$ , formally given by

$$-\nabla V(P) = \int \frac{Q - P}{|P - Q|^3} \rho(Q) dQ.$$

If  $P$  lies outside the support of  $\rho$ , the integral may be differentiated again, to yield Laplace's equation

$$\Delta V = 0,$$

where  $\Delta = \sum_{i=1}^3 \partial_i^2$ . However, if  $\rho(P) \neq 0$ , differentiation of the force field yields a divergent integral, because  $1/|P - Q|^3$  is not integrable. Poisson (1813) showed that, if  $\rho$  is constant in the neighborhood of  $P$ ,  $V$  nevertheless satisfies Poisson's equation at the point  $P$ :

$$-\Delta V = 4\pi\rho, \tag{6}$$

Indeed, one may split the density into two parts: a constant density in a ball around  $P$ , and a density which vanishes in a neighborhood of  $P$ . The first part yields a potential which may be computed exactly: it is quadratic near  $P$ ; the second yields a solution of Laplace's equation. Gauss [31] then proved that Poisson's equation is valid if the density is continuously differentiable. After investigations by Riemann, Dirichlet and Clausius, Hölder (1882) [41] proved that the second derivatives of the potential are continuous, and that Poisson's equation holds, under the *Hölder condition of order  $\alpha$*

$$|\rho(P) - \rho(Q)| \leq C|P - Q|^\alpha, \tag{7}$$

for some  $\alpha \in (0, 1)$ . In fact, the second derivatives of  $V$  also satisfy a Hölder condition. Furthermore, if  $\rho$  is merely continuous, the first-order derivatives of  $V$  satisfy a Hölder condition for any  $\alpha$ . The argument was streamlined by Neumann [62].<sup>6</sup> This substantiates Poisson's idea that the potential should be well-approximated by a quadratic function near every point where  $\rho$  is well-approximated by a constant.

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<sup>6</sup>The continuity of  $\rho$  is not sufficient to ensure that  $V$  is twice continuously differentiable. Necessary and sufficient conditions for the existence of second derivatives were studied by Petri.

### 1.6.2 Emergence of the Dirichlet problem

At the same time it became clear that the Newtonian potential is merely one among all possible solutions of Poisson's equation; in fact, solutions may be parameterized by their values on the boundary of sufficiently smooth bounded domains  $\Omega \subset \mathbb{R}^3$ : this leads us to *Dirichlet problem*

$$\begin{cases} -\Delta V = 4\pi\rho & \text{in } \Omega \\ V = g & \text{on } \partial\Omega \end{cases} \quad (8)$$

It seemed at first sight that the Dirichlet problem should have a unique solution on the grounds that it should represent the equilibrium potential in  $\Omega$  when the potential is prescribed on the boundary and continuous. Dirichlet and Riemann worked on the assumption that  $V$  could be obtained by minimizing the *Dirichlet integral*

$$E[u, \Omega] = \int_{\Omega} |\nabla u(Q)|^2 dQ \quad (9)$$

among all sufficiently regular  $u$  which agree with  $g$  on  $\partial\Omega$ . Weierstrass pointed out that such an argument may fail for certain variational principles, and it was only with the advent of Hilbert spaces that a justification of this method could be made, for smooth domains. But then, if we find a function  $V$  which admits integrable first-order derivatives and minimizes Dirichlet's integral, how do we know that it has second-order derivatives and that it solves Poisson's equation? There is a second difficulty: the Dirichlet problem may have no continuous solution if the boundary presents a sharp inward spike ("Lebesgue spine"). Even for  $\rho = 0$ , the Poincaré *balayage* method, re-formulated and simplified by Perron into the *method of sub- and supersolutions*, only proves that, for continuous  $g$ , there is a unique solution which is continuous up to the boundary if  $\partial\Omega$  is well-behaved<sup>7</sup> but does not prove that the solution is smooth if the data ( $\Omega$ ,  $\rho$  and  $g$ ) are smooth.

The corresponding issues for equations with variable coefficients and nonlinearities also led to the need for regularity estimates: the Calculus of Variations and Conformal Mapping led to nonlinear elliptic equations such as the equation of minimal surfaces and Liouville's equation ( $\Delta u = e^u$ ) in two variables. Picard emphasized the advantages of iterative methods for PDEs.

---

<sup>7</sup>For instance, it is sufficient that  $\partial\Omega$  satisfy an exterior sphere condition. A necessary and sufficient condition is due to Wiener.

Now, if one wishes to solve iteratively an equation of the form

$$\Delta u = f(u)$$

to fix ideas, one should define a sequence of functions obtained by solving the Poisson equations

$$\Delta u_n = f(u_{n-1})$$

with  $n = 1, 2, \dots$ . In view of the above results, it seems appropriate to work in a space of functions the second derivatives of which satisfy a Hölder condition. The first results in this direction seem to be due to Bernstein (see [12]). The *continuity method* may be viewed as a outgrowth of these efforts. But the iterative approach only allows one to reach problems close to Poisson's equation. Other approaches, based on the reduction to an integral equation on the boundary, required detailed estimates on the Green's function for operators with variable coefficients. In the course of this development, estimates for second derivatives of solutions of PDEs with variable coefficients in  $n$  variables were obtained (Korn, E. Hopf, Giraud, Kellogg, Schauder, . . . , see [14, 36, 37, 43, 55, 42]).

Schauder's approach is different: it reduces the problem to a new fixed-point theorem: the Leray-Schauder theorem for compact operators; the compactness is provided by estimates of second derivatives in Hölder spaces. Schauder's proof bypasses the construction of Green's function for variable-coefficient operators, and opens the door to the solution of wide classes of nonlinear equations.

## 1.7 Outline of the paper

Section 2 collects several characterizations of Hölder spaces, and gives the main interpolation results which enable the passage from constant to variable coefficients.

Section 3 illustrates the main proof techniques on the case of the interior estimates for the Laplacian.

Section 4 deals with the passage from the model case (Laplacian on a ball) to variable coefficients and general domains.

Section 5 gives the main general-purpose Fuchsian estimates.

Section 6 collects the most important applications; self-contained proofs of the major topological tools are also included.

## 2 Hölder spaces

### 2.1 First definitions

Let  $\Omega \subset \mathbb{R}^n$  be a domain (*i.e.*, an open and connected set).

**DEFINITION 1** *A function  $u$  is Hölder-continuous at the point  $P$  of  $\Omega$ , with exponent  $\alpha \in (0, 1)$ , if*

$$[u]_{\alpha, \Omega, P} := \sup_{Q \in \Omega, Q \neq P} \frac{|u(P) - u(Q)|}{|P - Q|^\alpha} < \infty.$$

*It is Hölder-continuous over  $\Omega$ , or of class  $C^\alpha(\Omega)$  if it satisfies this condition for every  $P \in \Omega$ . We write  $[u]_{\alpha, \Omega} := \sup_P [u]_{\alpha, \Omega, P}$ .*

*It is of class  $C^\alpha(\Omega)$  if*

$$\|u\|_{C^\alpha(\Omega)} := \sup_{\Omega} |u| + [u]_{\alpha, \Omega}.$$

Functions of class  $C^\alpha$  are in particular uniformly continuous. If  $\partial\Omega$  is smooth, one can extend  $u$  by continuity to a continuous function on  $\overline{\Omega}$ ; for this reason, it is sometimes convenient to write  $C^\alpha(\overline{\Omega})$  for  $C^\alpha(\Omega)$  in this case, to emphasize that  $u$  is continuous up to the boundary.

It is easy to check that

$$[uv]_{\alpha, \Omega} \leq \|u\|_{C^\alpha(\Omega)} \|v\|_{C^\alpha(\Omega)}.$$

Higher-order Hölder spaces  $C^{k+\alpha}(\Omega)$  are defined in the natural way: first, write  $|\nabla^k u|$  for the sum of the absolute values of the derivatives of  $u$  of order  $k$ , and define  $[\nabla^k u]_{\alpha, \Omega}$  similarly. Let

$$\|u\|_{C^k(\Omega)} := \max_{0 \leq j \leq k} \sup_{\Omega} |\nabla^j u|,$$

and

$$\|u\|_{C^{k+\alpha}(\Omega)} := \|u\|_{C^k(\Omega)} + [\nabla^k u]_{\alpha, \Omega}.$$

In all these norms, the reference domain  $\Omega$  will be omitted whenever it is clear from the context.

## 2.2 Dyadic decomposition

The Hölder spaces defined above are all Banach spaces, but smooth functions are not dense in them: even in one dimension, if  $(f_m)$  is a sequence of smooth functions and  $f \in C^\alpha(\mathbb{R})$  is such that  $\|f - f_m\|_{C^\alpha(\mathbb{R})} \rightarrow 0$ , one proves easily that for any  $P$  and any  $\varepsilon > 0$ , there is a neighborhood of  $P$  on which  $|f(P) - f(Q)| \leq \varepsilon|P - Q|^\alpha$ . In other words,  $\lim_{Q \rightarrow P} |f(P) - f(Q)||P - Q|^{-\alpha} = 0$ . Any function  $f$  which does not satisfy this property cannot be approximated by smooth functions in the  $C^\alpha$  norm.

Nevertheless, there is a systematic way to decompose Hölder-continuous functions on  $\mathbb{R}^n$  into a uniformly convergent sum of smooth functions: define the Fourier transform of  $u$  by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx$$

and consider  $\varphi \in C_0^\infty(\mathbb{R})$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi = 1$  for  $|x| \leq 1$ ,  $\varphi = 0$  for  $|x| \geq 0$ . Define

$$\hat{u}_0 = \varphi(|\xi|)\hat{u}(\xi); \quad \hat{u}_j = [\varphi(2^{-j}|\xi|) - \varphi(2^{-(j-1)}|\xi|)]\hat{u}(\xi) \text{ for } j \geq 1.$$

We let  $\hat{v}_j = \hat{u}_0 + \cdots + \hat{u}_j$ .

DEFINITION 2 *The decomposition*

$$u = \sum_{j \geq 0} u_j$$

is the Littlewood-Paley (LP), or dyadic decomposition of  $u$  [75].

By Fourier inversion, we have

$$u_j = \psi_j * u \text{ with } \psi_j(x) = 2^{jn}\psi(2^jx),$$

where  $\psi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} [\varphi(|\xi|/2) - \varphi(|\xi|)] \exp(ix \cdot \xi) d\xi$ . Note that  $\hat{\psi}$  vanishes near the origin; in particular,  $\hat{\psi}_j(0) = \int_{\mathbb{R}^n} \psi_j dx = 0$ .

THEOREM 3 *Let  $0 < \alpha < 1$ .*

1. (Bernstein's inequality.) *There is a constant  $C$  such that, for any  $k$ ,*  

$$\sup_x (|\nabla^k u_j| + |\nabla^k v_j|) \leq C2^{jk} \sup_x |u(x)|.$$

2. If  $u \in C^\alpha(\mathbb{R}^n)$ , there is a constant  $C$  independent of  $j$  such that  $\sup_x |u_j(x)| \leq C2^{-j\alpha} \|u\|_{C^\alpha}$ .
3. Conversely, if the above inequality holds for every  $j \geq 1$ , then  $u \in C^\alpha(\mathbb{R}^n)$ .

PROOF. (1) On the one hand, we have  $|u_j(x)| \leq \|\psi\|_{L^1} \sup |u|$  and  $|v_j(x)| \leq \|\phi\|_{L^1} \sup |u|$ . On the other hand, if  $a$  is a multi-index of length  $k$ ,

$$\begin{aligned} |\nabla^a u_j(x)| &= \left| \int u(y) 2^{jk} \nabla^a \psi[2^j(x-y)] 2^{jn} dy \right| \\ &= C2^{jk} \sup |u|. \end{aligned}$$

The result follows.

- (2) Since  $\int \psi(y) dy = 0$ ,  $u_j$  may be written, for  $j \geq 1$ ,

$$u_j(x) = \int [u(x-y) - u(x)] 2^{jn} \psi(2^j y) dy = \int [u(x-z/2^j) - u(x)] \psi(z) dz.$$

If  $u \in C^\alpha$ , it follows that

$$|u_j(x)| \leq 2^{-j\alpha} [u]_\alpha \int |z|^\alpha |\psi(z)| dz,$$

QED.

- (3) Conversely, if the  $u_j$  are of order  $2^{-j\alpha}$ , the series  $u_0 + u_1 + \dots$  converges uniformly. Call its sum  $u$ ; it is readily seen that the  $u_j$  do give its LP decomposition. We may apply (1) to  $u_{j-1} + u_j + u_{j+1}$ , and obtain

$$\sup_x |\nabla u_j(x)| \leq C2^{j(1-\alpha)}.$$

Writing  $u = v_{j-1} + w_j$ , where  $w_j = u_j + u_{j+1} + \dots$ , we find that

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{j>k} |x-y| \sup |\nabla u_j| + 2 \sup |w_j| \\ &\leq C|x-y|(1 + \dots + 2^{(j-1)(1-\alpha)}) + C2^{-j\alpha} \\ &\leq C[2^{-j\alpha} + |x-y|2^{j(1-\alpha)}]. \end{aligned}$$

Choose  $j$  such that  $2^{-j} \leq |x-y| \leq 2^{-(j-1)}$ . A bound on  $[u]_\alpha$  follows.  $\square$

## 2.3 Weighted norms

Several of the results we shall prove estimate the Hölder norm of a function  $u$  on a ball of radius  $R$  in terms of bounds on the ball of radius  $2R$  with the same center. In order to exploit these inequalities in a systematic fashion, it is useful to define Hölder norms weighted by the distance to the boundary.

Let  $\Omega \neq \mathbb{R}^n$  and let  $d(P)$  denote the distance from  $P$  to  $\partial\Omega$ , and

$$d_{P,Q} = \min(d(P), d(Q)).$$

Let also  $\delta$  be a smooth function in all of  $\Omega$  which is equivalent to  $d$  for  $d$  sufficiently small.<sup>8</sup> Define, for  $k = 0, 1, \dots$ ,

$$\|u\|_{k,\Omega}^{\#} = \sum_{j=0}^k \sup_{\Omega} d^j |\nabla^j u|,$$

and

$$\|u\|_{k+\alpha,\Omega}^{\#} = \sum_{j=0}^k \|\delta^j u\|_{C^{j+\alpha}(\Omega)},$$

The spaces corresponding to these norms are called  $C_{\#}^k(\Omega)$ ,  $C_{\#}^{k+\alpha}(\Omega)$ . The space  $C_{*}^{k+\alpha}(\Omega)$  has the norm

$$\|u\|_{k+\alpha,\Omega}^{*} = \|u\|_{k,\Omega}^{*} + [u]_{k+\alpha,\Omega},$$

where

$$\|u\|_{k,\Omega}^{*} = \sum_{j=0}^k [u]_{j,\Omega}^{*},$$

with  $[u]_{k,\Omega}^{*} = \sup_{\Omega} d^k |\nabla^k u|$ , and

$$[u]_{k+\alpha,\Omega}^{*} = \sup_{P,Q \in \Omega} d_{P,Q}^{k+\alpha} \frac{|\nabla^k u(P) - \nabla^k u(Q)|}{|P - Q|^{\alpha}}.$$

We also need the further definitions:

$$[u]_{\alpha,\Omega}^{(\sigma)} = \sup_{P,Q \in \Omega} d_{P,Q}^{\alpha+\sigma} \frac{|u(P) - u(Q)|}{|P - Q|^{\alpha}}; \quad \|u\|_{\alpha,\Omega}^{(\sigma)} = \sup_{\Omega} |d^{\sigma} u| + [u]_{\alpha,\Omega}^{(\sigma)}.$$

As before, the mention of  $\Omega$  will be omitted whenever possible.

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<sup>8</sup>Such a function is easy to construct if  $\Omega$  is bounded and smooth. Note that even in this case,  $d$  is smooth only near the boundary; see section 2.5.

## 2.4 Interpolation inequalities

THEOREM 4 For any  $\varepsilon > 0$ , there is a constant  $C_\varepsilon$  such that

$$\begin{aligned} [u]_1^* &\leq \varepsilon [u]_2^* + C_\varepsilon \sup |u| \\ [u]_1^* &\leq \varepsilon [u]_{1+\alpha}^* + C_\varepsilon \sup |u| \\ [u]_2^* &\leq \varepsilon [u]_{2+\alpha}^* + C_\varepsilon [u]_1^* \\ [u]_{1+\alpha}^* &\leq \varepsilon [u]_2^* + C_\varepsilon \sup |u|. \end{aligned}$$

PROOF. Recall the elementary inequality, for  $C^2$  functions of one variable  $t \in [a, b]$ ,<sup>9</sup>

$$\sup |f'| \leq \frac{2}{b-a} \sup |f| + (b-a) \sup |f''|.$$

Fix  $\theta \in (0, 1/2)$ , and  $P \in \Omega$ . Let  $r = \theta d(P)$ . If  $Q \in B_r(P)$ , and  $Z \in \partial\Omega$ , we have

$$|Z - Q| \geq |Z - P| - |P - Q| \geq d(P)(1 - \theta) \geq \frac{1}{2}d(P) \geq r \geq |P - Q|.$$

It follows in particular that  $d(Q) \geq d(P)(1 - \theta) \geq \frac{1}{2}d(P)$ , hence

$$d_{P,Q} \geq \frac{1}{2}d(P).$$

Applying the elementary inequality to  $u$  restricted to the segment  $[P, P + re_i]$ ,<sup>10</sup> where  $e_i$  is the  $i$ -th basis vector, we find

$$|\partial_i u(P)| \leq \frac{2}{r} \sup_{B_r} |u| + r \sup_{B_r} |\partial_{ii} u|.$$

It follows that

$$\sup_{B_r} |\partial_{ii} u| \leq \sup d(Q)^{-2} \sup d(Q)^2 |\partial_{ii} u| \leq \frac{[u]_2^*}{d(P)^2(1 - \theta)^2}.$$

Therefore,

$$[u]_1^* = \sup_{B_r} |d(Q) \partial_i u(Q)| \leq \frac{2}{\theta} \sup |u| + \frac{\theta}{(1 - \theta)^2} [u]_2^*.$$

<sup>9</sup>For the proof, write  $f'(t) = f'(s) + \int_s^t f''(\tau) d\tau$ , where  $s$  satisfies  $f'(s) = (f(b) - f(a))/(b - a)$ .

<sup>10</sup>By the choice of  $r$ , this segment lies entirely within  $\Omega$ .

If we choose  $\theta$  so that  $\theta(1 - \theta)^{-2} \leq \varepsilon$ , we arrive at the first of the desired inequalities.

For the second inequality, we note that, using again the mean-value theorem, there is on the segment  $[P, P + re_i]$  some  $\tilde{P}$  such that  $|\partial_i u(\tilde{P})| \leq (2/r) \sup_{B_r} |u|$ . It follows that

$$\begin{aligned} |\partial_i u(P)| &\leq |\partial_i u(\tilde{P})| + |\partial_i u(P) - \partial_i u(\tilde{P})| \\ &\leq \frac{2}{r} \sup_{\Omega} |u| \\ &\quad + \left( \sup_{Q \in B_r(P)} d_{P,Q}^{-1-\alpha} \right) |P - \tilde{P}|^\alpha \sup_{Q \in B_r(P)} d_{P,Q}^{1+\alpha} \frac{|\nabla u(P) - \nabla u(Q)|}{|P - Q|^\alpha} \\ &\leq \frac{2}{r} \sup_{\Omega} |u| + (2/d(P))^{1+\alpha} (\theta d(P))^\alpha [u]_{1+\alpha}^*. \end{aligned}$$

Multiplying through by  $d(P) = r/\theta$ , we find the second inequality.

A similar argument gives the third and fourth inequalities.  $\square$

## 2.5 Properties of the distance function

We prove a few properties of the function  $d(x) = \text{dist}(x, \partial\Omega)$ , when  $\Omega$  is bounded with boundary of class  $C^{2+\alpha}$ . Without smoothness assumption on the boundary, all we can say is that  $d$  is Lipschitz; indeed, since the boundary is compact, there is, for every  $x$  a  $z \in \partial\Omega$  such that  $d(x) = |x - z|$ . If  $y$  is any other point in  $\Omega$ , we have  $d(y) \leq |y - z| \leq |y - x| + |x - z| = |y - x| + d(x)$ . It follows that  $|d(x) - d(y)| \leq |x - y|$ . For more regular  $\partial\Omega$ , we have the following results:

**THEOREM 5** *If  $\partial\Omega$  is bounded of class  $C^{2+\alpha}$ ,*

1. *there is a  $\delta > 0$  such that every point such that  $d(x) < \delta$  has a unique nearest point on the boundary;*
2. *in this domain,  $d$  is of class  $C^{2+\alpha}$ ; furthermore,  $|\nabla d| = 1$ , and*

$$-\Delta d = \sum_j \frac{\kappa_j}{1 - \kappa_j d},$$

*where  $\kappa_1, \dots, \kappa_{n-1}$  are the principal curvatures of  $\partial\Omega$ . In particular,  $-\Delta d/(n - 1)$  is equal to the mean curvature of the boundary.*

PROOF. We work near the origin, which we may take on  $\partial\Omega$ . Our proofs will give local information near the origin, which can be made global by a standard compactness argument.

Choose the coordinate axes so that  $\Omega$  is locally represented  $\{x_n > h(x')\}$ , where  $x' = (x_1, \dots, x_n)$  and  $h$  is of class  $C^{2+\alpha}$  with  $h(0) = 0$  and  $\nabla h(0) = 0$ . We may also assume that the axes are rotated so that the Hessian  $(\partial_{ij}h(0))$  is diagonal. Its eigenvalues are, by definition, the principal curvatures  $\kappa_1, \dots, \kappa_{n-1}$  of the boundary. Their average is, again by definition, the mean curvature of the boundary.

At any boundary point, the vector with components

$$(\nu_i) = (-\partial_1 h, \dots, -\partial_{n-1} h, 1) / \sqrt{1 + |\nabla h|^2}$$

is the *inward* normal to  $\partial\Omega$  at that point. One checks  $\partial_j \nu_i(0) = -\partial_{ij} h(0) = \kappa_j \delta_{ij}$  for  $i$  and  $j$  less than  $n$ . Thus,  $\nu$  is of class  $C^1$ . For any  $T > 0$  and  $y \in \mathbb{R}^{n-1}$ , both small, consider the point  $x(Y, T) = (Y, h(Y)) + T\nu(Y)$ ; this represents the point obtained by traveling the distance  $T$  into  $\Omega$ , starting from the boundary point  $(Y, h(Y))$ , and traveling along the normal. We write

$$\Phi : (Y, T) \mapsto x(Y, T).$$

We want to prove that all points in a neighborhood of the boundary are obtained by this process, in a unique manner: in other words,  $(Y, h(Y))$  is the unique closest point from  $x(Y, T)$  on the boundary, provided that  $T$  is positive and small. It suffices to argue for  $Y = 0$ ; in that case, since  $h$  is  $C^2$ , it is bounded below by an expression of the form  $a|Y|^2$ , which implies that for  $T$  sufficiently small, the sphere of radius  $T$  about  $x(Y, T)$  contains no point of the boundary except the origin.<sup>11</sup> We may now consider the new coordinate system  $(Y, T)$  thus defined. We compute, for  $Y = 0$ , but  $T$  not necessarily zero,

$$\frac{\partial x_i}{\partial Y_j} = \delta_{ij}(1 - \kappa_j T)$$

for  $i$  and  $j < n$ , while

$$\frac{\partial x_n}{\partial Y_j} = \frac{\partial x_i}{\partial T} = 0; \quad \frac{\partial x_n}{\partial T} = 1.$$

---

<sup>11</sup>Indeed, the equation of this sphere is  $x_n = T - \sqrt{T^2 - |Y|^2}$ , which, by inspection, is bounded below by  $a|Y|^2$  for  $2aT < 1$ .

The inverse function theorem shows that, near the origin, the map  $\Phi$  and its inverse are of class  $C^1$ , and that the Jacobian of  $\Phi^{-1}$  is, for  $Y = 0$ ,

$$\frac{\partial(Y, T)}{\partial x} = \text{diag}\left(\frac{1}{1 - \kappa_1 T}, \dots, \frac{1}{1 - \kappa_{n-1} T}, 1\right).$$

In fact,  $\Phi^{-1}$  is of class  $C^{1+\alpha}$ . Indeed,  $\Phi$  has this regularity, and the differential of  $\Phi^{-1}$  is given by  $[\Phi' \circ \Phi^{-1}]^{-1}$ , and the map  $A \mapsto A^{-1}$  on invertible matrices is a smooth map. Since  $\nu(Y)$ , which is equal to the gradient of  $d$ , is a  $C^{1+\alpha}$  function of  $Y$ , we see that it is also a  $C^{1+\alpha}$  function of the  $x$  coordinates. It follows that  $d$  is of class  $C^{2+\alpha}$ . The computation of the second derivatives of  $d$  is now a consequence of the computation of the first-order derivatives of  $\nu$ .

It follows from this discussion that  $T = d$  near the boundary, and that  $|\nabla d| = 1$ ; in fact  $\nabla d = \nu$ .  $\square$

## 2.6 Integral characterization of Hölder continuity

Let  $\Omega$  be a bounded domain. Write  $\Omega(x, r)$  for  $\Omega \cap B(x, r)$ . We assume that the measure of  $\Omega(x, r)$  is at least  $Ar^n$  for some positive constant  $A$ , if  $x \in \Omega$  and  $r \leq 1$ . This condition is easily verified if  $\Omega$  has a smooth boundary. Define the average of  $u$ :

$$u_{x,r} = |\Omega(x, r)|^{-1} \int_{\Omega(x,r)} u \, dx.$$

**THEOREM 6** *The space  $C^\alpha(\Omega)$  coincides with the space of (classes of) measurable functions which satisfy*

$$\int_{\Omega(x,r)} |u(y) - u_{x,r}|^2 \, dy \leq Cr^{n+2\alpha}$$

for  $0 < r < \text{diam } \Omega$ . The smallest constant  $C$ , denoted by  $\|u\|_{\mathcal{L}^{2,n+2\alpha}}$  is equivalent to the  $C^\alpha(\Omega)$  norm.

*Remark 1* If one defines  $\mathcal{L}^{p,\lambda}$  by the property:  $\int_{\Omega(x,r)} |u(y) - u_{x,r}|^p \, dy \leq Cr^\lambda$ , with  $n < \lambda < n + p$ , one obtains a characterization of the space  $C^{(\lambda-n)/p}$ .

**PROOF.** The integral estimate is clearly true for Hölder continuous functions. Let us therefore focus on the converse. We first prove that  $u$  is uniformly approximated by its averages, and then derive a modulus of continuity for  $u$ .

If  $x_0 \in \Omega$  and  $0 < \rho < r \leq 1$ , we have

$$\begin{aligned} A\rho^n |u_{x_0,\rho} - u_{x_0,r}|^2 &\leq \int_{\Omega(x_0,\rho)} |u_{x_0,\rho} - u_{x_0,r}|^2 dx \\ &\leq 2 \left( \int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^2 dx + \int_{\Omega(x_0,r)} |u - u_{x_0,r}|^2 dx \right) \\ &\leq C(r^\lambda + \rho^\lambda). \end{aligned}$$

Letting  $r_j = r2^{-j}$  and  $u_j = u_{x_0,\rho_j}$  for  $j \geq 0$ , we find

$$|u_{j+1} - u_j| \leq C2^{j(n-\lambda)/2} r^{(\lambda-n)/2} = C2^{-j\alpha} r^\alpha.$$

For almost every  $x_0$ , the Lebesgue differentiation theorem ensures that  $u_j \rightarrow u(x_0)$  as  $j \rightarrow \infty$ . It follows that

$$|u(x_0) - u_{x_0,r}| \leq \sum_j |u_{j+1} - u_j| \leq Cr^\alpha.$$

Since  $u_{x,r}$  is continuous in  $x$  and converges uniformly as  $r \rightarrow 0$ , it follows that  $u$  may be identified, after modification on a null set, with a continuous function.

To estimate its modulus of continuity, we need the following result:

LEMMA 7 *Let  $u \in \mathcal{L}^{2,n+2\alpha}$ ,  $x, y$  two points in  $\Omega$ , and  $r = |x - y|$ ; we have*

$$|u_{x,r} - u_{y,r}| \leq Cr^\alpha.$$

PROOF. We may assume  $r = |x - y| \leq 1$ . If  $z \in B_r(x)$ , we have  $|z - y| \leq r + |x - y| \leq 2r$ . Therefore  $\Omega(y, 2r) \supset \Omega(x, r)$ . It follows that  $\Omega(x, 2r) \cap \Omega(y, 2r) \supset \Omega(x, r)$  has measure  $Ar^n$  at least. We therefore have

$$\begin{aligned} &|\Omega(x, 2r) \cap \Omega(y, 2r)| |u_{x,2r} - u_{y,2r}| \\ &\leq \int_{\Omega(x,2r)} |u(z) - u_{x,2r}| dz + \int_{\Omega(y,2r)} |u(z) - u_{y,2r}| dz \\ &\leq \left[ \int_{\Omega(x,2r)} |u(z) - u_{x,2r}|^2 dz \right]^{1/2} |\Omega(x, 2r)|^{1/2} \\ &\quad + \left[ \int_{\Omega(y,2r)} |u(z) - u_{y,2r}|^2 dz \right]^{1/2} |\Omega(y, 2r)|^{1/2} \\ &\leq Cr^{\alpha+n/2} r^{n/2}. \end{aligned}$$

It follows that

$$|u_{x,2r} - u_{y,2r}| \leq CA^{-1}r^\alpha,$$

□

To conclude the proof of the theorem, it suffices to estimate  $|u(x) - u(y)|$  by  $|u(x) - u_{x,r}| + |u_{x,r} - u_{y,r}| + |u_{y,r} - u(y)| \leq 2Cr^\alpha + |u_{x,r} - u_{y,r}|$ . □

### 3 Interior estimates for the Laplacian

#### 3.1 Direct arguments from potential theory

Let  $n \geq 2$ , and let  $B_R(P)$  denote the open ball of radius  $R$  about  $P$ . Mention of the point  $P$  is omitted whenever this does not create confusion. The volume of  $B_R$  is  $\omega_n R^n$  and its surface  $n\omega_n R^{n-1}$ . The Newtonian potential in  $n$  dimensions is

$$g(P, Q) = \frac{|P - Q|^{2-n}}{(2-n)n\omega_n} \text{ for } n \geq 3$$

and

$$\frac{1}{2\pi} \ln |P - Q| \text{ for } n = 2.$$

It is helpful to note that

1. The derivatives of  $g$  of order  $k \geq 1$  w.r.t.  $P$  are  $O(|P - Q|^{2-n-k})$ .
2. The average of each of these second derivatives over the sphere  $\{Q : |P - Q| = \text{const.}\}$  vanishes.<sup>12</sup>

Next, consider, for  $f \in L^1 \cap L^\infty(\mathbb{R}^n)$ , the integral

$$u(P) = \int_{\mathbb{R}^n} g(P, Q)f(Q) dQ.$$

We wish to estimate  $u$  and its derivatives in terms of bounds on  $f$ . Because of the behavior of  $g$  as  $P \rightarrow Q$ ,  $g$  and its first derivatives are locally integrable, but its second derivative is not.

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<sup>12</sup>To check this, it is useful to note that the average of  $x_i^2/r^2$  over the unit sphere  $\{r = 1\}$  is equal to  $\frac{1}{n}$ , and similarly, using symmetry, the average of  $(x_i - y_i)(x_j - y_j)/|x - y|^2$  over the set  $\{|x - y| = \text{const.}\}$  vanishes for  $i \neq j$ .

It is easy to see that, if the point  $P$  lies outside the support of  $f$ ,  $u$  is smooth near  $P$  and satisfies  $\Delta u = 0$ . For this reason, it suffices to study the case in which the density  $f$  is supported in a neighborhood of  $P$ .

We prove in the next three theorems: (i) a pointwise bound on  $u$  and its first-order derivatives; (ii) a representation of the second-order derivatives which involves only locally integrable functions; (iii) a direct estimation of  $\nabla^2 u(P) - \nabla^2 u(Q)$  using this representation.

**THEOREM 8** *If  $f$  vanishes outside  $B_R(0)$ , we have*

$$\sup_{B_R} (|u| + |\nabla u|) \leq CR^2 \sup |f|,$$

and  $\nabla u$  is given by formally differentiating the integral defining  $u$ .

**PROOF.** Consider a cut-off function  $\varphi_\varepsilon(P, Q) := \varphi(|P - Q|/\varepsilon)$ , where  $\varphi(t)$  is smooth, takes its values between 0 and 1, vanishes for  $t \leq 1$  and equals 1 for  $t \geq 2$ . Considering the functions

$$u_\varepsilon(P) = \int g(P, Q) \varphi_\varepsilon(P, Q) f(Q) dQ,$$

which are smooth, it is easy to see that the  $\partial_i u_\varepsilon$  converge uniformly, as  $\varepsilon \downarrow 0$ , to  $\int \partial_i g(P, Q) f(Q) dQ$ . Similarly,  $u_\varepsilon$  converges to  $u$ . Therefore,  $u$  is continuously differentiable. Using the growth properties of  $g$  and its derivatives, we may estimate  $\partial_i u(P)$  by

$$C \int_{B_{2R}(P)} C|P - Q|^{1-n} \sup |f| dQ,$$

because  $B_R(0) \subset B_{2R}(P)$ . Taking polar coordinates centered at  $P$ , the result follows.  $\square$

The case of second derivatives is more delicate, since the second derivatives of  $g$  are not locally integrable. We know since Poisson that the integral defining  $u$  is smooth near  $P$  if  $f$  is constant in a neighborhood of  $P$ . This suggests a reduction to the case in which  $f$  vanishes at  $P$ . We therefore first prove, for such  $f$ , a representation of the second-order derivatives which circumvents the fact that the second-order derivatives of  $g$  are not integrable.

**THEOREM 9** *If  $f$  has support in a bounded neighborhood  $\Omega$  of the origin, with smooth boundary, and if  $f \in C^\alpha(\mathbb{R}^n)$  for some  $\alpha \in (0, 1)$ , then all second-order derivatives of  $u$  exist, and are equal to*

$$w_{ij} := \int_{\Omega} \partial_{ij}g(P, Q)[f(Q) - f(P)]dQ - f(P) \int_{\partial\Omega} \partial_i g n_j ds(Q),$$

where derivatives of  $g$  are taken with respect to its first argument, and  $n_j$  are the components of the outward normal to  $\partial\Omega$ .

**PROOF.** To establish the existence of second derivatives, we consider

$$v_{i\varepsilon}(P) = \int \partial_i g(P, Q) \varphi_\varepsilon(P, Q) f(Q) dQ,$$

which converges pointwise to  $\partial_i u(P)$ ; in fact, since  $1 - \varphi_\varepsilon$  is supported by a ball of radius  $2\varepsilon$ , a direct computation yields  $|u_i - v_{i\varepsilon}|(P) = O(\varepsilon \sup |f|)$ . Now, writing  $P = (x_i)$  and  $Q = (y_i)$ , we have

$$\begin{aligned} \partial_j v_{i\varepsilon}(P) &= \int_{\Omega} \partial_{x_j}(\varphi_\varepsilon \partial_{x_i} g)(P, Q) [f(Q) - f(P)] dQ \\ &\quad + f(P) \int_{\Omega} (\varphi_\varepsilon \partial_{x_i} g)(P, Q) dQ. \end{aligned}$$

Now, since  $\varphi_\varepsilon$  and  $g$  only depend on  $|P - Q|$ , we may replace  $\partial/\partial x_j$  by  $-\partial/\partial y_j$  and integrate by parts. This yields

$$\begin{aligned} \partial_j v_{i\varepsilon}(P) &= \int_{\Omega} \partial_{x_j}(\varphi_\varepsilon \partial_{x_i} g)(P, Q) [f(Q) - f(P)] dQ \\ &\quad - f(P) \int_{\partial\Omega} \varphi_\varepsilon \partial_{x_i} g(P, Q) n_j(Q) ds(Q). \end{aligned}$$

We may now estimate the difference  $\partial_j v_{i\varepsilon} - w_{ij}$  using the same method as for the first-order derivatives. It follows that  $\partial_{ij} u = w_{ij}$ .  $\square$

We now give the main estimate for second-order derivatives.

**THEOREM 10** *Let*

$$u(P) = \int_{B_{2R}(0)} g(P, Q) f(Q) dQ,$$

where  $f \in C^\alpha(B_{2R})$ , with  $0 < \alpha < 1$ . Then

$$\sup_{B_R} |\nabla^2 u| + [\nabla^2 u]_{\alpha, B_R} \leq C(\sup_{B_{2R}} |f| + R^\alpha [f]_{\alpha, B_{2R}}). \quad (10)$$

PROOF. We wish to estimate the regularity of  $\partial_{ij}u$ ; we therefore study  $|\partial_{ij}u(P) - \partial_{ij}u(P')|$ , for  $P, P'$  in  $B_R(0)$ , where the second derivatives are given by the expressions in the previous theorem. The main step is to decompose the first integrand in the resulting expression for  $w_{ij}(P) - w_{ij}(P')$  into

$$[f(Q) - f(P')][\partial_{ij}g(P, Q) - \partial_{ij}g(P', Q)] + [f(P') - f(P)]\partial_{ij}g(P, Q).$$

We therefore need to estimate the following quantities

- (I)  $f(P)[\partial_i g(P, Q) - \partial_i g(P', Q)]$  for  $Q \in \partial B_{2R}$ .
- (II)  $[f(P) - f(P')]\partial_i g(P', Q)$  for  $Q \in \partial B_{2R}$ .
- (III)  $[f(P') - f(P)]\partial_{ij}g(P, Q)$  for  $Q \in B_{2R}$ .
- (IV)  $[f(Q) - f(P')][\partial_{ij}g(P, Q) - \partial_{ij}g(P', Q)]$  for  $Q \in B_{2R}$ .

The first boundary term (I) is easy to estimate using the mean-value theorem:

$$|\partial_i g(P, Q) - \partial_i g(P', Q)| \leq |P - P'| \sup_{\xi \in [P, P']} |\nabla \partial_i g(\xi, Q)|.$$

Now, since  $Q \in \partial B_{2R}$ , and  $\xi \in B_R$ , we have  $|\xi - Q| \geq 2R - R = R$ , hence the supremum in the above formula is bounded by a multiple of  $R^{-n}$ . Integrating, we get a contribution  $O(|P - P'|/R)$ , which is *a fortiori*  $O(|P - P'|^\alpha/R^\alpha)$ .

Expression (II) is  $O(|P - P'|^\alpha)$  since  $f$  is of class  $C^\alpha$ .

To estimate (III) and (IV), let  $r_0 = |P - P'|$  and  $M$  be the midpoint of  $[P, P']$ . We distinguish two cases: (i) When  $|Q - M| > r_0$ , the distance from  $Q$  to any point on the segment  $[P, P']$  is comparable to  $|Q - M|$ ; this will enable a direct estimation of (IV) using the mean-value theorem, and of (III) by integration by parts. (ii) On the set on which  $|Q - M| \leq r_0$ , we may directly estimate the sum of (III) and (IV); the smallness of the region of integration compensates the singularity of the derivatives of  $g$ .

We begin with the first case: consider first the integral of (III) over the set

$$A := \{Q \in B_{2R} : |Q - M| > r_0\}.$$

Its boundary is included in  $\partial B_{2R}(0) \cup \partial B_{r_0}(M)$ . Integrating by parts and using the fact that, on this set,  $|P - Q|$  is bounded below by  $\min(R, r_0/2)$ , we find that (III) =  $O(|P - P'|^\alpha)$ . For the term (IV), integrated over the same set, we estimate  $\partial_{ij}g(P, Q) - \partial_{ij}g(P', Q)$  by  $C|P - P'| |\xi - Q|^{-n-1}$ , for

some  $\xi \in [P, P']$ . Using the Hölder continuity of  $f$ , the integral of (IV) is estimated by

$$Cr_0 \frac{|Q - P'|^\alpha}{|Q - \xi|^{n+1}}.$$

Its integral over  $A$  is estimated by its integral over

$$A' := \{Q : |Q - M| > r_0\}.$$

On  $A'$ ,

$$|Q - P'| \leq |Q - M| + |M - P'| = |Q - M| + \frac{1}{2}r_0 \leq \frac{3}{2}|Q - M|.$$

On the other hand,

$$|Q - \xi| \geq |Q - M| - |M - \xi| \geq |Q - M| - \frac{1}{2}r_0 \geq \frac{1}{2}|Q - M|.$$

Combining the two pieces of information, we find

$$\begin{aligned} \int_{A'} Cr_0 \frac{|Q - P'|^\alpha}{|Q - \xi|^{n+1}} dQ &\leq Cr_0 \int_{A'} |Q - M|^{\alpha-n-1} dQ \\ &= Cr_0 \int_{r_0}^{\infty} r^{\alpha-2} dr = C|P - P'|^\alpha. \end{aligned}$$

This completes the analysis of the integrals of (III) and (IV) over  $A$ .

It remains to consider (III) and (IV) over the part of  $B_{2R}$  on which  $|Q - M| \leq r_0$ . In this case,  $|P - Q| \leq |P - M| + |M - Q| \leq \frac{3}{2}r_0$ , and similarly for  $|P' - Q|$ . We therefore estimate directly the sum of (III) and (IV), namely

$$[f(Q) - f(P)]\partial_{ij}g(P, Q) - [f(Q) - f(P')]\partial_{ij}g(P', Q),$$

by

$$\begin{aligned} C[f]_{\alpha, B_{2R}} \int_{|Q-M| < r_0} (|Q - P|^{\alpha-n} + |Q - P'|^{\alpha-n}) dQ \\ \leq C[f]_{\alpha, B_{2R}} \int_0^{3r_0/2} |Q - P|^{\alpha-1} d|Q - P| \leq Cr_0^\alpha. \end{aligned}$$

Since  $r_0 = |P - P'|$ , this completes the proof.  $\square$

### 3.2 $C^{1+\alpha}$ estimates via the maximum principle

We give two estimates for function such that  $\Delta u$  is bounded. The result is essentially optimal, and relies only on the maximum principle. The choice of comparison functions is motivated by numerical approximations for second-order derivatives; in this sense, the argument may be compared with Nirenberg's "method of translations" for the proof of  $L^2$ -type estimates. Second-order estimates may also be derived by this method, but the choice of comparison functions is much more complicated.

We begin with the  $C^1$  estimate.

**THEOREM 11** *If  $\Delta u = f$  on  $K = \{|x_i| < s \text{ for } i = 1, \dots, n\}$ , then*

$$|\partial_n u(0)| \leq \frac{n}{s} \sup_K |u| + \frac{d}{2} \sup_K |f|.$$

**PROOF.** Let  $M = \sup_K |u|$ ,  $N = \sup_K |f|$ ,

$$v(x', x_n) = \frac{1}{2}[u(x', x_n) - u(x', -x_n)]$$

and

$$w(x', x_n) = \frac{M}{s^2}[|x'|^2 + x_n(ns - (n-1)x_n)] + \frac{1}{2}Nx_n(s - x_n).$$

Applying the maximum principle to  $w \pm v$  on  $K \cap \{0 < x_n < s\}$  we obtain

$$\frac{1}{2x_n}|u(x', x_n) - u(x', -x_n)| \leq \frac{M}{s^2}(ns - (n-1)x_n) + \frac{N}{2}(s - x_n).$$

Letting  $x_n \rightarrow 0$ , one finds the desired inequality.  $\square$

We now turn to the continuity of the gradient of  $u$ . The result implies interior  $C^{1+\alpha}$  regularity for every  $\alpha < 1$ .

**THEOREM 12** *Let  $\mu = \sup_K |\nabla u|$ . There is a constant  $k$  such that:*

$$\frac{1}{2}|\partial_i u(0, x_n) - \partial_i u(0, -x_n)| \leq \mu \frac{x_n}{s} + kx_n \ln \frac{x_n}{s}$$

for  $|x_n| \leq s/4$ .

PROOF. It suffices to prove the result for  $i = n$  and  $i = n - 1$ .

For the case  $i = n$ , we consider the function of  $n + 1$  variables  $(x', y, z)$  defined by

$$\phi(x', y, z) = \frac{1}{4}[u(x', y + z) - u(x', y - z) - u(x', -y + z) + u(x', -y - z)],$$

and the operator  $L = \sum_{i < n} \partial_i^2 + \frac{1}{2}(\partial_y^2 + \partial_z^2)$ , so that one checks  $|L\phi| \leq N$ . We then compare  $\phi$  with

$$W = \frac{4M}{s}yz + kyz \ln \frac{2s}{y+z},$$

where

$$k = \frac{4}{3}(N + \frac{8M}{s^2}(n-1)),$$

on the set  $K' = \{|x_i| < \frac{s}{2} \text{ if } i \leq n-1; \quad 0 < y, z < \frac{s}{4}\}$ . Since

$$LW = \frac{8M}{s^2}(n-1) + k \left[ -1 + \frac{yz}{(y+z)^2} \right] \leq \frac{8M}{s^2}(n-1) - \frac{3}{4}k = -N$$

on  $K'$ , the maximum principle yields

$$|\phi(0, y, z)| \leq y \left[ \frac{4\mu}{s} + k \ln \frac{2s}{y+z} \right].$$

Letting  $z \rightarrow 0$  gives the first inequality in the theorem.

For the case  $i = n - 1$ , we work with functions of  $n$  variables  $(\tilde{x}, y, z)$ , where  $\tilde{x} = (x_1, \dots, x_{n-2})$ , on the set  $K'' = \{|x_i| < \frac{s}{2} \text{ if } i \leq n-2; \quad 0 < y, z < \frac{s}{2}\}$ , and the auxiliary functions

$$\psi(\tilde{x}, y, z) = \frac{1}{4}[u(\tilde{x}, y, z) - u(\tilde{x}, y, -z) - u(\tilde{x}, -y, z) + u(\tilde{x}, -y, -z)],$$

and

$$\tilde{W} = \frac{4L}{s^2}|\tilde{x}|^2 + yz \left[ \frac{4\mu}{s} + \tilde{k} \ln \frac{2s}{y+z} \right],$$

where  $\tilde{k} = \frac{2}{3} [N + \frac{8M}{s^2}(n-2)]$ . One finds  $|\Delta\psi| \leq N$  and  $-\Delta\tilde{W} \geq N$  on  $K''$ , and the maximum principle yields the desired result as before.  $\square$

### 3.3 $C^{2+\alpha}$ estimates via Littlewood-Paley theory

LP decomposition provides a simple proof of the basic interior estimate for the constant-coefficient case. This is essentially due to the fact that the Fourier transform is rotation-invariant and has a simple scaling behavior. The argument is however tailored to isotropic situations. We give the argument in its simplest form, with no aim at generality.

**THEOREM 13** *Let  $\rho \in C^\alpha(\mathbb{R}^n)$  be such that  $\hat{\rho} = O(|\xi|^5)$  near  $\xi = 0$ . Then there is a  $u \in C^{2+\alpha}(\mathbb{R}^n)$  such that  $-\Delta u = \rho$ .*

*Remark 2* The condition  $\hat{\rho} = O(|\xi|^5)$  means that the first few moments of  $\rho$  vanish; it may be achieved by subtracting from  $\rho$  a smooth potential with prescribed multipolar moments.

**PROOF.** Consider the LP decomposition  $\rho_0 + \rho_1 + \dots$  of  $\rho$ . Define  $u_j$  by  $\hat{u}_j = \hat{\rho}_j/|\xi|^{-2}$ . Then  $u = u_0 + u_1 + \dots$  is well-defined, and the flatness assumption ensures that  $u_0$  and its first two derivatives are bounded. In particular,  $u_0$  is of class  $C^{2+\alpha}$ . Consider the Fourier transform of  $-\partial_{kl}u$ :

$$\xi_k \xi_l \hat{u}_j(\xi) = \frac{\xi_k \xi_l}{|\xi|^2} \hat{\rho}(\xi) \hat{\psi}(2^j |\xi|) = \hat{\rho}(\xi) \frac{(2^j \xi_k)(2^j \xi_l) \hat{\psi}(2^j |\xi|)}{(2^j |\xi|)^2}.$$

Recall that  $\hat{\psi}$  is flat at the origin, so that there is no singularity for  $\xi = 0$ . Applying point (2) of theorem 3 to the decomposition of  $\rho$  in which  $\psi$  would be replaced by  $\psi'$ , with  $\hat{\psi}'(\xi) = \xi_k \xi_l |\xi|^{-2} \hat{\psi}(|\xi|)$ , we find  $\sup |(\partial_{kl}u)_j| = O(2^{-(2+\alpha)j})$ . From the characterization of  $C^{2+\alpha}$  spaces, the result follows.  $\square$

### 3.4 Variational approach

We turn to a different approach, based on the integral characterization of Hölder spaces. The techniques involved have many other applications beyond the one discussed here; in particular, they allow a “direct approach” to regularity theory for minimizers of coercive functionals, without having to consider the Euler equation. We begin with a simple result.

**LEMMA 14** *For any  $u \in H^1(B_R(x_0))$ , and any  $r < 1$ ,  $\int_{B_r} |u - c|^2 dx$  is minimum when  $c = u_{x_0,r}$  and Poincaré’s inequality*

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 dx \leq C \int_{B_r(x_0)} |\nabla u|^2 dx$$

holds. If we assume in addition that  $u$  is harmonic, and  $0 < a < 1$ , we have

$$\int_{B_{ar}(x_0)} |\nabla u|^2 \leq c(a)r^{-2} \int_{ar < |x-x_0| < r} |u - u_{x_0,r}|^2 dx.$$

The r.h.s. is in particular estimated by  $c(a)r^{-2} \int_{B_r(x_0)} |u|^2$ .

PROOF. Poincaré's inequality is classical, see *e.g.* [11]. Let  $x_0 = 0$  and choose a smooth, nonnegative function  $\eta$  supported by  $B_r$ , equal to 1 for  $|x| \leq ar$ , such that  $|\nabla \eta| \leq C/r$ .<sup>13</sup> Multiply the Laplace equation by  $(u - m)\eta^2$ , where  $m$  is any constant. We find, integrating by parts,

$$\int_{B_r} [\eta^2 |\nabla u|^2 + 2(u - m)\nabla \eta \cdot \nabla u] dx = 0,$$

hence, estimating  $\nabla \eta$  by  $C/r$  and using Hölder's inequality, we find

$$\int_{\eta=1} |\nabla u|^2 dx \leq \int \eta^2 |\nabla u|^2 dx \leq \frac{C}{r^2} \int_{\nabla \eta \neq 0} |u - m|^2 dx.$$

Taking  $m$  to be the average of  $u$  on  $B_r$ , we obtain in particular the desired inequality. The inequality corresponding to  $m = 0$  is also occasionally useful.  $\square$

Since the derivatives of a harmonic function are themselves harmonic, this result implies that higher order derivatives are locally square-integrable; the Sobolev inequality then shows easily that any harmonic function is smooth. We now turn to a more precise estimate which enables one to compare a harmonic function and its spherical mean.

**THEOREM 15** *Let  $u$  be harmonic in the ball of radius  $R_0$  about  $x_0 \in \mathbb{R}^n$ . If  $0 < r < R < R_0$ , we have*

$$\sup_{B_{R/2}(x_0)} |u|^2 \leq CR^{-n} \int_{B_R(x_0)} |u|^2 dx \quad (11)$$

$$\int_{B_r(x_0)} |u|^2 dx \leq C\left(\frac{r}{R}\right)^n \int_{B_R(x_0)} |u|^2 dx \quad (12)$$

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 dx \leq C\left(\frac{r}{R}\right)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 dx \quad (13)$$

where  $C$  is independent of  $u$ ,  $r$ ,  $R$  and  $R_0$ .

<sup>13</sup>It suffices to find such a function  $\eta_0$  for the case  $r = 1$  and then let  $\eta(x) = \eta_0(x/r)$ .

PROOF. We may take  $x_0 = 0$ . It suffices to prove the first inequality for  $R = 1$  and scale variables. If  $k$  is an integer, we find, applying the preceding lemma repeatedly, we find

$$\int_{B_{1/2}} |\nabla^k u|^2 dx \leq c(k) \int_{B_1} u^2 dx.$$

The result follows by the Sobolev inequality.

Regarding the last two inequalities, it suffices to prove them for  $r \leq R/2$  since they are obvious for  $r \geq R$ . In that case, we have, using the first inequality,

$$\int_{B_r} |u|^2 dx \leq \omega_n r^n \sup_{B_{R/2}} |u|^2 \leq C \left(\frac{r}{R}\right)^n \int_{B_R} |u|^2 dx$$

as desired. Similarly, since the derivatives of  $u$  are also harmonic, Poincaré's inequality yields

$$\int_{B_r} |u - u_{x_0,r}|^2 dx \leq C \rho^2 \int_{B_r} |\nabla u|^2 dx \leq C \rho^2 \left(\frac{r}{R}\right)^n \int_{B_{3R/4}(x_0)} |\nabla u|^2 dx.$$

We conclude using lemma 14. □

We turn to the estimation of second derivatives of the solutions of Poisson's equation  $-\Delta v = f$ . It is equivalent to seek an estimate for the *first* derivatives of solutions of  $-\Delta u = \partial_k f$ . It turns out to be convenient to consider more generally the problem

$$\Delta u + \sum_k \partial_k f^k = 0, \tag{14}$$

where  $u \in H^1(B_R)$  and the  $f^k$  are of class  $C^\alpha$ . Recall that function of class  $C^\alpha$  correspond to the class  $\mathcal{L}^{2,\lambda}$  for  $\lambda = n + 2\alpha$ . This suggests the following theorem.

**THEOREM 16** *Assume that  $\mathbf{f} := (f_1, \dots, f_n) \in \mathcal{L}^{2,\lambda}$  with  $0 \leq \lambda < n + 2$ , and that  $u \in H^1(B_R)$  solves (14), then  $\nabla u$  is locally of class  $\mathcal{L}^{2,\lambda}$ . In particular, if the  $f^k$  are locally  $C^\alpha$ , with  $0 < \alpha < 1$ , so is  $\nabla u$ .*

PROOF. We must analyze the behavior of the integrals

$$F(r) := \int_{B_r(x_0)} |\nabla u - (\nabla u)_{x_0,r}|^2 dx,$$

defined for given  $x_0 \in B_R$  and  $r < R - |x_0|$ , as  $r \rightarrow 0$ . We first prove the estimate

$$F(\rho) \leq A(\rho/r)^{n+2}F(r) + Br^\lambda. \quad (15)$$

for  $\rho < r$ . We then deduce from it an estimate of the form  $F(r) = O(r^\lambda)$ , from which the result follows.

Consider the solution of  $\Delta v = 0$  in  $B_r(x_0)$  such that  $u - v \in H_0^1(B_r(x_0))$ . From theorem 15, we have the inequality

$$\int_{B_\rho} |\nabla v - (\nabla v)_{x_0, \rho}|^2 dx \leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla v - (\nabla v)_{x_0, \rho}|^2 dx.$$

For any  $w \in H_0^1(B_r(x_0))$ , we have, writing  $\partial_k f^k = \partial_k(f^k - f_{x_0, r}^k)$ ,

$$\int_{B_r} \nabla(u - v) \cdot \nabla w \, dx = - \int (\mathbf{f} - \mathbf{f}_{x_0, r}) \cdot \nabla w \, dx.$$

Taking  $w = u - v$ , we find

$$\int_{B_r} |\nabla(u - v)|^2 \leq \int_{B_r} |\mathbf{f} - \mathbf{f}_{x_0, r}|^2 dx.$$

From now on, we omit the mention of the point  $x_0$  in averages. Since  $\nabla u - (\nabla u)_\rho = [\nabla v - (\nabla v)_\rho] + [\nabla w - (\nabla w)_\rho]$ , and

$$\int_{B_\rho} |\nabla w - (\nabla w)_\rho|^2 dx \leq \int_{B_\rho} |\nabla w|^2 dx \leq \int_{B_r} |\nabla w|^2 dx,$$

we find

$$\begin{aligned} \int_{B_\rho} |\nabla u - (\nabla u)_\rho|^2 dx &\leq 2 \int_{B_\rho} |\nabla v - (\nabla v)_\rho|^2 dx + 2 \int_{B_r} |\nabla w|^2 dx \\ &\leq C\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla v - (\nabla v)_r|^2 dx + C \int_{B_r} |\mathbf{f} - \mathbf{f}_r|^2 dx. \end{aligned}$$

The second term is  $O(r^\lambda)$  thanks to the hypothesis on  $\mathbf{f}$ . We now estimate the first term in terms of  $F(r)$ . To this end, we need the following result:

LEMMA 17  $\int_{B_r} |\nabla v - (\nabla v)_r|^2 dx \leq \int_{B_r} |\nabla u - (\nabla u)_r|^2 dx.$

PROOF. Since  $u - v$  is in  $H_0^1$ , we have

$$\int_{B_r} |\nabla v|^2 dx \leq \int_{B_r} |\nabla u|^2 dx,$$

and, as soon as  $\mathbf{g}$  is constant

$$\int_{B_r} \nabla(v - u) \cdot \mathbf{g} dx = 0.$$

It follows that

$$\begin{aligned} & \int_{B_r} |\nabla v - (\nabla v)_r|^2 dx - \int_{B_r} |\nabla u - (\nabla u)_r|^2 dx \\ &= \int_{B_r} \nabla(v - u) \cdot (\nabla(v + u) - (\nabla(v + u))_r) dx \\ &= \int_{B_r} (|\nabla v|^2 - |\nabla u|^2) dx \leq 0, \end{aligned}$$

QED. □

The proof of inequality (15) is now complete. To conclude the proof of the theorem, we argue as follows: Fix  $\gamma \in (\lambda, n + 2)$ , and choose  $t \in (0, 1)$  such that

$$2At^{n+2} \leq t^\gamma.$$

Fix  $j$  such that  $t^{j+1}r < \rho \leq t^j r$ . We find, since  $F$  is non-decreasing,

$$\begin{aligned} F(\rho) &\leq F(t^j) \leq t^\gamma F(t^{j-1}r) + B(t^{j-1}r)^\lambda \\ &\leq t^\gamma [t^\gamma F(t^{j-2}r) + B(t^{j-2}r)^\lambda] + B(t^{j-1}r)^\lambda \\ &= t^{2\gamma} F(t^{j-2}r) + Br^\lambda t^{(j-1)\lambda} [1 + t^{\gamma-\lambda}] \\ &\leq \dots \leq t^{j\gamma} F(r) + Br^\lambda \frac{t^{(j-1)\lambda}}{1 - t^{\gamma-\lambda}} \\ &\leq t^{-\gamma} \left(\frac{\rho}{r}\right)^\gamma F(r) + \frac{Bt^{-2\lambda}}{1 - t^{\gamma-\lambda}} \left(\frac{\rho}{r}\right)^\lambda. \end{aligned}$$

Since  $\gamma > \lambda$ , this implies  $F(\rho) = O(\rho^\lambda)$  as desired. This gives the Hölder regularity of the gradient of  $u$ . □

*Remark 3* For more general problems, it is useful to note that the last part of the argument also applies in the more general situation in which  $F$  is a non-negative, non-decreasing function satisfying

$$F(\rho) \leq A[(\rho/r)^a + \varepsilon]F(r) + Br^b,$$

for  $0 < \rho < r \leq R$ , with  $a > b$ . Taking  $t$  as before, we find that if  $\varepsilon \leq t^a$ , then  $F$  satisfies an inequality of the form

$$F(\rho) \leq c(a, b, A)[(\rho/r)^b F(R) + B\rho^b].$$

*Remark 4* A similar argument may be applied to non-divergence operators with Hölder-continuous coefficients, using the previous remark. As expected, the argument consists in writing the operator as the sum of a constant-coefficient operator, and an operator with small coefficients.

## 3.5 Other methods

We briefly outline two other approaches.

### 3.5.1 A regularization method

For any standard mollifier  $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ , consider, for any  $u(x)$ ,

$$u_\varepsilon(x) = \rho_\varepsilon * u = \int \rho(z)u(x - \varepsilon z)dz.$$

It is easy to see that  $u(x) - u_\varepsilon(x) = O(\varepsilon^\alpha)$  if  $u \in C^\alpha(\mathbb{R}^n)$ , and that, similarly, the derivatives of  $u_\varepsilon$  with respect to  $\varepsilon$  or  $x$  are  $O(\varepsilon^{\alpha-1})$ . Conversely, if these derivatives are  $\leq M\varepsilon^{\alpha-1}$ , it turns out that one may estimate the Hölder constant of  $u$ : first of all,

$$|u(x) - u(x_\varepsilon)| \leq \varepsilon \int_0^1 \left| \frac{\partial u}{\partial \varepsilon} \right|(x, \varepsilon\sigma) d\sigma \leq M\varepsilon^\alpha \int_0^1 \frac{d\sigma}{\sigma^{1-\alpha}} = \frac{M\varepsilon^\alpha}{\alpha}.$$

We now find the estimate

$$\begin{aligned} |u(x) - u(y)| &= |u(x) - u(x_\varepsilon)| + |u(x_\varepsilon) - u(y_\varepsilon)| + |u(y_\varepsilon) - u(y)| \\ &= \frac{2}{\alpha}M\varepsilon^\alpha + |x - y| |\nabla_x u_\varepsilon(z)| \end{aligned}$$

for some  $z \in [x, y]$ . One then estimates  $|\nabla_x u_\varepsilon(z)|$  by  $M\varepsilon^{\alpha-1}$  and takes  $\varepsilon = |x - y|$ . One also proves that mixed derivatives of  $u_\varepsilon$  with respect to  $x$  and  $\varepsilon$  are controlled by the Hölder norm of second derivatives of  $u$  with respect to  $x$ .

One then considers equation  $-\Delta u = f(x) = f(x_0) + g(x)$ , where  $|g| \leq R^\alpha [f]_{\alpha, B_R(x_0)}$ . We have  $-\Delta u_\varepsilon = g_\varepsilon$ . If  $\nabla^2$  represents any second-order derivative, in  $x$  and  $\varepsilon$ , we find  $-\Delta \nabla^2 u_\varepsilon = \nabla^2 g_\varepsilon$ . Applying the interior  $C^1$  estimate theorem 11, and letting  $R = N\varepsilon$ , where  $N$  is to be chosen later, one gets

$$\varepsilon^{1-\alpha} |\nabla \partial_{ij} u(x_0, \varepsilon)| \leq C \{ N^{\alpha-1} [\nabla_x^2 u]_\alpha + N^{\alpha+1} R^{-\alpha} \sup_{B_{(1+N)\varepsilon}(x_0)} |g| \}$$

where one has estimated  $\sup_{B_R(x_0)} |\nabla^2 g|$  by  $\varepsilon^{-2} \sup_{B_{R+\varepsilon}(x_0)} |g|$ . This quantity is itself estimated by  $[f]_\alpha (R + \varepsilon)^\alpha$ . Taking  $N$  so large that  $CN^{\alpha-1} < 1/2$ , we find an estimate of  $[\nabla_x^2 u]_\alpha$ , as desired.

### 3.5.2 Blow-up method

We sketch the idea of the proof of the interior estimate for the Laplacian; a similar idea, with somewhat more complicated details, applies to other situations.

Assume there is no estimate of the form  $[\nabla^2 u]_\alpha \leq C[\Delta u]_\alpha$  for functions of class  $C^{2+\alpha}(\mathbb{R}^n)$ . In that case, there must be some sequence  $u_k$  such that

$$[\nabla^2 u_k]_\alpha = 1 > 2k[\Delta u_k]_\alpha.$$

We may therefore find indices  $i$  and  $j$ , and sequences  $x_k, a_k$  of vectors in  $\mathbb{R}^n$  such that

$$1 \geq \frac{1}{|a_k|^\alpha} |\partial_{ij} u_k(x_k) - u_k(x_k + a_k)| \geq \frac{1}{2} \geq k[\Delta u_k]_\alpha.$$

Subtracting an affine function, we may assume that  $u_k$  and its first-order derivatives vanish at the point  $x_k$ . Subtracting a quadratic function, we may also assume that  $\Delta u_k$  vanishes at  $x_k$ . Performing a ( $k$ -dependent) rotation of axes, we may also assume that  $a_k = (h_k, 0, \dots, 0)$ . Considering  $v_k(y) = h_k^{-(2+\alpha)} u_k(x_k + h_k y)$ , we see that  $[\Delta v_k]_\alpha = [\Delta u_k]_\alpha \rightarrow 0$ , while  $v_k$  and its first-order derivatives vanish at the origin and grows at most like  $|y|^{2+\alpha}$  at infinity. In addition, we have

$$|\partial_{ij} v_k(0) - v_k(e_1)| \geq \frac{1}{2}. \tag{16}$$

where  $e_1 = (1, 0, \dots, 0)$ . After extraction of a subsequence, we are left with a harmonic function which grows at most like  $|y|^{2+\alpha}$ , and satisfies equation (16). A variant of the Liouville property ensures that  $v$  is quadratic, which contradicts (16).

## 4 Perturbation of coefficients

### 4.1 Basic estimate

Working on a relatively compact subset  $\Omega'$  of  $\Omega$ , we may assume that  $[u]_{2+\alpha}^* < \infty$ ; since the constants in the various inequalities will not depend on the choice of  $\Omega$ , the full result will follow.

Consider  $x_0 \in \Omega$  and let  $r = \theta d(x_0)$  with  $\theta \leq 1/2$ . Let  $L_0 = \sum_{ij} a^{ij}(x_0) \partial_{ij}$  (the ‘‘tangential operator,’’ with coefficients ‘‘frozen’’ at  $x_0$ ). We define

$$F := L_0 u = \sum_{ij} (a^{ij}(x_0) - a^{ij}(x)) \partial_{ij} u - \sum b^i \partial_i u - cu + f.$$

We apply the constant-coefficient interior estimates on the ball  $B_r(x_0)$ . Let  $y_0 \neq x_0$  such that  $d(y_0) \geq d(x_0)$ .

If  $|x_0 - y_0| < r/2$ , we have

$$\left(\frac{r}{2}\right)^{2+\alpha} [\nabla^2 u]_{\alpha, x_0, y_0} \leq C(\sup |u| + \sup_{B_r} |r^2 F| + \sup_{B_r \times B_r} r^{2+\alpha} \frac{|F(x) - F(y)|}{|x - y|^\alpha}).$$

Therefore,

$$d(x_0)^{2+\alpha} [\nabla^2 u]_{\alpha, x_0, y_0} \leq C\theta^{-2-\alpha} (\sup |u| + \|F\|_{\alpha, B_r}^{(2)}). \quad (17)$$

If  $|x_0 - y_0| \geq r/2$ , we have

$$d(x_0)^{2+\alpha} [\nabla^2 u]_{\alpha, x_0, y_0} \leq 2[u]_2^* \frac{d(x_0)^\alpha}{|x_0 - y_0|^\alpha} \leq 2[u]_2^* \left(\frac{2}{\theta}\right)^\alpha. \quad (18)$$

The issue is therefore the estimation of  $\|F\|_{\alpha, B_r}^{(2)}$  in terms of norms of  $u$  and its derivatives over  $\Omega$ .

For clarity, we begin with three lemmas.

LEMMA 18  $\|uv\|_{\alpha, \Omega}^{(s+t)} \leq \|u\|_{\alpha, \Omega}^{(s)} \|v\|_{\alpha, \Omega}^{(t)}$ .

PROOF. Direct verification.  $\square$

LEMMA 19 *If  $r = \theta d(x, \partial\Omega)$ , with  $0 < \theta \leq 1/2$  (so that  $B_r(x) \subset \Omega$ ), we have*

$$\|\nabla^2 u\|_{\alpha, B_r}^{(2)} \leq 8[\theta^2 \|\nabla^2 u\|_{2, \Omega}^* + \theta^{2+\alpha} [u]_{2+\alpha, \Omega}^*] \quad (19)$$

$$\|f\|_{\alpha, B_r}^{(2)} \leq 8\theta^2 \|f\|_{\alpha, \Omega}^{(2)} \quad (20)$$

PROOF. We need to estimate, for  $y \in B_r(x)$ ,  $d(y, \partial B_r(x))$  and  $d_{x,y,B_r}$  in terms of the corresponding distances relative to  $\Omega$ . On the one hand,  $d(y, \partial B_r) \leq r - |x - y| \leq r = \theta d(x)$ . On the other hand, if  $z \in B_r(x)$  and  $d(y, \partial B_r(x)) \leq d(z, \partial B_r(x))$ , we have  $d_{y,z,B_r} \leq \theta d(x)$  and also  $d(y) \geq d(y, \partial B_r(x)) \geq (1 - \theta)d(x)$ ; it follows that  $d(x) \leq (1 - \theta)^{-1} d_{x,y,\Omega}$ . Therefore,

$$d(y, \partial B_r) \leq \theta d(x)$$

and

$$d_{y,z,B_r} \leq \frac{\theta}{1 - \theta} d_{y,z,\Omega}.$$

The two desired inequalities follow.  $\square$

LEMMA 20 *If  $x \in B_r(x_0)$  with  $r = \theta d(x_0)$ , with  $0 < \theta \leq 1/2$ , we have*

$$\|a(x) - a(x_0)\|_{\alpha, B_r}^{(0)} \leq C\theta^\alpha [a]_{\alpha, \Omega}^*.$$

PROOF. If  $d(x) \leq d(y)$  and  $|x - y| \leq r = \theta d(x_0)$  with  $\theta \leq 1$ ,

$$|a(x) - a(y)| \leq d(x)^\alpha \frac{|a(x) - a(y)|}{|x - y|^\alpha} \left(\frac{|x - y|}{d(x)}\right)^\alpha \leq C\theta^\alpha [a]_\alpha^{(0)}$$

since  $(1 - \theta)d(x_0) \leq d(x) \leq (1 + \theta)d(x_0)$ . Therefore, estimating  $|a(x) - a(x_0)|$  by  $r^\alpha [a]_{\alpha, \Omega}^*$ , we find the announced inequality.  $\square$

We now resume the proof of the estimate of  $[\nabla^2 u]_\alpha$ : first,

$$\begin{aligned} \|(a(x) - a(x_0))\nabla^2 u(x)\|_{\alpha, B_r}^{(2)} &\leq \|a(x) - a(x_0)\|_{\alpha, B_r}^{(0)} \|\nabla^2 u\|_{\alpha, B_r}^{(2)} \\ &\leq C\theta^{2+\alpha} \|a\|_{\alpha, \Omega}^{(0)} (\|\nabla^2 u(x)\|_{\alpha, \Omega}^* + \theta^\alpha [u]_{2+\alpha, \Omega}^*). \end{aligned}$$

Similarly,

$$\begin{aligned} \|b\nabla u(x)\|_{\alpha, B_r}^{(2)} &\leq 8\theta^2 \|b\nabla u\|_{\alpha, \Omega}^{(2)} \\ &\leq 8\theta^2 \|b\|_{\alpha, \Omega}^{(1)} \|\nabla u\|_{\alpha, \Omega}^{(1)} \\ &\leq C\theta^2 \|b\|_{\alpha, \Omega}^{(1)} \{\theta^{2\alpha} [u]_{2+\alpha, \Omega}^* + \sup |u|\}. \end{aligned}$$

Finally,

$$\begin{aligned}\|cu\|_{\alpha, B_r}^{(2)} &\leq 8\theta^2 \|cu\|_{\alpha, \Omega}^{(2)} \leq 8\theta^2 \|c\|_{\alpha, \Omega}^{(2)} \|u\|_{\alpha, \Omega}^{(0)} \\ &\leq 8\theta^2 \{\theta^{2\alpha} [u]_{2+\alpha, \Omega}^* + \sup |u|\}.\end{aligned}$$

It follows that

$$\|F\|_{\alpha, B_r}^{(2)} \leq C\theta^{2+2\alpha} [u]_{2+\alpha, \Omega}^* + c(\theta)(\sup |u| + \|f\|_{\alpha, \Omega}^{(2)}).$$

Therefore, using this inequality in (17) and (18), we find

$$d(x_0)^{2+\alpha} [u]_{2+\alpha, \Omega}^* \leq C\theta^\alpha [u]_{2+\alpha, \Omega}^* + c'(\theta)(\sup |u| + \|f\|_{\alpha, \Omega}^{(2)}).$$

The desired estimate on  $[u]_{2+\alpha, \Omega}^*$  follows.

## 4.2 Estimates up to the boundary

The potential-theoretic argument extends easily to the case of Poisson's equation on the half-ball for the following reason: if we apply the formula for the second-order derivatives of the Newtonian potential (theorem 9) to the case in which  $\Omega$  is the half-ball  $B_R \cap \{x_n > 0\}$ , we find that the contribution to the boundary integral of the part of the boundary on which  $x_n = 0$  vanishes if  $j < n$ , because the component  $n_j$  of the outward normal then vanishes. The subsequent argument therefore goes through without change, and yields the Hölder continuity up to the boundary of all second-order derivatives of  $u$  except  $\partial_{x_n}^2 u$ ; but the latter is given in terms of the former using Poisson's equation. We therefore obtain the  $C^{2+\alpha}$  estimates up to the boundary for the Newtonian potential of a density  $f$  of class  $C^\alpha$  in the half-ball.

To obtain regularity up to  $x_n = 0$  for the solution of the Dirichlet problem on the half-ball, we use *Schwarz' reflection principle*

**LEMMA 21** *Let  $f$  be of class  $C^\alpha$  in the closed half-ball. If  $u$  is of class  $C^2$  on the open half-ball of radius  $R$ , is continuous on the closed ball, satisfies  $\Delta u = f$  in the half-ball, and vanishes for  $x_n = 0$ , it may be extended to the entire ball as a solution of an equation of the form  $\Delta u = f_1$ . In particular,  $u$  is of class  $C^{2+\alpha}$  on any compact subset of the closed half-ball which does not meet the spherical part of its boundary.*

**PROOF.** Write  $x = (x', x_n)$ , and extend  $f$  to an even function  $f_1$  on the ball. Using the inequality  $a^\alpha + b^\alpha \leq 2(a+b)^\alpha$ , we see that  $f_1$  is of class  $C^\alpha$ .

Now, the Newtonian potential of  $f_1$  does not satisfy the Dirichlet boundary condition. We therefore consider

$$W(x) := \int_{B_R \cap \{x_n > 0\}} [g(x - y) - g(x - \tilde{y})] f(y) dy,$$

where  $\tilde{y} = (y', -y_n)$  is the reflection of  $y$  across  $\{x_n = 0\}$ . It is easy to see that  $\Delta W = 0$  in the half-ball, and that  $W = 0$  for  $x_n = 0$ . It is also of class  $C^{2+\alpha}$  by the variant of theorem 9 already indicated. We now consider  $V := u - W$ , which is harmonic in the half-ball, and vanishes for  $x_n = 0$ . Extend  $V$  to an *odd* function of  $x_n$  on the entire ball. Consider the solution of the Dirichlet problem on the ball with boundary data equal to  $V$ . This problem has a unique solution  $V^*$  by the Poincaré-Perron method—which is independent of Schauder theory. Since  $-V^*(x', -x_n)$  solves the same problem, we find that  $V^*$  must be odd with respect to  $x_n$ . Therefore  $V^*$  is also the solution of the Dirichlet problem on the half-ball, with boundary value given by  $V$  on the spherical part of the boundary, and value zero on the flat part of the boundary (where  $x_n = 0$ ). Therefore,  $V^*$  must be equal to  $V$  on the half-ball, and therefore on the ball as well. This proves that  $V = V^*$  has the required regularity up to  $x_n = 0$ , as desired.  $\square$

The perturbation from constant to variable coefficients then proceeds by a variant of the argument used for the interior estimates [35, 27, 1].

## 5 Fuchsian operators on $C^{2+\alpha}$ domains

We now consider operators satisfying an asymptotic scale invariance condition near the boundary. These operators arise naturally as local models near singularities through the process of Fuchsian Reduction [50]. We develop the basic estimates for such operators without condition on the sign of the lower-order terms. A typical example of the more precise theorems one obtains under such conditions is given in theorem 43. We distinguish two types of Fuchsian operators.

An operator  $A$  is said to be *of type (I)* (on a given domain  $\Omega$ ) if it can be written

$$A = \partial_i (d^2 a^{ij} \partial_j) + db^i \partial_i + c,$$

with  $(a^{ij})$  uniformly elliptic and of class  $C^\alpha$ , and  $b^i, c$  bounded.

*Remark 5* One can also allow terms of the type  $\partial_i (b^i u)$  in  $Au$ , if  $b^i$  is of class  $C^\alpha$ , but this refinement will not be needed here.

An operator is said to be of type (II) if it can be written

$$A = d^2 a^{ij} \partial_{ij} + db^i \partial_i + c,$$

with  $(a^{ij})$  uniformly elliptic and  $a^{ij}$ ,  $b^i$ ,  $c$  of class  $C^\alpha$ .

*Remark 6* One checks directly that types (I) and (II) are invariant under changes of coordinates of class  $C^{2+\alpha}$ . In particular, to check that an operator is of type (I) or (II), we may work indifferently in coordinates  $x$  or  $(T, Y)$  defined in section 2.5. All proofs will be performed in the  $(T, Y)$  coordinates; an operator is of type (II) precisely if it has the above form with  $d$  replaced by  $T$ , and the coefficients  $a^{ij}$ ,  $b^i$ ,  $c$  are of class  $C^\alpha$  as functions of  $T$  and  $Y$ ; a similar statement holds for type (I).

The basic results for type (I) operators are

**THEOREM 22** *If  $Ag = f$ , where  $f$  et  $g$  are bounded and  $A$  is of type (I) on  $\Omega'$ , then  $d\nabla g$  is bounded, and  $dg$  and  $d^2\nabla g$  belong to  $C^\alpha(\Omega' \cup \partial\Omega)$ .*

**THEOREM 23** *If  $Ag = df$ , where  $f$  and  $g$  are bounded,  $g = O(d^\alpha)$ , and  $A$  is of type (I) on  $\Omega'$ , then  $g \in C^\alpha(\Omega' \cup \partial\Omega)$  and  $dg \in C^{1+\alpha}(\Omega' \cup \partial\Omega)$ .*

These two results are proved in the next subsection. The main result for type (II) operators is:

**THEOREM 24** *If  $Ag = df$ , where  $f \in C^\alpha(\Omega' \cup \partial\Omega)$ ,  $g = O(d^\alpha)$ , and  $A$  is of type (II) on  $\Omega'$ , then  $d^2g$  belongs to  $C^{2+\alpha}(\Omega' \cup \partial\Omega)$ .*

**PROOF.** The assumptions ensure that  $a^{ij} \partial_{ij}(d^2f)$  is Hölder-continuous and that  $f$  is bounded;  $d^2f$  therefore solves a Dirichlet problem to which the Schauder estimates apply near  $\partial\Omega$ . Therefore  $d^2f$  is of class  $C^{2+\alpha}$  up to the boundary. Since we already know that  $f \in C^\alpha(\overline{\Omega}_\delta)$  and  $df$  is of class  $C^{1+\alpha}(\overline{\Omega}_\delta)$ , we have indeed  $f$  of class  $C_{\frac{1}{4}}^{2+\alpha}(\overline{\Omega}_{\delta'})$  for  $\delta' < \delta$ .  $\square$

Let  $\rho > 0$  and  $t \leq 1/2$ . Throughout the proofs, we shall use the sets

$$\begin{aligned} Q &= \{(T, Y) : 0 \leq T \leq 2 \text{ and } |y| \leq 3\rho\}, \\ Q_1 &= \{(T, Y) : \frac{1}{4} \leq T \leq 2 \text{ and } |y| \leq 2\rho\}, \\ Q_2 &= \{(T, Y) : \frac{1}{2} \leq T \leq 1 \text{ and } |y| \leq \rho/2\}, \\ Q_3 &= \{(T, Y) : 0 \leq T \leq \frac{1}{2} \text{ and } |y| \leq \rho/2\}. \end{aligned}$$

We may assume, by scaling coordinates, that  $Q \subset \Omega'$ . It suffices to prove the announced regularity on  $Q_3$ .

## 5.1 First “type (I)” result

We prove theorem 22.

Let  $Af = g$ , with  $A, f, g$  satisfying the assumptions of the theorem over  $Q$ , and let  $y_0$  be such that  $|y_0| \leq \rho$ .

For  $0 < \varepsilon \leq 1$ , and  $(T, Y) \in Q_1$ , let

$$f_\varepsilon(T, Y) = f(\varepsilon T, y_0 + \varepsilon Y),$$

and similarly for  $g$  and other functions. We have  $f_\varepsilon = (Ag)_\varepsilon = A_\varepsilon f_\varepsilon$ , where

$$A_\varepsilon = \partial_i(T^2 a_\varepsilon^{ij} \partial_j) + T b_\varepsilon^i \partial_i + c_\varepsilon$$

is also of type (I), with coefficient norms independent of  $\varepsilon$  and  $y_0$ , and is uniformly elliptic in  $Q_1$ .

Interior estimates give

$$\|g_\varepsilon\|_{C^{1+\alpha}(Q_2)} \leq M_1 := C_1(\|f_\varepsilon\|_{L^\infty(Q_1)} + \|g_\varepsilon\|_{L^\infty(Q_1)}). \quad (21)$$

The assumptions of the theorem imply that  $M_1$  is independent of  $\varepsilon$  and  $y_0$ .

We therefore find,

$$|\varepsilon \nabla g(\varepsilon T, y_0 + \varepsilon Y)| \leq M_1, \quad (22)$$

$$\varepsilon |\nabla g(\varepsilon T, y_0 + \varepsilon Y) - \nabla g(\varepsilon T', y_0)| \leq M_1(|T - T'| + |Y|)^\alpha \quad (23)$$

if  $\frac{1}{2} \leq T, T' \leq 1$  and  $|Y| \leq \rho/2$ . It follows in particular, taking  $Y = 0$ ,  $\varepsilon = t \leq 1$ ,  $T = 1$ , and recalling that  $|y_0| \leq \rho$ , that

$$|t \nabla g(t, y)| \leq M_1 \text{ if } |y| \leq \rho, t \leq 1. \quad (24)$$

This proves the first statement in the theorem.

Taking  $\varepsilon = 2t \leq 1$ ,  $T = 1/2$ , and letting  $y = y_0 + \varepsilon Y$ ,  $t' = \varepsilon T'$ ,

$$2t |\nabla g(t, y) - \nabla g(t', y_0)| \leq M_1(|t - t'| + |y - y_0|)^\alpha (2t)^{-\alpha}$$

for  $|y - y_0| \leq \rho t$  and  $t \leq t' \leq 2t \leq 1$ .

Let us prove that

$$|t^2 \nabla g(t, y) - t'^2 \nabla g(t', y_0)| \leq M_2(|t - t'| + |y - y_0|)^\alpha \quad (25)$$

for  $|y|, |y_0| \leq \rho$ , and  $0 \leq t \leq t' \leq \frac{1}{2}$ , which will prove

$$t^2 \nabla g \in C^\alpha(Q_3).$$

It suffices to prove this estimate in the two cases: (i)  $t = t'$  and (ii)  $y = y_0$ ; the result then follows from the triangle inequality. We distinguish three cases.

1. If  $t = t'$ , we need only consider the case  $|y - y_0| \geq \rho t$ . We then find

$$t^2 |\nabla g(t, y) - \nabla g(t, y_0)| \leq 2M_1 t \leq 2M_1 |y - y_0| / \rho.$$

2. If  $y = y_0$  and  $t \leq t' \leq 2t \leq 1$ , we have  $t + t' \leq 2t'$ , hence

$$\begin{aligned} & |t^2 \nabla g(t, y_0) - t'^2 \nabla g(t', y_0)| \\ & \leq t^2 |\nabla g(t, y_0) - \nabla g(t', y_0)| + |t - t'| (t + t') |\nabla g(t', y_0)| \\ & \leq M_1 2^{-1-\alpha} t^{1-\alpha} |t - t'|^\alpha + 2M_1 |t - t'| \\ & \leq M_2 |t - t'|^\alpha. \end{aligned}$$

3. If  $y = y_0$ , and  $2t \leq t' \leq 1/2$ , we have  $t + t' \leq 3(t' - t)$ , and

$$\begin{aligned} |t^2 \nabla g(t, y_0) - t'^2 \nabla g(t', y_0)| & \leq M_1 (t + t') \\ & \leq 3M_1 |t - t'|. \end{aligned}$$

This proves estimate (25).

On the other hand, since  $g$  and  $T\nabla g$  are bounded over  $Q_3$ ,

$$Tg \in \text{Lip}(Q_3) \subset C^\alpha(Q_3).$$

This completes the proof of theorem 22.

## 5.2 Second “type (I)” result

We prove theorem 23.

The argument is similar, except that  $M_1$  is now replaced by  $M_3\varepsilon^\alpha$ , with  $M_3$  independent of  $\varepsilon$  and  $y_0$ . It follows that

$$|t\nabla g(t, y)| \leq M_3 t^\alpha \text{ if } |y| \leq \rho, t \leq 1. \quad (26)$$

Taking  $\varepsilon = 2t \leq 1$ ,  $T = 1/2$ , and letting  $y = y_0 + \varepsilon Y$ ,  $t' = \varepsilon T'$ , and noting that  $\varepsilon^\alpha(|T - T'| + |Y|)^\alpha = (|t - t'| + |y - y_0|)^\alpha$ , we find

$$2t|\nabla g(t, y) - \nabla g(t', y_0)| \leq M_3(|t - t'| + |y - y_0|)^\alpha$$

for  $|y - y_0| \leq \rho t$  and  $t \leq t' \leq 2t \leq 1$ . Let us prove that

$$|t\nabla g(t, y) - t'\nabla g(t', y_0)| \leq M_4(|t - t'| + |y - y_0|)^\alpha \quad (27)$$

for  $|y|, |y_0| \leq \rho$ , and  $0 \leq t \leq t' \leq \frac{1}{2}$ , which will prove

$$T\nabla g \in C^\alpha(Q_3).$$

We again distinguish three cases.

1. If  $t = t'$ ,  $|y - y_0| \geq \rho t$ , we find

$$t|\nabla g(t, y) - \nabla g(t, y_0)| \leq 2M_3 t^\alpha \leq 2M_3(|y - y_0|/\rho)^\alpha.$$

2. If  $y = y_0$  and  $t \leq t' \leq 2t \leq 1$ , we have  $|t - t'| \leq t \leq t'$ , hence

$$\begin{aligned} |t\nabla g(t, y_0) - t'\nabla g(t', y_0)| &\leq \frac{1}{2}M_3|t - t'|^\alpha + |t - t'| |\nabla g(t', y_0)| \\ &\leq M_3|t - t'|^\alpha \left(\frac{1}{2} + t^{1-\alpha}t'^{\alpha-1}\right) \leq 2M_3|t - t'|^\alpha. \end{aligned}$$

3. If  $y = y_0$ , and  $2t \leq t' \leq 1/2$ , we have  $t \leq t' \leq 3(t' - t)$ , and

$$\begin{aligned} |t\nabla g(t, y_0) - t'\nabla g(t', y_0)| &\leq M_3(t^\alpha + t'^\alpha) \\ &\leq 2M_3(3|t - t'|)^\alpha. \end{aligned}$$

Estimate (27) therefore holds.

The same type of argument shows that

$$g \in C^\alpha(Q_3).$$

In fact, we have, with again  $\varepsilon = 2t$ ,  $\|g_\varepsilon\|_{C^\alpha(Q_2)} \leq M_5\varepsilon^\alpha$ , where  $M_5$  depends on the r.h.s. and the uniform bound assumed on  $f$ . This implies

$$|g(t, y) - g(t', y_0)| \leq M_5(|t - t'| + |y - y_0|)^\alpha,$$

if  $t \leq t' \leq 2t \leq 1$  and  $|y - y_0| \leq \rho t$ . The assumptions of the theorem yield in particular

$$|g(t, y)| \leq M_5 t^\alpha,$$

for  $t \leq 1/2$  and  $|y| \leq \rho$ .

If  $\rho t \leq |y - y_0| \leq \rho$ , and  $t \leq 1/2$ , we have

$$|g(t, y) - g(t, y_0)| \leq 2M_5 t^\alpha \leq 2M_5 \left( \frac{|y - y_0|}{\rho} \right)^\alpha.$$

If  $2t \leq t' \leq 1/2$  and  $y = y_0$ ,

$$|g(t, y_0) - g(t', y_0)| \leq M_5(t^\alpha + t'^\alpha) \leq 2M_5(3|t - t'|)^\alpha.$$

If  $t \leq t' \leq 2t \leq 1/2$ , we already have

$$|g(t, y_0) - g(t', y_0)| \leq M_5|t - t'|^\alpha.$$

The Hölder continuity of  $g$  follows.

Combining these pieces of information, we conclude that

$$g \in C_{\#}^{1+\alpha}(Q_3),$$

QED.

## 6 Applications

### 6.1 Method of continuity

The principle of the method of continuity consists in solving a problem (P) by embedding it into a one-parameter family  $(P_t)$  of problems, such that  $(P_0)$

admits a unique solution, and  $(P_1)$  coincides with problem  $(P)$ . One then proves that the set of parameter values for which  $(P_t)$  admits a unique solution is both open and closed in  $[0, 1]$ . The openness usually follows from the implicit function theorem in Hölder spaces, and the closedness from Ascoli's theorem; thus, both steps are made possible by Schauder estimates.

We give an example in which a simplified procedure based on the contraction mapping principle suffices.

**THEOREM 25** *Let  $L$  be an elliptic operator with  $C^\alpha$  coefficients and  $c \leq 0$ , in a bounded domain  $\Omega$  of class  $C^{2+\alpha}$ . Then, for any  $g \in C^{2+\alpha}(\overline{\Omega})$ ,  $Lu = f$  admits a solution in  $C^{2+\alpha}(\overline{\Omega})$  which is equal to  $g$  on  $\partial\Omega$ .*

**PROOF.** Considering  $u - g$ , we may restrict our attention to the case  $g = 0$ . We let  $L_t u = tLu + (1 - t)\Delta u$  and consider the problem  $(P_t)$  which consists in solving  $L_t u = f$  with Dirichlet conditions.  $L_t$  is a bounded operator from  $C^{2+\alpha}(\overline{\Omega}) \cup \{u = 0 \text{ on } \partial\Omega\}$  to  $C^\alpha(\overline{\Omega})$ . We know that  $L_0$  is invertible, and we wish to invert  $L_1$ . By the maximum principle, the assumption  $c \leq 0$  ensures that any solution of  $(P_t)$  satisfies  $\sup_x |u(x)| \leq C \sup_x |f(x)|$ , with a constant  $C$  independent of  $t$ . Therefore, if  $T$  is the set of  $t$  such that  $L_t$  is invertible, the Schauder estimates show that  $L_t^{-1}$  is bounded, and that its norm admits a bound  $m$  independent of  $t$ . This fact makes the rest of the proof simpler: let  $t \in T$ ; for any  $s$ , the equation  $L_t u = f$  is equivalent to  $u = L_t^{-1} f + M(t, s)u$  where

$$M(s, t)u = (s - t)L_t^{-1}(L_0 - L_1)u.$$

If  $|t - s| < \delta := [m(\|L_0\| + \|L_1\|)]^{-1}$ ,  $M(t, s)$  is a contraction, and  $(P_s)$  is uniquely solvable. Covering  $[0, 1]$  by a finite number of open intervals of length  $\delta$ , we find that  $L_t$  is invertible for every  $t$ . The result follows.  $\square$

For a typical example of the application of the method of continuity, see [2, th. 7.14].

## 6.2 Basic fixed-point theorems for compact operators

We prove several versions of the Schauder fixed-point theorem. The first ingredient in the proofs is the Brouwer fixed-point theorem:

**THEOREM 26** *A continuous mapping  $g : B \rightarrow B$ , where  $B$  is the closed unit ball in  $\mathbb{R}^n$ , has at least one fixed point.*

PROOF. We begin with the case of smooth  $g$ . Assume that  $g$  has no fixed point. Let  $\tilde{x} = x + a(x - g(x))$ , where  $a$  is the largest root of the (quadratic) equation  $|\tilde{x}|^2 = 1$ . The point  $\tilde{x}$  is on the intersection of the segment  $[x, g(x)]$  with the unit sphere, and is chosen so that  $x$  lies between  $\tilde{x}$  and  $g(x)$ . The map from  $B$  to its boundary defined by  $x \mapsto \tilde{x}$  is well-defined and smooth; in fact,

$$0 = |\tilde{x}|^2 - 1 = |x - g(x)|^2 a^2 + 2(x, x - g(x))a + |x|^2 - 1,$$

where  $(\cdot, \cdot)$  denotes the usual dot product. The discriminant of this quadratic is  $4[(x, x - g(x))^2 + (1 - |x|^2)|x - g(x)|^2]$ , which is nonnegative, and vanishes only if  $|x| = 1$  and  $(x, g(x)) = 1$ . Since  $g(x)$  has norm one at most, the Cauchy-Schwarz inequality implies that  $g(x) = x$ , which contradicts the hypothesis. Therefore, our quadratic equation has two distinct real roots, obviously smooth.

For  $|x| = 1$ , we find that  $a = 0$ , since  $(x, x - g(x)) \geq 0$ .

Define  $f : \mathbb{R} \times B \rightarrow \mathbb{R}^n$  by

$$f(t, x_1, \dots, x_n) = x + ta(x)(x - g(x)).$$

We find by inspection that (i) if  $|x| = 1$ ,  $f(t, x) = x$  and  $\partial_t f(t, x) = 0$ ; (ii)  $f(0, x) = x$  for every  $x$  in  $B$ ; (iii)  $|f(1, x)| = 1$  for every  $x$  in  $B$  (by construction of  $a$ ).

Write  $x_0$  for  $t$ , and define the determinants

$$D_i = \det(f_{x_0}, \dots, \hat{f}_{x_i}, \dots, f_{x_n}),$$

where  $i$  runs from 0 to  $n$ ; a hat indicates that the corresponding vector is omitted, and the subscripts denote derivatives. Define further

$$I(t) = \int_B D_0(t, x) dx.$$

We have  $I(0) = 1$  since  $f(0, x) = x$ . For  $t = 1$ , since  $f$  lies on the boundary of the unit sphere,  $f_{x_1}, \dots, f_{x_n}$  are all tangent to the sphere, and are linearly dependent; therefore,  $I(1) = 0$ .

We prove that  $I(t)$  is constant, which will generate a contradiction to the hypothesis that  $g$  has no fixed point. We need the

LEMMA 27  $\sum_{i=0}^n (-1)^i \partial_{x_i} D_i = 0$ .

PROOF. We have, for every  $i$ ,

$$\partial_{x_i} D_i = \sum_{j < i} (-1)^j C_{ij} + \sum_{j > i} (-1)^{j-1} C_{ij},$$

where

$$C_{ij} = \det(f_{x_i x_j}, f_{x_0}, \dots, \hat{f}_{x_i}, \dots, \hat{f}_{x_j}, \dots, f_{x_n}) = C_{ji}.$$

Therefore  $\sum_{i=0}^n (-1)^i \partial_{x_i} D_i = \sum_{i,j=0}^n (-1)^{i+j} C_{ij} \sigma_{ij}$ , where  $\sigma_{ij} = 1$  for  $j < i$ ,  $-1$  for  $j > i$ , and zero for  $i = j$ . Since  $(-1)^{i+j} C_{ij}$  is symmetric in  $i$  and  $j$ , and  $\sigma_{ij}$  is antisymmetric, the result follows.  $\square$

Now, for  $i > 0$ ,  $D_i$  vanishes on the boundary of  $B$  because  $\partial_t f = 0$  there. If  $n_i$  is the  $i$ -th component of the outward normal to  $B$ , we find

$$\int_B \partial_{x_i} D_i dx = \int_{\partial B} n_i D_i ds = 0.$$

(This may be proved without using Stokes' theorem, by integrating with respect to the  $x_i$  variable keeping the others fixed.) Using the lemma, we find

$$\frac{dI(t)}{dt} = \int_B \partial_t D_0 dx = \int_B \sum_{i>0} \pm \partial_{x_i} D_i dx = 0.$$

This completes the proof in the smooth case.

Finally, we extend the result to the case of continuous  $g$ . By the Stone-Weierstrass theorem, there is a sequence of polynomial (vector-valued) mappings  $p_n$  such that  $|g - p_n| \leq \varepsilon_n \rightarrow 0$  uniformly over  $B$ . Since  $p_n/(1 + \varepsilon_n)$  maps  $B$  to itself, there is a  $y_n$  such that  $p_n(y_n) = (1 + \varepsilon_n)y_n$ . Extracting a subsequence, we may assume  $y_n$  has a limit  $y$ . It follows that  $g(y) = y$ , QED.  $\square$

The Brouwer fixed-point theorem may be extended as follows:

**THEOREM 28** *Let  $K$  be the closed convex hull of a set of  $N$  vectors  $x_1, \dots, x_N$  in  $n$ -dimensional space. A continuous map from  $K$  to itself has a fixed point.*

PROOF. Let  $\bar{x} = \frac{1}{N} \sum_k x_k$ . Decreasing  $n$  if necessary, and re-labeling the  $x_k$ , we may assume that the  $(x_k - \bar{x})_{k \leq n}$  generate  $\mathbb{R}^n$ . We prove that  $K$  is homeomorphic to the unit ball, so that the result follows from the Brouwer fixed point theorem. First,  $\bar{x}$  is interior to  $K$ , because,  $\bar{x} + \sum_{k \leq n} \varepsilon_k (x_k - \bar{x})$  is a convex combination of the  $x_k$  if the  $\varepsilon_k$  are small enough. Let  $\varepsilon$  be such that  $B_\varepsilon(\bar{x}) \subset \text{int } K$ . Let, for any unit vector  $y$ ,  $s(y) = \sup\{s : \bar{x} + sy \in L\}$ . It is well-defined, and bounded; also,  $s(y) \geq \varepsilon$ . We need the following lemma.

LEMMA 29  $s(y)$  is continuous.

PROOF. If  $y_m \rightarrow y$  and  $s(y_m) \rightarrow s$  as  $m \rightarrow \infty$ , with  $\bar{x} + s(y_m)y_m \in K$  for all  $m$ , we find  $\bar{x} + sy \in K$ , hence  $s \leq s(y)$ . If  $s' < s(y)$ , define  $t = s'/s(y) \in [0, 1]$ ,  $(1-t)B_\varepsilon(\bar{x}) + ts(y)y$  is included in  $K$  (which is convex), and is a neighborhood of  $\bar{x} + s'y$ . This implies that  $\bar{x} + s'y_m \in K$  for  $m$  sufficiently large; it follows that  $s(y_m) \geq s'$  for  $m$  large. Therefore,  $s \geq s(y)$ .  $\square$

We now construct the required homeomorphism from  $B$  to  $K$  by letting  $x \mapsto xs(x/|x|)$ , which inverse  $x \mapsto x/s(x/|x|)$ . We just proved that these maps are continuous at all points other than 0; the continuity at the origin follows from the fact that  $s$  and  $1/s$  are bounded.  $\square$

We now turn to fixed-point theorems in infinite dimensions.

THEOREM 30 If  $K$  is a compact convex subset of a Banach space  $E$ , and  $T : K \rightarrow K$  is continuity, then  $T$  admits a fixed point.

PROOF. For any integer  $p$ , there is an integer  $N = N(p)$  and points  $x_1, \dots, x_N$  in  $K$  such that  $K \subset B(x_1, 1/p) \cap \dots \cap B(x_N, 1/p)$ . Let  $B_k = B(x_k, 1/p)$ . Consider the closed convex hull  $K_p$  of  $x_1, \dots, x_N$  which is a convex set which lies in some finite-dimensional subspace of  $E$ ; it is a subset of  $K$ . The map

$$F_p : x \mapsto \frac{\sum_k x_k d(x, K \setminus B_k)}{\sum_k d(x, K \setminus B_k)}$$

is well-defined and continuous (the denominator does not vanish because the  $B_k$  cover  $K$ ). Since any term on the numerator contributes to the sum only if  $|x - x_k| \leq 1/p$ , we have  $\|F_p(x) - x\|_E \leq 1/p$ .

The map  $F_p \circ T$  therefore admits a fixed point  $y_p$ :  $F_p(T(y_p)) = y_p$ . We may extract a subsequence  $y_{p'}$  which tends to  $y \in K$ . We have  $T(y_{p'}) \rightarrow T(y)$ , and  $\|F_{p'}(T(y_{p'})) - T(y_{p'})\|_E \rightarrow 0$ . It follows that  $T(y) = y$ .  $\square$

THEOREM 31 If  $K$  is a closed convex subset of a Banach space  $E$ , and  $T : K \rightarrow K$  is continuous, then, if  $T(K)$  has compact closure, then  $T$  admits a fixed point.

PROOF. One approach would consist in working in the closure of the convex hull of  $T(K)$ ; this requires first proving that this set is compact. A more direct argument is to apply the same method of proof as in the previous theorem, with the difference that  $K$  is replaced by the closure of  $T(K)$  in

the definition of  $F_p$ . The map  $F_p \circ T$  is continuous on the closed convex hull of  $x_1, \dots, x_N$ , and therefore has a fixed point  $y_p$  as before. We may extract a subsequence  $y_{p'}$  such that  $Ty_{p'}$  tends to some  $z$  in the closure of  $T(K)$ . Since  $\|F_{p'}(T(y_{p'})) - T(y_{p'})\|_E \rightarrow 0$ ,  $y_{p'}$  also tends to  $z$ . It follows that  $Tz = z$ .  $\square$

A useful variant is the following:

**THEOREM 32** *Let  $F$  be a continuous mapping from the closed unit ball in a Banach space  $E$ , with values in  $E$  and with precompact image. If  $\|x\|_E = 1$  implies  $\|T(x)\|_E < 1$ , then  $T$  has a fixed point.*

**PROOF.** It suffices to consider the mapping

$$S : x \mapsto T(x) / \max(1, \|T(x)\|_E),$$

which is continuous with precompact image from the unit ball to itself. It therefore possesses a fixed point  $y$ . If  $\|T(y)\|_E \geq 1$ , we find that  $y = T(y) / \|T(y)\|_E$  has norm 1; the assumption now yields  $\|T(y)\|_E < 1$  : contradiction. Therefore  $\|T(y)\|_E < 1$  and  $T(y) = y$ , QED.  $\square$

The next theorem asserts the existence of a fixed point as soon as we have an *a priori* bound. Let  $E$  denote a Banach space. Recall that a compact operator is an operator which maps bounded sets to relatively compact sets.

**THEOREM 33** *Let  $S : E \rightarrow E$  be compact, and assume that there is a  $r > 0$  such that if  $u$  solves  $u = \sigma S(u)$  for some  $\sigma \in [0, 1]$ ,  $\|u\|_E < r$ . Then  $S$  admits a fixed point in the ball of radius  $r$  in  $E$ .*

**PROOF.** Let  $T(u) = S(u)$  if  $\|S(u)\|_E \leq r$  and  $T(u) = rS(u) / \|S(u)\|_E$  otherwise. Then the previous theorem applies and yields a fixed point  $u$  for  $T$ . If  $\|S(u)\|_E \geq r$ ,  $\|T(u)\|_E = r$  and  $u = T(u) = \sigma S(u)$ , with  $\sigma = r / \|S(u)\|_E \in [0, 1]$ . Therefore,  $\|u\|_E < r$ . Since  $u = T(u)$ , we find  $\|T(u)\|_E < r$ , which is impossible. Therefore,  $\|S(u)\|_E < r$ , and  $u = T(u) = S(u)$ , QED.  $\square$

We note two useful variants:

**THEOREM 34** *Let  $T : \mathbb{R} \times E \rightarrow E$  be compact, and satisfy  $T(0, u) = 0$  for every  $u \in E$ . Let  $C_{\pm}$  denote the connected component of  $(0, 0)$  in the set*

$$\{(\lambda, u) \in \mathbb{R} \times E : u = T(\lambda, u) \text{ and } \pm \lambda \geq 0\}.$$

*Then  $C_+$  and  $C_-$  are both unbounded.*

For this result, see [54, 66].

**THEOREM 35** *Let  $T : [0, 1] \times E \rightarrow E$  be compact, and satisfy  $T(0, u) = 0$  for every  $u \in E$ . Assume that the relation  $u = T(\sigma, u)$  implies  $\|u\|_E < r$ . Then equation  $T(x, 1) = x$  has a solution.*

**PROOF.** Changing the norm on  $E$ , we may assume that  $r = 1$ .

Let  $\varepsilon > 0$ , and consider the mapping  $F_\varepsilon$  defined by

$$F_\varepsilon(x) = T\left(\frac{x}{\|x\|_E}, \frac{1 - \|x\|_E}{\varepsilon}\right) \text{ if } 1 - \varepsilon \leq \|x\|_E \leq 1,$$

and

$$F_\varepsilon(x) = T\left(\frac{x}{1 - \varepsilon}, 1\right) \text{ if } \|x\|_E \leq 1 - \varepsilon,$$

which is continuous with precompact image. Note that

$$F_\varepsilon(x) = T\left(\frac{x}{\max(1 - \varepsilon, \|x\|_E)}, \min(1, \frac{1 - \|x\|_E}{\varepsilon})\right).$$

If  $\|x\|_E = 1$ ,  $F_\varepsilon(x) = 0$ . Theorem 31 applies, and yields  $x_\varepsilon$  in the (open) unit ball such that  $F_\varepsilon(x_\varepsilon) = x_\varepsilon$ . For any integer  $k \geq 1$ , let  $y_p = x_{1/p}$ , and  $\sigma_p = \min(p(1 - \|y_p\|_E), 1)$ . Since the image of  $T$  is precompact and the  $\sigma_p$  are bounded, we may extract a subsequence such that  $(x_{p'}, \sigma_{p'})$  tends to a point  $(x_\infty, \sigma_\infty) \in E \times [0, 1]$ .

If  $\sigma_\infty < 1$ , all  $\sigma_{p'}$  are less than 1 for large  $p'$ , which means that  $1 - \|y_{p'}\|_E \geq 1/p'$ . It follows that  $\|x_\infty\| = 1$ . The relation  $x_\infty = T(x_\infty, \sigma_\infty)$  now implies that  $\|x_\infty\| < 1$ : contradiction.

Therefore,  $\sigma_\infty = 1$ . From the second expression for  $F_\varepsilon$ , it follows, by passing to the limit, that  $x_\infty = T(x_\infty, 1)$ , so that  $x \mapsto T(x, 1)$  has a fixed point, QED.  $\square$

### 6.3 Fixed-point theory and the Dirichlet problem

We now apply the abstract theorems we just proved.

We begin with an application of theorem 33. Let  $\alpha$  and  $\beta$  denote two numbers in  $(0, 1)$ . Consider the non-linear operator

$$A : u \mapsto \sum_{ij} a^{ij}(x, u, \nabla u) \partial_{ij} u + b(x, u, \nabla u),$$

where  $a^{ij}$  and  $b$  are of class  $C^\alpha$  in their arguments say, globally, to fix ideas.<sup>14</sup> Let  $g$  be a function of class  $C^{2+\alpha}(\overline{\Omega})$ . We wish to solve  $Au = 0$  in  $\Omega$ , with  $u = g$  on the boundary.

To  $A$ , we associate linear operators  $A_v$ , parameterized by a function  $v$ :

$$A_v : u \mapsto \sum_{ij} a^{ij}(x, v, \nabla v) \partial_{ij} u + b(x, v, \nabla v),$$

and an operator  $T$  defined for  $v \in C^{1+\beta}(\overline{\Omega})$ , by  $T(v) = u$ , where  $u$  is the solution of the Dirichlet problem for equation

$$A_v u = 0$$

in  $\Omega$ , with  $u = g$  on the boundary. Since  $b(x, v, \nabla v)$  is easily seen to be of class  $C^{\alpha\beta}$ , the Schauder estimates ensure that  $u$  thus defined belongs to  $C^{2+\alpha\beta}(\overline{\Omega})$ . Note that  $u = \sigma T(u)$  means that  $\sum_{ij} a^{ij}(x, u, \nabla v) \partial_{ij} u + \sigma b(x, u, \nabla u) = 0$  in  $\Omega$ , and  $u = \sigma g$  on the boundary.

**THEOREM 36** *If there is a  $\beta \in (0, 1)$  such that solutions in  $C^{2+\alpha\beta}$  of equation  $A(u) = 0$  in  $\Omega$ , with  $u = \sigma g$  on the boundary admit an a priori bound of the form  $\|u\|_{C^{1+\beta}} \leq M$ , with  $M$  independent of  $u$  and  $\sigma \in [0, 1]$ , then equation  $A(u) = 0$  admits at least one solution with  $u = g$  on the boundary.*

**PROOF.** Operator  $T$  maps bounded sets of  $C^{1+\beta}$  to bounded sets of  $C^{2+\alpha\beta}$ , which, by Ascoli's theorem, are relatively compact in  $C^{1+\beta}$ . If  $v_n \rightarrow v$  in  $C^{1+\beta}$ , the functions  $u_n = T(v_n)$  are bounded in  $C^{2+\alpha\beta}$  by Schauder estimates, and therefore, admit a convergent subsequence  $u_{n'} \rightarrow u$  in the  $C^2$  topology, and *a fortiori* in  $C^{1+\beta}$ . Since

$$\sum_{ij} a^{ij}(x, v_n, \nabla v_n) \partial_{ij} u_n + b(x, v_n, \nabla v_n) = 0,$$

it follows that  $A_v(u) = 0$ . Therefore  $T$  is continuous and compact. The result now follows from theorem 33.  $\square$

We now turn to an application of theorem 35, which arises naturally if we wish  $\sigma$  to enter in the definition of  $A_v$ —which gives some flexibility in the perturbation argument. We simply define  $u = T(v, \sigma)$  by solving

$$\sum_{ij} a^{ij}(x, v, \nabla v, \sigma) \partial_{ij} u + b(x, v, \nabla v, \sigma) = 0,$$

---

<sup>14</sup>In many cases, the argument below automatically yields *a priori* bounds for  $u$  and its derivatives, so that one may truncate the nonlinearities for large values of their arguments.

with  $u = \sigma g$  on the boundary. Here again, the existence of an *a priori*  $C^{1+\beta}$  bound enables one to conclude that  $T(v, 1)$  has a fixed point.

## 6.4 Eigenfunctions and applications

Since the inverse of the Laplacian (with Dirichlet boundary condition) is compact, Riesz-Fredholm theory (see [11]) ensures that the Laplacian admits a sequence of real eigenvalues of finite multiplicity, tending to  $+\infty$ . The Fredholm alternative holds:  $\Delta u + \lambda u = f$  is solvable if and only if  $f$  is orthogonal to the eigenspace corresponding to the eigenvalue  $\lambda$ .

We mention two important techniques related to Schauder theory: bifurcation from a simple eigenvalue (see [68, 74], and the Krein-Rutman theorem (see [52, 66, 74]).

### 6.4.1 Bifurcation from a simple eigenvalue

Consider, to fix ideas, the problem

$$-\Delta u + \lambda u = u^2 \text{ on } \Omega,$$

with Dirichlet boundary conditions. Assume we have an eigenfunction  $\phi_0$  for the simple eigenvalue  $\lambda_0$ :

$$-\Delta \phi_0 + \lambda_0 \phi_0 = 0,$$

with  $\phi_0 = 0$  on the boundary. Let  $Q[u] = \int_{\Omega} u \phi_0 dx$  and  $P[u] = u - \phi_0 Q[u]$ . We seek a family  $(\mu(\varepsilon), v(\varepsilon))$  such that our nonlinear problem admits the solutions  $(\lambda, u)$ , where

$$u = \varepsilon \phi_0 + \varepsilon^2 v(\varepsilon); \quad \lambda = \lambda_0 + \varepsilon \mu(\varepsilon).$$

In other words, we seek a curve of solutions which is tangent to the eigenspace for the eigenvalue  $\lambda_0$ . If  $\varepsilon \mu$  is small, it is easy to see that  $-\Delta + \lambda$  is invertible on the orthogonal complement of this eigenspace. Projecting on the orthogonal complement of  $\phi_0$ , we find

$$v = (-\Delta + \lambda)^{-1} P[(\phi_0 + \varepsilon v)^2],$$

which may be solved for  $v$  as a function of  $\mu$ , by the implicit function theorem. This gives a map  $v = \Psi[\varepsilon, \mu]$ . Projecting the equation on  $\phi_0$  now yields an equation for  $\mu(\varepsilon)$ :

$$\mu(\varepsilon) = Q[(\phi_0 + \varepsilon \Psi[\varepsilon, \mu])^2],$$

which may be solved for  $\mu(\varepsilon)$ , again by an implicit function theorem. We find  $\mu(\varepsilon) = Q[\phi_0^2] + O(\varepsilon)$ . For variants of this argument, see *e.g.* [45, Ch. 5].

### 6.4.2 Krein-Rutman theorem

We wish to generalize to infinite dimensions a classical property of matrices with nonnegative entries.

We first need a variant of theorem 34, which follows from it using an extension theorem due to Dugundji (see [28, 66, 74]).

**THEOREM 37** *Let  $K$  be a closed convex cone with vertex 0, and let  $T : \mathbb{R}^+ \times K \rightarrow K$  be compact, and assume  $T(0, u) = 0$  for every  $u$ . Then the connected component of  $(0, 0)$  in the set of all solutions  $(\lambda, u)$  of  $u = T(\lambda, u)$  is unbounded.*

As a consequence, we derive the “compression of a cone” theorem:

**THEOREM 38** *Let  $K$  be a closed convex cone with vertex 0 and non-empty interior, with the property*

$$K \cap (-K) = \{0\}.$$

*Let  $L$  denote a bounded linear operator on  $E$  which maps  $K \setminus \{0\}$  to the interior of  $K$ . Then there is a unit vector in  $K$  and a positive real  $\mu$  such that  $Lx_0 = \mu$ .*

*Remark 7 A typical application: let  $E = C^{1+\alpha}(\Omega)$ , with  $\Omega$  bounded and smooth, take for  $L$  the inverse of an elliptic operator, such as  $-\Delta + c(x)$ , with  $c \geq 0$ , and for  $K$  the closure of  $\{u \in E : u > 0 \text{ in } \Omega, \text{ and } \partial u / \partial n < 0 \text{ on } \partial\Omega\}$ , where  $\partial / \partial n$  denotes the outward normal derivative. As usual, the compactness is ensured by the Schauder estimates. The fact that  $L$  is a “compression” of the cone  $K$ , i.e. sends  $K \setminus \{0\}$  to the interior of  $K$ , follows from the Hopf maximum principle. Note that the conclusion  $x_0 \in K$  gives directly the information that the first eigenfunction is positive throughout  $\Omega$ .*

**PROOF.** In this proof only, we write  $u \geq v$  when  $u - v \in K$ . Fix  $u \in K \setminus \{0\}$ ; in particular,  $Lu$ , which is interior to  $K$ , cannot be equal to 0. There is a positive  $M$  such that  $Lu \geq u/M$ , for otherwise, we would have  $Lu - u/M \notin K$  for all  $M > 0$ , and, letting  $M \rightarrow \infty$ , we would find  $Lu \notin \text{int } K$ .

For any  $\varepsilon > 0$ , consider the compact operator defined by  $T_\varepsilon(\lambda, x) = \lambda L(x + \varepsilon u)$ . Let  $C_\varepsilon$  be the connected component of  $(0, 0)$  in  $\mathbb{R}_+ \times K$  of the set of solutions of  $x = T_\varepsilon(\lambda, x)$ ; we know that it is unbounded. For such a solution, we have, since  $x \in K$ ,  $x = \lambda Lx + \lambda \varepsilon u \geq \lambda \varepsilon u$ . Since  $K$  is invariant under  $L$ , we find  $Lx \geq \lambda \varepsilon Lu \geq \lambda \varepsilon u/M$ . We also have  $x \geq \lambda Lx$ ; therefore,  $x \geq \lambda^2 \varepsilon u/M$ , and  $Lx \geq (\lambda/M)^2 \varepsilon u$ . By induction, we find  $Lx \geq (\lambda/M)^n \varepsilon u$  for every  $n \geq 1$ . If  $\lambda > M$ , we find, letting  $n \rightarrow \infty$ , that  $\varepsilon u \leq 0$ , which means  $u \in -K$ . Since  $u \in K$ , and  $u \neq 0$ , this is impossible. Therefore,  $C_\varepsilon$  lies in  $[0, M] \times K$ . Since  $C_\varepsilon$  is unbounded and contains  $(0, 0)$ , there is, for every  $\varepsilon > 0$ , a unit vector  $x_\varepsilon \in K$  such that

$$x_\varepsilon = \lambda_\varepsilon L(x_\varepsilon + \varepsilon u) \text{ and } 0 \leq \lambda_\varepsilon \leq M.$$

Since  $L$  is compact, there is a sequence  $\varepsilon_n \rightarrow 0$  and a  $(\mu, x_0) \in [0, M] \times K$  such that  $x_{\varepsilon_n} \rightarrow x_0$  and  $\lambda_{\varepsilon_n} \rightarrow \mu$ . It follows that  $x_0 = \mu Lx_0$  and  $\|x_0\|_E = 1$ . Since  $x_0 \neq 0$ , we must have  $\mu > 0$  and also  $x_0 \in \text{int } K$ . This completes the proof.  $\square$

## 6.5 Method of sub- and super-solutions

Consider the problem

$$-\Delta u = f(u) \tag{28}$$

with Dirichlet boundary conditions on a smooth bounded domain  $\Omega$ , and  $f$  smooth, such that  $f$  and  $df/du$  are both bounded.<sup>15</sup> We assume that we are given two *ordered sub- and super-solutions*  $v$  and  $w$ :  $v$  and  $w$  are of class  $C^2(\bar{\Omega})$ , vanish on  $\partial\Omega$  and satisfy, over  $\Omega$ ,

$$v \leq w; \quad -\Delta v \leq f(v); \quad -\Delta w \geq f(w).$$

We then have:

**THEOREM 39** *Problem (28) admits two solutions  $\underline{u}$  and  $\bar{u}$  such that*

$$v \leq \underline{u} \leq \bar{u} \leq w.$$

*In addition, if  $u$  is any solution of (28) which lies between  $v$  and  $w$ , then necessarily  $\underline{u} \leq u \leq \bar{u}$ .*

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<sup>15</sup>The boundedness condition is not as restrictive as it seems: for instance, if  $u$  represents a concentration, it must lie between 0 and 1, and  $f$  may be redefined outside  $[0, 1]$  so that it is bounded.

*Remark 8* For more results of this kind, see e.g. [67, 68].

PROOF. Choose a constant  $m$  such that  $g(u) := f(u) + mu$  is strictly increasing. Define inductively two sequences  $(v_j)_{j \geq 0}$  and  $(w_j)_{j \geq 0}$  by the relations:  $v_0 = v$ ;  $w_0 = w$ ;

$$-\Delta v_j + mv_j = g(v_{j-1}); \quad -\Delta w_j + mw_j = g(w_{j-1}) \text{ for } j \geq 1,$$

and  $v_j = w_j = 0$  on  $\partial\Omega$ . We have  $(-\Delta + m)(v_1 - v_0) \geq g(v_0) - g(v_0) = 0$ , which implies  $v_1 \geq v_0$  by the maximum principle.<sup>16</sup> Since  $(-\Delta + m)(v_{j+1} - v_j) = g(v_j) - g(v_{j-1})$ , we find by induction  $(-\Delta + m)(v_{j+1} - v_j) \geq 0$ , hence  $v_{j+1} - v_j \geq 0$ . Therefore, the sequence  $(v_j)$  is non-decreasing. Similarly,  $(w_j)$  is non-increasing. In addition,  $(-\Delta + m)(w_0 - v_0) = g(w_0) - g(v_0) \geq 0$ , and  $(-\Delta + m)(w_j - v_j) = g(w_{j-1}) - g(v_{j-1})$  for  $j \geq 1$ . It follows that  $w_0 \geq v_0$  and, by induction,  $w_j \geq v_j$ . We conclude that  $\underline{u} := \lim_{j \rightarrow \infty} v_j$  and  $\bar{u} := \lim_{j \rightarrow \infty} w_j$  exist and satisfy

$$v_0 \leq v_1 \leq \cdots \underline{u} \leq \bar{u} \leq \cdots w_1 \leq w_0.$$

By construction, the  $v_j$  are bounded. Therefore,  $(-\Delta + m)v_j$  is bounded independently of  $j$ . Consider now any ball  $B_r$  such that  $\bar{B}_{2r} \subset \Omega$ , and fix  $\alpha \in (0, 1)$ . The interior  $C^{1+\alpha}$  Schauder estimates ensure first that the  $v_j$  are, for  $j \geq 1$  bounded in  $C^1(B_{3r/2})$ , independently of  $j$ . This implies in particular a  $C^\alpha$  bound on  $g(v_j)$ . The  $C^{2+\alpha}$  Schauder estimates now ensure that the  $v_j$  are bounded in  $C^2(B_r)$  for  $j \geq 1$ , and that their second derivatives are equicontinuous. It follows that one may extract a subsequence  $v_{j'}$  which converges to  $\underline{u}$  in  $C^2(B_r)$ . It follows that  $(-\Delta + m)\underline{u} = f(\underline{u}) + m\underline{u}$ ; so that  $\underline{u}$  solves (28). A similar argument applies to  $\bar{u}$ . Finally, if  $u$  is a solution such that  $v_0 \leq u \leq w_0$ , we have  $(-\Delta + m)(v_0 - u) \leq g(v_0) - g(u)$  and  $(-\Delta + m)(v_j - u) = g(v_{j-1}) - g(u)$  for  $j \geq 1$ . It follows, by induction, that  $v_j \leq u$  for all  $j$ . Similarly,  $w_j \geq u$  for all  $j$ . Passing to the limit, we find  $\underline{u} \leq u \leq \bar{u}$ .  $\square$

## 6.6 Asymptotics near isolated singularities or at infinity

We give three simple examples where Schauder estimates help understand the behavior of solutions at infinity or at isolated singularities.

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<sup>16</sup>See e.g. [11] for a simple proof.

### 6.6.1 Liouville property

Regularity theory gives a simple proof of the Liouville property for scale-invariant equations. Consider for instance the  $p$ -Laplace equation  $A_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$ , where  $p > 1$ . We have [44] an interior  $C^1$  estimate of the form

$$\|u\|_{C^1(B_1)} \leq C \sup_{B_2} |u|.$$

Applying it to  $u(Rx)$ , we find, since  $\nabla(u(Rx)) = R(\nabla u)(Rx)$ ,

$$\sup_{B_R} |\nabla u| \leq \frac{C}{R} \sup_{B_{2R}} |u|.$$

Letting  $R \rightarrow \infty$ , it follows immediately that any solution which is bounded on all of  $\mathbb{R}^n$  is constant. A more subtle result of this type is: any nonnegative solution on  $\mathbb{R}^n \setminus 0$  is necessarily constant [51, p. 602].

### 6.6.2 Asymptotics at infinity

If  $u$  solves  $Lu = f$  on an exterior domain  $\{|x| > \rho\}$ , where the coefficients of  $L$  tend to constants at infinity, one may hope to apply the above scaling argument on balls  $B_R(x_R)$ , where, say,  $|x_R| \geq 2R \rightarrow \infty$ . In this way, it is possible to obtain weighted estimates at infinity, which are useful in solving the constraints equations in General Relativity [22] or in asymptotics for solutions of the Ginzburg-Landau equation [65].

### 6.6.3 Asymptotics near isolated singularities

The  $C^{1+\alpha}$  Schauder-type estimates for the  $p$ -Laplace equation  $A_p u = 0$  may be used to determine the behavior at the origin of positive solutions in a punctured neighborhood of the origin. For instance, if  $n \geq 2$  and  $p < n$  and

$$\mu(r) = \frac{p-1}{n-p} (n\omega_n)^{-1/(p-1)} r^{(p-n)/(p-1)},$$

resp.  $\mu(r) = (n\omega_n)^{-1/(n-1)} \ln(1/r)$  for  $p = n$ , then any solution which is bounded above and below by positive multiples of  $\mu(|x|)$  must in fact be of the form  $\gamma\mu(|x|) + O(1)$  for some constant  $\gamma$ . In fact,

$$-A_p u = \gamma|\gamma|^{p-2} \delta_0,$$

in the sense of distributions, where  $\delta_0$  is the Dirac distribution at the origin.

Regularity estimates enter the argument as follows: to consider the family of functions  $u_r(y) = u(ry)/\mu(r)$ , which, by Schauder-type  $C^{1+\alpha}$  estimates, satisfies a compactness condition on annular domains. Letting  $r \rightarrow 0$  along a suitable sequence, we find that  $u_r$  tends to a solution  $v$  of  $A_p v = 0$  outside the origin, and we may arrange so that  $v(y)/\mu(|y|)$  has an interior maximum  $\gamma$ . At such a maximum, the gradient of  $v$  is proportional to the gradient of  $\mu$  and thus does not vanish, so that the equation is in fact uniformly elliptic near the point of maximum; this makes it possible to conclude that  $v/\mu$  is in fact constant, using the strong maximum principle (as pointed out in [35, p. 263], the difference  $w = u - \gamma\mu$  solves a linear elliptic equation). See [44, 51] for details and further results. For  $p = n$ , one can see that  $u - \gamma\mu$  has a limit at the origin; this fact has found recent applications [3, 23]. For similar results for semilinear equations, see [33, 18].

## 6.7 Asymptotics for boundary blow-up

We give a typical application of Fuchsian reduction to elliptic problems [48, 49]. The proof structure hinges on general properties of the Fuchsian Reduction process and is therefore liable of application to many other situations.

### 6.7.1 Main result and structure of proof

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 3$ , be a bounded domain of class  $C^{2+\alpha}$ , where  $0 < \alpha < 1$ . Consider the Loewner-Nirenberg equation in the form

$$-\Delta u + n(n-2)u^{\frac{n+2}{n-2}} = 0. \quad (29)$$

It is known [56, 5, 7, 6, 57] that this equation admits a maximal solution  $u_\Omega$ , which is positive and smooth inside  $\Omega$ ; it is the limit of the increasing sequence  $(u_m)_{m \geq 1}$  of solutions of (29) which are equal to  $m$  on the boundary. It arises in many contexts [5, 56]. We note for later reference the monotonicity property: if  $\Omega \subset \Omega'$ , then any classical solution in  $\Omega'$  restricts to a classical solution in  $\Omega$ , so that

$$u_{\Omega'} \leq u_\Omega; \quad (30)$$

it follows easily from the maximality of  $u_\Omega$ . The *hyperbolic radius* of  $\Omega$  is the function

$$v_\Omega := u_\Omega^{-2/(n-2)};$$

it vanishes on  $\partial\Omega$ . Let  $d(x)$  denote the distance of  $x$  to  $\partial\Omega$ . It is of class  $C^{2+\alpha}$  near  $\partial\Omega$ . We prove

**THEOREM 40** *If  $\Omega$  is of class  $C^{2+\alpha}$ , then  $v_\Omega \in C^{2+\alpha}(\bar{\Omega})$ , and*

$$v_\Omega(x) = 2d(x) - d(x)^2[H(x) + o(1)]$$

as  $d(x) \rightarrow 0$ , where  $H(x)$  is the mean curvature at the point of  $\partial\Omega$  closest to  $x$ .

This result is optimal, since  $H$  is of class  $C^\alpha$  on the boundary. It follows from theorem 40 that  $v_\Omega$  is a *classical solution* of

$$v_\Omega \Delta v_\Omega = \frac{n}{2}(|\nabla v_\Omega|^2 - 4),$$

even though  $u_\Omega$  cannot be interpreted as a weak solution of (29), insofar as  $u_\Omega^{\frac{n+2}{n-2}} \sim (2d)^{-1-n/2} \notin L^1(\Omega)$ .

We now give an idea of the proof.

We begin by performing a Fuchsian reduction, that is, we introduce the degenerate equation solved by a renormalized unknown, which governs the higher-order asymptotics of the solution; in this case, a convenient renormalized unknown is

$$w := (v_\Omega - 2d)/d^2.$$

It follows from general arguments, see the overview in [49, 50], that the equation for  $w$  has a very special structure: the coefficient of the derivatives of order  $k$  is divisible by  $d^k$  for  $k = 0, 1$  and  $2$ , and the nonlinear terms all contain a factor of  $d$ . Such an equation is said to be *Fuchsian*.

In the present case, one finds

$$\frac{2v^{n/2}}{n-2} \{-\Delta u_\Omega + n(n-2)u_\Omega^{(n+2)/(n-2)}\} = Lw + 2\Delta d - M_w(w), \quad (31)$$

where

$$L := d^2\Delta + (4-n)d\nabla d \cdot \nabla + (2-2n),$$

and  $M_w$  is a linear operator with  $w$ -dependent coefficients, defined by

$$M_w(f) := \frac{nd^2}{2(2+dw)} [2f\nabla d \cdot \nabla w + d\nabla w \cdot \nabla f] - 2df\Delta d.$$

The proof now consists in a careful bootstrap argument in which better and better information on  $w$  results in better and better properties of the degenerate linear operator  $L - M_w$ . A key step is the inversion of the analogue of  $L$  in the half-space, which plays the role of the Laplacian in the usual Schauder theory.

Equation (31) needs only to be studied in the neighborhood of the boundary. Let us therefore introduce  $C^{2+\alpha}$  thin domains  $\Omega_\delta = \{0 < d < \delta\}$ , such that  $d \in C^{2+\alpha}(\overline{\Omega}_\delta)$ , and  $\partial\Omega_\delta = \partial\Omega \cup \Gamma$  consists of two hypersurfaces of class  $C^{2+\alpha}$ .

Recall that

$$\|u\|_{C_{\#}^{k+\alpha}(\overline{\Omega}_\delta)} := \sum_{j=0}^k \|d^j u\|_{C^{j+\alpha}(\overline{\Omega}_\delta)}.$$

The proof proceeds in five steps, corresponding to five theorems: first, a comparison argument combined with Schauder estimates gives

**THEOREM 41**  *$w$  and  $d^2\nabla w$  are bounded near  $\partial\Omega$ .*

Theorem 41 ensures that  $L - M_w$  is of type (I). Theorem 22 then implies that  $d\nabla w$  is bounded near the boundary; going back to the definition of  $M_w$ , we find  $M_w(w) = O(d)$ ; this yields the next theorem:

**THEOREM 42**  *$d\nabla w$  and  $M_w(w)/d$  are bounded near  $\partial\Omega$ .*

At this stage, we have  $Lw + 2\Delta w = O(d)$ . In order to use theorem 22, we need to subtract from  $w$  a function  $w_0$  such that  $Lw_0 + 2\Delta = 0$  with controlled boundary behavior, and  $w - w_0 = O(d)$ ; the function  $w_0$  is constructed in:

**THEOREM 43** *If  $\delta$  is sufficiently small, there is a  $w_0 \in C_{\#}^{2+\alpha}(\overline{\Omega}_\delta)$  such that*

$$Lw_0 + 2\Delta d = 0 \tag{32}$$

*in  $\Omega_\delta$ . Furthermore*

$$w_0|_{\partial\Omega} = -H, \tag{33}$$

*where  $H = -(\Delta d)/(n-1)$  is the mean curvature of the boundary.*

Incidentally, we see how the curvature of the boundary enters into the asymptotics. We now use a comparison function of the form  $w_0 + Ad$ , where  $A$  is a constant, to bound  $w - w_0$ :

THEOREM 44 *Near the boundary,*

$$\tilde{w} := w - w_0 = O(d).$$

At this stage, we know that

$$L\tilde{w} = O(d) \text{ and } \tilde{w} = O(d)$$

near  $\partial\Omega$ . Theorem 23 yields that  $\tilde{w}$  is in  $C_{\#}^{1+\alpha}(\overline{\Omega}_{\delta})$ , for  $\delta$  small enough. It follows that  $M_w(w) \in C^{\alpha}(\overline{\Omega}_{\delta})$ . We may now use theorem 24 to conclude that  $d^2w$  is of class  $C^{2+\alpha}$  near the boundary. Since  $\tilde{w} = O(d)$ ,  $w|_{\partial\Omega}$  is equal to  $-H$ . This completes the proof of theorem 40.

We write henceforth  $u$  and  $v$  for  $u_{\Omega}$  and  $v_{\Omega}$  respectively. The rest of this section is devoted to the proofs of the above theorems.

It remains to prove theorems 41, 43 and 44.

Theorem 41 is proved in section 6.8.4 by a comparison argument combined with regularity estimates, as in section 6.6.3.

Theorem 43 is proved in three steps: first, one decomposes  $L$  into a sum  $L_0 + L_1$  in a coordinate system adapted to the boundary, where  $L_0$  is the analogue of  $L$  in a half-space in the new coordinates (section 6.8.1); next, one solves  $Lf = g + O(d^{\alpha})$  in this coordinate system for any function of class  $C^{\alpha}$ —such as  $-2\Delta d$ —by inverting a model operator closely related to  $L_0$  (section 6.8.2); finally, we patch the results to obtain a function  $w_0$  such that  $Lw_0 = g$  (section 6.8.3).

Theorem 44 is proved in section 6.8.4 by a second comparison argument.

## 6.8 First comparison argument

Since  $\partial\Omega$  is  $C^{2+\alpha}$ , it satisfies a uniform interior and exterior sphere condition, and there is a positive  $r_0$  such that any  $P \in \Omega$  such that  $d(P) \leq r_0$  admits a unique nearest point  $Q$  on the boundary, and such that there are two points  $C$  and  $C'$  on the line determined by  $P$  and  $Q$ , such that

$$B_{r_0}(C) \subset \Omega \subset \mathbb{R}^n \setminus B_{r_0}(C'),$$

these two balls being tangent to  $\partial\Omega$  at  $Q$ . We now define two functions  $u_i$  and  $u_e$ . Let

$$u_i(M) = \left(r_0 - \frac{CM^2}{r_0}\right)^{1-n/2} \text{ and } u_e(M) = \left(\frac{C'M^2}{r_0} - r_0\right)^{1-n/2}.$$

$u_i$  and  $u_e$  are solutions of equation (29) in  $B_{r_0}(C)$  and  $\mathbb{R} \setminus B_{r_0}(C')$  respectively.

If we replace  $r_0$  by  $r_0 - \varepsilon$  in the definition of  $u_e$ , we obtain a classical solution of (29) in  $\Omega$ , which is therefore dominated by  $u_\Omega$ . It follows that

$$u_e \leq u_\Omega \text{ in } \Omega.$$

The monotonicity property (30) yields

$$u_\Omega \leq u_i \text{ in } B_{r_0}(C).$$

In particular, the inequality

$$u_e(M) \leq u_\Omega(M) \leq u_i(M)$$

holds if  $M$  lies on the semi-open segment  $[P, Q)$ . Since  $Q$  is then also the point of the boundary closest to  $M$ , we have  $QM = d(M)$ ,  $CM = r_0 - d$  and  $C'M = r_0 + d$ ; it follows that

$$(2d + \frac{d^2}{r_0})^{1-n/2} \leq u_\Omega(M) \leq (2d - \frac{d^2}{r_0})^{1-n/2}.$$

Since  $u_\Omega = (2d + d^2w)^{1-n/2}$ , it follows that

$$|w| \leq \frac{1}{r_0} \text{ if } d \leq r_0.$$

Next, consider  $P \in \Omega$  such that  $d(P) = 2\sigma$ , with  $3\sigma < r_0$ . For  $x$  in the closed unit ball  $\overline{B}_1$ , let

$$P_\sigma := P + \sigma x; \quad u_\sigma(x) := \sigma^{(n-2)/2} u(P_\sigma).$$

One checks that  $u_\sigma$  is a classical solution of (29) in  $\overline{B}_1$ . Since  $d \mapsto 2d \pm \frac{1}{r_0}d^2$  is increasing for  $d < r_0$ , and  $d(P_\sigma)$  varies between  $\sigma$  and  $3\sigma$  if  $x$  varies in  $\overline{B}_1$ , we have

$$(6 + \frac{9\sigma}{r_0})^{1-n/2} \leq u_\sigma(M) \leq (2 - \frac{\sigma}{r_0})^{1-n/2}.$$

This provides a uniform bound for  $u_\sigma$  on  $B_1$ . Applying interior regularity estimates as in [44, 7], we find that  $\nabla u_\sigma$  is uniformly bounded for  $x = 0$ . Recalling that  $\sigma = \frac{1}{2}d(P)$ , we find that

$$d^{\frac{n}{2}-1}u \text{ and } d^{\frac{n}{2}}\nabla u \text{ are bounded near } \partial\Omega.$$

It follows that  $u^{-n/(n-2)} = O(d^{n/2})$ , and since  $d^2w = -2d + u^{-2/(n-2)}$ , we have

$$d^2\nabla w = -2(1 + dw)\nabla d - \frac{2}{n-2}u^{-n/(n-2)}\nabla u,$$

hence  $d^2\nabla w$  is bounded near  $\partial\Omega$ . This completes the proof of theorem 41.

### 6.8.1 Decomposition of $L$ in adapted coordinates

Since  $\partial\Omega$  is compact, there is a positive  $r_0$  such that in any ball of radius  $r_0$  centered at a point of  $\partial\Omega$ , one may introduce a coordinate system  $(Y, T)$  in which  $T = d$  is the last coordinate. The formulae of section 2.5 apply. It will be convenient to assume that the domain of this coordinate system contains a set of the form

$$0 < T < \theta \text{ and } |Y_j| < \theta \text{ for } j \leq n-1.$$

Let  $\partial_j = \partial_{x_j}$ , and write  $d_n$  and  $d_j$  for  $\partial d / \partial x_n$  and  $\partial d / \partial x_j$  respectively. Primes denote derivatives with respect to the  $Y$  variables:  $\partial'_j = \partial_{Y_j}$ ,  $\nabla' = \nabla_Y$ ,  $\Delta' = \sum_{j < n} \partial_j'^2$ , etc. We write  $\tilde{\nabla}d = (d_1, \dots, d_{n-1})$ . Recall that  $|\nabla d| = 1$ . We let throughout

$$D = T\partial_T.$$

The transformation formulae are

$$\begin{aligned} T &= d(x_1, \dots, x_n); & Y_j &= x_j \text{ for } j < n; \\ \partial_n &= d_n \partial_T; & \partial_j &= d_j \partial_T + \partial'_j. \end{aligned}$$

We recall that  $\Delta d = (1-n)H$ , where  $H$  is the mean curvature of  $\partial\Omega$ .

We further have

$$\begin{aligned} d\nabla d \cdot \nabla w &= (D + T\tilde{\nabla}d \cdot \nabla')w \\ |\nabla w|^2 &= w_T^2 + |\nabla' w|^2 + 2w_T \tilde{\nabla}d \cdot \nabla' w \\ \Delta w &= w_{TT} + \Delta' w + 2\tilde{\nabla}d \cdot \nabla' w_T + w_T \Delta d. \end{aligned}$$

It follows that

$$Lw = L_0 w + L_1 w,$$

where

$$L_0 w = (D+2)(D+1-n)w + T^2 \Delta' w,$$

and

$$L_1 w = (4-n)\tilde{\nabla}d \cdot \nabla'(Tw) + 2T\tilde{\nabla}d \cdot \nabla'(Dw) + T(Dw)\Delta d.$$

### 6.8.2 Solution of $Lf = k + O(d^\alpha)$

We now solve approximately equation  $Lf = k$  by solving exactly a model problem, related to the operator  $L_0$ .

Let  $C_{\text{per}}^\alpha$  denote the space of functions  $k(Y, T) \in C^\alpha(0 \leq T \leq \theta)$  which satisfy  $k(Y_j + 2\theta, T) = k(Y_j, T)$  for  $1 \leq j \leq n - 1$ . We prove the following theorem.

**THEOREM 45** *Let  $\theta > 0$ , and  $k(Y, T)$  of class  $C_{\text{per}}^\alpha$ . Then there is a function  $f$  such that*

1.  $L_0 f = k + O(d^\alpha)$ ,
2.  $f$  is of class  $C_{\#}^{2+\alpha}(0 \leq T \leq \theta)$ ,
3.  $f(Y, 0) = k(Y, 0)/(2 - 2n)$  and
4.  $L_1 f = O(d^\alpha)$ .

**PROOF.** Let

$$L'_0 = (D + 2)(D - 1) + T^2 \Delta' = L_0 + (n - 2)(D + 2).$$

We first solve the equation  $L'_0 f_0 = k$ .

**LEMMA 46** *There is a bounded linear operator*

$$G : C_{\text{per}}^\alpha \longrightarrow C_{\#}^{2+\alpha}(0 \leq T \leq \theta)$$

*such that  $f_0 := G[k]$  verifies*

1.  $L'_0 f_0 = k$ ,
2.  $f_0$  is of class  $C_{\#}^{2+\alpha}(0 \leq T \leq \theta)$ ,
3.  $f_0(Y, 0) + k(Y, 0)/2 = 0$ ,  $Df_0(Y, 0) = 0$  and
4.  $L_1 f_0 = O(d^\alpha)$ .

**PROOF.** One first constructs  $\tilde{k}$  such that  $(D - 1)\tilde{k} = -k$ , and  $\tilde{k}$  and  $D\tilde{k}$  are both  $C^\alpha$  up to  $T = 0$ . One may take

$$\tilde{k} = \int_1^\infty F_1[k](Y, T\sigma) \frac{d\sigma}{\sigma^2}.$$

where  $F_1$  is an extension operator, so that  $F_1[k] = k$  for  $T \leq \theta$ .

One checks that  $\tilde{k} = k$  for  $T = 0$ .

One then solves  $(\partial_{TT} + \Delta')h + \tilde{k} = 0$  with periodic boundary conditions, of period  $2\theta$ , in each of the  $Y_j$ , and  $h(Y, 0) = h_T(Y, \theta) = 0$ ; this yields

$$h \text{ is of class } C^{2+\alpha}(0 \leq T \leq \theta)$$

by the Schauder estimates. In particular,  $h_T$  is continuous up to  $T = 0$ , and  $Dh = 0$  for  $T = 0$  and  $T = \theta$ .

Since  $h = 0$  for  $T = 0$ , we also have  $\Delta'h = 0$  for  $T = 0$ . The equation for  $h$  therefore gives

$$h_{TT} = -\tilde{k} = -k \text{ for } T = 0.$$

In addition,

$$(\partial_{TT} + \Delta')Dh = D(\partial_{TT} + \Delta')h + 2h_{TT} = k - \tilde{k} + 2h_{TT},$$

which is  $C^\alpha$ . Since, on the other hand,  $Dh$  is of class  $C^1$  and  $Dh = 0$  for  $T = 0$  and  $T = \theta$ , we conclude, using again the Schauder estimates, that

$$Dh \text{ is of class } C^{2+\alpha}(0 \leq T \leq \theta).$$

We now define  $f_0$  by

$$f_0 := T^{-2}(D - 1)h = \partial_T \left( \frac{h}{T} \right) = \int_0^1 \sigma h_{TT}(Y, T\sigma) d\sigma. \quad (34)$$

Since  $f_0$  is itself uniquely determined by  $h$ , itself defined in terms of  $k$  we define a map  $G$  by

$$f_0 = G[k].$$

A direct computation yields  $L'_0 f_0 = k$ :

$$\begin{aligned} L'_0 f_0 &= (D + 2)(D - 1)T^{-2}(D - 1)h + (D - 1)\Delta'h \\ &= T^{-2}D(D - 3)(D - 1)h + (D - 1) \left\{ -T^{-2}D(D - 1)h - \tilde{k} \right\} \\ &= T^{-2}D(D - 1)(D - 3)h - T^{-2}(D - 3)D(D - 1)h - (D - 1)\tilde{k} \\ &= k. \end{aligned}$$

Let us now consider the regularity of  $f_0$  up to  $\partial\Omega$ , and the values of  $f_0$  and its derivatives on  $\partial\Omega$ .

Consider  $g_0 := T^2 f_0$ . Since  $g_0 = (D - 1)h \in C^{2+\alpha}(0 \leq T \leq \theta)$  and vanishes for  $T = 0$ , we have  $g_0 = \int_0^1 g_{0T}(Y, T\sigma) T d\sigma$ . It follows that

$$T f_0(Y, T) = \int_0^1 g_{0T}(Y, T\sigma) d\sigma \in C^{1+\alpha}(0 \leq T \leq \theta).$$

Since, on the other hand,  $G[k] = \int_0^1 \sigma h_{TT}(Y, T\sigma) d\sigma$ , we find  $f_0 \in C^\alpha(0 \leq T \leq \theta)$ , and

$$f_0(Y, 0) = \frac{1}{2} h_{TT}(Y, 0) = -\frac{1}{2} k(Y, 0).$$

We therefore have

$$f_0 \text{ is of class } C_{\#}^{2+\alpha}(0 \leq T \leq \theta).$$

Since

$$(D + 2)f_0 = T^{-2} D(D - 1)h = h_{TT},$$

we find  $Df_0(Y, 0) = h_{TT}(Y, 0) - 2f_0(Y, 0) = 0$ . By differentiation with respect to the  $Y$  variables, we obtain that  $\tilde{\nabla} d \cdot \nabla'(Tf_0)$  is of class  $C^\alpha$  and vanishes for  $T = 0$ . The same is true of  $T(Df_0)\Delta d$ . Similarly,

$$2T\tilde{\nabla} d \cdot \nabla' Df_0 = 2\tilde{\nabla} d \cdot \nabla'[\partial_T(T^2 f_0) - 2Tf_0]$$

is of class  $C^\alpha$ , and vanishes for  $T = 0$  because this is already the case for  $TDf_0$ . It follows that  $L_1 f_0$  is a  $C^\alpha$  function which vanishes for  $T = 0$ ; it is therefore  $O(d^\alpha)$  as desired.  $\square$

We are now ready to prove theorem 45. Let  $a$  be a constant, and  $f = G[ak]$ . We therefore have  $L'_0 f = ak$ , and, for  $T = 0$ ,  $f = -\frac{1}{2}ak$ . Since  $L_1 f \in C^\alpha$ , and  $L_1 f$  and  $Df$  both vanish for  $T = 0$ , it follows that, for  $T = 0$ ,

$$Lf - k = (L'_0 - (n - 2)(D + 2) + L_1)f - k = [a + (n - 2)a - 1]k.$$

Taking  $a = 1/(n - 1)$ , we find that  $f$  has the announced properties.  $\square$

### 6.8.3 Solution of $Lw_0 = g$

Let us now consider a function  $g$  of class  $C^\alpha(\overline{\Omega}_\delta)$ .

Recall that there is a positive  $r_0 < \delta$  such that any ball of radius  $r_0$ , centered at a point of the boundary, is contained in a domain in which we have a system of coordinates of the type  $(Y, T)$ . Let us cover (a neighborhood

of)  $\partial\Omega$  by a finite number of balls  $(V_\lambda)_{\lambda \in \Lambda}$  of radius  $r_1 < r_0$  and centers on  $\partial\Omega$ , and consider the balls  $(U_\lambda)_{\lambda \in \Lambda}$  of radius  $r_0$  with the same centers. Thus, we may assume that every  $U_\lambda$  is associated with a coordinate system  $(Y_\lambda, T_\lambda)$  of the type considered in section 2.5; taking  $r_1$  smaller if necessary, we may also assume that  $\bar{V}_\lambda \subset Q_\lambda \subset U_\lambda$ , where  $Q_\lambda$  has the form

$$Q_\lambda := \{(Y_{\lambda,1}, \dots, Y_{\lambda,n-1}, T_\lambda) : 0 \leq Y_{\lambda,j} \leq \theta \text{ for every } j, \text{ and } 0 < T_\lambda < \theta\}.$$

Consider a smooth partition of unity  $(\varphi_\lambda)$  and smooth functions  $(\Phi_\lambda)$ , such that

1.  $\sum_{\lambda \in \Lambda} \varphi_\lambda = 1$  near  $\partial\Omega$ ;
2.  $\text{supp } \varphi_\lambda \subset V_\lambda$ ;
3.  $\text{supp } \Phi_\lambda \subset U_\lambda \cap \{T < \theta\}$ ;
4.  $\Phi_\lambda = 1$  on  $V_\lambda$ .

In particular,  $\Phi_\lambda \varphi_\lambda = \varphi_\lambda$ .

The function  $g\varphi_\lambda$  is of class  $C^\alpha(\bar{Q}_\lambda)$ ; it may be extended by successive reflections to an element of  $C_{\text{per}}^\alpha$ , with period  $2\theta$  in the  $Y_\lambda$  variables; this extension will be denoted by the same symbol for simplicity.

Let us apply theorem 45, and consider, for every  $\lambda$ , the function  $w_\lambda := G[g\varphi_\lambda/(n-1)]$ . We have

$$Lw_\lambda = g\varphi_\lambda + R_\lambda,$$

in  $U_\lambda \cap \{T < \theta\}$ , where  $R_\lambda$  is Hölder continuous for  $T \leq \theta$ , and vanishes on  $\partial\Omega$ ; as a consequence,  $R_\lambda = O(d^\alpha)$ .

The function  $\Phi_\lambda w_\lambda$  is compactly supported in  $U_\lambda$ , and may be extended, by zero, to all of  $\Omega$ ; it is of class  $C_{\#}^{2+\alpha}(\bar{\Omega})$ . We may therefore consider

$$w_1 := \sum_{\lambda \in \Lambda} \Phi_\lambda w_\lambda,$$

which is supported near  $\partial\Omega$ . Now, near  $\partial\Omega$ ,

$$\begin{aligned} \sum_{\lambda} L(\Phi_\lambda w_\lambda) &= \sum_{\lambda} \Phi_\lambda L(w_\lambda) + 2d^2 \nabla \Phi_\lambda \cdot \nabla w_\lambda \\ &\quad + d^2 w_\lambda \Delta \Phi_\lambda + (4-n) w_\lambda d \nabla d \cdot \nabla \Phi_\lambda \\ &= \sum_{\lambda} g \Phi_\lambda \varphi_\lambda + R'_\lambda = g + f, \end{aligned}$$

where  $f = \sum_{\lambda} R'_{\lambda}$  has the same properties as  $R_{\lambda}$ . It therefore suffices to solve  $Lw_2 = f$  when  $f$  is a Hölder continuous function which vanishes on the boundary.

LEMMA 47 *For any  $f \in C^{\alpha}(\overline{\Omega})$ , there is, for  $\delta$  small enough, an element  $w_2 \in C_{\#}^{2+\alpha}(\overline{\Omega}_{\delta})$  such that*

$$Lw_2 = f \text{ and } w_2 = O(d^{\alpha}) \text{ near } \partial\Omega.$$

PROOF. Consider the solution  $w_{\varepsilon}$  of the Dirichlet problem  $Lw_{\varepsilon} = f$  on a domain of the form  $\{\varepsilon < d(x) < \delta\}$ , with zero boundary data. As before,  $\delta$  is taken small enough to ensure that  $d \in C^{2+\alpha}(\overline{\Omega}_{\delta})$ . Schauder theory gives  $w_{\varepsilon} \in C^{2+\alpha}(\{\varepsilon \leq d(x) \leq \delta\})$ . By assumption,  $|f| \leq ad^{\alpha}$  for some constant  $a$ . Let  $A > (\alpha + 2)(n - 1 - \alpha)$ . Since

$$-L(d^{\alpha}) = d^{\alpha}[(\alpha + 2)(n - 1 - \alpha) - \alpha d\Delta d],$$

$Ad(x)^{\alpha}$  is a super-solution if  $\delta$  is small, and the maximum principle gives us a uniform bound on  $w_{\varepsilon}/d^{\alpha}$ . By interior regularity, we obtain that, for a sequence  $\varepsilon_n \rightarrow 0$ , the  $w_{\varepsilon_n}$  converge in  $C^2$ , in every compact away from the boundary, to a solution  $w_2$  of  $Lw_2 = f$  with  $w_2 = O(d^{\alpha})$ . Since the right-hand side  $f$  is also  $O(d^{\alpha})$ , we obtain, by the “type (I)” theorem 23, that  $w_2$  of class  $C_{\#}^{1+\alpha}(\overline{\Omega}_{\delta})$ . Theorem 24 now ensures that  $w_2$  is in fact of class  $C_{\#}^{2+\alpha}(\overline{\Omega}_{\delta})$ , QED.  $\square$

It now suffices to take  $g = -2\Delta d$  and let

$$w_0 = w_1 - w_2.$$

By construction,  $Lw_0 + 2\Delta d = 0$  near the boundary, and  $w_0$  is of class  $C_{\#}^{2+\alpha}(\overline{\Omega}_{\delta})$  if  $\delta$  is small. In addition, we know from theorem 45 that  $w_1|_{\partial\Omega} = (2\Delta d)/(2n - 2)$ , which is equal to  $-H$  on  $\partial\Omega$ . Lemma 47 gives us  $w_2 = O(d^{\alpha})$ . We conclude that  $w_0|_{\partial\Omega} = -H$  on the boundary.

This completes the proof of theorem 43.

#### 6.8.4 Second comparison argument

At this stage, we have the following information, where  $\Omega_{\delta} = \{x : 0 < d(x) < \delta\}$ , for  $\delta$  small enough:

1.  $w$  and  $d\nabla w$  are bounded near  $\partial\Omega$ ;

2.  $w = w_0 + \tilde{w}$ , where  $L\tilde{w} = M_w(w) = O(d)$ , and
3.  $w_0$  is of class  $C_{\#}^{2+\alpha}(\overline{\Omega}_\delta)$  for  $\delta$  small enough.

We wish to estimate  $\tilde{w}$ . Write  $|M_w(w)| \leq cd$ , where  $c$  is constant.

For any constant  $A > 0$ , define

$$w_A := w_0 + Ad.$$

Since  $L(d) = 3(2-n)d + d^2\Delta d$ , we have

$$L(w_A - w) = L(Ad - \tilde{w}) \leq Ad[3(2-n) + d\Delta d] + cd.$$

Choose  $\delta$  so that, say,  $2(2-n) - d\Delta d \leq 0$  for  $d \leq \delta$ . Then, choose  $A$  so large that (i)  $w_0 + A\delta \geq w$  for  $d = \delta$ , and (ii)  $(2-n)A + c \leq 0$ . We then have

$$L(w_A - w) \leq 0 \text{ in } \Omega_\delta \text{ and } w_A - w \geq 0 \text{ for } d = \delta.$$

Next, choose  $\delta$  and a constant  $B$  such that  $nB + (2+Bd)\Delta d \geq 0$  on  $\Omega_\delta$ . We have, by direct computation,

$$L(d^{-2} + Bd^{-1}) = -(nB + 2\Delta d)d^{-1} - B\Delta d \leq 0$$

on  $\Omega_\delta$ . Therefore, for any  $\varepsilon > 0$ ,  $z_\varepsilon := \varepsilon[d^{-2} + Bd^{-1}] + w_A - w$  satisfies  $Lz_\varepsilon \leq 0$ , and the maximum principle ensures that  $z_\varepsilon$  has no negative minimum in  $\Omega_\delta$ . Now,  $z_\varepsilon$  tends to  $+\infty$  as  $d \rightarrow 0$ . Therefore,  $z_\varepsilon$  is bounded below by the least value of its negative part restricted to  $d = \delta$ . In other words, for  $d \leq \delta$ , we have, since  $w_A - w \geq 0$  for  $d = \delta$ ,

$$w_A - w + \varepsilon[d^{-2} + Bd^{-1}] \geq \varepsilon \min(\delta^{-2} + B\delta^{-1}, 0).$$

Letting  $\varepsilon \rightarrow 0$ , we obtain  $w_A - w \geq 0$  in  $\Omega_\delta$ .

Similarly, for suitable  $\delta$  and  $A$ ,  $w - w_{-A} \geq 0$  in  $\Omega_\delta$ .

We now know that  $w$  lies between  $w_0 + Ad$  and  $w_0 - Ad$  near  $\partial\Omega$ , hence  $|w - w_0| = O(d)$ , QED.

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