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# On maximal diagonalizable Lie subalgebras of the first Hochschild cohomology

Patrick Le Meur

March 26, 2009

## Abstract

Let  $A$  be a basic connected finite dimensional algebra over an algebraically closed field, with ordinary quiver without oriented cycles. Given a presentation of  $A$  by quiver and admissible relations, Assem and de la Peña have constructed an embedding of the space of additive characters of the fundamental group of the presentation into the first Hochschild cohomology group of  $A$ . We compare the embeddings given by the different presentations of  $A$ . In some situations, we characterise the images of these embeddings in terms of (maximal) diagonalizable subalgebras of the first Hochschild cohomology group (endowed with its Lie algebra structure).

## Introduction

Let  $A$  be a finite dimensional algebra over an algebraically closed field  $k$ . The representation theory of  $A$  deals with the study of (right)  $A$ -modules. So we assume that  $A$  is basic and connected and it admits presentations  $A \simeq kQ/I$  by its (unique) ordinary quiver  $Q$  and an ideal  $I$  of admissible relations. In the eighties, Martinez-Villa and de la Peña introduced the fundamental group  $\pi_1(Q, I)$  of  $(Q, I)$  ([17]). Like in topology, this group is defined using an equivalence relation  $\sim_I$  (called the homotopy relation) on the set of unoriented paths in  $Q$ . This group is part of the so-called covering techniques initiated in [6, 18]. In particular, it has led to the definition of simple connectedness and strong simple connectedness for an algebra ([2, 20]). Also, it has proved to be a very useful tool in representation theory. For example, it is proved in [19] that any domestic self-injective algebra admitting a Galois covering by a strongly simply connected locally bounded  $k$ -category is of quasi-tilted type. Note that in general, different presentations  $A \simeq kQ/I$  and  $A \simeq kQ/J$  may lead to non-isomorphic groups  $\pi_1(Q, I)$  and  $\pi_1(Q, J)$ .

The fundamental group  $\pi_1(Q, I)$  behaves much like the fundamental group of a topological space. For example, given a presentation  $\nu: kQ \twoheadrightarrow A$  (with kernel  $I$ ), Assem and de la Peña have defined an injective group homomorphism  $\theta_\nu: \mathbf{Hom}(\pi_1(Q, I), k^+) \hookrightarrow \mathbf{HH}^1(A)$ . Here  $\mathbf{HH}^1(A)$  is the first Hochschild cohomology group  $\mathbf{Ext}_{A^{op} \otimes A}^1(A, A)$  ([12]). This result is to be compared with the classical isomorphism  $\mathbf{Hom}(\pi_1(X), \mathbb{Z}) \xrightarrow{\sim} H^1(X; \mathbb{Z})$  relating the singular cohomology to the fundamental group of a path connected topological space  $X$ . It is known from [10] that  $\mathbf{HH}^1(A)$  has a structure of Lie algebra, isomorphic to the Lie algebra of derivations of  $A$  (with the commutator as Lie bracket) factored out by the ideal of inner derivations. With this presentation of  $\mathbf{HH}^1(A)$ , the derivations that lie in the image of  $\theta_\nu$ , have been characterized in terms of diagonalizable derivations ([9], see also [7]).

The aim of this text is to characterise maximal diagonalisable Lie subalgebras of  $\mathbf{HH}^1(A)$  using the subspaces  $\mathbf{Im}(\theta_\nu)$  associated to the different presentations  $\nu$  of  $A$ . Recall that, given a Lie algebra, the maximal diagonalizable (for the adjoint representation) subalgebras are related to Cartan subalgebras.

On the one hand, one can define a diagonalizability for elements in  $\mathbf{HH}^1(A)$  using the above notion of diagonalizable derivations. Also one can define the diagonalizability of a subset of  $\mathbf{HH}^1(A)$  (as the simultaneous diagonalizability of its elements). It appears that  $\mathbf{Im}(\theta_\nu)$  is diagonalizable, and that any diagonalizable subset of  $\mathbf{HH}^1(A)$  is contained in  $\mathbf{Im}(\theta_\nu)$  for some presentation  $\nu: kQ \twoheadrightarrow A$ .

On the other hand, given two presentations  $\nu: kQ \twoheadrightarrow A$  and  $\mu: kQ \twoheadrightarrow A$  with kernel  $I$  and  $J$  respectively, it is not easy to compare the groups  $\pi_1(Q, I)$  and  $\pi_1(Q, J)$  (and therefore  $\theta_\nu$  and  $\theta_\mu$ ). In some cases, this is possible, however. For example, assume that  $(\alpha, u)$  is a bypass in  $Q$  (that is,  $\alpha$  is an arrow and  $u$  is an oriented path which is parallel to  $\alpha$  and distinct from  $\alpha$ ), that  $\tau \in k$ , and that  $J = \varphi_{\alpha, u, \tau}(I)$ . Here  $\varphi_{\alpha, u, \tau}: kQ \xrightarrow{\sim} kQ$  is the automorphism, called a *transvection*, which maps  $\alpha$  to  $\alpha + \tau u$ , and which fixes any other arrow ([13]). In such a situation, if  $\alpha \sim_I u$  (or  $\alpha \sim_J u$ ), then there is a natural surjective group homomorphism  $\pi_1(Q, J) \twoheadrightarrow \pi_1(Q, I)$  (or  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ , respectively); if  $\alpha \sim_I u$  and  $\alpha \not\sim_J u$  then  $\pi_1(Q, I)$  equals  $\pi_1(Q, J)$  and the natural homomorphisms are the identity maps; and if  $\alpha \not\sim_I u$  and  $\alpha \not\sim_J u$ , then  $I = J$  and  $\pi_1(Q, I) = \pi_1(Q, J)$ . In each of these cases, we shall see that there is a simple relation between  $\theta_\nu$  and  $\theta_\mu$ .

In order to formulate our main result, we use the quiver  $\Gamma$  of the homotopy relations of the presentations of  $A$  ([13]). Its set of vertices is the set of the homotopy relations  $\sim_{\text{Ker}(\nu)}$  associated to all the presentations  $\nu: kQ \twoheadrightarrow A$ . Also, there is an arrow  $\sim_I \rightarrow \sim_J$  if there exists a transvection  $\varphi_{\alpha, u, \tau}$  such that  $J = \varphi_{\alpha, u, \tau}(I)$  and such that the natural surjective group homomorphism is a non isomorphism  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ . The quiver  $\Gamma$  has been introduced in order to find conditions under which an algebra admits a universal Galois covering. This existence is related to the existence of a unique source (that is, a vertex which is the target of no arrow). Actually, under one of the two following conditions,  $\Gamma$  does have a unique source ([15, Prop. 2.11] and [16, Cor. 4.4]):

- (H<sub>1</sub>)  $Q$  has no double bypass and  $k$  has characteristic zero (a *double bypass* is a 4-tuple  $(\alpha, u, \beta, v)$  where  $(\alpha, u)$  and  $(\beta, v)$  are bypasses such that the arrow  $\beta$  appears in the path  $u$ ).
- (H<sub>2</sub>)  $A$  is monomial (that is,  $A \simeq kQ/I_0$  with  $I_0$  an ideal generated by a set of paths) and  $Q$  has no multiple arrows.

Using these results, we prove the main theorem of the text.

**Theorem 1.** *Assume that at least one the two hypotheses (H<sub>1</sub>) or (H<sub>2</sub>) is satisfied. Then:*

- (i) *The maximal diagonalizable subalgebras of  $\text{HH}^1(A)$  are exactly the subalgebras of the form  $\text{Im}(\theta_\nu)$  where  $\nu: kQ \twoheadrightarrow A$  is a presentation such that  $\sim_{\text{Ker}(\nu)}$  is the unique source of  $\Gamma$ .*
- (ii) *If  $\mathcal{G}, \mathcal{G}'$  are two such subalgebras of  $\text{HH}^1(A)$ , then there exists an algebra automorphism  $\psi: A \xrightarrow{\sim} A$  inducing a Lie algebra automorphism  $\psi_*: \text{HH}^1(A) \xrightarrow{\sim} \text{HH}^1(A)$  such that  $\mathcal{G}' = \psi_*(\mathcal{G})$ .*

Note that the Lie algebra  $\text{HH}^1(A)$  has already been studied (see [11, 21], for instance).

The text is organised as follows. In Section 1 we recall all the definitions we will need and prove some useful lemmas. In Section 2, we introduce the notion of diagonalizability in  $\text{HH}^1(A)$ . In particular, we will prove that a subset of  $\text{HH}^1(A)$  is diagonalizable is and only if it is contained in  $\text{Im}(\theta_\nu)$  for some presentation  $\nu: kQ \twoheadrightarrow A$ . In Section 3 we compare the Lie algebra homomorphisms  $\theta_\nu$  for different presentations  $\nu$  of  $A$ , using the quiver  $\Gamma$ . Finally, in Section 4 we prove Theorem 1.

This text is part of the author's thesis ([14]) made at Université  $\frac{1}{2}$  Montpellier II under the supervision of Claude Cibils.

## 1 Preliminaries

### 1.1 Terminology and notations for quivers

Let  $Q$  be a quiver. We write  $Q_0$  and  $Q_1$  for the set of vertices and of arrows, respectively. We read (oriented) paths from the right to the left, that is, we view a path  $u$  as a morphism and the concatenation  $vu$  of two paths  $u$  and  $v$  such that the source of  $v$  equals the target of  $u$  as a composition of morphisms. Given  $x \in Q_0$ , the *trivial* path (of length 0, with source and target equal to  $x$ ) is denoted by  $e_x$ . Two paths are called *parallel* if they have the same source and the same target. An *oriented cycle* in  $Q$  is a non trivial path whose source and target are equal. If  $\alpha \in Q_1$  we consider its formal inverse  $\alpha^{-1}$  with source and target equal to the target and the source of  $\alpha$ , respectively. Hence, we get the double quiver  $\overline{Q}$  such that  $\overline{Q}_0 = Q_0$  and  $\overline{Q}_1 = Q_1 \cup \{\alpha^{-1} \mid \alpha \in Q_1\}$ . Then, a *walk* in  $Q$  is exactly an oriented path in  $\overline{Q}$ . Given a walk  $\gamma = \alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$  (with  $\alpha_i \in Q_1$ ,  $\varepsilon_i \in \{\pm 1\}$ ), its inverse  $\gamma^{-1}$  is by definition  $\alpha_1^{-\varepsilon_1} \dots \alpha_n^{-\varepsilon_n}$ .

### 1.2 Presentations by quiver and admissible relations

Let  $Q$  be a quiver. Its *path algebra*  $kQ$  is the  $k$ -algebra whose basis as a  $k$ -vector space is the set of paths in  $Q$  (including the trivial paths), and whose product is bilinearly induced by the concatenation of paths (if  $u, v$  are two paths such that the source of  $v$  is different from the target of  $u$ , then we set  $vu = 0$ ). The unit of  $kQ$  is  $\sum_{x \in Q_0} e_x$  and  $kQ$  is finite dimensional if and only if  $Q$  is finite (that is

$Q_0$  and  $Q_1$  are finite) and has no oriented cycles. We let  $kQ^+$  be the ideal of  $kQ$  generated by the arrows.

An *admissible ideal* of  $kQ$  is an ideal  $I$  such that  $(kQ^+)^N \subseteq I \subseteq (kQ^+)^2$  for some  $N \geq 2$ . In such a case, the elements of  $I$  are called *relations* and, following [17], a *minimal relation* of  $I$  is a relation  $\sum_{i=1}^s t_i u_i \neq 0$  such that  $t_1, \dots, t_s \in k^*$  and  $u_1, \dots, u_s$  are pairwise distinct paths in  $Q$ , and such that there is no non empty proper subset  $S \subset \{1, \dots, s\}$  satisfying  $\sum_{i \in S} t_i u_i \in I$ . In such a case,  $u_1, \dots, u_s$  are necessarily parallel. Note that  $I$  is generated by its minimal relations.

Recall (see [4]) that any finite dimensional  $k$ -algebra  $A$  is Morita equivalent to a basic one. If  $A$  is basic, then there exists a unique quiver  $Q$ , the *ordinary quiver* of  $A$ , and a surjective  $k$ -algebra homomorphism  $\nu: kQ \twoheadrightarrow A$  whose kernel is an admissible ideal of  $kQ$ . Also,  $\{\nu(e_x) \mid x \in Q_0\}$  is a complete set of primitive orthogonal idempotents of  $A$ . The homomorphism  $\nu$  is called a *presentation (by quiver and admissible relations)*. We have  $A \simeq kQ/\text{Ker}(\nu)$  and  $A$  is connected if and only if  $Q$  is

connected.

### 1.3 Presentation of $\mathrm{HH}^1(A)$

Let  $A$  be a basic finite dimensional  $k$ -algebra and let  $\{e_1, \dots, e_n\}$  be a complete set of primitive orthogonal idempotents. A *unitary derivation* ([7]) is a  $k$ -linear map  $d: A \rightarrow A$  such that  $d(ab) = ad(b) + d(a)b$  for any  $a, b \in A$  and such that  $d(e_i) = 0$  for every  $i$ . Let  $\mathrm{Der}_0(A)$  be set of unitary derivations. It is a Lie algebra for the commutator. In the sequel, all derivations will be unitary. So we shall call them derivations. Let  $E := \{\sum_{i=1}^n t_i e_i \mid t_1, \dots, t_n \in k\}$ . Then  $E$  is a semi-simple subalgebra of  $A$  and  $A = E \oplus \mathfrak{r}$  where  $\mathfrak{r}$  is the radical of  $A$ . Let  $\mathrm{Int}_0(A) := \{\delta_e: A \rightarrow A, a \in A \mapsto ea - ae \mid e \in E\}$ , this is an ideal of  $\mathrm{Der}_0(A)$ . Throughout this text, we shall use the following presentation proved in [7]:

**Theorem 1.** ([7])  $\mathrm{HH}^1(A) \simeq \mathrm{Der}_0(A)/\mathrm{Int}_0(A)$  as Lie algebras.

In the following lemma, we collect some useful properties on derivations.

**Lemma 1.1.** Let  $d \in \mathrm{Der}_0(A)$ , then  $d(e_j A e_i) \subseteq e_j A e_i$ . Assume that the ordinary quiver  $Q$  of  $A$  has no oriented cycles, then  $d(\mathfrak{r}) \subseteq \mathfrak{r}$  and  $d(\mathfrak{r}^2) \subseteq \mathfrak{r}^2$ .

**Proof:** Since  $d$  is unitary and since  $Q$  has no oriented cycles, we have  $d(\mathfrak{r}) \subseteq \mathfrak{r}$ . So,  $d(\mathfrak{r}^2) \subseteq \mathfrak{r}^2$ . ■

If  $\psi: A \xrightarrow{\sim} A$  is a  $k$ -algebra automorphism such that  $\psi(e_i) = e_i$  for every  $i$ , then the map  $d \mapsto \psi \circ d \circ \psi^{-1}$  induces a Lie algebra automorphism of  $\mathrm{HH}^1(A)$ , denoted by  $\psi_*: \mathrm{HH}^1(A) \xrightarrow{\sim} \mathrm{HH}^1(A)$ .

### 1.4 Fundamental groups of presentations

Let  $(Q, I)$  be a *bound quiver* (that is,  $Q$  is a finite quiver and  $I$  is an admissible ideal of  $kQ$ ). The *homotopy relation*  $\sim_I$  was defined in [17] as the equivalence class on the set of walks in  $Q$  generated by the following properties:

- (1)  $\alpha\alpha^{-1} \sim_I e_y$  and  $\alpha^{-1}\alpha \sim_I e_x$  for any arrow  $\alpha$  with source  $x$  and target  $y$ ,
- (2)  $wvu \sim_I wv'u$  if  $w, v, v', u$  are walks such that the concatenations  $wvu$  and  $wv'u$  are well defined and such that  $v \sim_I v'$ ,
- (3)  $u \sim_I v$  if  $u$  and  $v$  are paths in a same minimal relation (with a non zero scalar).

Note that if  $r_1, \dots, r_t$  are minimal relations generating the ideal  $I$ , then the condition (3) above may be replaced by the following one ([5]):

- (3')  $u \sim_I v$  if  $u$  and  $v$  are paths in  $Q$  appearing in  $r_i$  (with a non zero scalar) for some  $i \in \{1, \dots, t\}$ .

The  $\sim_I$ -equivalence class of a walk  $\gamma$  is denoted by  $[\gamma]_I$ . Let  $x_0 \in Q_0$ , following [17], the set of  $\sim_I$ -equivalence classes of walks with source and target  $x_0$  is denoted by  $\pi_1(Q, I, x_0)$ . The concatenation of walks endows this set with a group structure whose unit is  $[e_{x_0}]_I$ . This group is called the *fundamental group* of  $(Q, I)$  at  $x_0$ . If  $Q$  is connected, then the isomorphism class of  $\pi_1(Q, I, x_0)$  does not depend on the choice  $x_0$ . In such a case, we write  $\pi_1(Q, I)$  for  $\pi_1(Q, I, x_0)$ . If  $A$  is a basic connected finite dimensional  $k$ -algebra and if  $\nu: kQ \twoheadrightarrow A$  is a presentation, the group  $\pi_1(Q, \mathrm{Ker}(\nu))$  is called the *fundamental group of the presentation*  $\nu$ . The following example shows that two presentations of  $A$  may have non isomorphic fundamental groups.

*Example 1.2.* Let  $A = kQ/I$  where  $Q$  is the quiver:  $1 \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{a} \end{array} 2 \xrightarrow{c} 3$  and  $I = \langle ca \rangle$ . Set  $x_0 = 1$ .

Then  $\pi_1(Q, I) \simeq \mathbb{Z}$  is generated by  $[b^{-1}a]_I$ . On the other hand,  $A \simeq kQ/J$  where  $J = \langle ca - cb \rangle$ , and  $\pi_1(Q, J)$  is the trivial group.

In the sequel we shall use the following technical lemma.

**Lemma 1.3.** Let  $(Q, I)$  be a bound quiver where  $Q$  has no oriented cycles and let  $d: kQ \rightarrow kQ$  be a linear map such that  $d(I) \subseteq I$ , and  $d(u) = t_u u$  for some  $t_u \in k$ , for any path  $u$ . Let  $\equiv_I$  be the equivalence relation on the set of paths in  $Q$  generated by the condition (3) defining  $\sim_I$ . Then, the following implication holds for any paths  $u, v$ :

$$u \equiv_I v \text{ implies } t_u = t_v .$$

**Proof:** We use a non multiplicative version of Gröbner bases ([1], see also [8]). Fix an arbitrary total order  $u_1 < \dots < u_N$  on the set of paths in  $Q$  and let  $(u_1^*, \dots, u_N^*)$  be the basis of  $\mathrm{Hom}_k(kQ, k)$  dual to  $(u_1, \dots, u_N)$ . Following [15, Sect. 1], the Gröbner basis of  $I$  is the unique basis  $(r_1, \dots, r_t)$  defined by the three following properties:

- (i)  $r_j \in u_{i_j} + \text{Span}(u_i ; i < i_j)$  for some  $i_j$ , for every  $j$ ,
- (ii)  $u_{i_j}^*(r_{j'}) = 0$  unless  $j = j'$ ,
- (iii)  $i_1 < \dots < i_t$ .

It follows from these properties that:

$$(iv) \quad r = \sum_{j=1}^t u_{i_j}^*(r) r_j \text{ for any } r \in I.$$

Recall from [15, Sect. 1] that  $r_1, \dots, r_t$  are minimal relations of  $I$  so that  $\equiv_I$  is generated by the property (3') defining  $\sim_I$ . So we only need to prove that  $d(r_j) \in k.r_j$  for any  $j$ . We proceed by induction on  $j \in \{1, \dots, t\}$ . By assumption on  $d$  and thanks to (i), we have  $d(r_1) \in I \cap \text{Span}(u_i ; i \leq i_1)$ . Hence, (iii) and (iv) imply that  $d(r_1) \in k.r_1$ . Let  $j \in \{1, \dots, t-1\}$  and assume that  $d(r_1) \in k.r_1, \dots, d(r_j) \in k.r_j$ . By assumption on  $d$  and thanks to (i) and (ii), we have  $d(r_{j+1}) \in I \cap \text{Span}(u_i ; i \leq i_{j+1})$  and  $u_{i_l}^*(d(r_{j+1})) = 0$  if  $l \leq j$ . So, (iii) and (iv) imply that  $d(r_{j+1}) \in k.r_{j+1}$ . This finishes the induction and proves the lemma.  $\blacksquare$

### 1.5 Comparison of fundamental groups

Let  $A$  be a basic connected finite dimensional  $k$ -algebra with ordinary quiver  $Q$  without oriented cycles. We defined the transvections in the introduction. A *dilatation* ([13]) is an automorphism  $D: kQ \xrightarrow{\sim} kQ$  such that  $D(e_i) = e_i$  for any  $i$  and such that  $D(\alpha) \in k.\alpha$  for any  $\alpha \in Q_1$ . The following proposition will be useful in the sequel, it was proved in [15]:

**Proposition 1.4.** ([15, Prop. 2.5]) *Let  $I$  be an admissible ideal of  $kQ$ , let  $\varphi$  be an automorphism of  $kQ$  and let  $J = \varphi(I)$ . If  $\varphi$  is a dilatation, then  $\sim_I$  and  $\sim_J$  coincide. Assume that  $\varphi = \varphi_{\alpha, u, \tau}$ :*

- If  $\alpha \sim_I u$  and  $\alpha \sim_J u$  then  $\sim_I$  and  $\sim_J$  coincide.
- If  $\alpha \not\sim_I u$  and  $\alpha \sim_J u$  then  $\sim_J$  is generated by  $\sim_I$  and  $\alpha \sim_J u$ .
- If  $\alpha \not\sim_I u$  and  $\alpha \not\sim_J u$  then  $I = J$  and  $\sim_I$  and  $\sim_J$  coincide.

In particular, if  $\alpha \sim_J u$ , then the identity map on the set of walks in  $Q$  induces a surjective group homomorphism  $\pi_1(Q, I) \rightarrow \pi_1(Q, J)$ .

Here *generated* means: generated as an equivalence relation on the set of walks in  $Q$ , and satisfying the conditions (1) and (2) in the definition of the homotopy relation. If  $I, J$  are admissible ideals such that there exists  $\varphi_{\alpha, u, \tau}$  satisfying  $J = \varphi_{\alpha, u, \tau}(I)$ ,  $\alpha \not\sim_I u$  and  $\alpha \sim_J u$ , then we say that  $\sim_J$  is a *direct successor* of  $\sim_I$ . Proposition 1.4 allows one to define a quiver  $\Gamma$  associated to  $A$  as follows ([13, Def. 4.1]):

- $\Gamma_0 = \{\sim_I \mid I \text{ is an admissible ideal of } kQ \text{ such that } A \simeq kQ/I\}$ ,
- there is an arrow  $\sim \rightarrow \sim'$  if  $\sim_J$  is a direct successor of  $\sim_I$ .

*Example 1.5.* Let  $A$  be as in Example 1.2, then  $J = \varphi_{\alpha, cb, 1}(I)$  and  $\Gamma$  is equal to  $\sim_I \rightarrow \sim_J$ .

The quiver  $\Gamma$  is finite, connected and has not oriented cycles ([13, Rem. 3, Prop. 4.2]). Moreover, if  $\Gamma$  has a unique source  $\sim_{I_0}$  (that is, a vertex with no arrow ending at it) then the fundamental group of any admissible presentation of  $A$  is a quotient of  $\pi_1(Q, I_0)$ . It was proved in [15] and [16] that  $\Gamma$  has a unique source under one of the hypotheses  $(H_1)$  or  $(H_2)$  presented in the introduction. Moreover, the hypotheses  $(H_1)$  and  $(H_2)$  both ensure the following proposition which will be particularly useful to prove Theorem 1.

**Proposition 1.6.** ([15, Lem. 4.3] and [16, Prop. 4.3]) *Assume that at least one of the two hypotheses  $(H_1)$  or  $(H_2)$  is satisfied. Let  $\sim_{I_0}, \sim_I \in \Gamma$ , where  $\sim_{I_0}$  is the unique source of  $\Gamma$ . Then there exist a dilatation  $D$  and a sequence of transvections  $\varphi_{\alpha_1, u_1, \tau_1}, \dots, \varphi_{\alpha_l, u_l, \tau_l}$  such that:*

- $I = D\varphi_{\alpha_l, u_l, \tau_l} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$ .
- If we set  $I_i := \varphi_{\alpha_i, u_i, \tau_i} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$ , then  $\alpha_i \sim_{I_i} u_i$  for every  $i$ .

If  $\sim_I \sim \sim_{I_0}$ , then  $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_l}, \sim_I$  coincide.

### 1.6 Comparison of the fundamental groups and the Hochschild cohomology

Let  $A$  be a basic connected finite dimensional  $k$ -algebra. Assume that the ordinary quiver  $Q$  of  $A$  has no oriented cycles. Let  $x_0 \in Q_0$  and fix a maximal tree  $T$  of  $Q$ , that is, a subquiver of  $Q$  such that  $T_0 = Q_0$  and such that the underlying graph of  $T$  is a tree. With these data, Assem and de la Peña have defined an injective homomorphism of abelian groups  $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \hookrightarrow \text{HH}^1(A)$  associated to any admissible presentation  $\nu: kQ \rightarrow A$  ([3]). We recall the definition of  $\theta_\nu$  and refer the reader to [3] for more details. For any  $x \in Q_0$  there exists a unique walk  $\gamma_x$  in  $T$  with source  $x_0$ , with target  $x$  and of minimal length for these properties. Let  $\nu: kQ \rightarrow A$  be an admissible presentation and

let  $f \in \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+)$  be a group homomorphism. Then,  $f$  defines a derivation  $\tilde{f}: A \rightarrow A$  as follows:  $\tilde{f}(\nu(u)) = f([\gamma_y^{-1}u\gamma_x]_{\sim_{\text{Ker}(\nu)}})\nu(u)$  for any path  $u$  with source  $x$  and target  $y$ . The following proposition was proved in [3]:

**Proposition 1.7.** ([3]) *The map  $f \mapsto \tilde{f}$  induces an injective map of abelian groups:*

$$\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \hookrightarrow \text{HH}^1(A) .$$

Note that  $\theta_\nu$  is not surjective in general. Indeed, if  $A$  is the path algebra of the Kronecker quiver, then  $\text{Ker}(\nu) = 0$ ,  $\dim_k \text{Im}(\theta_\nu) = 1$ , and  $\dim_k \text{HH}^1(A) = 3$ . Note also that despite its definition, the homomorphism  $\theta_\nu$  does not depend on the choice of  $T$ . Indeed, let  $T'$  be another maximal tree, thus defining the walk  $\gamma'_x$  of minimal length in  $T'$  with source  $x_0$  and target  $x$ , for every vertex  $x$ . Given a group homomorphism  $f: \pi_1(Q, \text{Ker}(\nu)) \rightarrow k^+$  there is a new derivation  $\hat{f}: A \rightarrow A$  (instead of  $\tilde{f}$ ) obtained by applying the previous construction to  $T'$  (instead of to  $T$ ), that is  $\hat{f}(\nu(u)) = f([\gamma'_y{}^{-1}u\gamma'_x]_{\text{Ker}(\nu)})\nu(u)$  for every path  $u$  in  $Q$  from  $x$  to  $y$ . Now let  $e = \sum_{x \in Q_0} f([\gamma'_x{}^{-1}\gamma_x]_{\text{Ker}(\nu)})e_x \in A$ .

It is easily checked that  $\hat{f} - \tilde{f}$  is the inner derivation associated to  $e$ . In particular,  $\tilde{f}$  and  $\hat{f}$  have equal images in  $\text{HH}^1(A)$ . So the construction of  $\theta_\nu$  does not depend on the choice of the maximal tree  $T$ .

The product in  $k$  endows  $\text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+)$  with a commutative  $k$ -algebra structure. So it is also an abelian Lie algebra for the commutator. The following lemma proves that  $\theta_\nu$  preserves this structure. The proof is just a direct computation, so we omit it.

**Lemma 1.8.**  $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \hookrightarrow \text{HH}^1(A)$  *is a Lie algebra homomorphism. In particular,  $\text{Im}(\theta_\nu)$  is an abelian Lie subalgebra of  $\text{HH}^1(A)$ .*

Throughout this text,  $A$  will be a basic connected finite dimensional  $k$ -algebra with ordinary quiver  $Q$  without oriented cycles ( $Q_0 = \{1, \dots, n\}$ ). We fix a complete set  $\{e_1, \dots, e_n\}$  of primitive orthogonal idempotents of  $A$ . So  $A = E \oplus \mathfrak{r}$ , where  $E = k.e_1 \oplus \dots \oplus k.e_n$  and  $\mathfrak{r}$  is the radical of  $A$ . Without loss of generality, we assume that any presentation  $\nu: kQ \twoheadrightarrow A$  is such that  $\nu(e_i) = e_i$ . Finally, in order to use the Lie algebra homomorphisms  $\theta_\nu$ , we fix a maximal tree  $T$  in  $Q$ .

## 2 Diagonalizability in $\text{HH}^1(A)$

The aim of this section is to prove some useful properties on the subspaces  $\text{Im}(\theta_\nu)$  in terms of diagonalizability in  $\text{HH}^1(A)$ . Note that diagonalizability was introduced for derivations of  $A$  in [9]. For short, a *basis* of  $A$  is a basis  $\mathcal{B}$  of the  $k$ -vector space  $A$  such that:  $\mathcal{B} \subseteq \bigcup_{i,j} e_j A e_i$ , such that

$\{e_1, \dots, e_n\} \subseteq \mathcal{B}$ , and such that  $\mathcal{B} \setminus \{e_1, \dots, e_n\} \subseteq \mathfrak{r}$ . Note the following link between bases and presentations of  $A$ :

- If  $\nu: kQ \twoheadrightarrow A$  is a presentation of  $A$ , then there exists a basis  $\mathcal{B}$  such that  $\nu(\alpha) \in \mathcal{B}$  for any  $\alpha \in Q_1$  and such that any element of  $\mathcal{B}$  is of the form  $\nu(u)$  with  $u$  a path in  $Q$ . We say that this basis  $\mathcal{B}$  is *adapted to  $\nu$* .
- If  $\mathcal{B}$  is a basis of  $A$ , then there exists a presentation  $\nu: kQ \twoheadrightarrow A$  such that  $\nu(\alpha) \in \mathcal{B}$  for any  $\alpha \in Q_1$ . We say that the presentation  $\nu$  is *adapted to  $\mathcal{B}$* .

The property of being diagonalizable (as a linear map) is stable under the sum with an inner derivation as the following lemma shows. The proof is immediate.

**Lemma 2.1.** *Let  $u: A \rightarrow A$  be a linear map, let  $e \in E$  and let  $\mathcal{B}$  be a basis of  $A$ . Then  $u$  is diagonal with respect to the basis  $\mathcal{B}$  if and only if the same holds for  $u + \delta_e$ .*

The preceding lemma justifies the following definition.

**Definition 2.2.** Let  $f \in \text{HH}^1(A)$  and let  $d$  be a derivation representing  $f$ . Then  $f$  is called *diagonalizable* (and *diagonal* with respect to a basis  $\mathcal{B}$  of  $A$ ) if and only if  $d$  is diagonalizable (and diagonal with respect to  $\mathcal{B}$ , respectively).

The subset  $D \subseteq \text{HH}^1(A)$  is called diagonalizable if and only if any there exists a basis  $\mathcal{B}$  of  $A$  such that any  $f \in D$  is diagonal with respect to  $\mathcal{B}$ .

The following proposition gives a criterion for a subset  $D \subset \text{HH}^1(A)$  to be diagonalizable.

**Proposition 2.3.** *Let  $D \subseteq \text{HH}^1(A)$ . Then,  $D$  is diagonalizable if and only if every element of  $D$  is diagonalizable and  $[f, f'] = 0$  for any  $f, f' \in D$ .*

**Proof:** Clearly, if  $D$  is diagonalizable, then so is every element of  $D$  and  $[f, f'] = 0$  for every  $f, f' \in D$ . We prove the converse. For each  $f \in D$ , let  $d_f$  be a derivation representing  $f$ . So  $d_f$  is diagonal with respect to some basis and it suffices to prove that this basis may be assumed to be the same for all  $f \in D$ . Note that  $d_f$  induces a diagonalizable linear map  $d_f: e_j \mathfrak{r} e_i \rightarrow e_j \mathfrak{r} e_i$ , for every  $i, j$  (see Lemma 1.1). Also, for every  $f, f' \in D$ , there exist scalars  $t_i^{(f, f')} \in k$ , for  $i \in \{1, \dots, n\}$ , such that

$[d_f, d_{f'}]$  is the inner derivation  $\delta_{e^{(f,f')}}$ , where  $e^{(f,f')} = \sum_{i=1}^n t_i^{(f,f')} e_i$ . Now, let  $i, j \in \{1, \dots, n\}$ . Then, given  $f, f' \in D$ , we have two diagonalizable maps  $d_f, d_{f'}: e_j \tau e_i \rightarrow e_j \tau e_i$  whose commutator is equal to  $(t_j^{(f,f')} - t_i^{(f,f')}) I d_{e_j \tau e_i}$ . So this commutator must be zero. This shows that there exists a basis  $\mathcal{B}_{i,j}$  of  $e_j \tau e_i$  for which  $d_f: e_j \tau e_i \rightarrow e_j \tau e_i$  has a diagonal matrix. So any  $f \in D$  is diagonal with respect to the basis  $\mathcal{B} = \{e_1, \dots, e_n\} \cup \bigcup_{i,j} \mathcal{B}_{i,j}$  which does not depend on  $f$ . This proves that  $D$  is diagonalizable. ■

Our main example of diagonalizable subspace of  $\text{HH}^1(A)$  is  $\text{Im}(\theta_\nu)$ :

**Proposition 2.4.** *Let  $\nu: kQ \rightarrow A$  be a presentation. Then,  $\text{Im}(\theta_\nu)$  is diagonalizable.*

**Proof:** Let  $\mathcal{B}$  be a basis of  $A$  adapted to  $\nu$  and let  $I = \text{Ker}(\nu)$ . Then  $\theta_\nu(f)$  is diagonal with respect to  $\mathcal{B}$ , for every  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ . ■

In this section, we aim at proving that any diagonalizable subset of  $\text{HH}^1(A)$  is contained in  $\text{Im}(\theta_\nu)$  for some presentation  $\nu$ . It was proved in [9] that any diagonalizable derivation (with suitable technical conditions) defines an element of  $\text{HH}^1(A)$  lying in  $\text{Im}(\theta_\nu)$  for some  $\nu$ . We will use the following similar result.

**Lemma 2.5.** *Let  $f \in \text{HH}^1(A)$  be diagonalizable. Let  $\mathcal{B}$  be a basis with respect to which  $f$  is diagonal. Let  $\nu: kQ \rightarrow A$  be a presentation adapted to  $\mathcal{B}$ . Then  $f \in \text{Im}(\theta_\nu)$ .*

**Proof:** Let  $I = \text{Ker}(\nu)$  and let  $d: A \rightarrow A$  be a derivation representing  $f$ . We set  $\bar{r} := \nu(r)$ , for any  $r \in kQ$ . Let  $\alpha \in Q_1$ . By assumption on  $\mathcal{B}$ , there exists  $t_\alpha \in k$  such that  $d(\bar{\alpha}) = t_\alpha \bar{\alpha}$ . Let  $t_u := t_{\alpha_1} + \dots + t_{\alpha_n}$ , for any path  $u = \alpha_n \dots \alpha_1$  (with  $\alpha_i \in Q_1$ ). So  $d(\bar{u}) = t_u \bar{u}$ , because  $d$  is a derivation. More generally, if  $\gamma = \alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$  is a walk in  $Q$  (with  $\alpha_i \in Q_1$ ), let us set  $t_\gamma := \sum_{i=1}^n (-1)^{\varepsilon_i} t_{\alpha_i}$ , with the convention that  $t_\gamma = 0$  if  $\gamma$  is trivial. We now to prove that the map  $\gamma \mapsto t_\gamma$  defines a group homomorphism  $g: \pi_1(Q, I) \rightarrow k^+, [\gamma]_I \mapsto t_\gamma$  and that  $f = \theta_\nu(g)$ .

First, we prove that the group homomorphism  $g: \pi_1(Q, I) \rightarrow k^+$  is well defined. By definition of the scalar  $t_\gamma$ , we have:

- (i)  $t_{e_x} = 0$  for any  $x \in Q_0$  and  $t_{\gamma'\gamma} = t_{\gamma'} + t_\gamma$  for any walks  $\gamma, \gamma'$  such that the walk  $\gamma'\gamma$  is defined.
- (ii)  $t_{\alpha^{-1}\alpha} = t_{e_x}$  and  $t_{\alpha\alpha^{-1}} = t_{e_y}$  for any arrow  $x \xrightarrow{\alpha} y \in Q_1$ .
- (iii)  $t_{wv'u} = t_{wv'u}$  for any walks  $w, v, v', u$  such that  $t_v = t_{v'}$ , and such that the walks  $wvu, wv'u$  are defined.

In order to prove that  $g$  is well defined, it only remains to prove that  $t_u = t_v$  whenever  $u, v$  are paths in  $Q$  appearing in the same minimal relation of  $I$  (with non zero scalars). For this purpose, let  $d': kQ \rightarrow kQ$  be the linear map such that  $d'(u) = t_u u$  for any path  $u$  in  $k$ . Thus,  $d \circ \nu = \nu \circ d'$ . In particular,  $d'(I) \subseteq I$ . So we may apply Lemma 1.3 to  $d'$  and deduce that:

- (iv)  $t_u = t_v$  if  $u, v$  are paths in  $Q$  lying in the support of a same minimal relation of  $I$ .

From (ii), (iii) and (iv) we deduce that we have a well defined map  $g: \pi_1(Q, I) \rightarrow k, [\gamma]_I \mapsto t_\gamma$ . Moreover, (i) proves that  $g$  is a group homomorphism.

Now we prove that  $f = \theta_\nu(g)$ . For any path  $u$  with source  $x$  and target  $y$ , we have  $g([\gamma_y^{-1} u \gamma_x]_I) = t_u - t_{\gamma_y} + t_{\gamma_x}$ . Hence,  $\theta_\nu(g) \in \text{HH}^1(A)$  is represented by the derivation  $\tilde{g}: A \rightarrow A$  such that  $\tilde{g}(\bar{u}) = (t_u - t_{\gamma_y} + t_{\gamma_x}) \bar{u}$  for any path  $u$  with source  $x$  and target  $y$ . Let us set  $e := \sum_{x \in Q_0} t_{\gamma_x} e_x \in E$ .

Therefore,  $\tilde{g} + \delta_e = d$ . This proves that  $f = \theta_\nu(g)$ . ■

Now we can state the main result of this section. It is a direct consequence of Proposition 2.4 and of Lemma 2.5.

**Proposition 2.6.** *Let  $D \subseteq \text{HH}^1(A)$ . Then  $D$  is diagonalizable if and only if there exists a presentation  $\nu: kQ \rightarrow A$  such that  $D \subseteq \text{Im}(\theta_\nu)$ .*

Remark that Lemma 2.5 also gives a sufficient condition for  $\theta_\nu$  to be an isomorphism. Recall that  $A$  is called constricted if and only if  $\dim e_y A e_x = 1$  for any arrow  $x \rightarrow y$  (this implies that  $Q$  has no multiple arrows). In [5] it was proved that for such an algebra, two different presentations have the same fundamental group.

**Proposition 2.7.** *Assume that  $A$  is constricted. Let  $\nu: kQ \rightarrow A$  be any presentation of  $A$ . Then  $\theta_\nu: \text{Hom}(\pi_1(Q, I), k^+) \rightarrow \text{HH}^1(A)$  is an isomorphism. In particular,  $\text{HH}^1(A)$  is an abelian Lie algebra.*

**Proof:** Since  $\theta_\nu$  is one-to-one, we only need to prove that it is onto. Let  $\mathcal{B}$  be a basis of  $A$  adapted to  $\nu$ , let  $f \in \text{HH}^1(A)$  and let  $d: A \rightarrow A$  be a derivation representing  $f$ . Let  $x \xrightarrow{\alpha} y$  be an arrow. Then  $e_y A e_x = k \cdot \nu(\alpha)$  so that there exists  $t_\alpha \in k$  such that  $d(\nu(\alpha)) = t_\alpha \nu(\alpha)$ . Let  $u = \alpha_n \dots \alpha_1$  be any path in  $Q$  (with  $\alpha_i \in Q_1$ ). Since  $d$  is a derivation, we have  $d(\nu(u)) = (t_{\alpha_1} + \dots + t_{\alpha_n}) \nu(u)$ . As a consequence,  $d$  is diagonal with respect to  $\mathcal{B}$ . Moreover,  $\nu$  is adapted to  $\mathcal{B}$ . So Lemma 2.5 proves that  $f \in \text{Im}(\theta_\nu)$ . This proves that  $\theta_\nu$  is an isomorphism. So  $\text{HH}^1(A)$  is abelian. ■

### 3 Comparison of $\text{Im}(\theta_\nu)$ and $\text{Im}(\theta_\mu)$ for different presentations $\mu$ and $\nu$ of $A$

If two presentations  $\nu$  and  $\mu$  of  $A$  are related by a transvection or a dilatation, then there is a simple relation between the associated fundamental groups (see Proposition 1.4). In this section, we compare  $\theta_\nu$  and  $\theta_\mu$ . We first compare  $\theta_\nu$  and  $\theta_\mu$  when  $\mu = \nu \circ D$  with  $D$  a dilatation. Recall that if  $J = D(I)$  with  $D$  a dilatation, then  $\sim_I$  and  $\sim_J$  coincide, so that  $\pi_1(Q, I) = \pi_1(Q, J)$ .

**Proposition 3.1.** *Let  $\nu: kQ \rightarrow A$  be a presentation, let  $D: kQ \xrightarrow{\sim} kQ$  be a dilatation. Let  $\mu := \nu \circ D: kQ \rightarrow A$ . Let  $I = \text{Ker}(\mu)$  and  $J = \text{Ker}(\nu)$ , so that  $J = D(I)$ . Then  $\theta_\mu = \theta_\nu$ .*

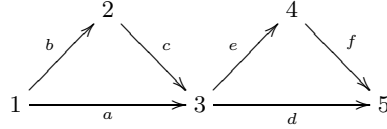
**Proof:** Let  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ . Then,  $\theta_\nu(f)$  and  $\theta_\mu(f)$  are represented by the derivations  $d_1$  and  $d_2$  respectively, such that for any arrow  $x \xrightarrow{a} y$ :

$$\begin{aligned} d_1(\nu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_J) \nu(\alpha) \\ d_2(\mu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) . \end{aligned}$$

Therefore,  $d_1(\nu(\alpha)) = d_2(\mu(\alpha))$  because  $D$  is a dilatation and because  $\sim_I$  and  $\sim_J$  coincide. This implies that  $d_1 = d_2$  and  $\theta_\nu(f) = \theta_\mu(f)$ .  $\blacksquare$

The following example shows that Proposition 3.1 does not necessarily hold true if  $\nu$  and  $\mu$  are two presentations of  $A$  such that  $\sim_{\text{Ker}(\nu)}$  and  $\sim_{\text{Ker}(\mu)}$  coincide.

*Example 3.2.* Assume that  $\text{char}(k) = 2$  and let  $A = kQ/I$  where  $Q$  is the quiver:



and  $I = \langle da, fecb, fea + dc b \rangle$ . Let  $T$  be the maximal tree such that  $T_1 = \{b, c, e, f\}$ . Let  $\nu: kQ \rightarrow A = kQ/I$  be the natural projection. Let  $\psi := \varphi_{a,cb,1} \varphi_{d,fe,1}$ . Thus,  $I = \psi(I)$ . Let  $\mu := \nu \circ \psi: kQ \rightarrow A$  so that  $\text{Ker}(\mu) = \text{Ker}(\nu) = I$ . Observe that  $\pi_1(Q, I)$  is the infinite cyclic group with generator  $[b^{-1}c^{-1}a]_I$ . So let  $f: \pi_1(Q, I) \rightarrow k^+$  be the unique group homomorphism such that  $f([b^{-1}c^{-1}a]_I) = 1$ . Then  $\theta_\nu(f)$  is represented by the following derivation:

$$\begin{aligned} d_1: \quad A &\longrightarrow A \\ \nu(x) &\longmapsto \nu(x) \text{ if } x \in \{a, d\} \\ \nu(x) &\longmapsto 0 \text{ if } x \in \{b, c, e, f\} . \end{aligned}$$

On the other hand,  $\theta_\mu(f)$  is represented by the derivation:

$$\begin{aligned} d_2: \quad A &\longrightarrow A \\ \nu(a) &\longmapsto \nu(a) + \nu(cb) \\ \nu(d) &\longmapsto \nu(d) + \nu(fe) \\ \nu(x) &\longmapsto 0 \text{ if } x \in \{b, c, e, f\} . \end{aligned}$$

It is easy to verify that  $d_2 - d_1$  is not an inner derivation. Hence,  $\theta_\nu \neq \theta_\mu$ .

Now we compare  $\theta_\nu$  and  $\theta_\mu$  when  $\mu = \nu \circ \varphi_{\alpha,u,\tau}$  and when the identity map on the set of walks in  $Q$  induces a surjective group homomorphism  $\pi_1(Q, \text{Ker}(\nu)) \twoheadrightarrow \pi_1(Q, \text{Ker}(\mu))$ .

**Proposition 3.3.** *Let  $\nu: kQ \rightarrow A$  be a presentation, let  $\varphi_{\alpha,u,\tau}: kQ \xrightarrow{\sim} kQ$  be a transvection and let  $\mu := \nu \circ \varphi_{\alpha,u,\tau}: kQ \rightarrow A$ . Set  $I = \text{Ker}(\nu)$  and  $J = \text{Ker}(\mu)$ , so that  $I = \varphi_{\alpha,u,\tau}(J)$ . Suppose that  $\alpha \sim_J u$  and let  $p: \pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$  be the quotient map (see Proposition 1.4). Then, the following diagram commutes:*

$$\begin{array}{ccc} \text{Hom}(\pi_1(Q, J), k^+) & & \\ \downarrow p^* & \searrow \theta_\mu & \\ & & \text{HH}^1(A) \\ & \nearrow \theta_\nu & \\ \text{Hom}(\pi_1(Q, I), k^+) & & \end{array}$$

where  $p^*: \text{Hom}(\pi_1(Q, J), k^+) \hookrightarrow \text{Hom}(\pi_1(Q, I), k^+)$  is the embedding induced by  $p$ . In particular,  $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_\nu)$ .



**Proof:** Recall that  $p$  is the map  $[\gamma]_I \mapsto [\gamma]_J$ . Let  $f \in \text{Hom}(\pi_1(Q, J), k^+)$ . So  $p^*(f)$  is the composition  $\pi_1(Q, I) \xrightarrow{p} \pi_1(Q, J) \xrightarrow{f} k$ . We know that  $\theta_\mu(f)$  and  $\theta_\nu(p^*(f))$  are represented by the derivations  $d_1$  and  $d_2$  respectively, such that for any arrow  $x \xrightarrow{a} y$ :

$$\begin{aligned} d_1(\mu(a)) &= f([\gamma_y^{-1} a \gamma_x]_J) \mu(a) = p^*(f)([\gamma_y^{-1} a \gamma_x]_I) \mu(a) \\ d_2(\nu(a)) &= p^*(f)([\gamma_y^{-1} a \gamma_x]_I) \nu(a) . \end{aligned}$$

Let us prove that  $d_1$  and  $d_2$  coincide on  $\nu(Q_1)$ . Let  $x \xrightarrow{a} y$  be an arrow. If  $a \neq \alpha$ , then  $\mu(a) = \nu(a)$  and the above characterizations of  $d_1$  and  $d_2$  imply that  $d_1(\nu(a)) = d_1(\mu(a)) = d_2(\nu(a))$ . Now assume that  $a = \alpha$  so that:  $\nu(a) = \mu(a) - \tau\mu(u)$  and  $[\gamma_y^{-1} a \gamma_x]_J = [\gamma_y^{-1} u \gamma_x]_J$  (recall that  $a = \alpha \sim_J u$ ). Thus:

$$\begin{aligned} d_1(\nu(a)) &= d_1(\mu(\alpha)) - \tau d_1(\mu(u)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_J) \mu(\alpha) - \tau f([\gamma_y^{-1} u \gamma_x]_J) \mu(u) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_J) (\mu(\alpha) - \tau \mu(u)) \\ &= p^*(f)([\gamma_y^{-1} \alpha \gamma_x]_I) \nu(\alpha) \\ &= d_2(\nu(\alpha)) = d_2(\nu(a)) . \end{aligned}$$

Hence,  $d_1$  and  $d_2$  are two derivations of  $A$  and they coincide on  $\nu(Q_1)$ . So  $d_1 = d_2$  and  $\theta_\mu(f) = \theta_\nu(p^*(f))$  for any  $f \in \text{Hom}(\pi_1(Q, J), k^+)$ .  $\blacksquare$

The following example shows that Proposition 3.3 does not necessarily hold true if  $\nu$  is a presentation of  $A$  and  $\psi: kQ \rightarrow kQ$  is an automorphism such that the identity map on the walks in  $Q$  induces a surjective group homomorphism  $\pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\nu \circ \psi))$ .

*Example 3.4.* Let  $A = kQ/I$  where  $\text{char}(k) = 2$ , where  $Q$  is the quiver of Example 3.2 and where  $I = \langle da, fea + dcb \rangle$ . Let  $\nu: kQ \rightarrow A$  be the natural projection with kernel  $I$ , let  $\psi := \varphi_{d,ef,1} \varphi_{a,cb,1}$  and let  $\mu := \nu \circ \psi: kQ \rightarrow A$ . Hence  $\text{Ker}(\mu) = \langle da + fecb, fea + dcb \rangle$ . Note that  $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$  is generated by  $[b^{-1}c^{-1}a]_I$  and that  $\pi_1(Q, \text{Ker}(\mu)) \simeq \mathbb{Z}/2\mathbb{Z}$  is generated by  $[b^{-1}c^{-1}a]_J$ . Note also that  $\sim_{\text{Ker}(\nu)}$  is weaker than  $\sim_{\text{Ker}(\mu)}$  so that the identity map on the set of walks in  $Q$  induces a surjective group homomorphism  $p: \pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\mu))$ . Let  $T$  be the maximal tree such that  $T_1 = \{b, c, e, f\}$ . Let  $f: \pi_1(Q, \text{Ker}(\mu)) \rightarrow k$  be the group homomorphism such that  $f([b^{-1}c^{-1}a]_J) = 1$ . On the one hand,  $\theta_\mu(f) \in \text{HH}^1(A)$  is represented by the derivation:

$$\begin{aligned} d_1: \quad A &\rightarrow A \\ \mu(x) &\mapsto \mu(x) \text{ if } x \in \{a, d\} \\ \mu(x) &\mapsto 0 \text{ if } x \in \{b, c, e, f\} . \end{aligned}$$

On the other hand,  $\theta_\nu(p^*(f)) \in \text{HH}^1(A)$  is represented by the derivation:

$$\begin{aligned} d_2: \quad A &\rightarrow A \\ \mu(a) &\mapsto \mu(a) + \mu(cb) \\ \mu(d) &\mapsto \mu(d) + \mu(fe) \\ \mu(x) &\mapsto 0 \text{ if } x \in \{b, c, e, f\} . \end{aligned}$$

One checks easily that  $d_2 - d_1$  is not inner so that  $\theta_\mu(f) \neq \theta_\nu(p^*(f))$ . Moreover,  $\text{Im}(\theta_\nu)$  and  $\text{Im}(\theta_\mu)$  are one dimensional (because  $\text{char}(k) = 2$ ,  $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$  and  $\pi_1(Q, \text{Ker}(\mu)) \simeq \mathbb{Z}/2\mathbb{Z}$ ) and  $d_1, d_2$  are not inner. Hence  $\text{Im}(\theta_\mu) \not\subseteq \text{Im}(\theta_\nu)$ .

Actually, Proposition 3.3 does not work here because the automorphism  $\psi: (kQ, J) \rightarrow (kQ, I)$  maps arrows to linear combination of paths which are not homotopic for  $\sim_I$ . For example,  $\psi(a) = a + cb$  whereas  $a \not\sim_I cb$  (recall that  $\pi_1(Q, I) \simeq \mathbb{Z}$  is generated by  $[b^{-1}c^{-1}a]_I$ ).

Finally, we compare  $\theta_\nu$  and  $\theta_\mu$  when  $\mu = \nu \circ \psi$  with  $\psi: kQ \xrightarrow{\sim} kQ$  an automorphism such that  $\text{Ker}(\nu) = \text{Ker}(\mu)$ .

**Proposition 3.5.** *Let  $\nu: kQ \rightarrow A$  be a presentation and let  $I = \text{Ker}(\nu)$ . Let  $\psi: kQ \xrightarrow{\sim} kQ$  be an automorphism such that  $\psi(e_i) = e_i$  for every  $i$  and such that  $\psi(I) = I$ . Let  $\mu := \nu \circ \psi: kQ \rightarrow A$  so that  $\text{Ker}(\mu) = I$ . Let  $\bar{\psi}: A \xrightarrow{\sim} A$  be the  $k$ -algebra automorphism such that  $\bar{\psi} \circ \mu = \mu \circ \psi$ . Then, the following diagram commutes:*

$$\begin{array}{ccc} & & \text{HH}^1(A) \\ & \nearrow \theta_\nu & \downarrow \sim \bar{\psi}_* \\ \text{Hom}(\pi_1(Q, I), k^+) & & \text{HH}^1(A) \\ & \searrow \theta_\mu & \end{array} .$$

In particular,  $\text{Im}(\theta_\mu)$  is equal to the image of  $\text{Im}(\theta_\nu)$  under the Lie algebra automorphism  $\bar{\psi}_* : \text{HH}^1(A) \xrightarrow{\sim} \text{HH}^1(A)$  induced by  $\bar{\psi} : A \xrightarrow{\sim} A$ .

**Proof:** Since  $\psi$  fixes the idempotents  $e_1, \dots, e_n$ , we know that  $\bar{\psi}_*$  is well defined. Let  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ . So  $\theta_\nu(f)$  and  $\theta_\mu(f)$  are represented by the derivations  $d_1$  and  $d_2$  respectively, such that for any arrow  $x \xrightarrow{\alpha} y$ :

$$\begin{aligned} d_1(\nu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \nu(\alpha) \\ d_2(\mu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) . \end{aligned}$$

In order to prove that  $\bar{\psi}_*(\theta_\nu(f)) = \theta_\mu(f)$  it suffices to prove that  $\bar{\psi} \circ d_1 = d_2 \circ \bar{\psi}$ . Let  $x \xrightarrow{\alpha} y$  be an arrow. Then:

$$\begin{aligned} d_2 \circ \bar{\psi}(\nu(\alpha)) &= d_2(\mu(\alpha)) && \text{because } \nu = \mu \circ \psi^{-1} \text{ and } \bar{\psi} \circ \mu = \mu \circ \psi \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) . \end{aligned}$$

On the other hand:

$$\begin{aligned} \bar{\psi} \circ d_1(\nu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \bar{\psi}(\nu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) && \text{because } \mu = \nu \circ \psi \text{ and } \bar{\psi} \circ \mu = \mu \circ \psi . \end{aligned}$$

Hence,  $\bar{\psi} \circ d_1$  and  $d_2 \circ \bar{\psi}$  are derivations of  $A$  which coincide on  $\nu(Q_1)$ . So  $\bar{\psi}_*(\theta_\nu(f)) = \theta_\mu(f)$  for any  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ .  $\blacksquare$

## 4 Proof of Theorem 1

In this section, we prove Theorem 1. We begin with the following useful lemma.

**Lemma 4.1.** *Assume that at least one of the two conditions  $(H_1)$  or  $(H_2)$  is satisfied. Let  $\nu : kQ \rightarrow A$  be a presentation whose kernel  $I_0$  is such that  $\sim_{I_0}$  is the unique source of  $\Gamma$ . Let  $\mu : kQ \rightarrow A$  be another presentation. Then, there exist  $\nu' : kQ \rightarrow A$  a presentation with kernel  $I_0$  and a  $k$ -algebra automorphism  $\psi : A \xrightarrow{\sim} A$  such that:*

- $\psi(e_i) = e_i$  for any  $i$ ,
- $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_{\nu'}) = \psi_*(\text{Im}(\theta_\nu))$ .

If moreover  $\sim_I$  and  $\sim_{I_0}$  coincide, then the above inclusion is an equality.

**Proof:** Let  $\bar{\nu} : kQ/I_0 \xrightarrow{\sim} A$  and  $\bar{\mu} : kQ/I \xrightarrow{\sim} A$  be the isomorphisms induced by  $\nu$  and  $\mu$  respectively. Hence,  $\bar{\mu}^{-1} \circ \bar{\nu} : kQ/I_0 \xrightarrow{\sim} kQ/I$  is an isomorphism which maps  $e_i$  to  $e_i$  for every  $i$ . Hence, there exists an automorphism  $\varphi : kQ \xrightarrow{\sim} kQ$  which maps  $e_i$  to  $e_i$  for every  $i$  and such that the following diagram commutes (see [14, Prop. 2.3.18], for instance):

$$\begin{array}{ccc} kQ & \xrightarrow{\varphi} & kQ \\ \downarrow & & \downarrow \\ kQ/I_0 & \xrightarrow{\bar{\mu}^{-1} \bar{\nu}} & kQ/I \end{array} .$$

where the vertical arrows are the natural projections. So, the following diagram is commutative:

$$\begin{array}{ccc} kQ & \xrightarrow{\varphi} & kQ \\ & \searrow \nu & \swarrow \mu \\ & & A \end{array}$$

Let us apply Proposition 1.6 to  $I$ . We keep the notations  $\alpha_i, u_i, \tau_i, I_i$  of that proposition. Let  $\psi := \varphi^{-1} D \varphi_{\alpha_1, u_1, \tau_1} \dots \varphi_{\alpha_l, u_l, \tau_l}$ . Thus,  $\psi(I_0) = I_0$  and  $\nu' := \nu \circ \psi : kQ \rightarrow A$  is a presentation with kernel  $I_0$ . Thanks to Proposition 3.5 from which we keep the notations, we know that:

$$\text{Im}(\theta_{\nu'}) = \bar{\psi}_*(\text{Im}(\theta_\nu)) . \quad (1)$$

Let us show that  $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_{\nu'})$ . By construction, we have  $\mu = \nu \varphi^{-1} = \nu' \varphi_{\alpha_1, u_1, \tau_1}^{-1} \dots \varphi_{\alpha_l, u_l, \tau_l}^{-1} D^{-1}$ . For simplicity, we use the following notations:  $\mu_0 := \nu'$  and  $\mu_i := \nu' \varphi_{\alpha_1, u_1, \tau_1}^{-1} \dots \varphi_{\alpha_i, u_i, \tau_i}^{-1}$  for  $i \in \{1, \dots, l\}$ . Note that  $I_i = \text{Ker}(\mu_i)$ . Since  $\alpha_i \sim_{I_i} u_i$  and  $\mu_i = \mu_{i-1} \circ \varphi_{\alpha_i, u_i, -\tau_i}$ , Proposition 3.3 implies that:

$$\text{Im}(\theta_{\mu_i}) \subseteq \text{Im}(\theta_{\mu_{i-1}}) \subseteq \dots \subseteq \text{Im}(\theta_{\mu_i}) \subseteq \text{Im}(\theta_{\mu_{i-1}}) \subseteq \dots \subseteq \text{Im}(\theta_{\mu_0}) = \text{Im}(\theta_{\nu'}) . \quad (2)$$

Moreover,  $\mu = \mu_m D^{-1}$  where  $D^{-1}$  is a dilatation. Hence, (1), (2) and Proposition 3.1 imply that:

$$\mathbf{Im}(\theta_\mu) = \mathbf{Im}(\theta_{\mu_l}) \subseteq \mathbf{Im}(\theta_{\nu'}) = \overline{\psi}_*(\mathbf{Im}(\theta_\nu)) . \quad (3)$$

Now assume that  $\sim_I$  is the unique source of  $\Gamma$ . Then Proposition 1.6 imply that the homotopy relations  $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_l}, \sim_I$  coincide. Therefore, for any  $i \in \{1, \dots, l\}$ , we have  $\mu_{i-1} = \mu_i \circ \varphi_{\alpha_i, u_i, \tau_i}$ , and  $\alpha_i \sim_{I_{i-1}} u_i$ . So Proposition 3.3, implies that  $\mathbf{Im}(\theta_{\mu_{i-1}}) \subseteq \mathbf{Im}(\theta_{\mu_i})$ . This proves that all the inclusions in (2) are equalities, and so is the inclusion in (3). ■

Now we can prove Theorem 1.

**Proof of Theorem 1:** (i) Let  $\mathcal{G}$  be a maximal diagonalizable subalgebra of  $\mathbf{HH}^1(A)$ . Thanks to Proposition 2.6, there exists a presentation  $\mu: kQ \rightarrow A$  such that  $\mathcal{G} \subseteq \mathbf{Im}(\theta_\mu)$ . On the other hand, Lemma 4.1, implies that there exists a presentation  $\nu: kQ \rightarrow A$  such that  $\sim_{\text{Ker}(\nu)}$  is the unique source of  $\Gamma$  and such that  $\mathbf{Im}(\theta_\mu) \subseteq \mathbf{Im}(\theta_\nu)$ . Hence,  $\mathcal{G} \subseteq \mathbf{Im}(\theta_\nu)$  where  $\mathbf{Im}(\theta_\nu)$  is a diagonalizable subalgebra of  $\mathbf{HH}^1(A)$ , thanks to Proposition 2.4. The maximality of  $\mathcal{G}$  forces  $\mathcal{G} = \mathbf{Im}(\theta_\nu)$ .

Conversely, let  $\mu: kQ \rightarrow A$  be a presentation such that  $\sim_{\text{Ker}(\mu)}$  is the unique source of  $\Gamma$ . Hence,  $\mathbf{Im}(\theta_\mu)$  is diagonalizable (thanks to Proposition 2.4) so there exists a maximal diagonalizable subalgebra  $\mathcal{G}$  of  $\mathbf{HH}^1(A)$  containing  $\mathbf{Im}(\theta_\mu)$ . Thanks to the above description, we know that  $\mathcal{G} = \mathbf{Im}(\theta_\nu)$  where  $\nu: kQ \rightarrow A$  is a presentation such that  $\sim_{\text{Ker}(\nu)}$  is the unique source of  $\Gamma$ . Moreover, Lemma 4.1 gives a  $k$ -algebra automorphism  $\psi: A \xrightarrow{\sim} A$  such that  $\mathbf{Im}(\theta_\mu) = \psi_*(\mathbf{Im}(\theta_\nu))$ . Since  $\psi_*$  is a Lie algebra automorphism of  $\mathbf{HH}^1(A)$ , the maximality of  $\mathcal{G} = \mathbf{Im}(\theta_\nu)$  implies that  $\mathbf{Im}(\theta_\mu)$  is maximal.

(ii) is a consequence of (i) and of Lemma 4.1. ■

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