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On the comparison of the Hochschild cohomology and the fundamental groups of and algebra

Patrick Le Meur ^{*†}

10th January 2007

Abstract

Let A be a basic connected finite dimensional k -algebra with ordinary quiver Q without oriented cycle. We compare the group morphisms $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \rightarrow HH^1(A)$ established in [3], for the different presentations $\nu: kQ \rightarrow A$ by quiver and admissible relations. We characterise the spaces $\text{Im}(\theta_\nu)$ in terms of diagonalisable subspaces of $HH^1(A)$. Assuming that A has no double bypass and k has characteristic zero, or that A is monomial and without multiple arrows, we characterise the maximal diagonalisable subalgebras of $HH^1(A)$ using the quiver of the homotopy relations of the presentations of A introduced in [11].

Introduction

Let A be a finite dimensional algebra over an algebraically closed field k . In the study of the derived category of A , the Hochschild cohomology ([9]) $HH^*(A)$ plays an important role because it is an invariant of the derived category (see [10]) and also because it carries a lot of structure since it is k -algebra, a Lie algebra and more generally a Gerstenhaber algebra (see [8]). In particular, $HH^1(A)$ is a Lie subalgebra of $HH^*(A)$. Recall that A is naturally an $A - A$ -bimodule (or a $A \otimes A$ -module) and that $HH^*(A) = \bigoplus_{n \geq 0} HH^n(A)$ where $HH^n(A) = \text{Ext}_{A \otimes A}^i(A, A)$.

The space $HH^1(A)$ was studied intensively (see for example [3], [5], [7], [16], [17], [18] and [19]). In particular, in [3], I. Assem and J. A. de la Peña have proved that if A is basic and connected, if the ordinary quiver Q of A has no oriented cycle and if $\nu: kQ \rightarrow A$ is a presentation, then there is an injective mapping of abelian groups $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \hookrightarrow HH^1(A)$. Here $\pi_1(Q, \text{Ker}(\nu))$ is the fundamental group of the ν defined in [15]. Later, in [7], D. Farkas, E. Green and E. N. Marcos have studied the elements of $\text{Im}(\theta_\nu)$ using the notion of diagonalisable derivation.

The aim of this text is to characterise the subspaces $\text{Im}(\theta_\nu)$ of $HH^1(A)$ in terms of Lie algebras and to compare the morphisms θ_ν associated to the different presentations ν of A . Indeed, following [7], it is possible to define a notion of diagonalisability for an element $HH^1(A)$ (which corresponds to the notion of diagonalisability of a derivation) and also of a subset of $HH^1(A)$ (which corresponds to the simultaneous diagonalisability of a set of derivations). It appears that $\text{Im}(\theta_\nu)$ is a diagonalisable subset of $HH^1(A)$ and that any diagonalisable subset of $HH^1(A)$ is contained in $\text{Im}(\theta_\nu)$ for some presentation ν . In order to compare the morphisms θ_ν associated to the different presentations of A , we will use the comparisons between the fundamental groups of the presentations of A made by the author in [14]. Recall that if $\nu: kQ \rightarrow A$ is a presentation with kernel I , then the fundamental group $\pi_1(Q, I)$ is defined as the quotient space of the set of unoriented paths (called walks) in Q by a homotopy relation \sim_I depending on I . Assuming that Q has no oriented cycle, it was proved in [14] that the homotopy relations of the presentations of A can be displayed as the vertices of finite, connected quiver Γ without oriented cycle and such that for any arrow $\sim_I \rightarrow \sim_J$ there is a natural surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$. In such a situation, if ν and μ are presentations of A such that $\text{Ker}(\nu) = I$ and $\text{Ker}(\mu) = J$, then we will see that $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_\nu)$. This property and the above description of the spaces $\text{Im}(\theta_\nu)$ as diagonalisable subspaces of $HH^1(A)$ lead naturally to the following question:

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Is it possible to characterise and compare the maximal diagonalisable subspaces of $HH^1(A)$?

The above explanations suggest that such a subspace of $HH^1(A)$ should be of the form $Im(\theta_\nu)$ for ν a presentation such that $\sim_{Ker(\nu)}$ is a source of Γ . We shall give a precise answer to the above question assuming one of the following hypotheses which ensure that Γ has a unique source (see [14, Prop. 2.11] and [13, Cor. 4.4]):

(H₁) Q has no double bypass and k has characteristic zero.

(H₂) A is monomial (i.e. $A \simeq kQ/I_0$ with I_0 an ideal generated by a set of paths) and Q has no multiple arrows.

More precisely we will prove the following theorem which is the main result of this text.

Theorem 1. *Assume that at least one the two hypotheses (H₁) and (H₂) is satisfied. Then:*

- (i) *The maximal diagonalisable subalgebras of $HH^1(A)$ are exactly the subalgebras of the form $Im(\theta_\nu)$ where $\nu: kQ \twoheadrightarrow A$ is a presentation such that $\sim_{Ker(\nu)}$ is the unique source of Γ .*
- (ii) *If $\mathcal{G}, \mathcal{G}'$ are two such subalgebras of $HH^1(A)$, then there exists an algebra automorphism $\psi: A \xrightarrow{\sim} A$ inducing a Lie algebra automorphism $\psi_*: HH^1(A) \xrightarrow{\sim} HH^1(A)$ and such that $\mathcal{G}' = \psi_*(\mathcal{G})$.*

The text is organised as follows. In Section 1 we recall all the definitions we will need and proof some useful lemmas. In Section 2, we introduce the notion of diagonalisability in $HH^1(A)$. In particular, we will prove that a subset of $HH^1(A)$ is diagonalisable is and only if it is contained in $Im(\theta_\nu)$ for some presentation $\nu: kQ \twoheadrightarrow A$. In Section 3 we compare the morphisms θ_ν for different presentations ν of A , using the quiver Γ . Finally, in Section 4 we prove Theorem 1.

This text is part of the author's thesis ([12]) made at Université Montpellier II under the supervision of Claude Cibils.

1 Preliminaries

Terminology and notations for quivers. If Q is a quiver, $Q_0 = \{1, \dots, n\}$ (resp. Q_1) will denote the set of vertices (resp. of arrows). The source (resp. target) of $\alpha \in Q_1$ will be denoted by $s(\alpha)$ (resp. by $t(\alpha)$). The **stationary** path (i.e. of length 0) with source and target $i \in Q_0$ will be denoted by e_x . The (oriented) paths in Q are read from the right to the left, i.e. the path $\alpha_n \dots \alpha_1$ with source x and target y denotes the sequence of arrows $\alpha_1, \dots, \alpha_n$ such that $s(\alpha_1) = x$, $s(\alpha_{i+1}) = t(\alpha_i)$ for every i , and $t(\alpha_n) = y$. Two paths are called **parallel** if they have the same source and the same target. If $u = \alpha_m \dots \alpha_1$ and $v = \beta_n \dots \beta_1$ are paths in Q , the concatenation vu is defined if $t(\alpha_m) = s(\beta_1)$ and is equal to $vu = \beta_n \dots \beta_1 \alpha_m \dots \alpha_1$, with the convention $vu = u$ (resp. $vu = v$) if v (resp. u) is stationary. An **oriented cycle** in Q is a non stationary path whose source and target are equal. If $\alpha \in Q_1$ we consider its formal inverse α^{-1} with source $s(\alpha^{-1}) := t(\alpha)$ and target $t(\alpha^{-1}) := s(\alpha)$. A **walk** in Q with source x and target y is either the stationary path e_x (in which case $y = x$) or a sequence $\alpha_1, \dots, \alpha_n$ of arrows and formal inverses of arrows of Q such that $s(\alpha_1) = x$, $s(\alpha_{i+1}) = t(\alpha_i)$ for every i and $t(\alpha_n) = y$, in such a case, the walk is denoted by $\alpha_n \dots \alpha_1$. The concatenation of walks in Q is defined similarly to the concatenation of paths. If $\gamma = \alpha_n \dots \alpha_1$ is walk, we will denote by γ^{-1} the walk $\alpha_1^{-1} \dots \alpha_n^{-1}$ with the convention $(\alpha^{-1})^{-1}$ for any arrow α . A **bypass** (see [2]) is a pair (α, u) where $\alpha \neq u$, $\alpha \in Q_1$ and u is a path in Q parallel to α . A **double bypass** (see [11]) is a 4-tuple (α, u, β, v) where (α, u) and (β, v) are bypasses such that the arrow β appears in the path u .

Admissible presentations. If Q is a finite quiver (i.e. Q_0 and Q_1 are finite), its **path algebra** kQ is the k -algebra whose basis as a k -vector space is the set of paths in Q (including the stationary paths) and whose product is bilinearly induced by the concatenation of paths. The unit of kQ is $\sum_{i=1}^n e_i$ and kQ is finite dimensional if and only if Q has no oriented cycle. An **admissible ideal** of kQ is an ideal I verifying the following conditions:

1. I is generated by linear combinations of paths of length at least 2,
2. there exists $N \geq 2$ such that any path of length at least N lies in I .

In such a case, the elements of I are called **relations** and, following [15], a **minimal relation** of I is a relation $\sum_{i=1}^s t_i u_i \neq 0$ such that:

1. $t_1, \dots, t_l \in k^*$ and u_1, \dots, u_l are pairwise distinct paths in Q ,

2. if $S \subseteq \{1, \dots, n\}$ verifies $\sum_{i \in S} t_i u_i \in I$, then $S = \emptyset$ or $S = \{1, \dots, s\}$.

With the above notations, the paths u_1, \dots, u_n are necessarily parallel. Notice that I is generated by its minimal relations. Let A be a finite dimensional k -algebra. Then A is Morita equivalent to a **basic** finite dimensional k -algebra A' , where basic means that $A' = P_1 \oplus \dots \oplus P_n$ where P_1, \dots, P_n are pairwise non isomorphic indecomposable projective A' -modules. If A is basic, then (see [4]) there exists a quiver Q and a surjective k -algebra morphism $\nu: kQ \twoheadrightarrow A$ whose kernel is an admissible ideal of kQ . The quiver Q is then unique and called the **ordinary quiver** of A and $\{\nu(e_i) \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents. The morphism ν is called an **(admissible) presentation**. In such a situation, $A \simeq kQ/\text{Ker}(\nu)$ and A is connected if and only if Q is connected (i.e. the underlying graph of Q is connected).

Presentation of $HH^1(A)$. Let A be a basic finite dimensional k -algebra and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents. A **unitary derivation** (see [16]) is a k -linear mapping $d: A \rightarrow A$ such that $d(ab) = ad(b) + d(a)b$ for any $a, b \in A$ and such that $d(e_i) = e_i$ for every i . Let $\text{Der}_0(A)$ be set of unitary derivations. It is a Lie algebra for the bracket $[d, d'] = d \circ d' - d' \circ d$. In the sequel, all derivations will be unitary, hence, we shall call them derivations, for short. Let $E := \{\sum_{i=1}^n t_i e_i\}$.

Then E is a semi-simple subalgebra of A and $A = E \oplus \mathfrak{r}$ where \mathfrak{r} is the radical of A . Let $\text{Int}_0(A) := \{\delta_e: A \rightarrow A, a \in A \mapsto ea - ae \mid e \in E\}$, this is an ideal of $\text{Der}_0(A)$. Throughout this text, we shall use the following presentation established in [16]:

Theorem 1.1. (see [16]) $HH^1(A) \simeq \text{Der}_0(A)/\text{Int}_0(A)$ as Lie algebras.

In the following lemma, we collect some useful properties on derivations.

Lemma 1.2. Let $d \in \text{Der}_0(A)$, then $d(e_j A e_i) \subseteq e_j A e_i$. Assume that the ordinary quiver Q of A has no oriented cycle, then $d(\mathfrak{r}) \subseteq \mathfrak{r}$ and $d(\mathfrak{r}^2) \subseteq \mathfrak{r}^2$.

Proof: The first assertion is a direct consequence of the fact that d is a unitary derivation. Let $u \in e_j \mathfrak{r} e_i \setminus \mathfrak{r}^2$. Since Q has no oriented cycle, we have $i \neq j$. Hence $d(u) \in e_j A e_i = e_j \mathfrak{r} e_i$. This proves that $d(\mathfrak{r}) \subseteq \mathfrak{r}$. Since d is a derivation, we infer that $d(\mathfrak{r}^2) \subseteq \mathfrak{r}^2$. ■

If $\psi: A \xrightarrow{\sim} A$ is a k -algebra automorphism such that $\psi(e_i) = e_i$ for every i , then the mapping $d \mapsto \psi \circ d \circ \psi^{-1}$ defines a Lie algebra automorphism of $\text{Der}_0(A)$ and preserves $\text{Int}_0(A)$. Hence, it induces a Lie algebra automorphism which will be denoted by $\psi_*: HH^1(A) \xrightarrow{\sim} HH^1(A)$.

Fundamental groups of presentations. Let (Q, I) be a **bound quiver** (i.e. Q is a finite quiver and I is an admissible ideal of kQ). The **homotopy relation** \sim_I was defined in [15] as the equivalence class on the set of walks in Q generated by the following properties:

- (1) $\alpha \alpha^{-1} \sim_I e_y$ and $\alpha^{-1} \alpha \sim_I e_x$ for any arrow α with source x and target y ,
- (2) $wvu \sim_I wv'u$ if w, v, v', u are walks such that the concatenations wvu and $wv'u$ are well defined and such that $v \sim_I v'$,
- (3) $u \sim_I v$ if u and v are paths in Q appearing in a same minimal relation (with a non zero scalar).

Notice that if r_1, \dots, r_t are minimal relations generating I , then the condition (3) above may be replaced by the following one (see [5]):

- (3') $u \sim_I v$ if u and v are paths in Q appearing in r_i (with a non zero scalar) for some $i \in \{1, \dots, t\}$.

The \sim_I -equivalence class of a walk γ will be denoted by $[\gamma]_I$. Let $x_0 \in Q_0$, following [15], the set of \sim_I -equivalence classes of walks with source and target x_0 is denoted by $\pi_1(Q, I, x_0)$. The concatenation of walks endows this set with a group structure whose unit is $[e_{x_0}]_I$. This group is called the **fundamental group** of (Q, I) at x_0 . If Q is connected, then the isomorphism class of $\pi_1(Q, I, x_0)$ does not depend on the choice x_0 . In such a case, we shall write $\pi_1(Q, I)$ for $\pi_1(Q, I, x_0)$. If A is a basic connected finite dimensional k -algebra and if $\nu: kQ \twoheadrightarrow A$ is a presentation, the group $\pi_1(Q, \text{Ker}(\nu))$ will be called the **fundamental group of the presentation** ν . The following example shows that two presentations of A may have non isomorphic fundamental groups.

Example 1.3. Let $A = kQ/I$ where Q is the quiver: $1 \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{a} \end{array} 2 \xrightarrow{c} 3$ and $I = \langle ca \rangle$. Set $x_0 = 1$.

Then $\pi_1(Q, I) \simeq \mathbb{Z}$ is generated by $[b^{-1}c^{-1}a]_I$. On the other hand, $A \simeq kQ/J$ where $J = \langle ca - cb \rangle$ and $\pi_1(Q, J)$ is the trivial group.

In the sequel we will use the following technical lemma.

Lemma 1.4. *Let (Q, I) be a bound quiver where Q has no oriented cycle and let $d: kQ \rightarrow kQ$ be a linear mapping such that $d(I) \subseteq I$ and verifying $d(u) = t_u u$ for some $u \in k$, for any path u . Let \equiv_I be the equivalence relation on the set of paths in Q generated by the condition (3) defining \sim_I . Then, the following implication holds for any paths u, v :*

$$u \equiv_I v \Rightarrow t_u = t_v$$

Proof: We shall use a non multiplicative version of Gröbner bases (see [1] and [6]). Fix an arbitrary total order $u_1 < \dots < u_N$ on the set of paths in Q and let (u_1^*, \dots, u_N^*) be the basis of $(kQ)^*$ dual to (u_1, \dots, u_N) . Following [14, Sect. 1], the Gröbner basis of I is the unique basis (r_1, \dots, r_t) defined by the three following properties:

- (i) $r_j \in u_{i_j} + \text{Span}(u_i ; i < i_j)$ for some i_j , for every j ,
- (ii) $u_{i_j}^*(r_{j'}) = 0$ unless $j = j'$,
- (iii) $i_1 < \dots < i_t$.

It follows from these properties that:

$$(iv) \quad r = \sum_{j=1}^t u_{i_j}^*(r) r_j \text{ for any } r \in I.$$

Recall from [14, Sect. 1] that r_1, \dots, r_t are minimal relations of I so that \equiv_I is generated by the property (3') defining \sim_I . This property and the assumption on d imply that in order to prove the lemma, it suffices to prove that $d(r_j) \in k.r_j$ for any j . Let us prove this assertion by induction on $j \in \{1, \dots, t\}$. By assumption on d and thanks to (i), we have $d(r_1) \in I \cap \text{Span}(u_i ; i \leq i_1)$. Hence, (iii) and (iv) imply that $d(r_1) \in k.r_1$. Let $j \in \{1, \dots, t-1\}$ and assume that $d(r_1) \in k.r_1, \dots, d(r_j) \in k.r_j$. By assumption on d and thanks to (i) and (ii), we have $d(r_{j+1}) \in I \cap \text{Span}(u_i ; i \leq i_1)$ and $u_{i_l}^*(d(r_{j+1})) = 0$ if $l \leq j$. So, (iii) and (iv) imply that $d(r_{j+1}) \in k.r_{j+1}$. This finishes the induction and proves the lemma. ■

Comparison of fundamental groups. Let A be a basic connected finite dimensional k -algebra with ordinary quiver Q without oriented cycle. A **dilatation** (see [11]) is an automorphism $D: kQ \xrightarrow{\sim} kQ$ such that $D(e_i) = e_i$ for any i and such that $D(\alpha) \in k.\alpha$ for any $\alpha \in Q_1$. If (α, u) is a bypass and $\tau \in k$, the **transvection** (see [11]) $\varphi_{\alpha, u, \tau}: kQ \xrightarrow{\sim} kQ$ is the automorphism such that $\varphi_{\alpha, u, \tau}(e_i) = e_i$ for every i , such that $\varphi_{\alpha, u, \tau}(\alpha) = \alpha + \tau u$ and such that $\varphi_{\alpha, u, \tau}(\beta) = \beta$ for any arrow $\beta \neq \alpha$. These automorphisms allow to compare the fundamental groups of the presentations of an algebra. More precisely, we have the following proposition announced in [11] and proved in [14]:

Proposition 1.5. *([14, Prop. 2.5]) Let I be an admissible ideal of kQ , let φ be an automorphism of kQ and let $J = \varphi(I)$. If φ is a dilatation, then \sim_I and \sim_J coincide. Assume that $\varphi = \varphi_{\alpha, u, \tau}$:*

- If $\alpha \sim_I u$ and $\alpha \sim_J u$ then \sim_I and \sim_J coincide.
- If $\alpha \not\sim_I u$ and $\alpha \sim_J u$ then \sim_J is generated by \sim_I and $\alpha \sim_J u$.
- If $\alpha \not\sim_I u$ and $\alpha \not\sim_J u$ then $I = J$ and \sim_I and \sim_J coincide.

In particular, if $\alpha \sim_I u$, then the identity map on the set of walks of Q induces a surjective group morphism $\pi_1(Q, I) \rightarrow \pi_1(Q, J)$.

Here *generated* means: generated as an equivalence relation on the set of walks of Q , and verifying the conditions (1) and (2) above in the definition of the homotopy relation. If I, J are admissible ideals such that there exists $\varphi_{\alpha, u, \tau}$ verifying $J = \varphi_{\alpha, u, \tau}(I)$, $\alpha \not\sim_I u$ and $\alpha \sim_J u$, then we shall say that \sim_J is a **direct successor of \sim_I** . Proposition 1.5 allows one to define a quiver Γ associated to A as follows (see [11, Def. 4.1]):

- $\Gamma_0 = \{\sim_I \mid I \text{ is an admissible ideal of } kQ \text{ such that } A \simeq kQ/I\}$,
- there is an arrow $\sim \rightarrow \sim'$ if \sim_J is a direct successor of \sim_I .

Example 1.6. *Let A be as in Example 1.3, then $J = \varphi_{\alpha, cb, 1}$ and Γ is equal to $\sim_I \rightarrow \sim_J$.*

The quiver Γ is finite, connected and has not oriented cycle ([11, Rem. 3, Prop. 4.2]). Moreover, if Γ has a unique source \sim_{I_0} (i.e. a vertex with no arrow ending at it) then the fundamental group of any admissible presentation of A is a quotient of $\pi_1(Q, I_0)$. It was proved in [14] and [13] that Γ has a unique source under one of the hypotheses (H_1) or (H_2) presented in the introduction. Moreover, the hypotheses H_1 and H_2 both ensure the following proposition which will be particularly useful to prove Theorem 1.

Proposition 1.7. ([14, Lem. 4.3] and [13, Prop. 4.3]) Assume that at least one of the two hypotheses H_1 or H_2 is satisfied: Let I_0 be the admissible ideal such that \sim_{I_0} is the unique source of Γ and let $kQ \twoheadrightarrow A$ be a presentation with kernel I . Then there exist a dilatation D and a sequence of transvections $\varphi_{\alpha_1, u_1, \tau_1}, \dots, \varphi_{\alpha_n, u_n, \tau_n}$ such that:

- $I = D\varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$,
- if we set $I_i := \varphi_{\alpha_i, u_i, \tau_i} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$, then $\alpha_i \sim_{I_i} u_i$ for every i .

If \sim_I is the unique source of Γ (i.e. $\sim_I = \sim_{I_0}$) then the homotopy relations $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_n}, \sim_I$ coincide.

Comparison of the fundamental groups and the Hochschild cohomology. Let A be a basic connected finite dimensional k -algebra. Assume that the ordinary quiver Q of A has no oriented cycle. Let $x_0 \in Q_0$ and fix a maximal tree T of Q , i.e. a subquiver of Q such that $T_0 = Q_0$ and such that the underlying graph of T is a tree. With these data, I. Assem and J. A. de la Peña have defined in [3] an injective mapping of abelian groups $\theta_\nu: Hom(\pi_1(Q, Ker(\nu)), k^+) \hookrightarrow HH^1(A)$ associated to any admissible presentation $\nu: kQ \twoheadrightarrow A$. We recall here the definition of θ_ν and refer the reader to [3] for more details. For any $x \in Q_0$ there exists a unique walk γ_x in T with source x_0 , with target x and with length minimal for these properties. Let $\nu: kQ \twoheadrightarrow A$ be an admissible presentation and let $f \in Hom(\pi_1(Q, Ker(\nu)), k^+)$ be a group morphism. Then, f defines a unitary derivation $\tilde{f}: A \rightarrow A$ as follows: $\tilde{f}(\nu(u)) = f([\gamma_y^{-1}y\gamma_x]_{\sim_{Ker(\nu)}}) \nu(u)$ for any path u with source x and target y . The following proposition was proved in [3]:

Proposition 1.8. (see [3]) The mapping $f \mapsto \tilde{f}$ induces an injective mapping of abelian groups:

$$\theta_\nu: Hom(\pi_1(Ker(\nu)), k^+) \hookrightarrow HH^1(A)$$

The abelian group $Hom(\pi_1(Q, Ker(\nu)), k^+)$ naturally carries a structure of commutative k -algebra, hence a structure of abelian Lie algebra. The following lemma proves that θ_ν preserves this structure.

Lemma 1.9. $\theta_\nu: Hom(\pi_1(Ker(\nu)), k^+) \hookrightarrow HH^1(A)$ is a Lie algebra morphism. In particular, $Im(\theta_\nu)$ is an abelian Lie subalgebra of $HH^1(A)$.

Proof: Let $f, f' \in Hom(\pi_1(Ker(\nu)), k^+)$. Then $[\theta_\nu(f), \theta_\nu(f')] \in HH^1(A)$ is represented by the derivation $\tilde{f} \circ \tilde{f}' - \tilde{f}' \circ \tilde{f}: A \rightarrow A$. Let u be a path in Q . It follows from the definition of θ_ν that $\tilde{f} \circ \tilde{f}' - \tilde{f}' \circ \tilde{f}$ vanishes on u . Hence, $[\theta_\nu(f), \theta_\nu(f')] = 0$. \blacksquare

Lemma 1.9 shows that θ_ν is not surjective in general. Indeed, assume for example that $A = kQ$ where $Q = \overline{A_3}$. If $\nu: kQ \twoheadrightarrow A$ is an admissible presentation, then $Ker(\nu) = 0$ and $\pi_1(Q, Ker(\nu)) \simeq \mathbb{Z}$ so that $dim_k Im(\theta_\nu) = 1$. On the other hand, one easily verifies that $dim_k HH^1(A) = 2$.

Throughout this text, A will be a basic connected finite dimensional k -algebra with ordinary quiver Q without oriented cycle ($Q_0 = \{1, \dots, n\}$). We fix $\{e_1, \dots, e_n\}$ a complete set of primitive orthogonal idempotents of A . The semi-simple subalgebra $k.e_1 \oplus \dots \oplus k.e_n$ of A will be denoted by E . The radical of A will be denoted by τ . For short, we shall say ‘‘presentation’’ instead of ‘‘admissible presentation’’. Without loss of generality, we shall assume that any presentation $\nu: kQ \twoheadrightarrow A$ verifies $\nu(e_i) = e_i$. Finally, in order to use the morphisms θ_ν , we fix a maximal tree T of Q .

2 Diagonalisability in $HH^1(A)$

The aim of this section is to give prove some useful properties on the subspaces $Im(\theta_\nu)$ in terms of diagonalisability in $HH^1(A)$. Notice that diagonalisability was introduced for derivations of A in [7]. For short, a **basis** of A is a basis B of the k -vector space A such that: $B \subseteq \bigcup_{i,j} e_j A e_i$, such that $\{e_1, \dots, e_n\} \subseteq B$, and such that $B \setminus \{e_1, \dots, e_n\} \subseteq \tau$. We shall say that a linear mapping $u: A \rightarrow A$ is diagonalisable if and only if there exists a basis as above such that the matrix of u in this basis is diagonal. In such a case, B is called a diagonalisation basis of u . Notice the following link between bases and presentations of A :

- if $\nu: kQ \twoheadrightarrow A$ is a presentation of A , then there exists a basis B such that $\nu(\alpha) \in B$ for any $\alpha \in Q_1$ and such that any element of B is of the form $\nu(u)$ with u a path in Q . We shall say that this basis B is adapted to ν ,
- if B is a basis of A , then there exists a presentation $\nu: kQ \twoheadrightarrow A$ such that $\nu(\alpha) \in B$ for any $\alpha \in Q_1$. We shall say that the presentation ν is adapted to B .

Remark that if $\nu: kQ \rightarrow A$ is a presentation and if B is a basis of A adapted to ν , then the presentation ν is adapted to B . Notice also that diagonalisability is invariant up to a sum with an inner derivation as the following lemma shows. The proof is immediate.

Lemma 2.1. *Let $u: A \rightarrow A$ be a linear mapping, let $e \in E$ and let B be a basis of A . Then B is a diagonalisation basis for u if and only if the same holds for $u + \delta_e$.*

The preceding lemma justifies the following definition.

Definition 2.2. *Let $f \in HH^1(A)$ and let d be a derivation representing f . Then f is called diagonalisable if and only if d is diagonalisable. In such a case, a diagonalisation basis of f is a diagonalisation basis of d .*

The subset $D \subseteq HH^1(A)$ is called diagonalisable if and only if any there exists a basis B of A such that B is a diagonalisation basis of f for any $f \in D$.

It is well know that a family of endomorphisms (of a finite dimensional k -vector space) is simultaneously diagonalisable as soon as these endomorphisms commute with each other and are diagonalisable. The following proposition gives a similar characterisation for the diagonalisability of subsets of $HH^1(A)$.

Proposition 2.3. *Let $D \subseteq HH^1(A)$. Then, D is diagonalisable if and only if every element of D is diagonalisable and $[f, f'] = 0$ for any $f, f' \in D$.*

Proof: For each $f \in D$, let d_f be a derivation representing f . If there exists a common diagonalisation basis for the elements of D , then the same holds for $\{d_f \mid f \in D\}$ so that $[d_f, d_{f'}] = 0$ for any $f, f' \in D$. Consequently $[f, f'] = 0$ for any $f, f' \in D$. Now let us prove the converse. The vanishing of the Lie brackets implies that for any $f, f' \in D$ there exists $e^{(f, f')} \in E$ such that $[d_f, d_{f'}] = \delta_{e^{(f, f')}}$. Let us write $e^{(f, f')} = \sum_{i=1}^n t_i^{(f, f')} e_i$. Hence (see Lemma 1.2), for any i, j , the derivations d_f and $d_{f'}$ induce diagonalisable

linear mappings $e_j \tau e_i \rightarrow e_j \tau_i$ whose Lie bracket equals $u \mapsto (t_j^{(f, f')} - t_i^{(d, d')})u$. Recall that if M, N are square matrices such that M is diagonal, then the diagonal entries of $MN - NM$ are all zero, so that if moreover $MN - NM$ is a scalar matrix, then $MN - NM = 0$. Consequently, the restriction of $[d_f, d_{f'}]$ to $e_j \tau e_i$ is zero for any i, j . Since $[d_f, d_{f'}](e_i) = 0$ for any i , we deduce that:

- $d_f: e_j \tau e_i \rightarrow e_j \tau_i$ is diagonalisable for every $f \in D$ and every i, j ,
- $[d_f, d_{f'}] = 0$ for every $f, f' \in D$,
- $A = E \oplus \bigoplus_{i,j} e_j \tau e_i$

Therefore, the classical simultaneous diagonalisation of pairwise commuting diagonalisable endomorphisms of a finite dimensional k -vector space gives rise to a basis of B in which the matrix of d_f is diagonal for any $f \in D$. ■

Our main example of diagonalisable subspace of $HH^1(A)$ is $Im(\theta_\nu)$:

Proposition 2.4. *Let $\nu: kQ \rightarrow A$ be a presentation. Then, $Im(\theta_\nu)$ is diagonalisable.*

Proof: Let B be a basis of A adapted to ν and let $I = Ker(\nu)$. For any $f \in Hom(\pi_1(Q, I), k^+)$, the element $\theta_\nu(f) \in HH^1(A)$ is represented by the derivation \tilde{f} which, by definition, verifies $\tilde{f}(\nu(u)) \in k \cdot \nu(u)$ for any path u in Q . ■

In this section, we aim at proving that any diagonalisable subset of $HH^1(A)$ is contained in $Im(\theta_\nu)$ for some presentation ν . It was proved in [7] that any diagonalisable derivation (with suitable technical conditions) defines an element of $HH^1(A)$ lying in $Im(\theta_\nu)$ for some ν . We will use the following similar result.

Lemma 2.5. *Let $f \in HH^1(A)$ and let B be a diagonalisation basis of f . Let $\nu: kQ \rightarrow A$ be a presentation adapted to B . Then $f \in Im(\theta_\nu)$.*

Proof: Let $I = Ker(\nu)$ and let $d: A \rightarrow A$ be a derivation representing f . We shall write \bar{r} for $\nu(r)$, for any $r \in kQ$. Let $\alpha \in Q_1$, since B is a diagonalisation basis of f and since ν is adapted to B , there exists $t_\alpha \in k$ such that $d(\bar{\alpha}) = t_\alpha \bar{\alpha}$. For any path $u = \alpha_n \dots \alpha_1$ (with $\alpha_i \in Q_1$), let us set $t_u := t_{\alpha_1} + \dots + t_{\alpha_n}$ so that $d(\bar{u}) = t_u \bar{u}$, because d is a derivation. More generally, if $\gamma = \alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$ is a walk in Q (with $\alpha_i \in Q_1$), let us set $t_\gamma := \sum_{i=1}^n (-1)^{\varepsilon_i} t_{\alpha_i}$, with the convention that $t_\gamma = 0$ if γ is stationary. We are going to prove that the mapping $\gamma \mapsto t_\gamma$ defines a group morphism $g: \pi_1(Q, I) \rightarrow k^+, [\gamma]_I \mapsto t_\gamma$ and that $f = \theta_\nu(g)$.

First, let us prove that the morphism $g: \pi_1(Q, I) \rightarrow k^+$ is well defined. By definition of the scalar t_γ , we have:

(i) $t_{e_x} = 0$ for any $x \in Q_0$ and $t_{\gamma'\gamma} = t_{\gamma'}t_\gamma$ for any walks γ, γ' such that $t(\gamma) = s(\gamma')$.

(ii) $t_{\alpha^{-1}\alpha} = t_{e_x}$ and $t_{\alpha\alpha^{-1}} = t_{e_y}$ for any arrow $x \xrightarrow{\alpha} y \in Q_1$.

(iii) $t_{wv} = t_{wv'u}$ for any walks w, v, v', u verifying $t_v = t_{v'}$, $s(w) = t(v) = t(v')$ and $s(v) = s(v') = t(u)$.

In order to prove that g is well defined, it only remains to prove that $t_u = t_v$ whenever u, v are paths in Q lying in the support of the same minimal relation of I . In this purpose, let $d': kQ \rightarrow kQ$ be the linear mapping such that $d'(u) = t_u u$ for any path u in k . By definition of the scalar t_u , this implies that $d \circ \nu = \nu \circ d'$. In particular, $d'(I) \subseteq I$. So we may apply Lemma 1.4 to d' and deduce that:

(iv) $t_u = t_v$ if u, v are paths in Q lying in the support of a same minimal relation of I .

From (ii), (iii) and (iv) we deduce that we have a well defined mapping $g: \pi_1(Q, I) \rightarrow k, [\gamma]_I \mapsto t_\gamma$. Moreover, (i) proves that g is a group morphism.

Now let us prove that $f = \theta_\nu(g)$. For any path u with source x and target y , we have $g([\gamma_y^{-1}u\gamma_x]_I) = t_u - t_{\gamma_y} + t_{\gamma_x}$. Hence, $\theta_\nu(g) \in HH^1(A)$ is represented by the derivation $\tilde{g}: A \rightarrow A$ such that $\tilde{g}(\bar{u}) = (t_u - t_{\gamma_y} + t_{\gamma_x})\bar{u}$ for any path u with source x and target y . Let us set $e := \sum_{x \in Q_0} t_{\gamma_x} e_x \in E$. Therefore, $\tilde{g} + \delta_e = d$. This proves that $f = \theta_\nu(g)$. \blacksquare

Now we can state the main result of this section. It is a direct consequence of Proposition 2.4 and of Lemma 2.5.

Proposition 2.6. *Let $D \subseteq HH^1(A)$. Then D is diagonalisable if and only if there exists a presentation $\nu: kQ \rightarrow A$ such that $D \subseteq \text{Im}(\theta_\nu)$.*

Remark that Lemma 2.5 also gives a sufficient condition for θ_ν to be an isomorphism. Recall that A is called constricted if and only if $\dim e_y A e_x = 1$ for any arrow $x \rightarrow y$ (this implies that Q has no multiple arrows). In [5] it was proved that for such an algebra, two different presentations have the same fundamental group.

Proposition 2.7. *Assume that A is constricted. Let $\nu: kQ \rightarrow A$ be any presentation of A . Then $\theta_\nu: \text{Hom}(\pi_1(Q, I), k^+) \rightarrow HH^1(A)$ is an isomorphism. In particular, $HH^1(A)$ is an abelian Lie algebra.*

Proof: Since θ_ν is one-to-one, we only need to prove that it is onto. Let B be a basis of A adapted to ν , let $f \in HH^1(A)$ and let $d: A \rightarrow A$ be a derivation representing f . Let $x \xrightarrow{\alpha} y$ be an arrow. Then $e_y A e_x = K \cdot \nu(\alpha)$ so that there exists $t_\alpha \in k$ such that $d(\nu(\alpha)) = t_\alpha \nu(\alpha)$. Let $u = \alpha_n \dots \alpha_1$ be any path in Q (with $\alpha_i \in Q_1$). Since d is a derivation, we have $d(\nu(u)) = (t_{\alpha_1} + \dots + t_{\alpha_n})\nu(u)$. As a consequence, B is a diagonalisation basis for d . Moreover, ν is adapted to B because B is adapted to ν . So Lemma 2.5 proves that $f \in \text{Im}(\theta_\nu)$. This proves that θ_ν is an isomorphism. Finally $HH^1(A)$ is abelian because $\text{Hom}(\pi_1(Q, I), k^+)$ is abelian. \blacksquare

3 Comparison of $\text{Im}(\theta_\nu)$ and $\text{Im}(\theta_\mu)$ for different presentations μ and ν of A

In Proposition 1.5 we have recalled that if two presentations ν and μ of A are linked by a transvection or a dilatation, then there is a simple relation between the associated fundamental groups. In this section, we shall use this relation to compare θ_ν and θ_μ . Our first result compares θ_ν and θ_μ when $\mu = \nu \circ D$ with D a dilatation. Recall from Proposition 1.5 that if $J = D(I)$ with D a dilatation, then \sim_I and \sim_J coincide, so that $\pi_1(Q, I) = \pi_1(Q, J)$.

Proposition 3.1. *Let $\nu: kQ \rightarrow A$ be a presentation, let $D: kQ \xrightarrow{\sim} kQ$ be a dilatation, let $\mu := \nu \circ D: kQ \rightarrow A$. Set $I = \text{Ker}(\mu)$ and $J = \text{Ker}(\nu)$, so that $J = D(I)$. Then $\theta_\mu = \theta_\nu$.*

Proof: Let $f \in \text{Hom}(\pi_1(Q, I), k^+)$. Then, $\theta_\nu(f)$ and $\theta_\mu(f)$ are represented by the derivations d_1 and d_2 respectively, such that for any arrow $x \xrightarrow{\alpha} y$:

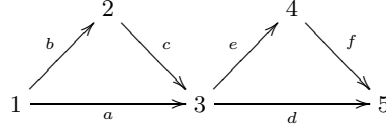
$$\begin{aligned} d_1(\nu(\alpha)) &= f([\gamma_y^{-1}\alpha\gamma_x]_J) \nu(\alpha) \\ d_2(\mu(\alpha)) &= f([\gamma_y^{-1}\alpha\gamma_x]_I) \mu(\alpha) \end{aligned}$$

On the other hand, for such an arrow $x \xrightarrow{\alpha} y$, there exists $t_\alpha \in k$ such that $D(\alpha) = t_\alpha \alpha$ which implies, thanks to the above property of d_1 and d_2 and thanks to the fact that \sim_I and \sim_J coincide, that $d_1(\mu(\alpha)) = d_2(\mu(\alpha))$. Hence, d_1 and d_2 are two derivations of A and they coincide on $\mu(Q_1)$. So $d_1 = d_2$

and $\theta_\nu(f) = \theta_\mu(f)$, for any $f \in \text{Hom}(\pi_1(Q, I), k^+)$. ■

The following example shows that the conclusion of Proposition 3.1 does not necessarily hold if ν and μ are two presentations of A such that $\sim_{\text{Ker}(\nu)}$ and $\sim_{\text{Ker}(\mu)}$ coincide.

Example 3.2. Assume that $\text{car}(k) = 2$ and let $A = kQ/I$ where Q is the quiver:



and $I = \langle ba, vu, fea + dcb \rangle$. Let T be the maximal tree such that $T_1 = \{b, c, e, f\}$. Let $\nu: kQ \rightarrow A = kQ/I$ be the natural projection. Let $\psi := \varphi_{a,cb,1}\varphi_{d,fe,1}$. Thus, $I = \psi(I)$. Let $\mu := \nu \circ \psi: kQ \rightarrow A$ so that $\text{Ker}(\mu) = \text{Ker}(\nu) = I$. Remark that $\pi_1(Q, I)$ is the infinite cyclic group with generator $[b^{-1}c^{-1}a]_I$. So let $f: \pi_1(Q, I) \rightarrow k^+$ be the unique group morphism such that $f([b^{-1}c^{-1}a]_I) = 1$. Then $\theta_\nu(f)$ is represented by the following derivation:

$$\begin{aligned}
 d_1: \quad A &\longrightarrow A \\
 \nu(x) &\longmapsto \nu(x) \text{ if } x \in \{a, d\} \\
 \nu(x) &\longmapsto 0 \text{ if } x \in \{b, c, e, f\}
 \end{aligned}$$

On the other hand, $\theta_\mu(f)$ is represented by the derivation:

$$\begin{aligned}
 d_2: \quad A &\longrightarrow A \\
 \nu(a) &\longmapsto \nu(a) + \nu(cb) \\
 \nu(d) &\longmapsto \nu(d) + \nu(fe) \\
 \nu(x) &\longmapsto 0 \text{ if } x \in \{b, c, e, f\}
 \end{aligned}$$

It is easy to verify that $d_2 - d_1$ is not inner. Hence, $\theta_\nu \neq \theta_\mu$.

Now we compare θ_ν and θ_μ when $\mu = \nu \circ \varphi_{\alpha, u, \tau}$ and when the identity map on the set of walks in Q induces a surjective group morphism $\pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\mu))$.

Proposition 3.3. Let $\nu: kQ \rightarrow A$ be a presentation, let $\varphi_{\alpha, u, \tau}: kQ \xrightarrow{\sim} kQ$ be a transvection and let $\mu := \nu \circ \varphi_{\alpha, u, \tau}: kQ \rightarrow A$. Set $I = \text{Ker}(\nu)$ and $J = \text{Ker}(\mu)$, so that $I = \varphi_{\alpha, u, \tau}(J)$. Suppose that $\alpha \sim_J u$ and let $p: \pi_1(Q, I) \rightarrow \pi_1(Q, J)$ be the quotient mapping (see Proposition 1.5). Then, the following diagram commutes:

$$\begin{array}{ccc}
 \text{Hom}(\pi_1(Q, J), k^+) & & \\
 \downarrow p^* & \searrow \theta_\mu & \\
 & & HH^1(A) \\
 & \nearrow \theta_\nu & \\
 \text{Hom}(\pi_1(Q, I), k^+) & &
 \end{array}$$

where $p^*: \text{Hom}(\pi_1(Q, J), k^+) \hookrightarrow \text{Hom}(\pi_1(Q, I), k^+)$ is the embedding induced by p . In particular, $\text{Im}(\theta_\nu) \subseteq \text{Im}(\theta_\mu)$.

Proof: Recall that p is the mapping $[\gamma]_I \mapsto [\gamma]_J$. Let $f \in \text{Hom}(\pi_1(Q, J), k^+)$. So $p^*(f)$ is the composition $\pi_1(Q, I) \xrightarrow{p} \pi_1(Q, J) \xrightarrow{f} k$. We know that $\theta_\mu(f)$ and $\theta_\nu(p^*(f))$ are represented by the derivations d_1 and d_2 respectively, such that for any arrow $x \xrightarrow{a} y$:

$$\begin{aligned}
 d_1(\mu(a)) &= f([\gamma_y^{-1}a\gamma_x]_J) \mu(a) = p^*(f)([\gamma_y a \gamma_x]_I) \mu(a) \\
 d_2(\nu(a)) &= p^*(f)([\gamma_y^{-1}a\gamma_x]_I) \nu(a)
 \end{aligned}$$

Let us prove that d_1 and d_2 coincide on $\nu(Q_1)$. Let $x \xrightarrow{a} y$ be an arrow. If $a \neq \alpha$, then $\mu(a) = \nu(a)$ and the above characterisations of d_1 and d_2 imply that $d_1(\nu(a)) = d_1(\mu(a)) = d_2(\nu(a))$. Now assume that $a = \alpha$ so that: $\nu(a) = \mu(a) - \tau\mu(u)$ and $[\gamma_y^{-1}a\gamma_x]_J = [\gamma_y^{-1}u\gamma_x]_J$ (recall that $a = \alpha \sim_J u$). Thus:

$$\begin{aligned}
 d_1(\nu(a)) &= d_1(\mu(\alpha)) - \tau d_1(\mu(u)) \\
 &= f([\gamma_y^{-1}\alpha\gamma_x]_J) \mu(\alpha) - \tau f([\gamma_y^{-1}u\gamma_x]_J) \mu(u) \\
 &= f([\gamma_y^{-1}\alpha\gamma_x]_J) (\mu(\alpha) - \tau \mu(u)) \\
 &= p^*(f)([\gamma_y^{-1}\alpha\gamma_x]_I) \nu(\alpha) \\
 &= d_2(\nu(\alpha)) = d_2(\nu(a))
 \end{aligned}$$

Hence, d_1 and d_2 are two derivations of A and they coincide on $\nu(Q_1)$. This proves that $d_1 = d_2$ and that $\theta_\mu(f) = \theta_\nu(p^*(f))$ for any $f \in \text{Hom}(\pi_1(Q, J), k^+)$. \blacksquare

The following example shows that the conclusions of Proposition 3.3 do not necessarily hold if ν is a presentation of A and $\psi: kQ \rightarrow kQ$ is an automorphism such that the identity map on the walks in Q induces a surjective group morphism $\pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\nu \circ \psi))$.

Example 3.4. Let $A = kQ/I$ where $\text{car}(k) = 2$, where Q is the quiver of Example 3.2 and where $I = \langle da, fea + dcb \rangle$. Let $\nu: kQ \rightarrow A$ be the natural projection with kernel I , let $\psi := \varphi_{d,ef,1}\varphi_{a,cb,1}$ and let $\mu := \nu \circ \psi: kQ \rightarrow A$. Hence $\text{Ker}(\mu) = \langle da + fecb, fea + dcb \rangle$. Notice that $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$ is generated by $[b^{-1}c^{-1}a]_I$ and that $\pi_1(Q, \text{Ker}(\mu)) \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by $[b^{-1}c^{-1}a]_J$. Notice also that $\sim_{\text{Ker}(\nu)}$ is weaker than $\sim_{\text{Ker}(\mu)}$ so that the identity map on the set of walks in Q induces a surjective group morphism $p: \pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\mu))$. Let T be the maximal tree such that $T_1 = \{b, c, e, f\}$. Let $f: \pi_1(Q, \text{Ker}(\mu)) \rightarrow k$ be the group morphism such that $f([b^{-1}c^{-1}a]_J) = 1$ and let us verify that $\theta_\mu(f) \neq \theta_\nu(p^*(f))$. On the one hand, $\theta_\mu(f) \in HH^1(A)$ is represented by the derivation:

$$\begin{aligned} d_1: \quad A &\rightarrow A \\ \mu(x) &\mapsto \mu(x) \text{ if } x \in \{a, d\} \\ \mu(x) &\mapsto 0 \text{ if } x \in \{b, c, e, f\} \end{aligned}$$

On the other hand, $\theta_\nu(p^*(f)) \in HH^1(A)$ is represented by the derivation:

$$\begin{aligned} d_2: \quad A &\rightarrow A \\ \mu(a) &\mapsto \mu(a) + \mu(cb) \\ \mu(d) &\mapsto \mu(d) + \mu(fe) \\ \mu(x) &\mapsto 0 \text{ if } x \in \{b, c, e, f\} \end{aligned}$$

and one checks easily that $d_2 - d_1$ is not inner so that $\theta_\mu(f) \neq \theta_\nu(p^*(f))$. Moreover, $\text{Im}(\theta_\nu)$ and $\text{Im}(\theta_\mu)$ are one dimensional (because $\text{car}(k) = 2$, $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$ and $\pi_1(Q, \text{Ker}(\mu)) \simeq \mathbb{Z}/2\mathbb{Z}$) and d_1, d_2 are not inner. Hence $\text{Im}(\theta_\mu) = k \cdot \theta_\mu(f) \not\subseteq \text{Im}(\theta_\nu) = k \cdot \theta_\nu(p^*(f))$.

Finally, we compare θ_ν and θ_μ when $\mu = \nu \circ \psi$ with $\psi: kQ \xrightarrow{\sim} kQ$ an automorphism such that $\text{Ker}(\nu) = \text{Ker}(\mu)$.

Proposition 3.5. Let $\nu: kQ \rightarrow A$ be a presentation and let $I = \text{Ker}(\nu)$. Let $\psi: kQ \xrightarrow{\sim} kQ$ be an automorphism such that $\psi(e_i) = e_i$ for every i and such that $\psi(I) = I$. Let $\mu := \nu \circ \psi: kQ \rightarrow A$ so that $\text{Ker}(\mu) = I$. Let $\bar{\psi}: A \xrightarrow{\sim} A$ be the k -algebra automorphism making commute the following diagram:

$$\begin{array}{ccc} kQ & \xrightarrow{\psi} & kQ \\ \mu \downarrow & & \downarrow \mu \\ A & \xrightarrow{\bar{\psi}} & A \end{array}$$

Then, the following diagram commutes:

$$\begin{array}{ccc} & & HH^1(A) \\ & \nearrow \theta_\nu & \downarrow \sim \bar{\psi}_* \\ \text{Hom}(\pi_1(Q, I), k^+) & & \\ & \searrow \theta_\mu & \\ & & HH^1(A) \end{array}$$

In particular, $\text{Im}(\theta_\mu)$ is equal to the image of $\text{Im}(\theta_\nu)$ under the Lie algebra automorphism $\bar{\psi}_*: HH^1(A) \xrightarrow{\sim} HH^1(A)$ induced by $\bar{\psi}: A \xrightarrow{\sim} A$.

Proof: Since ψ fixes the idempotents e_1, \dots, e_n , $\bar{\psi}_*$ is well defined. Let $f \in \text{Hom}(\pi_1(Q, I), k^+)$. So $\theta_\nu(f)$ and $\theta_\mu(f)$ are represented by the derivations d_1 and d_2 respectively, such that for any arrow $x \xrightarrow{\alpha} y$:

$$\begin{aligned} d_1(\nu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \nu(\alpha) \\ d_2(\mu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) \end{aligned}$$

In order to prove that $\overline{\psi}_*(\theta_\nu(f)) = \theta_\mu(f)$ it suffices to prove that $\overline{\psi} \circ d_1 = d_2 \circ \overline{\psi}$. Let $x \xrightarrow{\alpha} y$ be an arrow. Then:

$$\begin{aligned} d_2 \circ \overline{\psi}(\nu(\alpha)) &= d_2 \circ \overline{\psi} \circ \mu \circ \psi^{-1}(\alpha) && \text{because } \nu = \mu \circ \psi^{-1} \\ &= d_2 \circ \mu \circ \psi(\psi^{-1}(\alpha)) && \text{because } \overline{\psi} \circ \mu = \mu \circ \psi \\ &= d_2(\mu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) \end{aligned}$$

On the other hand:

$$\begin{aligned} \overline{\psi} \circ d_1(\nu(\alpha)) &= \overline{\psi}(f([\gamma_y^{-1} \alpha \gamma_x]_I) \nu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \overline{\psi}(\nu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \overline{\psi} \circ \mu \circ \psi^{-1}(\alpha) && \text{because } \mu = \nu \circ \psi \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu \circ \psi \circ \psi^{-1}(\alpha) && \text{because } \overline{\psi} \circ \mu = \mu \circ \psi \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) \end{aligned}$$

Hence, $\overline{\psi} \circ d_1$ and $d_2 \circ \overline{\psi}$ are derivations of A and they coincide on $\nu(Q_1)$. This proves that $\overline{\psi}_*(\theta_\nu(f)) = \theta_\mu(f)$ for any $f \in \text{Hom}(\pi_1(Q, I, k^+))$. \blacksquare

4 Proof of Theorem 1

In this section, we will prove Theorem 1. We begin with the following useful lemma.

Lemma 4.1. *Assume that at least one of the two conditions (H_1) or (H_2) is satisfied. Let $\nu: kQ \rightarrow A$ be a presentation with kernel I_0 such that \sim_{I_0} is the unique source of Γ , and let $\mu: kQ \rightarrow A$ be a presentation. Then, there exist $\nu': kQ \rightarrow A$ a presentation with kernel I_0 and a k -algebra automorphism $\psi: A \xrightarrow{\sim} A$ verifying $\psi(e_i) = e_i$ for any i and such that:*

$$\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_{\nu'}) = \psi_*(\text{Im}(\theta_\nu))$$

If moreover $\sim_I = \sim_{I_0}$, then the above inclusion is an equality.

Proof: Let $\overline{\nu}: kQ/I_0 \xrightarrow{\sim} A$ and $\overline{\mu}: kQ/I \xrightarrow{\sim} A$ be the isomorphisms induced by ν and μ respectively. Hence, $\overline{\mu}^{-1} \circ \overline{\nu}: kQ/I_0 \xrightarrow{\sim} kQ/I$ is an isomorphism which maps e_i to e_i for every i . Hence (see for example [12, Prop. 2.3.18]), there exists an automorphism $\varphi: kQ \xrightarrow{\sim} kQ$ mapping e_i to e_i for every i and such that the following diagram commutes:

$$\begin{array}{ccc} kQ & \xrightarrow{\varphi} & kQ \\ \downarrow & & \downarrow \\ kQ/I_0 & \xrightarrow{\overline{\mu}^{-1} \overline{\nu}} & kQ/I \end{array}$$

where the vertical arrows are the natural projections. In particular, the following diagram is commutative:

$$\begin{array}{ccc} kQ & \xrightarrow{\varphi} & kQ \\ \nu \searrow & & \swarrow \mu \\ & A & \end{array}$$

Now let us apply Proposition 1.7 to I . With the same notations, let $\psi := \varphi^{-1} D\varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$. Thus, $\psi(I_0) = I_0$ and $\nu' := \nu \circ \psi: kQ \rightarrow A$ is a presentation with kernel I_0 . Thanks to Proposition 3.5 from which we keep the notations, we know that:

$$\text{Im}(\theta_{\nu'}) = \overline{\psi}_*(\text{Im}(\theta_\nu)) \tag{1}$$

Now let us show that $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_{\nu'})$. By construction, we have $\mu = \nu \circ \psi^{-1} = \nu' \circ \varphi_{\alpha_1, u_1, \tau_1}^{-1} \dots \varphi_{\alpha_n, u_n, \tau_n}^{-1} D^{-1}$. For simplicity, we shall use the following notations: $\mu_0 := \nu'$ and $\mu_i := \nu' \circ \varphi_{\alpha_1, u_1, \tau_1}^{-1} \dots \varphi_{\alpha_i, u_i, \tau_i}^{-1}$ for $i \in \{1, \dots, n\}$. Therefore, $\text{Ker}(\mu_i) = I_i$ for every i . Let $i \in \{1, \dots, n\}$, so:

- $\mu_{i-1}: kQ \rightarrow A$ and $\mu_i: kQ \rightarrow A$ are two presentations of A ,
- $\mu_i = \mu_{i-1} \circ \varphi_{\alpha_i, u_i, \tau_i}^{-1}$,
- $\text{Ker}(\mu_i) = I_i$, $\text{Ker}(\mu_{i-1}) = I_{i-1}$ and $\alpha_i \sim_{I_i} u_i$.

From Proposition 3.3 we deduce that:

$$\text{Im}(\theta_{\mu_n}) \subseteq \text{Im}(\theta_{\mu_{m-1}}) \subseteq \dots \subseteq \text{Im}(\theta_{\mu_i}) \subseteq \text{Im}(\theta_{\mu_{i-1}}) \subseteq \dots \subseteq \text{Im}(\theta_{\mu_0}) = \text{Im}(\theta_{\nu'}) \quad (2)$$

Moreover, $\mu = \mu_m D^{-1}$ where D^{-1} is a dilatation. Hence, (1), (2) and Proposition 3.1 imply that:

$$\text{Im}(\theta_{\mu}) = \text{Im}(\theta_{\mu_n}) \subseteq \text{Im}(\theta_{\nu'}) = \overline{\psi}_*(\text{Im}(\theta_{\nu})) \quad (3)$$

Now assume that \sim_I is the unique source of Γ . Then Proposition 1.7 imply that the homotopy relations $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_n}, \sim_I$ coincide. Therefore, for any $i \in \{1, \dots, n\}$, we have:

- $\mu_{i-1} = \mu_i \circ \varphi_{\alpha_i, u_i, \tau_i}$,
- $\text{Ker}(\mu_i) = I_i, \text{Ker}(\mu_{i-1}) = I_{i-1}$ and $\alpha_i \sim_{I_{i-1}} u_i$,

These two properties imply, thanks to Proposition 3.3, that $\text{Im}(\theta_{\mu_{i-1}}) \subseteq \text{Im}(\theta_{\mu_i})$. This proves that all the inclusions in (2) are equalities. And so does the inclusion in (3). ■

Now we can prove Theorem 1.

Proof of Theorem 1: (i) Let \mathcal{G} be a maximal diagonalisable subalgebra of $HH^1(A)$. Thanks to Proposition 2.6, there exists a presentation $\mu: kQ \twoheadrightarrow A$ such that $\mathcal{G} \subseteq \text{Im}(\theta_{\mu})$. On the other hand, Lemma 4.1, implies that there exists a presentation $\nu: kQ \twoheadrightarrow A$ such that $\sim_{\text{Ker}(\nu)}$ is the unique source of Γ and verifying $\text{Im}(\theta_{\mu}) \subseteq \text{Im}(\theta_{\nu})$. Hence, $\mathcal{G} \subseteq \text{Im}(\theta_{\nu})$ where $\text{Im}(\theta_{\nu})$ is a diagonalisable subalgebra of $HH^1(A)$, thanks to Proposition 2.4. The maximality of \mathcal{G} forces $\mathcal{G} = \text{Im}(\theta_{\nu})$ where $\sim_{\text{Ker}(\nu)}$ is the unique source of Γ .

Conversely, let $\mu: kQ \twoheadrightarrow A$ be a presentation such that $\sim_{\text{Ker}(\mu)}$ is the unique source of Γ . Hence, $\text{Im}(\theta_{\mu})$ is diagonalisable (thanks to Proposition 2.4) so there exists \mathcal{G} a maximal diagonalisable subalgebra of $HH^1(A)$ containing $\text{Im}(\theta_{\mu})$. Thanks to the above description, we know that $\mathcal{G} = \text{Im}(\theta_{\nu})$ where $\nu: kQ \twoheadrightarrow A$ is a presentation such that $\sim_{\text{Ker}(\nu)}$ is the unique source of Γ . Moreover, Lemma 4.1 gives a k -algebra automorphism $\psi: A \xrightarrow{\sim} A$ such that $\text{Im}(\theta_{\mu}) = \psi_*(\text{Im}(\theta_{\nu}))$. Since ψ_* is a Lie algebra automorphism of $HH^1(A)$, the maximality of $\mathcal{G} = \text{Im}(\theta_{\nu})$ implies that $\text{Im}(\theta_{\mu})$ is maximal.

(ii) is a consequence of (i) and of Lemma 4.1. ■

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