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# On the comparison of the Hochschild cohomology and the fundamental groups of and algebra

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## Abstract

Let  $A$  be a basic connected finite dimensional  $k$ -algebra with ordinary quiver  $Q$  without oriented cycle. We compare the group morphisms  $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \rightarrow HH^1(A)$  established in [3], for the different presentations  $\nu: kQ \twoheadrightarrow A$  by quiver and admissible relations. We characterise the spaces  $\text{Im}(\theta_\nu)$  in terms of diagonalisable subspaces of  $HH^1(A)$ . Assuming that  $A$  has no double bypass and  $k$  has characteristic zero, or that  $A$  is monomial and without multiple arrows, we characterise the maximal diagonalisable subalgebras of  $HH^1(A)$  using the quiver of the homotopy relations of the presentations of  $A$  introduced in [11].

## Introduction

Let  $A$  be a finite dimensional algebra over an algebraically closed field  $k$ . In the study of the derived category of  $A$ , the Hochschild cohomology ([9])  $HH^*(A)$  plays an important role because it is an invariant of the derived category (see [10]) and also because it carries a lot of structure since it is  $k$ -algebra, a Lie algebra and more generally a Gerstenhaber algebra (see [8]). In particular,  $HH^1(A)$  is a Lie subalgebra of  $HH^*(A)$ . Recall that  $A$  is naturally an  $A - A$ -bimodule (or a  $A \otimes A$ -module) and that  $HH^*(A) = \bigoplus_{n \geq 0} HH^n(A)$  where  $HH^n(A) = \text{Ext}_{A \otimes A}^i(A, A)$ .

The space  $HH^1(A)$  was studied intensively (see for example [3], [5], [7], [16], [17], [18] and [19]). In particular, in [3], I. Assem and J. A. de la Peña have proved that if  $A$  is basic and connected, if the ordinary quiver  $Q$  of  $A$  has no oriented cycle and if  $\nu: kQ \twoheadrightarrow A$  is a presentation, then there is an injective mapping of abelian groups  $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \hookrightarrow HH^1(A)$ . Here  $\pi_1(Q, \text{Ker}(\nu))$  is the fundamental group of the  $\nu$  defined in [15]. Later, in [7], D. Farkas, E. Green and E. N. Marcos have studied the elements of  $\text{Im}(\theta_\nu)$  using the notion of diagonalisable derivation.

The aim of this text is to characterise the subspaces  $\text{Im}(\theta_\nu)$  of  $HH^1(A)$  in terms of Lie algebras and to compare the morphisms  $\theta_\nu$  associated to the different presentations  $\nu$  of  $A$ . Indeed, following [7], it is possible to define a notion of diagonalisability for an element  $HH^1(A)$  (which corresponds to the notion of diagonalisability of a derivation) and also of a subset of  $HH^1(A)$  (which corresponds to the simultaneous diagonalisability of a set of derivations). It appears that  $\text{Im}(\theta_\nu)$  is a diagonalisable subset of  $HH^1(A)$  and that any diagonalisable subset of  $HH^1(A)$  is contained in  $\text{Im}(\theta_\nu)$  for some presentation  $\nu$ . In order to compare the morphisms  $\theta_\nu$  associated to the different presentations of  $A$ , we will use the comparisons between the fundamental groups of the presentations of  $A$  made by the author in [14]. Recall that if  $\nu: kQ \twoheadrightarrow A$  is a presentation with kernel  $I$ , then the fundamental group  $\pi_1(Q, I)$  is defined as the quotient space of the set of unoriented paths (called walks) in  $Q$  by a homotopy relation  $\sim_I$  depending on  $I$ . Assuming that  $Q$  has no oriented cycle, it was proved in [14] that the homotopy relations of the presentations of  $A$  can be displayed as the vertices of finite, connected quiver  $\Gamma$  without oriented cycle and such that for any arrow  $\sim_I \rightarrow \sim_J$  there is a natural surjective group morphism  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ . In such a situation, if  $\nu$  and  $\mu$  are presentations of  $A$  such that  $\text{Ker}(\nu) = I$  and  $\text{Ker}(\mu) = J$ , then we will see that  $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_\nu)$ . This property and the above description of the spaces  $\text{Im}(\theta_\nu)$  as diagonalisable subspaces of  $HH^1(A)$  lead naturally to the following question:

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*Is it possible to characterise and compare the maximal diagonalisable subspaces of  $HH^1(A)$ ?*

The above explanations suggest that such a subspace of  $HH^1(A)$  should be of the form  $Im(\theta_\nu)$  for  $\nu$  a presentation such that  $\sim_{Ker(\nu)}$  is a source of  $\Gamma$ . We shall give a precise answer to the above question assuming one of the following hypotheses which ensure that  $\Gamma$  has a unique source (see [14, Prop. 2.11] and [13, Cor. 4.4]):

(H<sub>1</sub>)  $Q$  has no double bypass and  $k$  has characteristic zero.

(H<sub>2</sub>)  $A$  is monomial (i.e.  $A \simeq kQ/I_0$  with  $I_0$  an ideal generated by a set of paths) and  $Q$  has no multiple arrows.

More precisely we will prove the following theorem which is the main result of this text.

**Theorem 1.** *Assume that at least one the two hypotheses (H<sub>1</sub>) and (H<sub>2</sub>) is satisfied. Then:*

- (i) *The maximal diagonalisable subalgebras of  $HH^1(A)$  are exactly the subalgebras of the form  $Im(\theta_\nu)$  where  $\nu: kQ \twoheadrightarrow A$  is a presentation such that  $\sim_{Ker(\nu)}$  is the unique source of  $\Gamma$ .*
- (ii) *If  $\mathcal{G}, \mathcal{G}'$  are two such subalgebras of  $HH^1(A)$ , then there exists an algebra automorphism  $\psi: A \xrightarrow{\sim} A$  inducing a Lie algebra automorphism  $\psi_*: HH^1(A) \xrightarrow{\sim} HH^1(A)$  and such that  $\mathcal{G}' = \psi_*(\mathcal{G})$ .*

The text is organised as follows. In Section 1 we recall all the definitions we will need and proof some useful lemmas. In Section 2, we introduce the notion of diagonalisability in  $HH^1(A)$ . In particular, we will prove that a subset of  $HH^1(A)$  is diagonalisable is and only if it is contained in  $Im(\theta_\nu)$  for some presentation  $\nu: kQ \twoheadrightarrow A$ . In Section 3 we compare the morphisms  $\theta_\nu$  for different presentations  $\nu$  of  $A$ , using the quiver  $\Gamma$ . Finally, in Section 4 we prove Theorem 1.

This text is part of the author's thesis ([12]) made at Université Montpellier II under the supervision of Claude Cibils.

## 1 Preliminaries

**Terminology and notations for quivers.** If  $Q$  is a quiver,  $Q_0 = \{1, \dots, n\}$  (resp.  $Q_1$ ) will denote the set of vertices (resp. of arrows). The source (resp. target) of  $\alpha \in Q_1$  will be denoted by  $s(\alpha)$  (resp. by  $t(\alpha)$ ). The **stationary** path (i.e. of length 0) with source and target  $i \in Q_0$  will be denoted by  $e_x$ . The (oriented) paths in  $Q$  are read from the right to the left, i.e. the path  $\alpha_n \dots \alpha_1$  with source  $x$  and target  $y$  denotes the sequence of arrows  $\alpha_1, \dots, \alpha_n$  such that  $s(\alpha_1) = x$ ,  $s(\alpha_{i+1}) = t(\alpha_i)$  for every  $i$ , and  $t(\alpha_n) = y$ . Two paths are called **parallel** if they have the same source and the same target. If  $u = \alpha_m \dots \alpha_1$  and  $v = \beta_n \dots \beta_1$  are paths in  $Q$ , the concatenation  $vu$  is defined if  $t(\alpha_m) = s(\beta_1)$  and is equal to  $vu = \beta_n \dots \beta_1 \alpha_m \dots \alpha_1$ , with the convention  $vu = u$  (resp.  $vu = v$ ) if  $v$  (resp.  $u$ ) is stationary. An **oriented cycle** in  $Q$  is a non stationary path whose source and target are equal. If  $\alpha \in Q_1$  we consider its formal inverse  $\alpha^{-1}$  with source  $s(\alpha^{-1}) := t(\alpha)$  and target  $t(\alpha^{-1}) := s(\alpha)$ . A **walk** in  $Q$  with source  $x$  and target  $y$  is either the stationary path  $e_x$  (in which case  $y = x$ ) or a sequence  $\alpha_1, \dots, \alpha_n$  of arrows and formal inverses of arrows of  $Q$  such that  $s(\alpha_1) = x$ ,  $s(\alpha_{i+1}) = t(\alpha_i)$  for every  $i$  and  $t(\alpha_n) = y$ , in such a case, the walk is denoted by  $\alpha_n \dots \alpha_1$ . The concatenation of walks in  $Q$  is defined similarly to the concatenation of paths. If  $\gamma = \alpha_n \dots \alpha_1$  is walk, we will denote by  $\gamma^{-1}$  the walk  $\alpha_1^{-1} \dots \alpha_n^{-1}$  with the convention  $(\alpha^{-1})^{-1}$  for any arrow  $\alpha$ . A **bypass** (see [2]) is a pair  $(\alpha, u)$  where  $\alpha \neq u$ ,  $\alpha \in Q_1$  and  $u$  is a path in  $Q$  parallel to  $\alpha$ . A **double bypass** (see [11]) is a 4-tuple  $(\alpha, u, \beta, v)$  where  $(\alpha, u)$  and  $(\beta, v)$  are bypasses such that the arrow  $\beta$  appears in the path  $u$ .

**Admissible presentations.** If  $Q$  is a finite quiver (i.e.  $Q_0$  and  $Q_1$  are finite), its **path algebra**  $kQ$  is the  $k$ -algebra whose basis as a  $k$ -vector space is the set of paths in  $Q$  (including the stationary paths) and whose product is bilinearly induced by the concatenation of paths. The unit of  $kQ$  is  $\sum_{i=1}^n e_i$  and  $kQ$  is finite dimensional if and only if  $Q$  has no oriented cycle. An **admissible ideal** of  $kQ$  is an ideal  $I$  verifying the following conditions:

1.  $I$  is generated by linear combinations of paths of length at least 2,
2. there exists  $N \geq 2$  such that any path of length at least  $N$  lies in  $I$ .

In such a case, the elements of  $I$  are called **relations** and, following [15], a **minimal relation** of  $I$  is a relation  $\sum_{i=1}^s t_i u_i \neq 0$  such that:

1.  $t_1, \dots, t_l \in k^*$  and  $u_1, \dots, u_l$  are pairwise distinct paths in  $Q$ ,

2. if  $S \subseteq \{1, \dots, n\}$  verifies  $\sum_{i \in S} t_i u_i \in I$ , then  $S = \emptyset$  or  $S = \{1, \dots, s\}$ .

With the above notations, the paths  $u_1, \dots, u_n$  are necessarily parallel. Notice that  $I$  is generated by its minimal relations. Let  $A$  be a finite dimensional  $k$ -algebra. Then  $A$  is Morita equivalent to a **basic** finite dimensional  $k$ -algebra  $A'$ , where basic means that  $A' = P_1 \oplus \dots \oplus P_n$  where  $P_1, \dots, P_n$  are pairwise non isomorphic indecomposable projective  $A'$ -modules. If  $A$  is basic, then (see [4]) there exists a quiver  $Q$  and a surjective  $k$ -algebra morphism  $\nu: kQ \twoheadrightarrow A$  whose kernel is an admissible ideal of  $kQ$ . The quiver  $Q$  is then unique and called the **ordinary quiver** of  $A$  and  $\{\nu(e_i) \mid i \in Q_0\}$  is a complete set of primitive orthogonal idempotents. The morphism  $\nu$  is called an **(admissible) presentation**. In such a situation,  $A \simeq kQ/\text{Ker}(\nu)$  and  $A$  is connected if and only if  $Q$  is connected (i.e. the underlying graph of  $Q$  is connected).

**Presentation of  $HH^1(A)$ .** Let  $A$  be a basic finite dimensional  $k$ -algebra and let  $\{e_1, \dots, e_n\}$  be a complete set of primitive orthogonal idempotents. A **unitary derivation** (see [16]) is a  $k$ -linear mapping  $d: A \rightarrow A$  such that  $d(ab) = ad(b) + d(a)b$  for any  $a, b \in A$  and such that  $d(e_i) = e_i$  for every  $i$ . Let  $\text{Der}_0(A)$  be set of unitary derivations. It is a Lie algebra for the bracket  $[d, d'] = d \circ d' - d' \circ d$ . In the sequel, all derivations will be unitary, hence, we shall call them derivations, for short. Let  $E := \{\sum_{i=1}^n t_i e_i\}$ . Then  $E$  is a semi-simple subalgebra of  $A$  and  $A = E \oplus \mathfrak{r}$  where  $\mathfrak{r}$  is the radical of  $A$ . Let  $\text{Int}_0(A) := \{\delta_e: A \rightarrow A, a \in A \mapsto ea - ae \mid e \in E\}$ , this is an ideal of  $\text{Der}_0(A)$ . Throughout this text, we shall use the following presentation established in [16]:

**Theorem 1.1.** (see [16])  $HH^1(A) \simeq \text{Der}_0(A)/\text{Int}_0(A)$  as Lie algebras.

In the following lemma, we collect some useful properties on derivations.

**Lemma 1.2.** Let  $d \in \text{Der}_0(A)$ , then  $d(e_j A e_i) \subseteq e_j A e_i$ . Assume that the ordinary quiver  $Q$  of  $A$  has no oriented cycle, then  $d(\mathfrak{r}) \subseteq \mathfrak{r}$  and  $d(\mathfrak{r}^2) \subseteq \mathfrak{r}^2$ .

**Proof:** The first assertion is a direct consequence of the fact that  $d$  is a unitary derivation. Let  $u \in e_j \mathfrak{r} e_i \setminus \mathfrak{r}^2$ . Since  $Q$  has no oriented cycle, we have  $i \neq j$ . Hence  $d(u) \in e_j A e_i = e_j \mathfrak{r} e_i$ . This proves that  $d(\mathfrak{r}) \subseteq \mathfrak{r}$ . Since  $d$  is a derivation, we infer that  $d(\mathfrak{r}^2) \subseteq \mathfrak{r}^2$ . ■

If  $\psi: A \xrightarrow{\sim} A$  is a  $k$ -algebra automorphism such that  $\psi(e_i) = e_i$  for every  $i$ , then the mapping  $d \mapsto \psi \circ d \circ \psi^{-1}$  defines a Lie algebra automorphism of  $\text{Der}_0(A)$  and preserves  $\text{Int}_0(A)$ . Hence, it induces a Lie algebra automorphism which will be denoted by  $\psi_*: HH^1(A) \xrightarrow{\sim} HH^1(A)$ .

**Fundamental groups of presentations.** Let  $(Q, I)$  be a **bound quiver** (i.e.  $Q$  is a finite quiver and  $I$  is an admissible ideal of  $kQ$ ). The **homotopy relation**  $\sim_I$  was defined in [15] as the equivalence class on the set of walks in  $Q$  generated by the following properties:

- (1)  $\alpha \alpha^{-1} \sim_I e_y$  and  $\alpha^{-1} \alpha \sim_I e_x$  for any arrow  $\alpha$  with source  $x$  and target  $y$ ,
- (2)  $wvu \sim_I wv'u$  if  $w, v, v', u$  are walks such that the concatenations  $wvu$  and  $wv'u$  are well defined and such that  $v \sim_I v'$ ,
- (3)  $u \sim_I v$  if  $u$  and  $v$  are paths in  $Q$  appearing in a same minimal relation (with a non zero scalar).

Notice that if  $r_1, \dots, r_t$  are minimal relations generating  $I$ , then the condition (3) above may be replaced by the following one (see [5]):

- (3')  $u \sim_I v$  if  $u$  and  $v$  are paths in  $Q$  appearing in  $r_i$  (with a non zero scalar) for some  $i \in \{1, \dots, t\}$ .

The  $\sim_I$ -equivalence class of a walk  $\gamma$  will be denoted by  $[\gamma]_I$ . Let  $x_0 \in Q_0$ , following [15], the set of  $\sim_I$ -equivalence classes of walks with source and target  $x_0$  is denoted by  $\pi_1(Q, I, x_0)$ . The concatenation of walks endows this set with a group structure whose unit is  $[e_{x_0}]_I$ . This group is called the **fundamental group** of  $(Q, I)$  at  $x_0$ . If  $Q$  is connected, then the isomorphism class of  $\pi_1(Q, I, x_0)$  does not depend on the choice  $x_0$ . In such a case, we shall write  $\pi_1(Q, I)$  for  $\pi_1(Q, I, x_0)$ . If  $A$  is a basic connected finite dimensional  $k$ -algebra and if  $\nu: kQ \twoheadrightarrow A$  is a presentation, the group  $\pi_1(Q, \text{Ker}(\nu))$  will be called the **fundamental group of the presentation**  $\nu$ . The following example shows that two presentations of  $A$  may have non isomorphic fundamental groups.

**Example 1.3.** Let  $A = kQ/I$  where  $Q$  is the quiver:  $1 \begin{matrix} \xrightarrow{b} \\ \xleftarrow{a} \end{matrix} 2 \xrightarrow{c} 3$  and  $I = \langle ca \rangle$ . Set  $x_0 = 1$ .

Then  $\pi_1(Q, I) \simeq \mathbb{Z}$  is generated by  $[b^{-1}c^{-1}a]_I$ . On the other hand,  $A \simeq kQ/J$  where  $J = \langle ca - cb \rangle$  and  $\pi_1(Q, J)$  is the trivial group.

In the sequel we will use the following technical lemma.

**Lemma 1.4.** *Let  $(Q, I)$  be a bound quiver where  $Q$  has no oriented cycle and let  $d: kQ \rightarrow kQ$  be a linear mapping such that  $d(I) \subseteq I$  and verifying  $d(u) = t_u u$  for some  $u \in k$ , for any path  $u$ . Let  $\equiv_I$  be the equivalence relation on the set of paths in  $Q$  generated by the condition (3) defining  $\sim_I$ . Then, the following implication holds for any paths  $u, v$ :*

$$u \equiv_I v \Rightarrow t_u = t_v$$

**Proof:** We shall use a non multiplicative version of Gröbner bases (see [1] and [6]). Fix an arbitrary total order  $u_1 < \dots < u_N$  on the set of paths in  $Q$  and let  $(u_1^*, \dots, u_N^*)$  be the basis of  $(kQ)^*$  dual to  $(u_1, \dots, u_N)$ . Following [14, Sect. 1], the Gröbner basis of  $I$  is the unique basis  $(r_1, \dots, r_t)$  defined by the three following properties:

- (i)  $r_j \in u_{i_j} + \text{Span}(u_i ; i < i_j)$  for some  $i_j$ , for every  $j$ ,
- (ii)  $u_{i_j}^*(r_{j'}) = 0$  unless  $j = j'$ ,
- (iii)  $i_1 < \dots < i_t$ .

It follows from these properties that:

$$(iv) \quad r = \sum_{j=1}^t u_{i_j}^*(r) r_j \text{ for any } r \in I.$$

Recall from [14, Sect. 1] that  $r_1, \dots, r_t$  are minimal relations of  $I$  so that  $\equiv_I$  is generated by the property (3') defining  $\sim_I$ . This property and the assumption on  $d$  imply that in order to prove the lemma, it suffices to prove that  $d(r_j) \in k.r_j$  for any  $j$ . Let us prove this assertion by induction on  $j \in \{1, \dots, t\}$ . By assumption on  $d$  and thanks to (i), we have  $d(r_1) \in I \cap \text{Span}(u_i ; i \leq i_1)$ . Hence, (iii) and (iv) imply that  $d(r_1) \in k.r_1$ . Let  $j \in \{1, \dots, t-1\}$  and assume that  $d(r_1) \in k.r_1, \dots, d(r_j) \in k.r_j$ . By assumption on  $d$  and thanks to (i) and (ii), we have  $d(r_{j+1}) \in I \cap \text{Span}(u_i ; i \leq i_1)$  and  $u_{i_l}^*(d(r_{j+1})) = 0$  if  $l \leq j$ . So, (iii) and (iv) imply that  $d(r_{j+1}) \in k.r_{j+1}$ . This finishes the induction and proves the lemma. ■

**Comparison of fundamental groups.** Let  $A$  be a basic connected finite dimensional  $k$ -algebra with ordinary quiver  $Q$  without oriented cycle. A **dilatation** (see [11]) is an automorphism  $D: kQ \xrightarrow{\sim} kQ$  such that  $D(e_i) = e_i$  for any  $i$  and such that  $D(\alpha) \in k.\alpha$  for any  $\alpha \in Q_1$ . If  $(\alpha, u)$  is a bypass and  $\tau \in k$ , the **transvection** (see [11])  $\varphi_{\alpha, u, \tau}: kQ \xrightarrow{\sim} kQ$  is the automorphism such that  $\varphi_{\alpha, u, \tau}(e_i) = e_i$  for every  $i$ , such that  $\varphi_{\alpha, u, \tau}(\alpha) = \alpha + \tau u$  and such that  $\varphi_{\alpha, u, \tau}(\beta) = \beta$  for any arrow  $\beta \neq \alpha$ . These automorphisms allow to compare the fundamental groups of the presentations of an algebra. More precisely, we have the following proposition announced in [11] and proved in [14]:

**Proposition 1.5.** ([14, Prop. 2.5]) *Let  $I$  be an admissible ideal of  $kQ$ , let  $\varphi$  be an automorphism of  $kQ$  and let  $J = \varphi(I)$ . If  $\varphi$  is a dilatation, then  $\sim_I$  and  $\sim_J$  coincide. Assume that  $\varphi = \varphi_{\alpha, u, \tau}$ :*

- If  $\alpha \sim_I u$  and  $\alpha \sim_J u$  then  $\sim_I$  and  $\sim_J$  coincide.
- If  $\alpha \not\sim_I u$  and  $\alpha \sim_J u$  then  $\sim_J$  is generated by  $\sim_I$  and  $\alpha \sim_J u$ .
- If  $\alpha \not\sim_I u$  and  $\alpha \not\sim_J u$  then  $I = J$  and  $\sim_I$  and  $\sim_J$  coincide.

In particular, if  $\alpha \sim_I u$ , then the identity map on the set of walks of  $Q$  induces a surjective group morphism  $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$ .

Here *generated* means: generated as an equivalence relation on the set of walks of  $Q$ , and verifying the conditions (1) and (2) above in the definition of the homotopy relation. If  $I, J$  are admissible ideals such that there exists  $\varphi_{\alpha, u, \tau}$  verifying  $J = \varphi_{\alpha, u, \tau}(I)$ ,  $\alpha \not\sim_I u$  and  $\alpha \sim_J u$ , then we shall say that  $\sim_J$  is a **direct successor** of  $\sim_I$ . Proposition 1.5 allows one to define a quiver  $\Gamma$  associated to  $A$  as follows (see [11, Def. 4.1]):

- $\Gamma_0 = \{\sim_I \mid I \text{ is an admissible ideal of } kQ \text{ such that } A \simeq kQ/I\}$ ,
- there is an arrow  $\sim \rightarrow \sim'$  if  $\sim_J$  is a direct successor of  $\sim_I$ .

**Example 1.6.** *Let  $A$  be as in Example 1.3, then  $J = \varphi_{\alpha, cb, 1}$  and  $\Gamma$  is equal to  $\sim_I \rightarrow \sim_J$ .*

The quiver  $\Gamma$  is finite, connected and has not oriented cycle ([11, Rem. 3, Prop. 4.2]). Moreover, if  $\Gamma$  has a unique source  $\sim_{I_0}$  (i.e. a vertex with no arrow ending at it) then the fundamental group of any admissible presentation of  $A$  is a quotient of  $\pi_1(Q, I_0)$ . It was proved in [14] and [13] that  $\Gamma$  has a unique source under one of the hypotheses  $(H_1)$  or  $(H_2)$  presented in the introduction. Moreover, the hypotheses  $H_1$  and  $H_2$  both ensure the following proposition which will be particularly useful to prove Theorem 1.

**Proposition 1.7.** ([14, Lem. 4.3] and [13, Prop. 4.3]) Assume that at least one of the two hypotheses  $H_1$  or  $H_2$  is satisfied: Let  $I_0$  be the admissible ideal such that  $\sim_{I_0}$  is the unique source of  $\Gamma$  and let  $kQ \twoheadrightarrow A$  be a presentation with kernel  $I$ . Then there exist a dilatation  $D$  and a sequence of transvections  $\varphi_{\alpha_1, u_1, \tau_1}, \dots, \varphi_{\alpha_n, u_n, \tau_n}$  such that:

- $I = D\varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$ ,
- if we set  $I_i := \varphi_{\alpha_i, u_i, \tau_i} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$ , then  $\alpha_i \sim_{I_i} u_i$  for every  $i$ .

If  $\sim_I$  is the unique source of  $\Gamma$  (i.e.  $\sim_I = \sim_{I_0}$ ) then the homotopy relations  $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_n}, \sim_I$  coincide.

**Comparison of the fundamental groups and the Hochschild cohomology.** Let  $A$  be a basic connected finite dimensional  $k$ -algebra. Assume that the ordinary quiver  $Q$  of  $A$  has no oriented cycle. Let  $x_0 \in Q_0$  and fix a maximal tree  $T$  of  $Q$ , i.e. a subquiver of  $Q$  such that  $T_0 = Q_0$  and such that the underlying graph of  $T$  is a tree. With these data, I. Assem and J. A. de la Peña have defined in [3] an injective mapping of abelian groups  $\theta_\nu: \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+) \hookrightarrow HH^1(A)$  associated to any admissible presentation  $\nu: kQ \twoheadrightarrow A$ . We recall here the definition of  $\theta_\nu$  and refer the reader to [3] for more details. For any  $x \in Q_0$  there exists a unique walk  $\gamma_x$  in  $T$  with source  $x_0$ , with target  $x$  and with length minimal for these properties. Let  $\nu: kQ \twoheadrightarrow A$  be an admissible presentation and let  $f \in \text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+)$  be a group morphism. Then,  $f$  defines a unitary derivation  $\tilde{f}: A \rightarrow A$  as follows:  $\tilde{f}(\nu(u)) = f([\gamma_y^{-1}y\gamma_x]_{\sim_{\text{Ker}(\nu)}}) \nu(u)$  for any path  $u$  with source  $x$  and target  $y$ . The following proposition was proved in [3]:

**Proposition 1.8.** (see [3]) The mapping  $f \mapsto \tilde{f}$  induces an injective mapping of abelian groups:

$$\theta_\nu: \text{Hom}(\pi_1(\text{Ker}(\nu)), k^+) \hookrightarrow HH^1(A)$$

The abelian group  $\text{Hom}(\pi_1(Q, \text{Ker}(\nu)), k^+)$  naturally carries a structure of commutative  $k$ -algebra, hence a structure of abelian Lie algebra. The following lemma proves that  $\theta_\nu$  preserves this structure.

**Lemma 1.9.**  $\theta_\nu: \text{Hom}(\pi_1(\text{Ker}(\nu)), k^+) \hookrightarrow HH^1(A)$  is a Lie algebra morphism. In particular,  $\text{Im}(\theta_\nu)$  is an abelian Lie subalgebra of  $HH^1(A)$ .

**Proof:** Let  $f, f' \in \text{Hom}(\pi_1(\text{Ker}(\nu)), k^+)$ . Then  $[\theta_\nu(f), \theta_\nu(f')] \in HH^1(A)$  is represented by the derivation  $\tilde{f} \circ \tilde{f}' - \tilde{f}' \circ \tilde{f}: A \rightarrow A$ . Let  $u$  be a path in  $Q$ . It follows from the definition of  $\theta_\nu$  that  $\tilde{f} \circ \tilde{f}' - \tilde{f}' \circ \tilde{f}$  vanishes on  $u$ . Hence,  $[\theta_\nu(f), \theta_\nu(f')] = 0$ .  $\blacksquare$

Lemma 1.9 shows that  $\theta_\nu$  is not surjective in general. Indeed, assume for example that  $A = kQ$  where  $Q = \overline{A_3}$ . If  $\nu: kQ \twoheadrightarrow A$  is an admissible presentation, then  $\text{Ker}(\nu) = 0$  and  $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$  so that  $\dim_k \text{Im}(\theta_\nu) = 1$ . On the other hand, one easily verifies that  $\dim_k HH^1(A) = 2$ .

Throughout this text,  $A$  will be a basic connected finite dimensional  $k$ -algebra with ordinary quiver  $Q$  without oriented cycle ( $Q_0 = \{1, \dots, n\}$ ). We fix  $\{e_1, \dots, e_n\}$  a complete set of primitive orthogonal idempotents of  $A$ . The semi-simple subalgebra  $k.e_1 \oplus \dots \oplus k.e_n$  of  $A$  will be denoted by  $E$ . The radical of  $A$  will be denoted by  $\mathfrak{r}$ . For short, we shall say “presentation” instead of “admissible presentation”. Without loss of generality, we shall assume that any presentation  $\nu: kQ \twoheadrightarrow A$  verifies  $\nu(e_i) = e_i$ . Finally, in order to use the morphisms  $\theta_\nu$ , we fix a maximal tree  $T$  of  $Q$ .

## 2 Diagonalisability in $HH^1(A)$

The aim of this section is to give prove some useful properties on the subspaces  $\text{Im}(\theta_\nu)$  in terms of diagonalisability in  $HH^1(A)$ . Notice that diagonalisability was introduced for derivations of  $A$  in [7]. For short, a **basis** of  $A$  is a basis  $B$  of the  $k$ -vector space  $A$  such that:  $B \subseteq \bigcup_{i,j} e_j A e_i$ , such that  $\{e_1, \dots, e_n\} \subseteq B$ , and such that  $B \setminus \{e_1, \dots, e_n\} \subseteq \mathfrak{r}$ . We shall say that a linear mapping  $u: A \rightarrow A$  is diagonalisable if and only if there exists a basis as above such that the matrix of  $u$  in this basis is diagonal. In such a case,  $B$  is called a diagonalisation basis of  $u$ . Notice the following link between bases and presentations of  $A$ :

- if  $\nu: kQ \twoheadrightarrow A$  is a presentation of  $A$ , then there exists a basis  $B$  such that  $\nu(\alpha) \in B$  for any  $\alpha \in Q_1$  and such that any element of  $B$  is of the form  $\nu(u)$  with  $u$  a path in  $Q$ . We shall say that this basis  $B$  is adapted to  $\nu$ ,
- if  $B$  is a basis of  $A$ , then there exists a presentation  $\nu: kQ \twoheadrightarrow A$  such that  $\nu(\alpha) \in B$  for any  $\alpha \in Q_1$ . We shall say that the presentation  $\nu$  is adapted to  $B$ .

Remark that if  $\nu: kQ \rightarrow A$  is a presentation and if  $B$  is a basis of  $A$  adapted to  $\nu$ , then the presentation  $\nu$  is adapted to  $B$ . Notice also that diagonalisability is invariant up to a sum with an inner derivation as the following lemma shows. The proof is immediate.

**Lemma 2.1.** *Let  $u: A \rightarrow A$  be a linear mapping, let  $e \in E$  and let  $B$  be a basis of  $A$ . Then  $B$  is a diagonalisation basis for  $u$  if and only if the same holds for  $u + \delta_e$ .*

The preceding lemma justifies the following definition.

**Definition 2.2.** *Let  $f \in HH^1(A)$  and let  $d$  be a derivation representing  $f$ . Then  $f$  is called diagonalisable if and only if  $d$  is diagonalisable. In such a case, a diagonalisation basis of  $f$  is a diagonalisation basis of  $d$ .*

*The subset  $D \subseteq HH^1(A)$  is called diagonalisable if and only if any there exists a basis  $B$  of  $A$  such that  $B$  is a diagonalisation basis of  $f$  for any  $f \in D$ .*

It is well know that a family of endomorphisms (of a finite dimensional  $k$ -vector space) is simultaneously diagonalisable as soon as these endomorphisms commute with each other and are diagonalisable. The following proposition gives a similar characterisation for the diagonalisability of subsets of  $HH^1(A)$ .

**Proposition 2.3.** *Let  $D \subseteq HH^1(A)$ . Then,  $D$  is diagonalisable if and only if every element of  $D$  is diagonalisable and  $[f, f'] = 0$  for any  $f, f' \in D$ .*

**Proof:** For each  $f \in D$ , let  $d_f$  be a derivation representing  $f$ . If there exists a common diagonalisation basis for the elements of  $D$ , then the same holds for  $\{d_f \mid f \in D\}$  so that  $[d_f, d_{f'}] = 0$  for any  $f, f' \in D$ . Consequently  $[f, f'] = 0$  for any  $f, f' \in D$ . Now let us prove the converse. The vanishing of the Lie brackets implies that for any  $f, f' \in D$  there exists  $e^{(f, f')} \in E$  such that  $[d_f, d_{f'}] = \delta_{e^{(f, f')}}$ . Let us write  $e^{(f, f')} = \sum_{i=1}^n t_i^{(f, f')} e_i$ . Hence (see Lemma 1.2), for any  $i, j$ , the derivations  $d_f$  and  $d_{f'}$  induce diagonalisable

linear mappings  $e_j \tau e_i \rightarrow e_j \tau e_i$  whose Lie bracket equals  $u \mapsto (t_j^{(f, f')} - t_i^{(d, d')})u$ . Recall that if  $M, N$  are square matrices such that  $M$  is diagonal, then the diagonal entries of  $MN - NM$  are all zero, so that if moreover  $MN - NM$  is a scalar matrix, then  $MN - NM = 0$ . Consequently, the restriction of  $[d_f, d_{f'}]$  to  $e_j \tau e_i$  is zero for any  $i, j$ . Since  $[d_f, d_{f'}](e_i) = 0$  for any  $i$ , we deduce that:

- $d_f: e_j \tau e_i \rightarrow e_j \tau e_i$  is diagonalisable for every  $f \in D$  and every  $i, j$ ,
- $[d_f, d_{f'}] = 0$  for every  $f, f' \in D$ ,
- $A = E \bigoplus \bigoplus_{i,j} e_j \tau e_i$

Therefore, the classical simultaneous diagonalisation of pairwise commuting diagonalisable endomorphisms of a finite dimensional  $k$ -vector space gives rise to a basis of  $B$  in which the matrix of  $d_f$  is diagonal for any  $f \in D$ . ■

Our main example of diagonalisable subspace of  $HH^1(A)$  is  $Im(\theta_\nu)$ :

**Proposition 2.4.** *Let  $\nu: kQ \rightarrow A$  be a presentation. Then,  $Im(\theta_\nu)$  is diagonalisable.*

**Proof:** Let  $B$  be a basis of  $A$  adapted to  $\nu$  and let  $I = Ker(\nu)$ . For any  $f \in Hom(\pi_1(Q, I), k^+)$ , the element  $\theta_\nu(f) \in HH^1(A)$  is represented by the derivation  $\tilde{f}$  which, by definition, verifies  $\tilde{f}(\nu(u)) \in k \cdot \nu(u)$  for any path  $u$  in  $Q$ . ■

In this section, we aim at proving that any diagonalisable subset of  $HH^1(A)$  is contained in  $Im(\theta_\nu)$  for some presentation  $\nu$ . It was proved in [7] that any diagonalisable derivation (with suitable technical conditions) defines an element of  $HH^1(A)$  lying in  $Im(\theta_\nu)$  for some  $\nu$ . We will use the following similar result.

**Lemma 2.5.** *Let  $f \in HH^1(A)$  and let  $B$  be a diagonalisation basis of  $f$ . Let  $\nu: kQ \rightarrow A$  be a presentation adapted to  $B$ . Then  $f \in Im(\theta_\nu)$ .*

**Proof:** Let  $I = Ker(\nu)$  and let  $d: A \rightarrow A$  be a derivation representing  $f$ . We shall write  $\bar{r}$  for  $\nu(r)$ , for any  $r \in kQ$ . Let  $\alpha \in Q_1$ , since  $B$  is a diagonalisation basis of  $f$  and since  $\nu$  is adapted to  $B$ , there exists  $t_\alpha \in k$  such that  $d(\bar{\alpha}) = t_\alpha \bar{\alpha}$ . For any path  $u = \alpha_n \dots \alpha_1$  (with  $\alpha_i \in Q_1$ ), let us set  $t_u := t_{\alpha_1} + \dots + t_{\alpha_n}$  so that  $d(\bar{u}) = t_u \bar{u}$ , because  $d$  is a derivation. More generally, if  $\gamma = \alpha_n^{\varepsilon_n} \dots \alpha_1^{\varepsilon_1}$  is a walk in  $Q$  (with  $\alpha_i \in Q_1$ ), let us set  $t_\gamma := \sum_{i=1}^n (-1)^{\varepsilon_i} t_{\alpha_i}$ , with the convention that  $t_\gamma = 0$  if  $\gamma$  is stationary. We are going to prove that the mapping  $\gamma \mapsto t_\gamma$  defines a group morphism  $g: \pi_1(Q, I) \rightarrow k^+, [\gamma]_I \mapsto t_\gamma$  and that  $f = \theta_\nu(g)$ .

First, let us prove that the morphism  $g: \pi_1(Q, I) \rightarrow k^+$  is well defined. By definition of the scalar  $t_\gamma$ , we have:

- (i)  $t_{e_x} = 0$  for any  $x \in Q_0$  and  $t_{\gamma'\gamma} = t_{\gamma'}t_\gamma$  for any walks  $\gamma, \gamma'$  such that  $t(\gamma) = s(\gamma')$ .
  - (ii)  $t_{\alpha^{-1}\alpha} = t_{e_x}$  and  $t_{\alpha\alpha^{-1}} = t_{e_y}$  for any arrow  $x \xrightarrow{\alpha} y \in Q_1$ .
  - (iii)  $t_{wvu} = t_{wv'u}$  for any walks  $w, v, v', u$  verifying  $t_v = t_{v'}$ ,  $s(w) = t(v) = t(v')$  and  $s(v) = s(v') = t(u)$ .
- In order to prove that  $g$  is well defined, it only remains to prove that  $t_u = t_v$  whenever  $u, v$  are paths in  $Q$  lying in the support of the same minimal relation of  $I$ . In this purpose, let  $d': kQ \rightarrow kQ$  be the linear mapping such that  $d'(u) = t_u u$  for any path  $u$  in  $k$ . By definition of the scalar  $t_u$ , this implies that  $d \circ \nu = \nu \circ d'$ . In particular,  $d'(I) \subseteq I$ . So we may apply Lemma 1.4 to  $d'$  and deduce that:
- (iv)  $t_u = t_v$  if  $u, v$  are paths in  $Q$  lying in the support of a same minimal relation of  $I$ .

From (ii), (iii) and (iv) we deduce that we have a well defined mapping  $g: \pi_1(Q, I) \rightarrow k, [\gamma]_I \mapsto t_\gamma$ . Moreover, (i) proves that  $g$  is a group morphism.

Now let us prove that  $f = \theta_\nu(g)$ . For any path  $u$  with source  $x$  and target  $y$ , we have  $g([\gamma_y^{-1}u\gamma_x]_I) = t_u - t_{\gamma_y} + t_{\gamma_x}$ . Hence,  $\theta_\nu(g) \in HH^1(A)$  is represented by the derivation  $\tilde{g}: A \rightarrow A$  such that  $\tilde{g}(\overline{u}) = (t_u - t_{\gamma_y} + t_{\gamma_x})\overline{u}$  for any path  $u$  with source  $x$  and target  $y$ . Let us set  $e := \sum_{x \in Q_0} t_{\gamma_x} e_x \in E$ . Therefore,  $\tilde{g} + \delta_e = d$ . This proves that  $f = \theta_\nu(g)$ . ■

Now we can state the main result of this section. It is a direct consequence of Proposition 2.4 and of Lemma 2.5.

**Proposition 2.6.** *Let  $D \subseteq HH^1(A)$ . Then  $D$  is diagonalisable if and only if there exists a presentation  $\nu: kQ \rightarrow A$  such that  $D \subseteq \text{Im}(\theta_\nu)$ .*

Remark that Lemma 2.5 also gives a sufficient condition for  $\theta_\nu$  to be an isomorphism. Recall that  $A$  is called constricted if and only if  $\dim e_y A e_x = 1$  for any arrow  $x \rightarrow y$  (this implies that  $Q$  has no multiple arrows). In [5] it was proved that for such an algebra, two different presentations have the same fundamental group.

**Proposition 2.7.** *Assume that  $A$  is constricted. Let  $\nu: kQ \rightarrow A$  be any presentation of  $A$ . Then  $\theta_\nu: \text{Hom}(\pi_1(Q, I), k^+) \rightarrow HH^1(A)$  is an isomorphism. In particular,  $HH^1(A)$  is an abelian Lie algebra.*

**Proof:** Since  $\theta_\nu$  is one-to-one, we only need to prove that it is onto. Let  $B$  be a basis of  $A$  adapted to  $\nu$ , let  $f \in HH^1(A)$  and let  $d: A \rightarrow A$  be a derivation representing  $f$ . Let  $x \xrightarrow{\alpha} y$  be an arrow. Then  $e_y A e_x = K.\nu(\alpha)$  so that there exists  $t_\alpha \in k$  such that  $d(\nu(\alpha)) = t_\alpha \nu(\alpha)$ . Let  $u = \alpha_n \dots \alpha_1$  be any path in  $Q$  (with  $\alpha_i \in Q_1$ ). Since  $d$  is a derivation, we have  $d(\nu(u)) = (t_{\alpha_1} + \dots + t_{\alpha_n})\nu(u)$ . As a consequence,  $B$  is a diagonalisation basis for  $d$ . Moreover,  $\nu$  is adapted to  $B$  because  $B$  is adapted to  $\nu$ . So Lemma 2.5 proves that  $f \in \text{Im}(\theta_\nu)$ . This proves that  $\theta_\nu$  is an isomorphism. Finally  $HH^1(A)$  is abelian because  $\text{Hom}(\pi_1(Q, I), k^+)$  is abelian. ■

### 3 Comparison of $\text{Im}(\theta_\nu)$ and $\text{Im}(\theta_\mu)$ for different presentations $\mu$ and $\nu$ of $A$

In Proposition 1.5 we have recalled that if two presentations  $\nu$  and  $\mu$  of  $A$  are linked by a transvection or a dilatation, then there is a simple relation between the associated fundamental groups. In this section, we shall use this relation to compare  $\theta_\nu$  and  $\theta_\mu$ . Our first result compares  $\theta_\nu$  and  $\theta_\mu$  when  $\mu = \nu \circ D$  with  $D$  a dilatation. Recall from Proposition 1.5 that if  $J = D(I)$  with  $D$  a dilatation, then  $\sim_I$  and  $\sim_J$  coincide, so that  $\pi_1(Q, I) = \pi_1(Q, J)$ .

**Proposition 3.1.** *Let  $\nu: kQ \rightarrow A$  be a presentation, let  $D: kQ \xrightarrow{\sim} kQ$  be a dilatation, let  $\mu := \nu \circ D: kQ \rightarrow A$ . Set  $I = \text{Ker}(\mu)$  and  $J = \text{Ker}(\nu)$ , so that  $J = D(I)$ . Then  $\theta_\mu = \theta_\nu$ .*

**Proof:** Let  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ . Then,  $\theta_\nu(f)$  and  $\theta_\mu(f)$  are represented by the derivations  $d_1$  and  $d_2$  respectively, such that for any arrow  $x \xrightarrow{\alpha} y$ :

$$\begin{aligned} d_1(\nu(\alpha)) &= f([\gamma_y^{-1}\alpha\gamma_x]_J) \nu(\alpha) \\ d_2(\mu(\alpha)) &= f([\gamma_y^{-1}\alpha\gamma_x]_I) \mu(\alpha) \end{aligned}$$

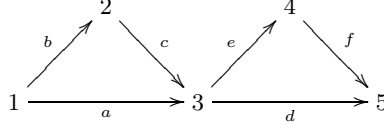
On the other hand, for such an arrow  $x \xrightarrow{\alpha} y$ , there exists  $t_\alpha \in k$  such that  $D(\alpha) = t_\alpha \alpha$  which implies, thanks to the above property of  $d_1$  and  $d_2$  and thanks to the fact that  $\sim_I$  and  $\sim_J$  coincide, that  $d_1(\mu(\alpha)) = d_2(\mu(\alpha))$ . Hence,  $d_1$  and  $d_2$  are two derivations of  $A$  and they coincide on  $\mu(Q_1)$ . So  $d_1 = d_2$



and  $\theta_\nu(f) = \theta_\mu(f)$ , for any  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ . ■

The following example shows that the conclusion of Proposition 3.1 does not necessarily hold if  $\nu$  and  $\mu$  are two presentations of  $A$  such that  $\sim_{\text{Ker}(\nu)}$  and  $\sim_{\text{Ker}(\mu)}$  coincide.

**Example 3.2.** Assume that  $\text{car}(k) = 2$  and let  $A = kQ/I$  where  $Q$  is the quiver:



and  $I = \langle ba, vu, fea + dcb \rangle$ . Let  $T$  be the maximal tree such that  $T_1 = \{b, c, e, f\}$ . Let  $\nu: kQ \twoheadrightarrow A = kQ/I$  be the natural projection. Let  $\psi := \varphi_{a,cb,1}\varphi_{d,fe,1}$ . Thus,  $I = \psi(I)$ . Let  $\mu := \nu \circ \psi: kQ \twoheadrightarrow A$  so that  $\text{Ker}(\mu) = \text{Ker}(\nu) = I$ . Remark that  $\pi_1(Q, I)$  is the infinite cyclic group with generator  $[b^{-1}c^{-1}a]_I$ . So let  $f: \pi_1(Q, I) \rightarrow k^+$  be the unique group morphism such that  $f([b^{-1}c^{-1}a]_I) = 1$ . Then  $\theta_\nu(f)$  is represented by the following derivation:

$$\begin{aligned} d_1: \quad A &\longrightarrow A \\ \nu(x) &\longmapsto \nu(x) \text{ if } x \in \{a, d\} \\ \nu(x) &\longmapsto 0 \text{ if } x \in \{b, c, e, f\} \end{aligned}$$

On the other hand,  $\theta_\mu(f)$  is represented by the derivation:

$$\begin{aligned} d_2: \quad A &\longrightarrow A \\ \nu(a) &\longmapsto \nu(a) + \nu(cb) \\ \nu(d) &\longmapsto \nu(d) + \nu(fe) \\ \nu(x) &\longmapsto 0 \text{ if } x \in \{b, c, e, f\} \end{aligned}$$

It is easy to verify that  $d_2 - d_1$  is not inner. Hence,  $\theta_\nu \neq \theta_\mu$ .

Now we compare  $\theta_\nu$  and  $\theta_\mu$  when  $\mu = \nu \circ \varphi_{\alpha,u,\tau}$  and when the identity map on the set of walks in  $Q$  induces a surjective group morphism  $\pi_1(Q, \text{Ker}(\nu)) \twoheadrightarrow \pi_1(Q, \text{Ker}(\mu))$ .

**Proposition 3.3.** Let  $\nu: kQ \twoheadrightarrow A$  be a presentation, let  $\varphi_{\alpha,u,\tau}: kQ \xrightarrow{\sim} kQ$  be a transvection and let  $\mu := \nu \circ \varphi_{\alpha,u,\tau}: kQ \twoheadrightarrow A$ . Set  $I = \text{Ker}(\nu)$  and  $J = \text{Ker}(\mu)$ , so that  $I = \varphi_{\alpha,u,\tau}(J)$ . Suppose that  $\alpha \sim_J u$  and let  $p: \pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$  be the quotient mapping (see Proposition 1.5). Then, the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}(\pi_1(Q, J), k^+) & & \\ \downarrow p^* & \searrow \theta_\mu & \\ & & HH^1(A) \\ & \nearrow \theta_\nu & \\ \text{Hom}(\pi_1(Q, I), k^+) & & \end{array}$$

where  $p^*: \text{Hom}(\pi_1(Q, J), k^+) \hookrightarrow \text{Hom}(\pi_1(Q, I), k^+)$  is the embedding induced by  $p$ . In particular,  $\text{Im}(\theta_\nu) \subseteq \text{Im}(\theta_\mu)$ .

**Proof:** Recall that  $p$  is the mapping  $[\gamma]_I \mapsto [\gamma]_J$ . Let  $f \in \text{Hom}(\pi_1(Q, J), k^+)$ . So  $p^*(f)$  is the composition  $\pi_1(Q, I) \xrightarrow{p} \pi_1(Q, J) \xrightarrow{f} k$ . We know that  $\theta_\mu(f)$  and  $\theta_\nu(p^*(f))$  are represented by the derivations  $d_1$  and  $d_2$  respectively, such that for any arrow  $x \xrightarrow{a} y$ :

$$\begin{aligned} d_1(\mu(a)) &= f([\gamma_y^{-1}a\gamma_x]_J) \mu(a) = p^*(f)([\gamma_y a\gamma_x]_I) \mu(a) \\ d_2(\nu(a)) &= p^*(f)([\gamma_y^{-1}a\gamma_x]_I) \nu(a) \end{aligned}$$

Let us prove that  $d_1$  and  $d_2$  coincide on  $\nu(Q_1)$ . Let  $x \xrightarrow{a} y$  be an arrow. If  $a \neq \alpha$ , then  $\mu(a) = \nu(a)$  and the above characterisations of  $d_1$  and  $d_2$  imply that  $d_1(\nu(a)) = d_1(\mu(a)) = d_2(\nu(a))$ . Now assume that  $a = \alpha$  so that:  $\nu(a) = \mu(a) - \tau\mu(u)$  and  $[\gamma_y^{-1}a\gamma_x]_J = [\gamma_y^{-1}u\gamma_x]_J$  (recall that  $a = \alpha \sim_J u$ ). Thus:

$$\begin{aligned} d_1(\nu(a)) &= d_1(\mu(\alpha)) - \tau d_1(\mu(u)) \\ &= f([\gamma_y^{-1}\alpha\gamma_x]_J) \mu(\alpha) - \tau f([\gamma_y^{-1}u\gamma_x]_J) \mu(u) \\ &= f([\gamma_y^{-1}\alpha\gamma_x]_J) (\mu(\alpha) - \tau \mu(u)) \\ &= p^*(f)([\gamma_y^{-1}\alpha\gamma_x]_I) \nu(\alpha) \\ &= d_2(\nu(\alpha)) = d_2(\nu(a)) \end{aligned}$$

Hence,  $d_1$  and  $d_2$  are two derivations of  $A$  and they coincide on  $\nu(Q_1)$ . This proves that  $d_1 = d_2$  and that  $\theta_\mu(f) = \theta_\nu(p^*(f))$  for any  $f \in \text{Hom}(\pi_1(Q, J), k^+)$ .  $\blacksquare$

The following example shows that the conclusions of Proposition 3.3 do not necessarily hold if  $\nu$  is a presentation of  $A$  and  $\psi: kQ \rightarrow kQ$  is an automorphism such that the identity map on the walks in  $Q$  induces a surjective group morphism  $\pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\nu \circ \psi))$ .

**Example 3.4.** Let  $A = kQ/I$  where  $\text{car}(k) = 2$ , where  $Q$  is the quiver of Example 3.2 and where  $I = \langle da, fea + dcb \rangle$ . Let  $\nu: kQ \rightarrow A$  be the natural projection with kernel  $I$ , let  $\psi := \varphi_{d,ef,1}\varphi_{a,cb,1}$  and let  $\mu := \nu \circ \psi: kQ \rightarrow A$ . Hence  $\text{Ker}(\mu) = \langle da + fecb, fea + dcb \rangle$ . Notice that  $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$  is generated by  $[b^{-1}c^{-1}a]_I$  and that  $\pi_1(Q, \text{Ker}(\mu)) \simeq \mathbb{Z}/2\mathbb{Z}$  is generated by  $[b^{-1}c^{-1}a]_J$ . Notice also that  $\sim_{\text{Ker}(\nu)}$  is weaker than  $\sim_{\text{Ker}(\mu)}$  so that the identity map on the set of walks in  $Q$  induces a surjective group morphism  $p: \pi_1(Q, \text{Ker}(\nu)) \rightarrow \pi_1(Q, \text{Ker}(\mu))$ . Let  $T$  be the maximal tree such that  $T_1 = \{b, c, e, f\}$ . Let  $f: \pi_1(Q, \text{Ker}(\mu)) \rightarrow k$  be the group morphism such that  $f([b^{-1}c^{-1}a]_J) = 1$  and let us verify that  $\theta_\mu(f) \neq \theta_\nu(p^*(f))$ . On the one hand,  $\theta_\mu(f) \in HH^1(A)$  is represented by the derivation:

$$\begin{aligned} d_1: \quad A &\rightarrow A \\ \mu(x) &\mapsto \mu(x) \text{ if } x \in \{a, d\} \\ \mu(x) &\mapsto 0 \text{ if } x \in \{b, c, e, f\} \end{aligned}$$

On the other hand,  $\theta_\nu(p^*(f)) \in HH^1(A)$  is represented by the derivation:

$$\begin{aligned} d_2: \quad A &\rightarrow A \\ \mu(a) &\mapsto \mu(a) + \mu(cb) \\ \mu(d) &\mapsto \mu(d) + \mu(fe) \\ \mu(x) &\mapsto 0 \text{ if } x \in \{b, c, e, f\} \end{aligned}$$

and one checks easily that  $d_2 - d_1$  is not inner so that  $\theta_\mu(f) \neq \theta_\nu(p^*(f))$ . Moreover,  $\text{Im}(\theta_\nu)$  and  $\text{Im}(\theta_\mu)$  are one dimensional (because  $\text{car}(k) = 2$ ,  $\pi_1(Q, \text{Ker}(\nu)) \simeq \mathbb{Z}$  and  $\pi_1(Q, \text{Ker}(\mu)) \simeq \mathbb{Z}/2\mathbb{Z}$ ) and  $d_1, d_2$  are not inner. Hence  $\text{Im}(\theta_\mu) = k \cdot \theta_\mu(f) \not\subseteq \text{Im}(\theta_\nu) = k \cdot \theta_\nu(p^*(f))$ .

Finally, we compare  $\theta_\nu$  and  $\theta_\mu$  when  $\mu = \nu \circ \psi$  with  $\psi: kQ \xrightarrow{\sim} kQ$  an automorphism such that  $\text{Ker}(\nu) = \text{Ker}(\mu)$ .

**Proposition 3.5.** Let  $\nu: kQ \rightarrow A$  be a presentation and let  $I = \text{Ker}(\nu)$ . Let  $\psi: kQ \xrightarrow{\sim} kQ$  be an automorphism such that  $\psi(e_i) = e_i$  for every  $i$  and such that  $\psi(I) = I$ . Let  $\mu := \nu \circ \psi: kQ \rightarrow A$  so that  $\text{Ker}(\mu) = I$ . Let  $\bar{\psi}: A \xrightarrow{\sim} A$  be the  $k$ -algebra automorphism making commute the following diagram:

$$\begin{array}{ccc} kQ & \xrightarrow{\psi} & kQ \\ \mu \downarrow & & \downarrow \mu \\ A & \xrightarrow{\bar{\psi}} & A \end{array}$$

Then, the following diagram commutes:

$$\begin{array}{ccc} & & HH^1(A) \\ & \nearrow \theta_\nu & \downarrow \sim \bar{\psi}_* \\ \text{Hom}(\pi_1(Q, I), k^+) & & \\ & \searrow \theta_\mu & \\ & & HH^1(A) \end{array}$$

In particular,  $\text{Im}(\theta_\mu)$  is equal to the image of  $\text{Im}(\theta_\nu)$  under the Lie algebra automorphism  $\bar{\psi}_*: HH^1(A) \xrightarrow{\sim} HH^1(A)$  induced by  $\bar{\psi}: A \xrightarrow{\sim} A$ .

**Proof:** Since  $\psi$  fixes the idempotents  $e_1, \dots, e_n$ ,  $\bar{\psi}_*$  is well defined. Let  $f \in \text{Hom}(\pi_1(Q, I), k^+)$ . So  $\theta_\nu(f)$  and  $\theta_\mu(f)$  are represented by the derivations  $d_1$  and  $d_2$  respectively, such that for any arrow  $x \xrightarrow{a} y$ :

$$\begin{aligned} d_1(\nu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \nu(\alpha) \\ d_2(\mu(\alpha)) &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) \end{aligned}$$

In order to prove that  $\overline{\psi}_*(\theta_\nu(f)) = \theta_\mu(f)$  it suffices to prove that  $\overline{\psi} \circ d_1 = d_2 \circ \overline{\psi}$ . Let  $x \xrightarrow{\alpha} y$  be an arrow. Then:

$$\begin{aligned} d_2 \circ \overline{\psi}(\nu(\alpha)) &= d_2 \circ \overline{\psi} \circ \mu \circ \psi^{-1}(\alpha) && \text{because } \nu = \mu \circ \psi^{-1} \\ &= d_2 \circ \mu \circ \psi(\psi^{-1}(\alpha)) && \text{because } \overline{\psi} \circ \mu = \mu \circ \psi \\ &= d_2(\mu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) \end{aligned}$$

On the other hand:

$$\begin{aligned} \overline{\psi} \circ d_1(\nu(\alpha)) &= \overline{\psi}(f([\gamma_y^{-1} \alpha \gamma_x]_I) \nu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \overline{\psi}(\nu(\alpha)) \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \overline{\psi} \circ \mu \circ \psi^{-1}(\alpha) && \text{because } \mu = \nu \circ \psi \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu \circ \psi \circ \psi^{-1}(\alpha) && \text{because } \overline{\psi} \circ \mu = \mu \circ \psi \\ &= f([\gamma_y^{-1} \alpha \gamma_x]_I) \mu(\alpha) \end{aligned}$$

Hence,  $\overline{\psi} \circ d_1$  and  $d_2 \circ \overline{\psi}$  are derivations of  $A$  and they coincide on  $\nu(Q_1)$ . This proves that  $\overline{\psi}_*(\theta_\nu(f)) = \theta_\mu(f)$  for any  $f \in \text{Hom}(\pi_1(Q, I, k^+))$ .  $\blacksquare$

## 4 Proof of Theorem 1

In this section, we will prove Theorem 1. We begin with the following useful lemma.

**Lemma 4.1.** *Assume that at least one of the two conditions  $(H_1)$  or  $(H_2)$  is satisfied. Let  $\nu: kQ \rightarrow A$  be a presentation with kernel  $I_0$  such that  $\sim_{I_0}$  is the unique source of  $\Gamma$ , and let  $\mu: kQ \rightarrow A$  be a presentation. Then, there exist  $\nu': kQ \rightarrow A$  a presentation with kernel  $I_0$  and a  $k$ -algebra automorphism  $\psi: A \xrightarrow{\sim} A$  verifying  $\psi(e_i) = e_i$  for any  $i$  and such that:*

$$\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_{\nu'}) = \psi_*(\text{Im}(\theta_\nu))$$

If moreover  $\sim_I = \sim_{I_0}$ , then the above inclusion is an equality.

**Proof:** Let  $\overline{\nu}: kQ/I_0 \xrightarrow{\sim} A$  and  $\overline{\mu}: kQ/I \xrightarrow{\sim} A$  be the isomorphisms induced by  $\nu$  and  $\mu$  respectively. Hence,  $\overline{\mu}^{-1} \circ \overline{\nu}: kQ/I_0 \xrightarrow{\sim} kQ/I$  is an isomorphism which maps  $e_i$  to  $e_i$  for every  $i$ . Hence (see for example [12, Prop. 2.3.18]), there exists an automorphism  $\varphi: kQ \xrightarrow{\sim} kQ$  mapping  $e_i$  to  $e_i$  for every  $i$  and such that the following diagram commutes:

$$\begin{array}{ccc} kQ & \xrightarrow{\varphi} & kQ \\ \downarrow & & \downarrow \\ kQ/I_0 & \xrightarrow{\overline{\mu}^{-1} \circ \overline{\nu}} & kQ/I \end{array}$$

where the vertical arrows are the natural projections. In particular, the following diagram is commutative:

$$\begin{array}{ccc} kQ & \xrightarrow{\varphi} & kQ \\ \searrow \nu & & \swarrow \mu \\ & A & \end{array}$$

Now let us apply Proposition 1.7 to  $I$ . With the same notations, let  $\psi := \varphi^{-1} D \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$ . Thus,  $\psi(I_0) = I_0$  and  $\nu' := \nu \circ \psi: kQ \rightarrow A$  is a presentation with kernel  $I_0$ . Thanks to Proposition 3.5 from which we keep the notations, we know that:

$$\text{Im}(\theta_{\nu'}) = \overline{\psi}_*(\text{Im}(\theta_\nu)) \tag{1}$$

Now let us show that  $\text{Im}(\theta_\mu) \subseteq \text{Im}(\theta_{\nu'})$ . By construction, we have  $\mu = \nu \varphi^{-1} = \nu' \varphi_{\alpha_1, u_1, \tau_1}^{-1} \dots \varphi_{\alpha_n, u_n, \tau_n}^{-1} D^{-1}$ . For simplicity, we shall use the following notations:  $\mu_0 := \nu'$  and  $\mu_i := \nu' \varphi_{\alpha_1, u_1, \tau_1}^{-1} \dots \varphi_{\alpha_i, u_i, \tau_i}^{-1}$  for  $i \in \{1, \dots, n\}$ . Therefore,  $\text{Ker}(\mu_i) = I_i$  for every  $i$ . Let  $i \in \{1, \dots, n\}$ , so:

- $\mu_{i-1}: kQ \twoheadrightarrow A$  and  $\mu_i: kQ \twoheadrightarrow A$  are two presentations of  $A$ ,
- $\mu_i = \mu_{i-1} \circ \varphi_{\alpha_i, u_i, \tau_i}^{-1}$ ,
- $\text{Ker}(\mu_i) = I_i$ ,  $\text{Ker}(\mu_{i-1}) = I_{i-1}$  and  $\alpha_i \sim_{I_i} u_i$ .

From Proposition 3.3 we deduce that:

$$Im(\theta_{\mu_n}) \subseteq Im(\theta_{\mu_{n-1}}) \subseteq \dots \subseteq Im(\theta_{\mu_i}) \subseteq Im(\theta_{\mu_{i-1}}) \subseteq \dots \subseteq Im(\theta_{\mu_0}) = Im(\theta_{\nu'}) \quad (2)$$

Moreover,  $\mu = \mu_m D^{-1}$  where  $D^{-1}$  is a dilatation. Hence, (1), (2) and Proposition 3.1 imply that:

$$Im(\theta_\mu) = Im(\theta_{\mu_n}) \subseteq Im(\theta_{\nu'}) = \overline{\psi}_*(Im(\theta_\nu)) \quad (3)$$

Now assume that  $\sim_I$  is the unique source of  $\Gamma$ . Then Proposition 1.7 imply that the homotopy relations  $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_n}, \sim_I$  coincide. Therefore, for any  $i \in \{1, \dots, n\}$ , we have:

$$\begin{aligned} \cdot \quad & \mu_{i-1} = \mu_i \circ \varphi_{\alpha_i, u_i, \tau_i}, \\ \cdot \quad & Ker(\mu_i) = I_i, Ker(\mu_{i-1}) = I_{i-1} \text{ and } \alpha_i \sim_{I_{i-1}} u_i, \end{aligned}$$

These two properties imply, thanks to Proposition 3.3, that  $Im(\theta_{\mu_{i-1}}) \subseteq Im(\theta_{\mu_i})$ . This proves that all the inclusions in (2) are equalities. And so does the inclusion in (3). ■

Now we can prove Theorem 1.

**Proof of Theorem 1:** (i) Let  $\mathcal{G}$  be a maximal diagonalisable subalgebra of  $HH^1(A)$ . Thanks to Proposition 2.6, there exists a presentation  $\mu: kQ \twoheadrightarrow A$  such that  $\mathcal{G} \subseteq Im(\theta_\mu)$ . On the other hand, Lemma 4.1, implies that there exists a presentation  $\nu: kQ \twoheadrightarrow A$  such that  $\sim_{Ker(\nu)}$  is the unique source of  $\Gamma$  and verifying  $Im(\theta_\mu) \subseteq Im(\theta_\nu)$ . Hence,  $\mathcal{G} \subseteq Im(\theta_\nu)$  where  $Im(\theta_\nu)$  is a diagonalisable subalgebra of  $HH^1(A)$ , thanks to Proposition 2.4. The maximality of  $\mathcal{G}$  forces  $\mathcal{G} = Im(\theta_\nu)$  where  $\sim_{Ker(\nu)}$  is the unique source of  $\Gamma$ .

Conversely, let  $\mu: kQ \twoheadrightarrow A$  be a presentation such that  $\sim_{Ker(\mu)}$  is the unique source of  $\Gamma$ . Hence,  $Im(\theta_\mu)$  is diagonalisable (thanks to Proposition 2.4) so there exists  $\mathcal{G}$  a maximal diagonalisable subalgebra of  $HH^1(A)$  containing  $Im(\theta_\mu)$ . Thanks to the above description, we know that  $\mathcal{G} = Im(\theta_\nu)$  where  $\nu: kQ \twoheadrightarrow A$  is a presentation such that  $\sim_{Ker(\nu)}$  is the unique source of  $\Gamma$ . Moreover, Lemma 4.1 gives a  $k$ -algebra automorphism  $\psi: A \xrightarrow{\sim} A$  such that  $Im(\theta_\mu) = \psi_*(Im(\theta_\nu))$ . Since  $\psi_*$  is a Lie algebra automorphism of  $HH^1(A)$ , the maximality of  $\mathcal{G} = Im(\theta_\nu)$  implies that  $Im(\theta_\mu)$  is maximal.

(ii) is a consequence of (i) and of Lemma 4.1. ■

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