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► **To cite this version:**

Patrick Le Meur. The universal cover of a monomial triangular algebra without multiple arrows. *Journal of Algebra and Its Applications*, 2008, 7 (4), pp.443–469. 10.1142/S0219498808002850 . hal-00123559v2

HAL Id: hal-00123559

<https://hal.science/hal-00123559v2>

Submitted on 6 Mar 2008

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The universal cover of a monomial triangular algebra without multiple arrows

Patrick Le Meur ^{*†}

6th March 2008

Abstract

Let A be a basic connected finite dimensional algebra over an algebraically closed field k . Assuming that A is monomial and that the ordinary quiver Q of A has no oriented cycle and no multiple arrows, we prove that A admits a universal cover with group the fundamental group of the underlying space of Q .

Introduction

Let A be a finite dimensional k -algebra where k is an algebraically closed field. In order to study the category $\text{mod}(A)$ of (left) A -modules, one may assume that A is basic and connected. In [10] (see also [2]), C. Riedtmann has introduced the covering techniques which reduce the study of part of $\text{mod}(A)$ to the easier one of $\text{mod}(\mathcal{C})$, where $\mathcal{C} \rightarrow A$ is a Galois covering and \mathcal{C} is locally bounded. These techniques are based on the coverings of translation quivers and their fundamental group, and therefore, have been particularly efficient for representation-finite and standard algebras A : In this case, P. Gabriel ([5]) has constructed a universal Galois covering of A , whose properties have led to a precise description of the standard form of a representation-finite algebra ([3]). Unfortunately, the above construction of [5] cannot be proceeded in the representation-infinite case precisely because the Auslander-Reiten quiver is no longer connected.

In [9], R. Martinez-Villa and J. A. de la Peña have constructed a Galois covering $k\tilde{Q}/\tilde{I} \rightarrow A$ associated with each presentation $kQ/I \simeq A$ (by quiver and admissible relations), for any algebra A . This Galois covering is induced by the universal cover $(\tilde{Q}, \tilde{I}) \rightarrow (Q, I)$ with fundamental group $\pi_1(Q, I)$ of the bound quiver (Q, I) , as a generalisation of the universal cover of a translation quiver defined in [2] and [10]. Like in topology, the group $\pi_1(Q, I)$ is defined by means of a homotopy relation \sim_I on the set of unoriented paths of Q . When A is representation-finite and standard, this Galois covering coincides with the one constructed by P. Gabriel. Therefore, it is a natural candidate for a universal cover of A in the general case. Alas, different presentations may have non-isomorphic fundamental groups. So there may exist many candidates for a universal cover. As an example, let $A = kQ / \langle da \rangle$, where Q is the quiver:



Then, $\pi_1(Q, \langle da \rangle) \simeq \mathbb{Z}$. On the other hand, $A \simeq kQ / \langle da - dcb \rangle$, and $\pi_1(Q, \langle da - dcb \rangle) = 1$. Notice that A is tilted of euclidean type and therefore belongs to a quite well-understood class of algebras. This illustrates the fact that except for representation-finite algebras there are quite a few classes of algebras for which the existence of a universal cover is known.

In this text, we prove the existence of a universal cover for certain monomial algebras, that is, quotients of paths algebras of quivers by a monomial ideal (*i.e.* generated by a set of paths). More precisely, we prove the following main result.

Theorem 1. *Let $A = kQ/I_0$, where Q is a quiver without oriented cycle and without multiple arrows, and I_0 is a monomial admissible ideal of kQ . Let $\hat{\mathcal{C}} \rightarrow kQ/I_0$ be the Galois covering with group $\pi_1(Q)$ defined by the presentation $kQ/I_0 \simeq A$ (see [9]). Then $\hat{\mathcal{C}} \rightarrow A$ is a universal cover of A in the following sense. For any Galois covering $\mathcal{C} \rightarrow A$*

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with group G and with \mathcal{C} connected and locally bounded, there exists a commutative "factorisation diagram":

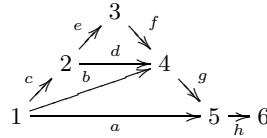
$$\begin{array}{ccc} \widehat{\mathcal{C}} & & \mathcal{C} \\ \downarrow & \searrow & \downarrow \\ A & \xrightarrow{\sim} & A \end{array}$$

where the bottom arrow is an isomorphism of k -algebras, extending the identity map on the set Q_0 of vertices, and where $\widehat{\mathcal{C}} \rightarrow \mathcal{C}$ is a Galois covering with group $N \triangleleft \pi_1(Q)$ such that there exists an exact sequence of groups: $1 \rightarrow N \rightarrow \pi_1(Q) \rightarrow G \rightarrow 1$.

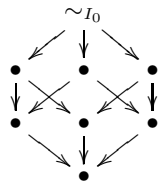
The author gratefully acknowledges an anonymous referee for pointing out the following example from [6, 3.2]. It shows that the assumption on multiple arrows cannot be removed: Let $A = k \left(\begin{array}{c} \cdot \xrightarrow{x} \cdot \\ \cdot \xrightarrow{x'} \cdot \end{array} \cdot \begin{array}{c} \cdot \xrightarrow{y} \cdot \\ \cdot \xrightarrow{y'} \cdot \end{array} \right) / I_0$, where $I_0 = \langle yx', y'x \rangle$. Then, $\pi_1(Q, I_0) = \pi_1(Q) \simeq \mathbb{Z} * \mathbb{Z}$, and if $\text{char}(k) \neq 2$, then $A \simeq kQ/I$ where $I = \langle yx - y'x', yx' - y'x \rangle$, and $\pi_1(Q, I) \simeq \mathbb{Z}/2\mathbb{Z}$. Now, let $\mathcal{C} \rightarrow A$ be the Galois covering with group $\pi_1(Q, I)$ defined by the presentation $A \simeq kQ/I$. Then, with the notations of Theorem 1, it is easy to show that there is no k -linear functor $\widehat{\mathcal{C}} \rightarrow \mathcal{C}$. Thus, in this example, A admits no universal cover in the sense of Theorem 1. From the very definition of monomial algebras, one would expect that they all have a universal cover. The above counter-example shows that this is not always the case. It is all the more surprising as the involved algebra is gentle, so its representation theory is fairly well-known.

We now explain the strategy of the proof of Theorem 1. Our main tool is the quiver Γ of the homotopy relations \sim_I of the presentations $kQ/I \simeq A$. It was introduced in [8] to prove the existence of a universal cover for algebras over a zero characteristic field, and whose ordinary quiver have no double bypass. In general, if $kQ/I \simeq A$ and $kQ/J \simeq A$, there is no simple relation between $\pi_1(Q, I)$ and $\pi_1(Q, J)$ (and therefore between the associated Galois coverings of A) unless A is representation-finite (in which case A is schurian, so that $\pi_1(Q, I) = \pi_1(Q, J)$). This is the main difficulty in proving the existence of a universal cover. Hopefully, such a relation exists when I and J are related by a transvection $\varphi_{\alpha, u, \tau}$, that is, $J = \varphi_{\alpha, u, \tau}(I)$, where (α, u) is a bypass (meaning that α is an arrow and u is a path parallel to and different from α), $\tau \in k$ and $\varphi_{\alpha, u, \tau} \in \text{Aut}(kQ)$ is the automorphism which maps α to $\alpha + \tau u$ and which fixes any other arrow. In such a case, there is a natural quotient relation between $\pi_1(Q, I)$ and $\pi_1(Q, J)$. Besides, the Galois coverings of A with groups $\pi_1(Q, J)$ and $\pi_1(Q, I)$ are the vertical arrows of a factorisation diagram like in Theorem 1, and the associated exact sequence of groups is given by the above quotient relation. The quiver Γ is then defined as follows. Its vertices are the homotopy relations \sim_I of all the presentations $kQ/I \simeq A$, and there is an arrow $\sim \rightarrow \sim'$ if there exist presentations $kQ/I \simeq A$ and $kQ/J \simeq A$, and a transvection φ such that: $\sim = \sim_I$, $\sim' = \sim_J$, $J = \varphi(I)$, and $\pi_1(Q, J)$ is a strict quotient of $\pi_1(Q, I)$. The quiver Γ is then finite, connected, and it has no oriented cycle. Notice that Γ is reduced to a point (with no arrow) when A is schurian (and in particular when A is representation-finite). But, usually, Γ has many vertices and many arrows. We refer the reader to Section 1 for more details.

Roughly speaking, the existence of a universal cover is related to the existence of a unique source in Γ . More precisely, assume that there exists a presentation $kQ/I_0 \simeq A$ (which in our case will be the monomial presentation) such that for any other presentation $kQ/I \simeq A$, there exists a sequence of ideals $I_0, I_1 = \varphi_1(I_0), \dots, I_n = \varphi_n(I_{n-1}) = I$, where $\varphi_1, \dots, \varphi_n$ are transvections defining a path $\sim_{I_0} \rightarrow \sim_{I_1} \rightarrow \dots \rightarrow \sim_{I_n} = \sim_I$ in Γ . Then, the Galois covering of A with group $\pi_1(Q, I_0)$ associated to the presentation $kQ/I_0 \simeq A$ is a universal cover of A . As an example, assume that $A = kQ/I_0$, where Q is the quiver



and $I_0 = \langle ha, gb, dc \rangle$. Then Γ has the following shape:



We do not specify all the vertices, but one can check that for every $\sim_I \in \Gamma$, the group $\pi_1(Q, I)$ is free over $3 - l$ generators, where l is the length of any path from \sim_{I_0} to \sim_I (so two presentations may have distinct homotopy

relations yet isomorphic fundamental groups). For the needs of the proof, we construct a specific total order on the set of bypasses of Q . In our example, this order is: $(d, fe) < (b, fec) < (b, dc) < (a, gfec) < (a, gdc) < (a, gb)$. Now, let $I = \langle ha + hgfec, gb + gfec, dc \rangle$, then:

1. $I = \varphi_{a,gfec,1}\varphi_{b,fec,1}(I_0)$. Moreover, $\varphi_{a,gfec,1}\varphi_{b,fec,1}$ is the unique automorphism of kQ which transforms I_0 into I , and which maps every arrow α to the sum of α and a linear combination of paths of length greater than 1, none of which lying in I_0 (indeed: $gfec, fec \notin I_0$). For that reason, we set $\psi_I := \varphi_{a,gfec,1}\varphi_{b,fec,1}$.
2. The equality $\psi_I = \varphi_{a,gfec,1}\varphi_{b,fec,1}$ expresses ψ_I as a product of transvections $\varphi_{a,gfec,1}, \varphi_{b,fec,1}$. The associated sequence of bypasses is decreasing ($(a, gfec) > (b, fec)$). Actually, the sequence $\varphi_{a,gfec,1}, \varphi_{b,fec,1}$ is unique for this property.
3. We have a path $I_0 \rightarrow \varphi_{b,fec,1}(I_0) \rightarrow \varphi_{a,gfec,1}\varphi_{b,fec,1}(I_0) = I$ in Γ .

In this example, it is easy to check that for any other presentation $A \simeq kQ/J$, the ideal J defines a unique automorphism ψ_J (as in 1.) which decomposes uniquely (as in 2.), giving rise to a path from \sim_{I_0} to \sim_J (as in 3.).

The proof of Theorem 1 mimicks the three steps we proceeded in that example. Indeed, we prove the three following technical points:

1. If $kQ/I \simeq kQ/I_0$, then there exists a unique product ψ_I of transvections such that $\psi_I(I_0) = I$, and such that ψ_I maps every arrow α to the sum of α and a linear combination of paths of length greater than 1, none of which lying in I_0 .
2. There exists a suitable ordering on the set of bypasses such that if $\psi \in \text{Aut}(kQ)$ is a product of transvections, then ψ can be written uniquely as $\psi = \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$ with $\tau_1, \dots, \tau_n \in k^*$ and $(\alpha_n, u_n) > \dots > (\alpha_1, u_1)$.
3. If $kQ/I \simeq kQ/I_0$, the unique ordered sequence of transvections given by 1. and 2. yield a path in Γ starting at \sim_{I_0} and ending at \sim_I . Also, this sequence gives rise to the factorisation diagram of Theorem 1.

The text is therefore organised as follows. In Section 1 we recall all the notions that we need to prove Theorem 1. In Section 2, we prove some combinatorial facts on the paths in a quiver. These lead to the order and to the decomposition of the second point above. In Section 3 we prove the first point above. Finally, in Section 4 we prove the last point and Theorem 1.

1 Basic definitions

A k -**category** is a category \mathcal{C} whose objects class \mathcal{C}_0 is a set, whose space of morphisms from x to y (denoted by ${}_y\mathcal{C}_x$) is a k -vector space for any $x, y \in \mathcal{C}_0$ and whose composition of morphisms is k -bilinear. All functors between k -categories will be assumed to be k -linear functors. In particular, $\text{Aut}(\mathcal{C})$ will denote the group of k -linear automorphism of \mathcal{C} , and $\text{Aut}_0(\mathcal{C})$ will denote by for the subgroup $\{\psi \in \text{Aut}(\mathcal{C}) \mid \psi(x) = x \text{ for any } x \in \mathcal{C}_0\}$ of $\text{Aut}(\mathcal{C})$. The k -category \mathcal{C} is called **connected** if it cannot be written as the disjoint union of two full subcategories. An ideal I of \mathcal{C} is the data of subspaces ${}_yI_x \subseteq {}_y\mathcal{C}_x$ (for any $x, y \in \mathcal{C}_0$) such that $fgh \in I$ whenever f, g, h are composable morphisms in \mathcal{C} such that $g \in I$. The k -category \mathcal{C} is called **locally bounded** provided that: 1) for any $x \in \mathcal{C}_0$, the vector spaces $\bigoplus_{y \in \mathcal{C}_0} {}_y\mathcal{C}_x$ and

$\bigoplus_{y \in \mathcal{C}_0} {}_x\mathcal{C}_y$ are finite dimensional, 2) ${}_x\mathcal{C}_x$ is a local algebra for any $x \in \mathcal{C}_0$, 3) distinct objects are not isomorphic. Let A be a finite dimensional k -algebra and let $\{e_1, \dots, e_n\}$ be a complete set of primitive orthogonal idempotents. Then A is also a k -category: $A_0 := \{e_1, \dots, e_n\}$, ${}_iAe_j := e_jAe_i$ and the product of A induces the composition of morphisms. Notice that different choices for the idempotents e_1, \dots, e_n give rise to isomorphic k -categories. Also, A is connected (resp. basic) as a k -algebra if and only if it is connected (resp. locally bounded) as a k -category. In the sequel we shall make no distinction between a finite dimensional k -algebra and its associated k -category. If \mathcal{C} is a locally bounded k -category, the radical of \mathcal{C} is the ideal \mathcal{RC} of \mathcal{C} such that: ${}_y\mathcal{RC}_x$ is the space of non-isomorphisms $x \rightarrow y$ in \mathcal{C} , for any $x, y \in \mathcal{C}_0$. The ideal of \mathcal{C} generated by compositions gf where f and g lie in \mathcal{RC} will be denoted by $\mathcal{R}^2\mathcal{C}$.

A **Galois covering with group** G of \mathcal{C} (by \mathcal{C}') is a functor $F: \mathcal{C}' \rightarrow \mathcal{C}$ endowed with a group morphism $G \rightarrow \text{Aut}(\mathcal{C}')$ and such that: 1) the induced action of G on \mathcal{C}'_0 is free, 2) $F \circ g = F$ for any $g \in G$, 3) for any k -linear functor $F': \mathcal{C}' \rightarrow \mathcal{C}''$ such that $F' \circ g = F'$ for any $g \in G$, there exists a unique $\overline{F'}: \mathcal{C} \rightarrow \mathcal{C}''$ such that $\overline{F'} \circ F = F'$ (in other words, F is a quotient of \mathcal{C}' by G in the category of k -categories). For short, the Galois covering F is called connected if \mathcal{C}' is connected and locally bounded (this implies that \mathcal{C} is connected and locally bounded). For more details on Galois coverings (in particular for the connections with representations theory), we refer the reader to [2].

Quivers, paths, bypasses. A quiver is a 4-tuple $Q = (Q_1, Q_0, s, t)$ where Q_1 and Q_0 are sets and $s, t: Q_1 \rightarrow Q_0$ are maps. The elements of Q_1 (resp. of Q_0) are called the arrows (resp. the vertices) of Q . If $\alpha \in Q_1$, the vertex $s(\alpha)$ (resp. $t(\alpha)$) is called the source (resp. the target) of α . The quiver Q is called **locally finite** if and only if any vertex is the source (resp. the target) of finitely many arrows. For example, if \mathcal{C} is a locally bounded k -category, the **ordinary**

quiver of \mathcal{C} is the locally finite quiver Q such that: $Q_0 := \mathcal{C}_0$ and for any $x, y \in \mathcal{C}_0$, the number of arrows starting at x and arriving at y is equal to $\dim_k {}_y\mathcal{R}\mathcal{C}_x / {}_y\mathcal{R}^2\mathcal{C}_y$. A path in Q of length n ($n \geq 0$) with source $x \in Q_0$ (or starting at x) and target $y \in Q_0$ (or arriving at y) is a sequence of arrows $\alpha_1, \dots, \alpha_n$ such that: $x = y$ if $n = 0$, $s(\alpha_1) = x$, $s(\alpha_{i+1}) = t(\alpha_i)$ for any $i \in \{1, \dots, n-1\}$ and $t(\alpha_n) = y$. If $n \geq 1$ this path will be written $\alpha_n \dots \alpha_1$ and called non trivial. If $n = 0$ this path will be written e_x and called stationary at x . The length of this path is $|u| := n$. The mappings s, t are naturally extended to paths in Q . If u and v are paths, the concatenation vu is defined if and only if $t(u) = s(v)$ by the following rule: 1) $vu = v$ is u is stationary, 2) $vu = u$ is v is stationary, 3) $vu = \beta_m \dots \beta_1 \alpha_n \dots \alpha_1$ if $v = \beta_m \dots \beta_1$ and $u = \alpha_n \dots \alpha_1$ (with $\alpha_i, \beta_j \in Q_1$). Two paths in Q are called **parallel** whenever they have the same source and the same target. An **oriented cycle** in Q is a non trivial path whose source and target are equal. The quiver Q is said to have **multiple arrows** if and only if there exist in Q distinct parallel arrows. If Q has no oriented cycle and if (α, u, β, v) is a double bypass (see the introduction) there exists two unique paths u_1, u_2 such that $u = u_2 \beta u_1$. In such a situation, the path $u_2 v u_1$ will be called obtained from $u = u_2 \beta u_1$ after replacing β by v .

Admissible presentations (see [2, 2.1]). A quiver Q defines the **path category** kQ such that $(kQ)_0 = Q_0$, such that ${}_y kQ_x$ is the k -vector space with basis the family of paths starting at x and arriving at y , and the composition in kQ is induced by the concatenation of paths. For short, a **normal form** for $r \in {}_y kQ_x$ is an equality $r = \sum_{i=1}^n t_i u_i$ where $t_1, \dots, t_n \in k^*$ and u_1, \dots, u_n are pairwise distinct paths in Q . With this notation, **the support** of r is the set $\text{supp}(r) := \{u_1, \dots, u_n\}$ ($\text{supp}(0) = \emptyset$). A **subexpression** of r is a linear combination $\sum_{i \in E} t_i u_i$ with $E \subseteq \{1, \dots, n\}$.

Later, we will need the following fact: if $r = r_1 + \dots + r_n \in {}_y kQ_x$ is such that $\text{supp}(r_1), \dots, \text{supp}(r_n)$ are pairwise disjoint, then $r_{i_1} + \dots + r_{i_t}$ is a subexpression of r , for any indices $1 \leq i_1 < \dots < i_t \leq n$. An ideal I of kQ is called **admissible** provided that: 1) any morphism in I is a linear combination of paths of length at least 2, 2) the factor category kQ/I is locally bounded. A morphism in I is called a **relation** (of I). In particular, a **minimal relation** of I (see [9]) is a non zero relation r of I such that 0 and r are the only subexpressions of r which are relations. With this definition, any relation of I is the sum of minimal relations with pairwise disjoint supports. A **monomial relation** is a path u lying in I and I is called monomial if it is generated by a set of monomial relations. A pair (Q, I) where Q is a locally finite and I is an admissible ideal of kQ is called a **bound quiver**. In such a case, kQ/I is locally bounded and it is connected if and only if Q is connected (i.e. the underlying graph of Q is connected). Conversely, if \mathcal{C} is a locally bounded k -category, then there exists an isomorphism $kQ/I \xrightarrow{\sim} \mathcal{C}$ where (Q, I) is a bound quiver such that Q is the ordinary quiver of \mathcal{C} . Such an isomorphism is called **admissible presentation of \mathcal{C}** . If the ideal I is monomial, the admissible presentation and \mathcal{C} are called monomial. Notice that \mathcal{C} may have different admissible presentations.

Fundamental group of a presentation (see [9]). Let (Q, I) be a bound quiver and let $x_0 \in Q_0$. For every arrow $x \xrightarrow{\alpha} y \in Q_1$ we define its formal inverse α^{-1} with source $s(\alpha^{-1}) = y$ and target $t(\alpha^{-1}) = x$. This defines a new quiver $\overline{Q} = (Q_0, Q_1 \cup \{\alpha^{-1} \mid \alpha \in Q_1\}, s, t)$. With these notations, a **walk in Q** is a path in \overline{Q} . The concatenation of walks in Q is by definition the concatenation of paths in \overline{Q} . The **homotopy relation** of (Q, I) is the equivalence relation on the set of walks in Q , denoted by \sim_I and generated by the following properties:

1. $\alpha \alpha^{-1} \sim_I e_y$ and $\alpha^{-1} \alpha \sim_I e_x$ for any arrow $x \xrightarrow{\alpha} y$ in Q ,
2. $u \sim_I v$ for any $u, v \in \text{supp}(r)$ where r is a minimal relation of I ,
3. $wvu \sim_I wv'u$ for any walks w, v, v', u such that $v \sim_I v'$ and such that the concatenations wvu and $wv'u$ are well-defined (i.e. \sim_I is compatible with the concatenation).

The \sim_I -equivalence class of a walk γ will be denoted by $[\gamma]_I$. Let $\pi_1(Q, I, x_0)$ be the set of equivalence classes of walks in Q with source and target equal to x_0 . The concatenation of walks endows this set with a group structure (with unit e_{x_0}) and this group is called the **fundamental group of (Q, I) at x_0** . If Q is connected, the isomorphism class of this group does not depend on $x_0 \in Q_0$ and $\pi_1(Q, I, x_0)$ is denoted by $\pi_1(Q, I)$. If \mathcal{C} is a connected locally bounded k -category and if $kQ/I \simeq \mathcal{C}$ is an admissible presentation, the fundamental group $\pi_1(Q, I)$ is called the fundamental group of this presentation.

Dilatations, transvections (see [8]). Let Q be a quiver. A **dilatation** of kQ is an automorphism $D \in \text{Aut}_0(kQ)$ such that $D(\alpha) \in k^* \alpha$ for any $\alpha \in Q_1$. The dilatations of kQ form a subgroup \mathcal{D} of $\text{Aut}_0(kQ)$. Let (α, u) be a bypass in Q and let $\tau \in k$. This defines $\varphi_{\alpha, u, \tau} \in \text{Aut}_0(kQ)$ as follows: $\varphi_{\alpha, u, \tau}(\alpha) = \alpha + \tau u$ and $\varphi_{\alpha, u, \tau}(\beta) = \beta$ for any arrow $\beta \neq \alpha$. The automorphism $\varphi_{\alpha, u, \tau}$ is called a **transvection**. The composition of transvections is ruled as follows. Let $\varphi_{\alpha, u, \tau}$ and $\varphi_{\alpha, u, \tau'}$, then $\varphi_{\alpha, u, \tau} \varphi_{\alpha, u, \tau'} = \varphi_{\alpha, u, \tau + \tau'}$ and $\varphi_{\alpha, u, \tau}^{-1} = \varphi_{\alpha, u, -\tau}$. If (α, u, β, v) and (β, v, α, u) are not a double bypasses, then $\varphi_{\alpha, u, \tau} \varphi_{\beta, v, \nu} = \varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}$. If (α, u, β, v) is a double bypass and if Q has no oriented cycle, then $\varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau} = \varphi_{\alpha, u, \tau} \varphi_{\alpha, w, \tau \nu} \varphi_{\beta, v, \nu}$, where w is the path obtained from u after replacing β by v . The subgroup of $\text{Aut}_0(kQ)$ generated by all the transvections is denoted by \mathcal{T} . The dilatations and the transvections are useful to compare admissible presentations of an algebra because of the following proposition:

Proposition 1.1. (see [8, Prop. 2.1, Prop. 2.2]) Let $kQ/I \simeq A$ and $kQ/J \simeq A$ be admissible presentations of the basic finite dimensional algebra A . If Q has no oriented cycle, then there exists $\psi \in \text{Aut}_0(kQ)$ such that $\psi(I) = J$. Moreover, \mathcal{T} is a normal subgroup of $\text{Aut}_0(kQ)$ and $\text{Aut}_0(kQ) = \mathcal{T}\mathcal{D} = \mathcal{D}\mathcal{T}$.

The dilatations and the transvections were introduced because they allow comparisons between the fundamental groups of presentations of the same locally bounded k -category. Notice that if I, J are admissible ideals of kQ such that $\gamma \sim_I \gamma' \Rightarrow \gamma \sim_J \gamma'$ for any walks γ, γ' , then the identity map on the set of walks induces a surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$. In particular, if \sim_I and \sim_J coincide, then $\pi_1(Q, I) = \pi_1(Q, J)$.

Proposition 1.2. (see [8, Prop. 2.5]) Let I be an admissible ideal of kQ (with Q without oriented cycle), let $\varphi \in \text{Aut}_0(kQ)$ and set $J = \varphi(I)$. If φ is a dilatation, then \sim_I and \sim_J coincide. If $\varphi = \varphi_{\alpha, u, \tau}$ is a transvection, then:

1. if $\alpha \sim_I u$ and $\alpha \sim_J u$ then \sim_I and \sim_J coincide.
2. if $\alpha \not\sim_I u$ and $\alpha \sim_J u$ then \sim_J is generated by \sim_I and $\alpha \sim_J u$.
3. if $\alpha \not\sim_I u$ and $\alpha \not\sim_J u$ then $I = J$ and \sim_I and \sim_J coincide.

If there exists a transvection φ such that $\varphi(I) = J$ and such the second point above occurs, then \sim_J is called a **direct successor** of \sim_I .

Here the expression “ \sim_J is generated by \sim_I and $\alpha \sim_J u$ ” means that \sim_J is the equivalence relation on the set of walks in Q , compatible with the concatenation and generated by the two following properties: 1) $\gamma \sim_I \gamma' \Rightarrow \gamma \sim_J \gamma'$, 2) $\alpha \sim_J u$. Following [8, Def. 2.7], if A is a basic connected finite dimensional algebra with ordinary quiver Q without oriented cycle, we define **the quiver Γ of the homotopy relations of A** to be the quiver such that $\Gamma_0 = \{\sim_I \mid kQ/I \simeq A\}$ and such that there exists arrow $\sim_I \rightarrow \sim_J$ if and only if \sim_J is a direct successor of \sim_I . Recall ([8, Rem. 5, Prop. 2.8]) that Γ is finite, connected, without oriented cycle and such that for any (oriented) path with source \sim_I and target \sim_J , the identity map on the set of walks in Q induces a surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$.

Gröbner bases Let E be a k -vector space with an ordered basis (e_1, \dots, e_n) , let (e_1^*, \dots, e_n^*) be the associated dual basis of E^* , and let F be a subspace of E . A Gröbner basis (see [1] for the usual definition) of F is a basis (r_1, \dots, r_d) such that:

1. $r_j \in e_{i_j} + \text{Span}(e_l \mid l < i_j)$ for some i_j , for any $j \in \{1, \dots, r\}$,
2. $i_1 < i_2 < \dots < i_r$,
3. $e_{i_j}^*(r_{j'}) = 0$ for any $j \neq j'$.

It is well known that F admits a unique Gröbner basis. Also, $r \in F$ if and only if: $r = \sum_{j=1}^d e_{i_j}^*(r) r_j$. In the sequel, we will use this notion in the following setting: E is the vector space with basis (for some order to be defined) the family of non trivial paths in a finite quiver Q without oriented cycles and F is the underlying subspace of E associated to an admissible ideal I of kQ . Notice that in this setting, the Gröbner basis of F is made of minimal relations of I . Also, if $r \in E$ and if u is a non trivial path, then: $u \in \text{supp}(r) \Leftrightarrow u^*(r) \neq 0$.

Until the end of the text, Q will denote a finite quiver without oriented cycle and without multiple arrows.

2 Combinatorics on the paths in a quiver

Recall from the previous section that if (α, u, β, v) is a double bypass and if τ, ν are scalars, then $\psi := \varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}$ is equal to $\varphi_{\alpha, u, \tau} \varphi_{\alpha, w, \tau \nu} \varphi_{\beta, v, \nu}$ where w is the path obtained from u by replacing β by v . Remark that $\varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}(\alpha) = \alpha + \tau u + \tau \nu w$. Hence, the paths $(u$ and $w)$ appearing in $\varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}(\alpha) - \alpha$ are exactly those paths θ such that (α, θ) is a bypass appearing in one of the transvections of the product $\varphi_{\alpha, u, \tau} \varphi_{\alpha, w, \tau \nu} \varphi_{\beta, v, \nu}$. Moreover, the scalars $(\tau$ and $\tau \nu)$ appearing with these paths are exactly the scalars of the corresponding transvections in this product. So, the computation of $\varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}(\alpha)$ can be done just by looking for the occurrences of α in the product $\varphi_{\alpha, u, \tau} \varphi_{\alpha, w, \tau \nu} \varphi_{\beta, v, \nu}$. From this point of view, the decomposition $\psi = \varphi_{\alpha, u, \tau} \varphi_{\alpha, w, \tau \nu} \varphi_{\beta, v, \nu}$ is more useful than the decomposition $\psi = \varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}$. The aim of this section is to show that this phenomenon is a general one. In this purpose the useful notion of derivation of a path and a total order on the set of bypasses will be introduced.

2.1 Derivation of paths

Definition 2.1. Let $u = \alpha_n \dots \alpha_1$ and v be paths in Q . Then v is called *derived* of u (of order t) if there exist indices $1 \leq i_1 < \dots < i_t \leq n$ and bypasses $(\alpha_{i_1}, v_1), \dots, (\alpha_{i_t}, v_t)$ such that v is obtained from u by replacing α_{i_l} by v_l , for each l :

$$v = \alpha_n \dots \alpha_{i_t+1} v_t \alpha_{i_t-1} \dots \alpha_{i_1+1} v_1 \alpha_{i_1-1} \dots \alpha_1$$

Remark 2.2. If $\alpha \in Q_1$, then u is derived of α if and only if (α, u) is a bypass.

With the above definition, the following lemma is easily verified using the fact that Q has no multiple arrows and no oriented cycle.

Lemma 2.3. 1. If v is derived of u with both orders t and t' , then $t=t'$.

2. If v is derived of u of order t then there exists a sequence of paths $u_0 = u, u_1, \dots, u_t = v$ such that u_i is derived of u_{i-1} of order 1 for any i .
3. If v is derived of u of order t , then $|v| \geq |u| + t$.
4. If v is derived of u of order t and if w is derived of v of order t' , then w is derived of u of order at least t .
5. Let u, v, w be paths verifying:
 - v is derived of u ,
 - w is derived of v ,
 - w is derived of u of order 1,

then we have:

$$u = u_2\alpha u_1, \quad v = u_2\theta u_1, \quad w = u_2\theta' u_1$$

where u_1, u_2 are paths, (α, θ) is a bypass and θ' is derived of θ .

6. If v (resp. v') is derived of u (resp. of u') of order t (resp. t'), then $v'v$ is derived of $u'u$ of order $t' + t$, whenever these compositions of paths are well defined.

The following example shows that, in the preceding lemma, the inequality in the 4-th point may be an equality.

Example 2.4. Let (α, u, β, v) be a double bypass. Let u_1, u_2 be the paths such that $u = u_2\beta u_1$. Then u is derived of α of order 1, $w := u_2v u_1$ is derived of u of order 1 and w is derived of u of order 1.

2.2 Order between paths, order between bypasses

Now, we construct a total order on the set of non trivial paths in Q . This construction is a particular case of the one introduced in [4]. Also it depends on an arbitrary order \triangleleft on Q_1 . We assume that this order \triangleleft is fixed for this subsection. We shall write \triangleleft for the lexicographical order induced by \triangleleft on the set of nontrivial paths in Q . For details on the correctness of the following definition we refer the reader to [4].

Definition 2.5. For $\alpha \in Q_1$, set:

$$W(\alpha) = \text{Card}(B(\alpha)) \quad \text{where} \quad B(\alpha) = \{(\alpha, u) \mid (\alpha, u) \text{ is a bypass in } Q\}$$

For $u = \alpha_n \dots \alpha_1$ a path in Q (with $\alpha_i \in Q_1$), let us set:

$$W(u) = W(\alpha_n) + \dots + W(\alpha_1)$$

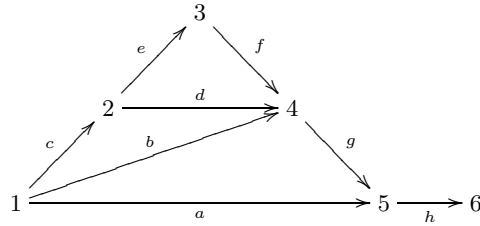
These data define a total order $<$ on the set of non trivial paths in Q as follows:

$$u < v \Leftrightarrow \begin{cases} W(u) < W(v) \\ \text{or} \\ W(u) = W(v) \text{ and } u \triangleleft v \end{cases}$$

We shall write $<$ for the lexicographical order induced by $<$ on the set of couples of paths.

Remark 2.6. If u and v are (non trivial) paths such that vu is well defined, then $W(vu) = W(u) + W(v)$.

Example 2.7. Let Q be the following quiver without oriented cycle and without multiple arrows:



and let \triangleleft be any total order on Q_1 . Then, $B(a) = \{(a, bg), (a, gcd), (a, gfe)\}$, $B(b) = \{(b, cd), (b, fec)\}$, $B(d) = \{(d, fe)\}$ and $B(x) = \emptyset$ for $x \in Q_1 \setminus \{a, b, d\}$. In particular, the paths with source 1 and target 5 are ordered as follows:

$$gfec < gcd < gb < a$$

Lemma 2.8. 1. If u, v, u', v' are paths such that $v < u$ and $v' < u'$ then $v'v < u'u$ whenever these compositions are well defined.

2. If (α, u) is a bypass, then $W(u) < W(\alpha)$. So $u < \alpha$.

3. If v is derived of u , then $v < u$.

4. If (α, u, β, v) is a double bypass and if w is the path obtained from u after replacing β by v , then:

$$(\beta, v) < (\alpha, w) < (\alpha, u)$$

Proof: 1) is a direct consequence of Definition 2.5 and Remark 2.6.

2) Let us write $u = a_n \dots a_1$ with $a_i \in Q_1$ for each i (hence $a_i \neq a_j$ if $i \neq j$ because Q has no oriented cycle). Therefore:

- . $B(a_1), \dots, B(a_n)$ are pairwise disjoint,
- . $W(u) = W(a_1) + \dots + W(a_n)$

Notice that if $(a_i, v) \in B(a_i)$, then $(\alpha, a_n \dots a_{i+1}va_{i-1} \dots a_1) \in B(\alpha)$. Thus, we have a well defined mapping:

$$\begin{aligned} \theta: \quad B(a_1) \sqcup \dots \sqcup B(a_n) &\longrightarrow B(\alpha) \\ (a_i, v) &\longmapsto (\alpha, a_n \dots a_{i+1}va_{i-1} \dots a_1) \end{aligned}$$

This mapping is one-to-one, indeed:

- . if $\theta(a_i, v) = \theta(a_i, v')$ with $(a_i, v), (a_i, v') \in B(a_i)$ then:

$$a_n \dots a_{i+1}va_{i-1} \dots a_1 = a_n \dots a_{i+1}v'a_{i-1} \dots a_1$$

and therefore $(a_i, v) = (a_i, v')$,

- . if $\theta(a_i, v) = \theta(a_j, v')$ with $(a_i, v) \in B(a_i)$, $(a_j, v') \in B(a_j)$ and $j < i$, then:

$$a_n \dots a_{i+1}va_{i-1} \dots a_1 = a_n \dots a_{j+1}v'a_{j-1} \dots a_1$$

So:

$$va_{i-1} \dots a_1 = a_i \dots a_{j+1}v'a_{j-1} \dots a_1$$

Since v and a_i are parallel and since Q has no oriented cycle, we infer that $v = a_i$ which is impossible because $(a_i, v) \in B(a_i)$.

On the other hand, θ is not onto. Indeed, if there exists $(a_i, v) \in B(a_i)$ verifying $\theta(a_i, v) = (\alpha, u)$, then:

$$a_n \dots a_1 = u = a_n \dots a_{i+1}va_{i-1} \dots a_1$$

which implies $a_i = v$, a contradiction. Since θ is one-to-one and not onto, we deduce that:

$$W(\alpha) = \text{Card}(B(\alpha)) > \text{Card}(B(a_1) \sqcup \dots \sqcup B(a_n)) = W(u)$$

This proves that $W(u) < W(\alpha)$ and that $u < \alpha$.

3) is a direct consequence of 1) and of 2).

4) Let us write $u = u_2\beta u_1$ (with u_1, u_2 paths) so that $w = u_2vu_1$. From 2), we have:

$$W(\alpha) > W(u) = W(u_1) + W(\beta) + W(u_2) \geq W(\beta)$$

So $\beta < \alpha$ and therefore $(\beta, v) < (\alpha, w)$. Using 2) again, we also have:

$$W(w) = W(u_2) + W(v) + W(u_1) < W(u_2) + W(\beta) + W(u_1) = W(u)$$

So $w < u$ and therefore $(\alpha, w) < (\alpha, u)$ ■

Unless otherwise specified, $<$ will always denote an order on the set of paths as in Definition 2.5.

2.3 Image of a path by a product of transvections

In this paragraph, we apply the previous constructions to find an easy way to compute $\psi(u)$ when $\psi \in \mathcal{T}$ and u is a path in Q . We begin with the following lemma on the description of $\psi(\alpha)$ when $\psi \in \mathcal{T}$ and $\alpha \in Q_1$. Recall that Q has no multiple arrows and no oriented cycle.

Lemma 2.9. *Let $\psi \in \mathcal{T}$ and let $\alpha \in Q_1$. Then $\psi(\alpha) - \alpha$ is a linear combination of paths parallel to α and of length greater than or equal to 2. In particular, $\alpha \in \text{supp}(\psi(\alpha))$ and $\alpha^*(\psi(\alpha)) = 1$.*

Proof: The conclusion is immediate if ψ is a transvection because Q has no multiple arrows. The conclusion in the general case is obtained using an easy induction on the number of transvections whose product equal ψ . \blacksquare

The preceding lemma gives the following description of $\psi(u)$ when $\psi \in \mathcal{T}$ and u is a path. We omit the proof which is immediate thanks to Lemma 2.9 and to point 6) of Lemma 2.3.

Proposition 2.10. *Let $\psi \in \mathcal{T}$ and let $u = \alpha_n \dots \alpha_1$ be a path in Q (with $\alpha_i \in Q_1$ for any i). For each i , let:*

$$\psi(\alpha_i) = \alpha_i + \sum_{j=1}^{m_i} \lambda_{i,j} u_{i_j}$$

be a normal form for $\psi(\alpha_i)$. Then $\text{supp}(\psi(u))$ is the set of the paths in Q described as follows. Let $r \in \{0, \dots, n\}$. Let $1 \leq i_1 < \dots < i_r \leq n$ be indices. For each $l \in \{1, \dots, r\}$, let $j_l \in \{1, \dots, m_{i_l}\}$. Then the following path obtained from u after replacing α_{i_l} by u_{j_l} for each l belongs to $\text{supp}(\psi(u))$:

$$\alpha_n \dots \alpha_{i_r+1} u_{j_r} \alpha_{i_r-1} \dots \alpha_{i_l+1} u_{j_l} \alpha_{i_l-1} \dots \alpha_{i_1+1} u_{j_1} \alpha_{i_1-1} \dots \alpha_1$$

Moreover, this path appears in $\psi(u)$ with coefficient:

$$\lambda_{i_1, j_1} \dots \lambda_{i_r, j_r}$$

As a consequence, $\psi(u) - u$ is a linear combination of paths derived of u .

Example 2.11. *The previous proposition does not hold if Q has multiple arrows. For example, if Q is the Kronecker*

quiver $1 \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{b} \end{array} 2$ and if $\psi = \varphi_{a,b,1} \varphi_{b,a,-1} \varphi_{a,b,1}$, then $\psi(a) = b$ and $\psi(b) = -a$.

Remark 2.12. *If (α, u) is a bypass and if $v \in \text{supp}(\psi(u) - u)$, then (α, v) is also a bypass and $(\alpha, v) < (\alpha, u)$.*

Now we are able to state the main result of this paragraph. It describes $\psi(\alpha)$ ($\alpha \in Q_1$) using a particular writing of ψ as a product of transvections. Notice that the following proposition formalises the phenomenon observed at the beginning of the section.

Proposition 2.13. *Let $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ be an increasing sequence of bypasses, let $\tau_1, \dots, \tau_n \in k^*$ and set $\psi = \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$. For any $\alpha \in Q_1$, there is a normal form for $\psi(\alpha)$:*

$$\psi(\alpha) = \alpha + \sum_{i \text{ such that } \alpha = \alpha_i} \tau_i u_i$$

Proof: Let us prove that the conclusion of the proposition is true using an induction on $n \geq 1$. By definition of a transvection, the proposition holds of $n = 1$. Assume that $n \geq 2$ and that the conclusion of the proposition holds if we replace $\psi = \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$ by $\varphi_{\alpha_{n-1}, u_{n-1}, \tau_{n-1}} \dots \varphi_{\alpha_1, u_1, \tau_1}$. Therefore, for $\alpha \in Q_1$, we have a normal form:

$$\varphi_{\alpha_{n-1}, u_{n-1}, \tau_{n-1}} \dots \varphi_{\alpha_1, u_1, \tau_1}(\alpha) = \alpha + \sum_{i \leq n-1, \alpha = \alpha_i} \tau_i u_i$$

So:

$$\psi(\alpha) = \varphi_{\alpha_n, u_n, \tau_n}(\alpha) + \sum_{i \leq n-1, \alpha = \alpha_i} \tau_i \varphi_{\alpha_n, u_n, \tau_n}(u_i) \quad (i)$$

Let $i \in \{1, \dots, n-1\}$. Thanks to Lemma 2.8, the inequality $(\alpha_i, u_i) < (\alpha_n, u_n)$ implies that $(\alpha_i, u_i, \alpha_n, u_n)$ is not a double bypass. Thus, α_n does not appear in the path u_i . This proves that:

$$(\forall i \in \{1, \dots, n-1\}) \varphi_{\alpha_n, u_n, \tau_n}(u_i) = u_i \quad (ii)$$

The definition of $\varphi_{\alpha_n, u_n, \tau_n}$, together with (i) and (ii), imply the equality:

$$\psi(\alpha) = \alpha + \sum_{\alpha = \alpha_i} \tau_i u_i \quad (iii)$$

It only remains to prove that the equality (iii) is a normal form. Remark that all the scalars which appear in the right-hand side of (iii) are non zero. Moreover, if $i \in \{1, \dots, n\}$ verifies $\alpha = \alpha_i$, then $\alpha \neq u_i$, because (α, u_i) is a bypass. Finally, if $1 \leq i < j \leq n$ verify $\alpha = \alpha_i = \alpha_j$, then $(\alpha, u_i) = (\alpha_i, u_i) < (\alpha_j, u_j) = (\alpha, u_j)$ so $u_i \neq u_j$. Therefore, (iii) is a normal form for $\psi(\alpha)$. \blacksquare

When $\psi \in \mathcal{T}$ is like in Proposition 2.13, we shall say that ψ is written as a decreasing product of transvections. Later we will prove that any $\psi \in \mathcal{T}$ can be written uniquely as a decreasing product of transvections. The description in Proposition 2.13 will be particularly useful in the sequel. We end this paragraph with two propositions concerning the description of $\psi(r)$ when $\psi \in \mathcal{T}$ and r is a linear combination of paths. The following proposition gives conditions for $\psi^{-1}(r')$ to be a subexpression of r when r' is a subexpression of $\psi(r)$.

Proposition 2.14. *Let $\psi \in \mathcal{T}$, let $r \in {}_y k Q_x$ and let r' be a subexpression of $\psi(r)$. Let \simeq be the equivalence relation on the set of paths in Q generated by:*

$$v \in \text{supp}(\psi(u)) \Rightarrow u \simeq v$$

Assume that for any $u, v \in \text{supp}(\psi(r))$ verifying $u \simeq v$ we have:

$$u \in \text{supp}(r') \Leftrightarrow v \in \text{supp}(r')$$

Then $\psi^{-1}(r')$ is a subexpression of r .

Proof: Let \simeq' be the trace of \simeq on $\text{supp}(r)$ and let us write $\text{supp}(r) = c_1 \sqcup \dots \sqcup c_n$ as a disjoint union of its \simeq' -classes. This partition of $\text{supp}(r)$ defines a decomposition of $r = r_1 + \dots + r_n$ where r_i is the subexpression of r verifying $\text{supp}(r_i) = c_i$. For each i , let us fix a normal form:

$$r_i = \sum_{j=1}^{n_i} t_{i,j} u_{i,j}$$

so that we have the following normal form for r :

$$r = \sum_{i=1}^n \sum_{j=1}^{n_i} t_{i,j} u_{i,j}$$

Let us set $r'_i := \psi(r_i)$. In order to prove that $\psi^{-1}(r')$ is a subexpression of r , we will prove that there exist indices $1 \leq i_1 < \dots < i_t \leq n$ verifying $r' = r'_{i_1} + \dots + r'_{i_t}$ (so that $\psi^{-1}(r') = r'_{i_1} + \dots + r'_{i_t}$). In this purpose, we will successively prove the following facts:

- 1) $u, v \in \text{supp}(r'_i) \Rightarrow u \simeq v$, for any i ,
 - 2) $\text{supp}(r'_1), \dots, \text{supp}(r'_n)$ are pairwise disjoint,
 - 3) for each i , r'_i is a subexpression of $\psi(r)$,
 - 4) if $i \in \{1, \dots, n\}$ verifies $\text{supp}(r') \cap \text{supp}(r'_i) \neq \emptyset$, then $\text{supp}(r'_i) \subseteq \text{supp}(r')$,
- 1) Let $i \in \{1, \dots, n\}$ and let $u, v \in \text{supp}(r'_i)$. So there exist $u', v' \in \text{supp}(r_i)$ such that $u \in \text{supp}(\psi(u'))$ and $v \in \text{supp}(\psi(v'))$. By definition of \simeq and of r_i , we deduce that:

$$u, v \in \text{supp}(r'_i) \Rightarrow u \simeq v \tag{i}$$

2) Let $i, j \in \{1, \dots, n\}$ be such that there exists $v \in \text{supp}(r'_i) \cap \text{supp}(r'_j)$. So there exist $u \in \text{supp}(r_i)$ and $u' \in \text{supp}(r_j)$ such that $v \in \text{supp}(\psi(u))$ and $v \in \text{supp}(\psi(u'))$. This implies that $u \simeq v \simeq u'$. Since $u \in c_i = \text{supp}(r_i)$ and $u' \in c_j = \text{supp}(r_j)$, we deduce that $c_i = c_j$ and therefore $i = j$. So:

$$i \neq j \Rightarrow \text{supp}(r'_i) \cap \text{supp}(r'_j) = \emptyset \tag{ii}$$

3) We have $\psi(r) = r'_1 + \dots + r'_n$ so (ii) implies that:

$$r'_i \text{ is a subexpression of } \psi(r) \text{ for any } i \tag{iii}$$

4) Let $i \in \{1, \dots, n\}$ and assume that there exists $u \in \text{supp}(r'_i) \cap \text{supp}(r')$. If $v \in \text{supp}(r'_i)$ then $u \simeq v$ thanks to (i). So, by assumption on r' , we have $v \in \text{supp}(r')$. This proves that:

$$\text{supp}(r'_i) \cap \text{supp}(r') \neq \emptyset \Rightarrow \text{supp}(r'_i) \subseteq \text{supp}(r') \tag{iv}$$

Now, we can prove that $\psi^{-1}(r')$ is a subexpression of r . Thanks to (iii), the elements r', r'_1, \dots, r'_n are subexpressions of $\psi(r)$. So (iv) and the equality $\psi(r) = r'_1 + \dots + r'_n$ imply that there exist indices $1 \leq i_1 < \dots < i_t \leq n$ such that $r' = r'_{i_1} + \dots + r'_{i_t}$. So $\psi^{-1}(r') = r_{i_1} + \dots + r_{i_t}$. This proves that $\psi^{-1}(r')$ is a subexpression of r . \blacksquare

The last proposition of this subsection gives a sufficient condition on $u \in \text{supp}(r)$ to verify $u \in \text{supp}(\psi(r))$.

Proposition 2.15. *Let $\psi \in \mathcal{T}$, let $r \in {}_y k Q_x$ and let $u \in \text{supp}(r)$. Then, at least one of the two following facts is verified:*

1. $u \in \text{supp}(\psi(r))$,
2. there exists $v \in \text{supp}(r)$ such that $u \neq v$ and such that $u \in \text{supp}(\psi(v))$.

As a consequence, if u is not derived of v for any $v \in \text{supp}(r)$, then:

$$u \in \text{supp}(\psi(r)) \quad \text{and} \quad u^*(\psi(r)) = u^*(r)$$

Proof: Let us fix a normal form $r = \sum_{i=1}^n t_i u_i$ where we may assume that $u = u_1$. Let us assume that $u \notin \text{supp}(\psi(r))$, i.e. $u^*(\psi(r)) = 0$. Recall from Proposition 2.10 that $u^*(\psi(u)) = 1$, so:

$$0 = u^*(\psi(r)) = t_1 + \sum_{i=2}^n t_i u^*(\psi(u_i)) \quad (i)$$

Therefore, there exists $i_0 \in \{2, \dots, n\}$ such that $u^*(\psi(u_{i_0})) \neq 0$. So:

$$u_{i_0} \in \text{supp}(r), \quad u_{i_0} \neq u_1 = u \quad \text{and} \quad u_1^*(\psi(u_{i_0})) \neq 0$$

This proves the first assertion of the proposition. Now let us assume that u is not derived of v for any $v \in \text{supp}(r)$. Let $i \in \{2, \dots, n\}$. Since $u = u_1 \neq u_i$, Proposition 2.10 gives the following implications:

$$u \in \text{supp}(\psi(u_i)) \Rightarrow u \in \text{supp}(\psi(u_i) - u_i) \Rightarrow u \text{ is derived of } u_i$$

By assumption on u , this implies that $u^*(\psi(u_i)) = 0$ for any $i \geq 2$. Using (i), we deduce the announced conclusion: $u^*(\psi(r)) = t_1 = u^*(r) \neq 0$ ■

2.4 Ordering products of transvections

In Proposition 2.13 we have seen that $\psi(\alpha)$ may be computed easily when $\psi \in \mathcal{T}$ and $\alpha \in Q_1$ provided that ψ is written as a decreasing product of transvections. The main result of this subsection proves that any $\psi \in \mathcal{T}$ can be uniquely written that way. Recall that $<$ is an order on the set of non trivial paths in Q defined in Definition 2.5. The following notations will be useful.

Definition 2.16. *Let (α, u) be a bypass. We set $\mathcal{T}_{<(\alpha, u)}$ and $\mathcal{T}_{\leq(\alpha, u)}$ to be the subgroups of \mathcal{T} generated by the following sets of transvections:*

$$\begin{aligned} & \{\varphi_{\beta, v, \tau} \mid (\beta, v) < (\alpha, u) \text{ and } \tau \in k\} \quad \text{for } \mathcal{T}_{<(\alpha, u)} \\ & \{\varphi_{\beta, v, \tau} \mid (\beta, v) \leq (\alpha, u) \text{ and } \tau \in k\} \quad \text{for } \mathcal{T}_{\leq(\alpha, u)} \end{aligned}$$

Also, we define $\mathcal{T}_{(\alpha, u)}$ to be the following subgroup of \mathcal{T} :

$$\mathcal{T}_{(\alpha, u)} = \{\varphi_{\alpha, u, \tau} \mid \tau \in k\}$$

Remark 2.17. $\mathcal{T}_{(\alpha, u)}$ is indeed a subgroup of \mathcal{T} because $\varphi_{\alpha, u, \tau} \varphi_{\alpha, u, \tau'} = \varphi_{\alpha, u, \tau + \tau'}$ for any $\tau, \tau' \in k$. Actually, the following mapping is an isomorphism of abelian groups:

$$\begin{aligned} k & \longrightarrow \mathcal{T}_{(\alpha, u)} \\ \tau & \longmapsto \varphi_{\alpha, u, \tau} \end{aligned}$$

- . $\mathcal{T}_{\leq(\alpha, u)}$ is generated by $\mathcal{T}_{<(\alpha, u)} \cup \mathcal{T}_{(\alpha, u)}$.
- . If $(\alpha, u) < (\beta, v)$, then $\mathcal{T}_{\leq(\alpha, u)} \subseteq \mathcal{T}_{\leq(\beta, v)}$ and $\mathcal{T}_{<(\alpha, u)} \subseteq \mathcal{T}_{<(\beta, v)}$.
- . $\mathcal{T} = \bigcup_{(\alpha, u)} \mathcal{T}_{\leq(\alpha, u)}$ and if (α_m, u_m) is the greatest bypass in Q , then $\mathcal{T} = \mathcal{T}_{\leq(\alpha_m, u_m)}$ (recall that Q has finitely many bypasses because it has no oriented cycle).

The following lemma proves that any $\psi \in \mathcal{T}$ is a decreasing product of transvections.

Lemma 2.18. $\mathcal{T}_{<(\alpha, u)}$ is a normal subgroup of $\mathcal{T}_{\leq(\alpha, u)}$, for any bypass (α, u) .

- . Let $(a_1, v_1) < \dots < (a_N, v_N)$ be the (finite) increasing sequence of all the bypasses in Q . Then:
 - $\mathcal{T}_{<(a_i, v_i)} = \mathcal{T}_{\leq(a_{i-1}, v_{i-1})}$ if $i \geq 1$,
 - $\mathcal{T}_{<(a_1, v_1)} = 1$,
 - $\mathcal{T}_{\leq(a_i, v_i)} = \mathcal{T}_{(a_i, v_i)} \mathcal{T}_{(a_{i-1}, v_{i-1})} \dots \mathcal{T}_{(a_1, v_1)}$.

Proof: Thanks to Remark 2.17, it is sufficient to prove that if $\tau, \nu \in k$ and if $(\beta, v), (\alpha, u)$ are bypasses such that $(\beta, v) < (\alpha, u)$, then:

$$\varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau} \in \varphi_{\alpha, u, \tau} \mathcal{T}_{<(\alpha, u)} \quad (\star)$$

There are two situations whether (α, u, β, v) is a double bypass or not. If (α, u, β, v) is a double bypass, then Section 1 gives:

$$\varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau} = \varphi_{\alpha, u, \tau} \varphi_{\alpha, w, \tau \nu} \varphi_{\beta, v, \nu}$$

where w is the path obtained from u after replacing β by v . Moreover, Lemma 2.8 implies that $(\beta, v) < (\alpha, w) < (\alpha, u)$. Therefore, (\star) is satisfied when (α, u, β, v) is a double bypass. If (α, u, β, v) is not a double bypass, then Section 1 gives (notice that thanks to Lemma 2.8 and to the inequality $(\beta, v) < (\alpha, u)$ we know that (β, v, α, u) is not a double bypass):

$$\varphi_{\alpha, u, \tau} \varphi_{\beta, v, \nu} = \varphi_{\beta, v, \nu} \varphi_{\alpha, u, \tau}$$

So (\star) is also satisfied when (α, u, β, v) is not a double bypass. ■

Using the preceding lemma and Proposition 2.13, it is now possible to prove that any $\psi \in \mathcal{T}$ is uniquely a decreasing product of transvections.

Proposition 2.19. *Let (α, u) be a bypass and let $\psi \in \mathcal{T}_{\leq(\alpha, u)}$. Then, there exist a non negative integer n , a sequence of bypasses $(\alpha_1, u_1), \dots, (\alpha_n, u_n)$ and non zero scalars $\tau_1, \dots, \tau_n \in k^*$ verifying:*

- (i) $\psi = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_1, u_1, \tau_1}$,
- (ii) $(\alpha_1, u_1) < \dots < (\alpha_n, u_n) \leq (\alpha, u)$.

Moreover, the integer n and the sequence $(\alpha_1, u_1, \tau_1), \dots, (\alpha_n, u_n, \tau_n)$ are unique for these properties.

Proof: The existence is given by Lemma 2.18. So it suffices to characterise the triples (α_i, u_i, τ_i) using ψ only. Let A, B and T be the following sets:

$$\begin{aligned} A &:= \{\alpha \in Q_1 \mid \psi(\alpha) \neq \alpha\} \\ B &:= \{(\alpha, u) \mid (\alpha, u) \text{ is a bypass, } \alpha \in A \text{ and } u \in \text{supp}(\psi(\alpha))\} \\ T &:= \{(\alpha, u, \tau) \mid (\alpha, u) \in B \text{ and } \tau = u^*(\psi(\alpha))\} \end{aligned}$$

Notice that the definition of A, B, T depend on ψ only (and not on the triples (α_i, u_i, τ_i)). Let $\beta \in Q_1$. Then Proposition 2.13 gives a normal form:

$$\psi(\beta) = \beta + \sum_{i \text{ such that } \beta = \alpha_i} \tau_i u_i$$

By definition of a normal form and because of (i) and (ii), the following equalities hold:

$$\begin{aligned} A &= \{\alpha_1, \dots, \alpha_n\} \\ B &= \{(\alpha_1, u_1), \dots, (\alpha_n, u_n)\} \\ T &= \{(\alpha_1, u_1, \tau_1), \dots, (\alpha_n, u_n, \tau_n)\} \end{aligned}$$

This proves that n and $(\alpha_1, u_1, \tau_1), \dots, (\alpha_n, u_n, \tau_n)$ are uniquely determined by the sets A, B, T (which depend on ψ only) and by the total order $<$. ■

3 Comparison of the presentations of a monomial algebra

Let $A = kQ/I_0$ with I_0 a monomial admissible ideal of kQ and let $kQ/I \simeq A$ be an admissible presentation of A . Thanks to Proposition 1.1, there exists ψ a product of transvections and of a dilatation such that $\psi(I_0) = I$. The aim of this section is to exhibit ψ_I the “simplest” possible among all the ψ ’s verifying $\psi(I_0) = I$. It will appear that ψ_I verifies a property which makes it unique. The construction of ψ_I will use specific properties of the Gröbner basis of I , due to the fact that I_0 is monomial. So, throughout the section, $<$ will denote a total order on the set of non trivial paths in Q , as in Definition 2.5. Before studying the Gröbner basis of I , it is useful to give some properties on the automorphisms $\psi \in \text{Aut}_0(kQ)$ verifying $\psi(I_0) = I$.

Lemma 3.1. *Let $D \in \mathcal{D}$ be a dilatation. Then $D(I_0) = I_0$. As a consequence, if $kQ/I \simeq A$ is an admissible presentation, then there exists $\psi \in \mathcal{T}$ such that $\psi(I_0) = I$.*

Proof: The first assertion is due to the fact that $D(u) \in k^*u$ for any path u and to the fact that I_0 is monomial. The second one is a consequence of the first one and of Proposition 1.1. ■

Lemma 3.2. *Let (α, u) be a bypass in Q . Then exactly one of the two following assertions is satisfied:*

- $\varphi_{\alpha,u,\tau}(I_0) = I_0$ for any $\tau \in k$.
- $\varphi_{\alpha,u,\tau}(I_0) \neq I_0$ for any $\tau \in k^*$.

Proof: Assume that $\tau \in k^*$ verifies $\varphi_{\alpha,u,\tau}(I_0) = I_0$ and let $\mu \in k$. Let $v \in I_0$ be a path. If α does not appear in v , then $\varphi_{\alpha,u,\nu}(v) = v \in I_0$. Assume that α appears in v , i.e. $v = v_2\alpha v_1$ with v_1, v_2 paths in which α does not appear (because Q has no oriented cycle). Therefore, $\varphi_{\alpha,u,\tau}(v) = v + \tau v_2 u v_1 \in I_0$. Thus, $v_2 u v_1 \in I_0$. This implies that $\varphi_{\alpha,u,\nu}(v) = v + \nu v_2 u v_1 \in I_0$. Since I_0 is monomial, $\varphi_{\alpha,u,\nu}(I_0) = I_0$. \blacksquare

Lemma 3.3. *Let $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ be an increasing sequence of bypasses, let $\tau_1, \dots, \tau_n \in k^*$ and set $\psi = \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$. Then:*

$$\psi(I_0) = I_0 \Leftrightarrow \varphi_{\alpha_i, u_i, \tau_i}(I_0) = I_0 \text{ for any } i$$

Proof: Assume that $\psi(I_0) = I_0$. Let $i \in \{1, \dots, n\}$, let $u = a_r \dots a_1 \in I_0$ be a path (with $a_i \in Q_1$) and fix $i \in \{1, \dots, n\}$. If $a_j \neq \alpha_i$ for any $j \in \{1, \dots, r\}$ then $\varphi_{\alpha_i, u_i, \tau_i}(u) = u \in I_0$. Now assume that there exists $j \in \{1, \dots, r\}$ such that $a_j = \alpha_i$ (j is necessarily unique because Q has no oriented cycle). Therefore:

$$\varphi_{\alpha_i, u_i, \tau_i}(u) = u + \tau_i a_r \dots a_{j+1} u_i a_{j-1} \dots a_1 \quad (i)$$

On the other hand, Proposition 2.10 and Proposition 2.13 imply that $a_r \dots a_{j+1} u_i a_{j-1} \dots a_1 \in \text{supp}(\psi(u))$. Thus (recall that $\psi(u) \in I_0$ and that I_0 is monomial):

$$a_r \dots a_{j+1} u_i a_{j-1} \dots a_1 \in I_0 \quad (ii)$$

From (i) and (ii) we deduce that $\varphi_{\alpha_i, u_i, \tau_i}(u) \in I_0$ for any path $u \in I_0$. So $\varphi_{\alpha_i, u_i, \tau_i}(I_0) = I_0$ for any i . The remaining implication is immediate. \blacksquare

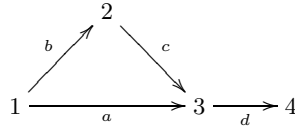
Remark 3.4. *The three preceding lemmas imply that the group $\text{Aut}_0(kQ, I_0)$ defined as follows:*

$$\text{Aut}_0(kQ, I_0) := \{\psi \in \text{Aut}(kQ) \mid \psi(x) = x \text{ for any } x \in Q_0, \text{ and } \psi(I_0) = I_0\}$$

is generated by the dilatations and by all the transvections preserving I_0 :

$$\text{Aut}_0(kQ, I_0) = \langle \mathcal{D} \cup \{\varphi \mid \varphi \text{ is a transvection such that } \varphi(I_0) = I_0\} \rangle$$

Example 3.5. *The preceding remark does not hold for any ideal I , even if kQ/I is monomial. For example, let Q be the quiver:*



and let $I = \langle da - dc \rangle$. Notice that $kQ/I \simeq kQ/I_0$ where $I_0 = \varphi_{a,cb,1}(I) = \langle da \rangle$. On the other hand:

1. $Id = \varphi_{a,cb,0}$ is the only transvection lying in $\text{Aut}_0(kQ, I)$,
2. for $t \in k \setminus \{0, 1\}$, the dilatation D_t such that $D_t(a) = ta$ and $D_t(x) = x$ for any other arrow x does not belong to $\text{Aut}_0(kQ, I)$,
3. $D_t \varphi_{a,cb,t} \in \text{Aut}_0(kQ, I)$ for any $t \in k^*$.

So $\text{Aut}_0(kQ, I)$ is not generated by \mathcal{D} and by the transvections it contains.

The following proposition gives the announced properties on the Gröbner bases of the admissible ideals I of kQ such that $kQ/I \simeq A$. Recall that for such an I , there exists $\psi \in \mathcal{T}$ such that $\psi(I_0) = I$ (see Lemma 3.1).

Proposition 3.6. *Let $\psi \in \mathcal{T}$ and let $I = \psi(I_0)$. Let B_0 (resp. B) be the Groebner basis of I_0 (resp. of I). Then B_0 is made of all the paths in Q which belong to I_0 . Moreover, the mapping:*

$$\begin{aligned} B &\longrightarrow B_0 \\ r &\longmapsto \max(\text{supp}(r)) \end{aligned} \quad (\star)$$

is well defined and bijective. For $u \in B_0$, let $r_u \in B$ be the inverse image of u under (\star) . Then $\text{supp}(r_u - u)$ is a set of paths derived of u .

Proof: Let $u_1 < \dots < u_n$ be the increasing sequence of all the non trivial paths in Q . Let (r_1, \dots, r_d) be the Gröbner basis of I and for each $j \in \{1, \dots, d\}$, let $i_j \in \{1, \dots, n\}$ be such that:

$$r_j \in u_{i_j} + \text{Span}(u_l \ ; \ l < i_j)$$

Since I_0 is monomial, B_0 is made of all the paths in Q belonging to I_0 .

Let $j \in \{1, \dots, d\}$. Since $u_{i_j} = \max(\text{supp}(r_j))$, the path u_{i_j} is not derived of u for any $u \in \text{supp}(r_j)$ (thanks to Lemma 2.8). So Proposition 2.15 implies that $u_{i_j} \in \text{supp}(\psi^{-1}(r_j)) \in I_0$. Because I_0 is monomial, this proves that $u_{i_j} \in I_0$. Therefore, the mapping (\star) is well defined. It is also one-to-one because of the definition of the Groebner basis of I . Let $u \in B_0$. Proposition 2.10 implies that $u = \max(\text{supp}(\psi(u)))$. Since $\psi(u) \in I$, there exists $j \in \{1, \dots, d\}$ such that $u = u_{i_j} = \max(\text{supp}(r_j))$. This proves that (\star) is onto and therefore bijective.

It remains to prove the last assertion of the proposition. This will be done by proving by induction on $j \in \{1, \dots, d\}$ that the following assertion is true:

$$H_j : \text{''supp}(r_j - u_{i_j}) \text{ is a set of paths derived of } u_{i_j}\text{''}$$

Remark that Proposition 2.10 implies that for any j :

$$u_{i_j} = \max(\text{supp}(\psi(u_{i_j}))) \quad \text{and} \quad u_{i_j}^*(\psi(u_{i_j})) = 1 \quad (i)$$

Moreover, $\psi(u_{i_j}) \in I$ because (\star) is well defined and because $\psi(I_0) = I$. Now begins the induction. Both r_1 and $\psi(u_{i_1})$ lie in I . Moreover, $u_{i_1} = \max(\text{supp}(r_1))$ by definition of u_{i_1} and $u_{i_1} = \max(\text{supp}(\psi(u_{i_1})))$ because of Proposition 2.10. So H_1 is true. Assume that $j \geq 2$ and that H_1, \dots, H_{j-1} are true. Since $\psi(u_{i_j}) \in I$ and because of (i), the following holds:

$$\psi(u_{i_j}) = r_j + \sum_{\substack{j' < j, \\ u_{i_{j'}} \in \text{supp}(\psi(u_{i_j}))}} u_{i_{j'}}^*(\psi(u_{i_j})) r_{j'}$$

So:

$$r_j - u_{i_j} = \psi(u_{i_j}) - u_{i_j} - \sum_{\substack{j' < j, \\ u_{i_{j'}} \in \text{supp}(\psi(u_{i_j}))}} u_{i_{j'}}^*(\psi(u_{i_j})) \left[(r_{j'} - u_{i_{j'}}) + u_{i_{j'}} \right] \quad (ii)$$

Notice that in the above equality:

(iii) $\text{supp}(\psi(u_{i_j}) - u_{i_j})$ is a set of paths derived of u_{i_j} (thanks to Proposition 2.10),

(iv) if $j' < j$ verifies $u_{i_{j'}} \in \text{supp}(\psi(u_{i_j}))$, then:

(v) $u_{i_{j'}}$ is derived of u_{i_j} (see (iii) above),

(vi) $\text{supp}(r_{j'} - u_{i_{j'}})$ is a set of paths derived of $u_{i_{j'}}$ (because $H_{j'}$ is true) and therefore derived of u_{i_j} (thanks to (v) and to Lemma 2.3).

The points (ii) – (vi) prove that H_j is true. Hence, H_j is true for any $j \in \{1, \dots, d\}$. This finishes the proof of the proposition. \blacksquare

Now it is possible to define precisely the automorphism ψ_I mentioned at the beginning of the section.

Proposition 3.7. *Let $kQ/I \simeq A$ be an admissible presentation. Then there exists a unique $\psi_I \in \mathcal{T}$ verifying the following conditions:*

- 1) $\psi_I(I_0) = I$,
- 2) if (α, u) is a bypass such that $u \in \text{supp}(\psi_I(\alpha))$ then $\varphi_{\alpha, u, \tau}(I_0) \neq I_0$ for any $\tau \in k^*$ (see Lemma 3.2).

Proof: • First, the existence of ψ_I . Thanks to Lemma 3.1, there exists $\psi \in \mathcal{T}$ verifying 1). Set:

$$\mathcal{A} := \{\psi \in \mathcal{T} \mid \psi(I_0) = I\}$$

and assume that for any $\psi \in \mathcal{A}$, the condition 2) is not verified. So, for any $\psi \in \mathcal{A}$, there is a finite (recall that Q has no oriented cycle) and non empty set of bypasses (see Lemma 3.2):

$$B_\psi = \left\{ (\alpha, u) \left| \begin{array}{l} (\alpha, u) \text{ is a bypass} \\ u \in \text{supp}(\psi(\alpha)) \\ \varphi_{\alpha, u, \tau}(I_0) = I_0 \text{ for any } \tau \in k \end{array} \right. \right\}$$

For each $\psi \in \mathcal{A}$, let $(\alpha_\psi, u_\psi) = \max B_\psi$ and let $\psi \in \mathcal{A}$ be such that:

$$(\alpha_\psi, u_\psi) = \min \{(\alpha_{\psi'}, u_{\psi'}) \mid \psi' \in \mathcal{A}\}$$

For simplicity, set $(\alpha, u) := (\alpha_\psi, u_\psi)$, $\tau := u^*(\psi(\alpha))$ $\psi' := \psi\varphi_{\alpha, u, -\tau}$. Notice that $\psi' \in \mathcal{A}$ because $(\alpha, u) \in B_\psi$. In order to get a contradiction, let us prove that $(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u)$. To do this, let us prove first that $(\alpha, u) \notin B_{\psi'}$. Thanks to Proposition 2.19, the following equality holds:

$$\psi = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_1, u_1, \tau_1}$$

where $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ and where $\tau_1, \dots, \tau_n \in k^*$. On the other hand, since $u^*(\psi(\alpha)) = \tau \neq 0$, Proposition 2.13 gives:

$$(\exists! i \in \{1, \dots, n\}) \quad (\alpha_i, u_i, \tau_i) = (\alpha, u, \tau)$$

Let us set:

$$\psi_1 := \varphi_{\alpha_{i-1}, u_{i-1}, \tau_{i-1}} \cdots \varphi_{\alpha_1, u_1, \tau_1} \in \mathcal{T}_{<(\alpha, u)}$$

Hence, the following equality holds:

$$\psi' = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_{i+1}, u_{i+1}, \tau_{i+1}} \varphi_{\alpha, u, \tau} \psi_1 \varphi_{\alpha, u, \tau}^{-1}$$

Since $\psi_1 \in \mathcal{T}_{<(\alpha, u)}$, Lemma 2.18 implies that $\varphi_{\alpha, u, \tau} \psi_1 \varphi_{\alpha, u, \tau}^{-1} \in \mathcal{T}_{<(\alpha, u)}$. Therefore, Proposition 2.19 gives the equality:

$$\varphi_{\alpha, u, \tau} \psi_1 \varphi_{\alpha, u, \tau}^{-1} = \varphi_{\beta_m, v_m, \nu_m} \cdots \varphi_{\beta_1, v_1, \nu_1}$$

where $(\beta_1, v_1) < \dots < (\beta_m, v_m) < (\alpha, u)$ and $\nu_1, \dots, \nu_m \in k^*$. As a consequence:

$$\psi' = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_{i+1}, u_{i+1}, \tau_{i+1}} \varphi_{\beta_m, v_m, \nu_m} \cdots \varphi_{\beta_1, v_1, \nu_1}$$

where $(\beta_1, v_1) < \dots < (\beta_m, v_m) < (\alpha, u) < (\alpha_{i+1}, u_{i+1}) < \dots < (\alpha_n, u_n)$ and where $\tau_{i+1}, \dots, \tau_n, \nu_1, \dots, \nu_m \in k^*$. In particular, Proposition 2.13 implies that $u \notin \text{supp}(\psi'(\alpha))$. Therefore, $(\alpha, u) \notin B_{\psi'}$ and in particular, $(\alpha, u) \neq (\alpha_{\psi'}, u_{\psi'}) = \max B_{\psi'}$. Thus, in order to prove that $(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u)$, it suffices to prove that the following implication holds for any bypass (β, v) :

$$v \in \text{supp}(\psi'(\beta)) \text{ and } (\alpha, u) < (\beta, v) \Rightarrow \varphi_{\beta, v, t}(I_0) \neq I_0 \text{ for any } \tau \in k^* \quad (i)$$

Let (β, v) be a bypass such that $v \in \text{supp}(\psi'(\beta))$ and such that $(\alpha, u) < (\beta, v)$. Since $\psi' = \psi\varphi_{\alpha, u, -\tau}$, the following holds:

$$\psi'(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \neq \alpha \\ \psi(\beta) - \tau\psi(u) & \text{if } \beta = \alpha \end{cases}$$

Therefore, $v \in \text{supp}(\psi'(\beta)) \subseteq \text{supp}(\psi(\beta)) \cup \text{supp}(\psi(u))$. Remark that if $v \in \text{supp}(\psi(u)) \setminus \text{supp}(\psi(\beta))$, then $\alpha = \beta$ and Proposition 2.10 implies that v is derived of u (we have $u \neq v$ because $\beta = \alpha$ and $(\alpha, u) < (\beta, v)$) and therefore $(\alpha, u) > (\alpha, v) = (\beta, v)$ whereas we assumed that $(\alpha, u) < (\beta, v)$. This proves that $v \in \text{supp}(\psi(\beta))$. Since $(\beta, v) > (\alpha, u) = (\alpha_\psi, u_\psi) = \max B_\psi$ we deduce that $\varphi_{\beta, v, \tau}(I_0) = I_0$ for any $\tau \in k$. This proves that the implication (i) is satisfied. Thus:

$$(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u) = (\alpha_\psi, u_\psi)$$

This contradicts the minimality of (α_ψ, u_ψ) and proves the existence of ψ .

• It remains to prove the uniqueness of ψ_I . Assume that $\psi, \psi' \in \mathcal{T}$ verify the conditions 1) and 2). In order to prove that $\psi = \psi'$, it is sufficient to prove that $\theta^*(\psi(\alpha)) = \theta^*(\psi'(\alpha))$ for any bypass (α, θ) . Let $\alpha \in Q_1$ and assume that there exists a minimal path θ such that (α, θ) is bypass and such that $\theta^*(\psi(\alpha)) \neq \theta^*(\psi'(\alpha))$. We may assume that $\theta^*(\psi(\alpha)) \neq 0$, i.e. $\theta \in \text{supp}(\psi(\alpha))$. Since ψ verifies 2), we deduce that there exist paths u and v such that:

$$u \in I_0, v \notin I_0 \text{ and } \varphi_{\alpha, \theta, 1}(u) = u + v \notin I_0$$

Notice that Proposition 2.10 gives:

$$\begin{cases} v^*(\psi(u)) = \theta^*(\psi(\alpha)) \text{ and } u^*(\psi(u)) = 1 \\ v^*(\psi'(u)) = \theta^*(\psi'(\alpha)) \text{ and } u^*(\psi'(u)) = 1 \end{cases} \quad (ii)$$

Moreover, $\psi(u), \psi'(u) \in I_0$ because $u \in I$. Therefore, Proposition 3.6 gives, the same notations concerning the Groebner bases, we have:

$$\begin{cases} \psi(u) = r_u + \sum_{w \in \mathcal{A}_\psi} w^*(\psi(u))r_w \\ \psi'(u) = r_u + \sum_{w \in \mathcal{A}_{\psi'}} w^*(\psi'(u))r_w \end{cases}$$

where \mathcal{A}_ψ is equal to:

$$\mathcal{A}_\psi := \{w \in \text{supp}(\psi(u)) \mid w \neq u \text{ and } w \in I_0\}$$

So:

$$\begin{cases} v^*(\psi(u)) = v^*(r_u) + \sum_{w \in \mathcal{A}_\psi} w^*(\psi(u))v^*(r_w) \\ v^*(\psi'(u)) = v^*(r_u) + \sum_{w \in \mathcal{A}_{\psi'}} w^*(\psi'(u))v^*(r_w) \end{cases} \quad (iii)$$

Let $w \in \mathcal{A}_\psi$ be such that $v^*(r_w) \neq 0$, i.e. $v \in \text{supp}(r_w)$. Remark that $v \in \text{supp}(r_w - w)$ because $v \notin I_0$ and $w \in I_0$. So:

- . v is derived of w (thanks to Proposition 3.6 and because $v \in \text{supp}(r_w - w)$).
- . v is derived of u of order 1 (because $\varphi_{\alpha, \theta, 1}(u) = u + v$).
- . w is derived of u (because $w \in \mathcal{A}_\psi$ and thanks to Proposition 2.10).

Using Lemma 2.3, these three facts imply that:

$$u = u_2\alpha u_1, \quad v = u_2\theta u_1 \text{ and } w = u_2\theta' u_1 \quad (iv)$$

where u_1, u_2 are paths and where θ' is a path derived of θ . In particular, (α, θ') is a bypass such that $\theta' < \theta$ (see Lemma 2.8). Therefore, the minimality of θ forces $\theta'^*(\psi(\alpha)) = \theta'^*(\psi'(\alpha))$. Moreover, (iv) and Proposition 2.10 imply that

$$w^*(\psi(u)) = \theta'^*(\psi(\alpha)) = \theta'^*(\psi'(\alpha)) = w^*(\psi'(u))$$

Therefore we have proved the following implication:

$$w \in \mathcal{A}_\psi \text{ and } v^*(r_w) \neq 0 \Rightarrow w^*(\psi(u))v^*(r_w) = w^*(\psi'(u))v^*(r_w) \quad (v)$$

After exchanging the roles of ψ and ψ' , the arguments used to prove (v) also give the following implication:

$$w \in \mathcal{A}_{\psi'} \text{ and } v^*(r_w) \neq 0 \Rightarrow w^*(\psi(u))v^*(r_w) = w^*(\psi'(u))v^*(r_w) \quad (vi)$$

Then, (iii), (v) and (vi) give $v^*(\psi(u)) = v^*(\psi'(u))$. This and (ii) imply that $\theta^*(\psi(\alpha)) = \theta^*(\psi'(\alpha))$, a contradiction. This proves that $\psi = \psi'$. \blacksquare

4 Proof of the main theorem

Let $A = kQ/I_0$ where I_0 is a monomial admissible ideal of kQ . The aim of this section is to prove that the quiver Γ of the homotopy relations of the admissible presentations of A admits \sim_{I_0} as unique source. This fact will be used in order to the existence of the universal cover of A . Notice that \sim_{I_0} is a source of Γ . Indeed, all minimal relations in I_0 are monomial relations so, for any $\sim_I \in \Gamma_0$ we have $\gamma \sim_{I_0} \gamma' \Rightarrow \gamma \sim_I \gamma'$. In order to prove that \sim_{I_0} is the unique source in Γ it will be proved that for any admissible presentation $kQ/I \simeq A$, the decomposition of ψ_I (given by Proposition 3.7) into a decreasing product of transvections (see Proposition 2.19) defines a path in Γ starting at \sim_{I_0} and ending at \sim_I . In this purpose, the following proposition will be useful.

Proposition 4.1. *Let $kQ/I \simeq A$ be an admissible presentation. Then, for any bypass (α, u) :*

$$u \in \text{supp}(\psi_I(\alpha)) \Rightarrow u \sim_I \alpha$$

Proof: For simplicity, set $\psi := \psi_I$. Thanks to Proposition 2.19 there is an equality:

$$\psi = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_1, u_1, \tau_1}$$

with $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ and $\tau_1, \dots, \tau_n \in k^*$. Thanks Proposition 2.13 it suffices to prove that $\alpha_i \sim_I u_i$ for any i . This will be done using a decreasing induction on $m \in \{1, \dots, n\}$. Let H_m be the assertion:

$$H_m : \text{"}\alpha_i \sim_I u_i \text{ for any } i \in \{m, m+1, \dots, n\}\text{"}$$

H_{n+1} is true because $\{i \mid n+1 \leq i \leq n\}$ is empty. So assume that H_{m+1} is true ($m \in \{1, \dots, n\}$). In order to prove that H_m is true, it thus suffices to prove that $\alpha_m \sim_I u_m$. From Proposition 2.13, the path u_m lies in $\text{supp}(\psi(\alpha_m))$. Hence, Proposition 3.7 provides a path $u \in I_0$ such that $\varphi_{\alpha_m, u_m, 1}(u) \notin I_0$. Therefore, there exist paths v_1, v_2 such that:

$$u = v_2\alpha_m v_1, \quad v := v_2 u_m v_1 \notin I_0 \text{ and } \varphi_{\alpha_m, u_m, 1}(u) = u + v \quad (i)$$

Since $\psi(u) \in I$, there exists a decomposition:

$$\psi(u) = r_1 + \dots + r_N$$

where r_1, \dots, r_N are minimal relations in I with pairwise disjoint supports. Remark that $u, v \in \text{supp}(\psi(u))$ thanks to Proposition 2.10 and to Proposition 2.13. Without loss of generality, it may be assumed that $v \in \text{supp}(r_1)$. Let $i \in \{1, \dots, N\}$ be such that $u \in \text{supp}(r_i)$. If $i = 1$ then $u \sim_I v$ and (i) gives $\alpha_m \sim_I u_m$. So assume that $i \neq 1$. Remark that $\psi^{-1}(r_1) \in I_0$ because $r_1 \in I$. Since I_0 is monomial, this also implies that $v \notin \text{supp}(\psi^{-1}(r_1))$. And thanks to Proposition 2.15, this proves that:

there exists $w \in \text{supp}(r_1)$ such that v is derived from w (ii)

Therefore:

- . w is derived of u since $w \in \text{supp}(r_1) \subseteq \text{supp}(\psi(u))$ (see Proposition 2.10, notice that $u \neq w$ because $u \notin \text{supp}(r_1)$),
- . v is derived of w (see (ii)),
- . v is derived of u of order 1 (because of (i)).

Thanks to Lemma 2.3, these three points imply that:

$$w = v_2 \theta v_1 \text{ and } u_m \text{ is derived of } \theta \tag{iii}$$

Since $w \in \text{supp}(\psi(u))$, the equalities $w = v_2 \theta v_1$, $u = v_2 \alpha_m v_1$ and Proposition 2.10 imply that $\theta \in \text{supp}(\psi(\alpha_m))$. Hence, there exists $j \in \{1, \dots, n\}$ such that:

$$(\alpha_m, \theta) = (\alpha_j, u_j)$$

Since u_m is derived of θ (see (iii)), this last equality gives $u_j = \theta > u_m$ (see Lemma 2.8) and therefore $j > m$. On the other hand, H_{m+1} is true, so:

$$\alpha_m = \alpha_j \sim_I u_j = \theta \tag{iv}$$

Finally, $v \sim_I w$, because r_1 is a minimal relation in I such that $v, w \in \text{supp}(r_1)$. This together with (i), (iii) and (iv) imply that $\alpha_m \sim_I u_m$. So H_m is true and the induction is finished. ■

Remark 4.2. *The preceding proposition proves that $\alpha \sim_I u$ for any $u \in \text{supp}(\psi(\alpha))$. On the other hand, \sim_{I_0} is weaker than \sim_I (i.e. $\gamma \sim_{I_0} \gamma' \Rightarrow \gamma \sim_I \gamma'$). These two properties are linked in general. Indeed, in [7, Prop. 4.2.35, Prop. 4.2.36] the author has proved that if I is an admissible ideal (non necessarily monomial) of kQ and if $\psi \in \mathcal{T}$ is such that $\alpha \sim_{\psi(I)} u$ for any bypass (α, u) such that $u \in \text{supp}(\psi(\alpha))$, then \sim_I is weaker than $\sim_{\psi(I)}$.*

Now it is possible to prove the existence of a path in Γ starting at \sim_{I_0} and ending at \sim_I , whenever $kQ/I \simeq A$.

Proposition 4.3. *Let $kQ/I \simeq A$ be an admissible presentation. Let $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ be the bypasses and $\tau_1, \dots, \tau_n \in k^*$ the scalars such that $\psi_I = \varphi_{\alpha_n, u_n, \tau_n} \dots \varphi_{\alpha_1, u_1, \tau_1}$ (see Proposition 2.19). For each $i \in \{1, \dots, n\}$, set:*

$$I_i := \varphi_{\alpha_i, u_i, \tau_i} \dots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$$

then, for each i , exactly one of the two following situations occurs:

- . $\sim_{I_{i-1}}$ and \sim_{I_i} coincide,
- . $\varphi_{\alpha_i, u_i, \tau_i}$ induces an arrow $\sim_{I_{i-1}} \rightarrow \sim_{I_i}$ in Γ .

In particular, there exists a path in Γ starting at \sim_{I_0} and ending at $\sim_{I_n} = \sim_I$.

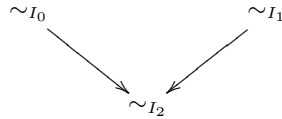
Proof: Let $i \in \{1, \dots, n\}$ and set $\psi_i := \varphi_{\alpha_i, u_i, \tau_i} \dots \varphi_{\alpha_1, u_1, \tau_1}$. Thus $I_i = \psi_i(I_0)$. Using Proposition 2.13 and Proposition 3.7 it is easily verified that $\psi_i = \psi_{I_i}$. Therefore, Proposition 4.1 applied to I_i gives $\alpha_i \sim_{I_i} u_i$. Since $I_i = \varphi_{\alpha_i, u_i, \tau_i}(I_{i-1})$, this proves that (see Proposition 1.2) either $\sim_{I_{i-1}}$ and \sim_{I_i} coincide or $\varphi_{\alpha_i, u_i, \tau_i}$ induces an arrow $\sim_{I_{i-1}} \rightarrow \sim_{I_i}$ in Γ . Thus, the vertices $\sim_{I_0}, \sim_{I_1}, \dots, \sim_{I_n} = \sim_I$ of Γ are the vertices of a path in Γ (maybe with repetitions) starting at \sim_{I_0} and ending at \sim_I . ■

The preceding proposition and the fact that Γ has no oriented cycle gives immediately the following corollary which was proved by the author in [8] in the case of algebras without double bypass over an algebraically closed field of characteristic zero.

Corollary 4.4. *Let Q be a quiver without oriented cycle and without multiple arrows. Let I_0 be an admissible and monomial ideal of kQ and let $A = kQ/I_0$. Then the quiver Γ of the homotopy relations of the admissible presentations of A admits \sim_{I_0} as unique source.*

The following example shows that the preceding corollary does not hold if Q has multiple arrows.

Example 4.5. Let $A = kQ/I_0$ where Q is the quiver $1 \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{b} \end{array} 2 \xrightarrow{c} 3$ and $I_0 = \langle ca \rangle$. Then Γ is equal to:



where $I_1 = \langle cb \rangle$ and $I_2 = \langle ca - cb \rangle$. In particular, Γ has two distinct sources. Notice however, that the mapping $a \mapsto b$, $b \mapsto a$, $c \mapsto c$ defines a group isomorphism $\pi_1(Q, I_0) \simeq \pi_1(Q, I_1)$. One has $\pi_1(Q, I_0) \simeq \pi_1(Q, I_1) \simeq \mathbb{Z}$ and $\pi_1(Q, I_2) = 1$.

Proposition 4.3 also allows one to prove Theorem 1. It extends [8, Thm. 2] to monomial triangular algebras without multiple arrows. Notice that Theorem 1 makes no assumption on the characteristic of k . Also recall that $\pi_1(Q, I_0) = \pi_1(Q)$.

Proof of Theorem 1: The proof is identical to the proof of [8, Thm. 2] except that one uses Proposition 4.3 instead of [8, Lem. 4.3]. ■

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