The universal cover of a monomial triangular algebra without multiple arrows

Patrick Le Meur

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Abstract

Let A be a basic connected finite dimensional algebra over an algebraically closed field k. Assuming that A is monomial and that the ordinary quiver Q of A has no oriented cycle and no multiple arrows, we prove that A admits a universal cover with group the fundamental group of the underlying space of Q.

Introduction

Let A be a finite dimensional k-algebra where k is an algebraically closed field. In order to study the category $mod(A)$ of (left) A-modules, one may assume that A is basic and connected. When $\mathcal{C} \to A$ is a Galois covering with $\mathcal{C}$ locally bounded, the covering techniques introduced in [3] and [12] allow one to reduce the study of (part of) $mod(A)$ to the study of $mod(\mathcal{C})$ and which is easier to handle (see for example [4], [6], [7]). Hence, Galois coverings of A behave like Galois coverings of topological spaces and one can wonder if A admits a a fundamental group and a universal Galois covering, i.e. a Galois covering $\mathcal{C} \to A$ with $\mathcal{C}$ connected and locally bounded and which is factorised by any other Galois covering. Such a universal Galois covering exists if A is of finite representation type (see [6]). Moreover, a fundamental group $\pi_1(Q, I)$ associated with any admissible presentation $kQ/I \simeq A$ with quiver and relations I was defined in [11]. This group is constructed using a homotopy relation $\sim_I$ like in the topological situation and is the group of a Galois covering of A defined by the presentation $kQ/I \simeq A$. However, the group $\pi_1(Q, I)$ depends on (Q, I) and there exist examples of algebras A with different presentations having non isomorphic fundamental groups. In a previous text ([9]), the author has compared the fundamental groups associated with different presentations of the same algebra. This study showed ([9, Thm. 1]) that: if A is triangular (i.e. its ordinary quiver Q has no oriented cycle) and without double bypass, and if k has characteristic zero then A admits a presentation $kQ/I_0 \simeq A$ such that the fundemantal group $\pi_1(Q, I)$ of any other presentation $kQ/I \simeq A$ is a quotient of $\pi_1(Q, I_0)$. Under the same hypotheses, this study showed ([9, Thm. 2]) that: the Galois covering with group $\pi_1(Q, I_0)$ of A defined by the presentation $kQ/I_0 \simeq A$ satisfies a factorisation property with respect to the Galois coverings of A similarly to the universal cover of an arcwise connected topological space. Recall ([2]) that a bypass in a quiver is a couple (α, u) where $\alpha \neq u$, α is an arrow of Q, and u is a path parallel to α. Recall also ([9]) that a double bypass is 4-tuple (α, u, β, v) where (α, u), (β, v) are bypasses such that the arrow β appears in the path u.

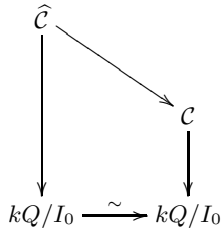
The aim of the present text is to extend [9, Thm. 2] to monomial algebras without using conditions on the characteristic of the field or on the double bypasses. Recall that A is called monomial if it admits a presentation $kQ/I_0 \simeq A$ where I_0 is an ideal of kQ generated by a set of paths. Notice that for such an algebra, [9, Thm. 1] cited above is obvious. Indeed: for any bound quiver (Q, I), the group $\pi_1(Q, I)$ is a quotient of the fundamental group $\pi_1(Q)$ of the underlying graph of Q, and these two groups are isomorphic if I is generated by a set of paths. On the other hand [9, Thm. 2] is less obvious because it carries a linear setting which is not encoded in the fundamental group. With this setting, we prove the following theorem which is the main result of this text:

Theorem 1. *Let Q be a quiver without otiented cycle and without multiple arrows. Let I_0 be a monomial ideal of kQ. Let $\widehat{\mathcal{C}} \to kQ/I_0$ be the Galois covering with group $\pi_1(Q)$ associated to the universal Galois*

* *adress:* Département de Mathématiques, Ecole normale supérieure de Cachan, 61 avenue du président Wilson, 94235 Cachan, France

† *e-mail:* plemeur@dptmaths.ens-cachan.fr

covering of (Q, I_0) (see [11]). For any Galois covering $\mathcal{C} \to kQ/I_0$ with group G and with $\mathcal{C}$ connected and locally bounded, there exists a commutative diagram of k-categories and k-linear functors:



where $kQ/I_0 \xrightarrow{\sim} kQ/I_0$ is an isomorphism of k-algebras, restricting to the identity map on the set Q_0 of vertices and where $\widehat{\mathcal{C}} \to \mathcal{C}$ is a Galois covering with group N a normal subgroup of $\pi_1(Q)$ such that there exists an exact sequence of groups: $1 \to N \to \pi_1(Q) \to G \to 1$.

As said before, the above theorem is interesting because it gives a class of algebras for which a universal Galois covering exists and which is not defined using double bypasses. Moreover, the proof of Theorem 1 allows us to prove, under the same assumptions, that the quiver Γ of the homotopy relations of A (see [9, Def. 2.4]) has a unique source. Recall that the unicity of the source of Γ was proved in [9] for algebras without double bypass over a zero characteristic field. In order to prove Theorem 1, we will compare the fundamental groups of the admissible presentations of kQ/I_0 using the fact that I_0 is monomial. More precisely, we will use the arguments of the proof of [9, Thm. 2]. This proof essentially uses the fact that for an admissible presentation $kQ/I \simeq kQ/I_0$, there exists a sequence of $\varphi_n, \ldots, \varphi_1$ of transvections of kQ such that:

(*i*) $I = \varphi_n \ldots \varphi_1(I_0)$,

(*ii*) if we set $I_i := \varphi_i \ldots \varphi_1(I_0)$, then $\pi_1(Q, I_i)$ is either isomorphic to or a quotient of $\pi_1(Q, I_{i-1})$.

Recall ([9, Sect. 1]) that a transvection is an automorphism $\varphi_{\alpha,u,\tau}$ of kQ defined by a bypass (α, u) and a scalar $\tau \in k$. In order to show the existence of the above sequence of transvections, we will prove the following facts:

1. There exists a suitable ordering on the set of bypasses such that if $\psi \in Aut(kQ)$ is a product of transvections, then ψ can be written uniquely as $\psi = \varphi_{\alpha_n,u_n\tau_n} \ldots \varphi_{\alpha_1,u_1,\tau_1}$ with $\tau_1, \ldots, \tau_n \in k^*$ and $(\alpha_n, u_n) > \ldots > (\alpha_1, u_1)$.
2. If $kQ/I \simeq kQ/I_0$ then there exists a unique product of transvections ψ_I verifying simple technical conditions and such that $\psi_I(I_0) = I$ (this step will intensively use the fact that I_0 is monomial).
3. If $kQ/I \simeq kQ/I_0$, the unique ordered sequence of transvection given by 1. and whose product equal ψ_I verify the conditions (*i*) and (*ii*) above.

The second step is particularly interesting. Indeed, in general, if I_0 and I are admissible ideals (without assuming monomiality) such that $kQ/I_0 \simeq kQ/I$, then, there are various automorphisms $\psi \in Aut(kQ)$ such that $\psi(I) = J$, whereas in the present situation, the monomiality of I_0 gives us a distinguished and uniquely characterised automorphism ψ_I such that $\psi_I(I_0) = I$.

The text is organised as follows. In Section 1 we recall all the notions that we will need to prove Theorem 1. In Section 2, we will prove some combinatoric facts on the paths in a quiver. These will lead to the order and to the decomposition of the first step above. In Section 3 we will prove the second step above. Finally, in Section 4 we will prove the last step and Theorem 1.

1 Basic definitions

A **k-category** is a category $\mathcal{C}$ whose objects class $\mathcal{C}_0$ is a set, whose space of morphisms from x to y (denoted by ${}_y\mathcal{C}_x$) is a k-vector space for any $x, y \in \mathcal{C}_0$ and whose composition of morphisms is k-bilinear. All functors between k-categories will be assumed to be k-linear functors. In particular, $Aut(\mathcal{C})$ will denote the group of k-linear automorphism of $\mathcal{C}$, and we shall denote by $Aut_0(\mathcal{C})$ for the subgroup $\{\psi \in Aut(\mathcal{C}) \mid \psi(x) = x \text{ for any } x \in \mathcal{C}_0\}$ of $Aut(\mathcal{C})$. Let $\mathcal{C}$ be a k-category, $\mathcal{C}$ is called **connected** if $\mathcal{C}$ cannot be written as the disjoint union of two full subcategories. An ideal I of $\mathcal{C}$ is the data of subspaces ${}_yI_x \subseteq {}_y\mathcal{C}_x$ (for any $x, y \in \mathcal{C}_0$) such that $fgh \in I$ whenever f, g, h are composable morphisms in $\mathcal{C}$ such that $g \in I$. We say that $\mathcal{C}$ is **locally bounded** provided that: 1) for any $x \in \mathcal{C}_0$, the vector spaces $\bigoplus_{y \in \mathcal{C}_0} {}_y\mathcal{C}_x$

and $\bigoplus_{y\in\mathcal{C}_0} {}_x\mathcal{C}_y$ are finite dimensional, 2) ${}_x\mathcal{C}_x$ is a local algebra for any $x \in \mathcal{C}_0$, 3) distinct objects are not isomorphic. Let A be a finite dimensional k-algebra and let $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents. Then A is also a k-category as follows: $A_0 := \{e_1, \ldots, e_n\}$, ${}_{e_i}A_{e_i} := e_jAe_i$ and the composition of morphisms is induced by the product in A. Notice that different choices for the idempotents $e_1, \ldots, e_n$ give rise to isomorphic k-categories. With this setting, the k-algebra A is connected (resp. basic) if and only if the k-category A is connected (resp. locally bounded) as a k-category. In the sequel we shall make no distinction between a finite dimensional k-algebra and its associated k-category. If $\mathcal{C}$ is a locally bounded k-category, the radical of $\mathcal{C}$ is the ideal $\mathcal{RC}$ of $\mathcal{C}$ such that: ${}_y\mathcal{RC}_x$ is the space of non-isomorphisms $x \to y$ in $\mathcal{C}$, for any $x, y \in \mathcal{C}_0$. The ideal of $\mathcal{C}$ generated by compositions gf where f and g lie in $\mathcal{RC}$ will be denoted by $\mathcal{R}^2\mathcal{C}$.

A **Galois covering with group** G of $\mathcal{C}$ (by $\mathcal{C}'$) is a functor $F\colon \mathcal{C}' \to \mathcal{B}$ endowed with a group morphism $G \to Aut(\mathcal{C}')$ and such that: 1) the induced action of G on $\mathcal{C}'_0$ is free, 2) $F \circ g = F$ for any $g \in G$, 3) for any k-linear functor $F'\colon \mathcal{C}' \to \mathcal{C}''$ such that $F' \circ g = F'$ for any $g \in G$, there exists a unique $\overline{F'}\colon \mathcal{C} \to \mathcal{C}''$ such that $\overline{F'} \circ F = F'$ (in other words, F is a quotient of $\mathcal{C}'$ by $\mathcal{C}$ in the category of k-categories). We shall say that F is connected if $\mathcal{C}'$ is connected connected and locally bounded (this implies that $\mathcal{C}$ is connected and locally bounded). For more details on Galois coverings (in particular for the connections with representations theory), we refer the reader to [3].

Quivers, paths, bypasses. A quiver is a 4-tuple $Q = (Q_1, Q_0, s, t)$ where Q_1 and Q_0 are sets and $s, t\colon Q_1 \to Q_0$ are maps. The elements of Q_1 (resp. of Q_0) are called the arrows (resp. the vertices) of Q. If $\alpha \in Q_1$, the vertex $s(\alpha)$ (resp. $t(\alpha)$) is called the source (resp. the target) of α. The quiver Q is called **locally finite** if and only if any vertex is the source point (resp. the target) of finitely many arrows. For example, if $\mathcal{C}$ is a locally bounded k-category, the **ordinary quiver of** $\mathcal{C}$ is the locally finite quiver Q such that: $Q_0 := \mathcal{C}_0$ and for any $x, y \in \mathcal{C}_0$, the number of arrows starting at x and arriving at y is equal to $dim_k\ {}_y\mathcal{RC}_x/\ {}_y\mathcal{R}^2\mathcal{C}_y$. A path in Q of length n ($n \geqslant 0$) with source $x \in Q_0$ (or starting at x) and target $y \in Q_0$ (or arriving at y) is a sequence of arrows $\alpha_1, \ldots, \alpha_n$ such that: $x = y$ if $n = 0$, $s(\alpha_1) = x$, $s(\alpha_{i+1}) = t(\alpha_i)$ for any $i \in \{1, \ldots, n-1\}$ and $t(\alpha_n) = y$. If $n \geqslant 1$ this path will be written $\alpha_n \ldots \alpha_1$ and called non trivial. If $n = 0$ this path will be written e_x and called stationary at x. The length of this path is $|u| := n$. The mappings s, t are naturally extended to paths in Q. If u and v are paths, the concatenation vu is defined if and only if $t(u) = s(v)$ by the following rule: 1) $vu = v$ is u is stationary, 2) $vu = u$ is v is stationary, 3) $vu = \beta_m \ldots \beta_1\alpha_n \ldots \alpha_1$ if $v = \beta_m \ldots \beta_1$ and $u = \alpha_n \ldots \alpha_1$ (with $\alpha_i, \beta_j \in Q_1$). Two paths in Q are called **parallel** whenever they have the same source and the same target. An **oriented cycle** in Q is a non trivial path whose source and target are equal. We shall say that Q has **multiple arrows** if and only if there exist in Q distinct parallel arrows. A **bypass** in Q is a couple (α, u) where $\alpha \in Q_1$ and where u is a path parallel to and different from α. A **double bypass** in Q is 4-tuple (α, u, β, v) where (α, u) and (β, v) are bypasses and such that the arrow β appears in the path u. In other words, there exists paths u_1, u_2 such that $u = u_2\beta u_1$. When Q has no oriented cycle, the paths u_1, u_2 are unique for this property, and the path u_2vu_1 will be called obtained from $u = u_2\beta u_1$ after replacing β by v.

Admissible presentations (see [3, 2.1]). A quiver Q defines the **path category** kQ such that $(kQ)_0 = Q_0$, such that ${}_ykQ_x$ is the k-vector space with basis the family of paths starting at x and arriving at y, and the composition in kQ is induced by the concatenation of paths. If $r \in {}_ykQ_x$, **a normal form** for r is an equality $r = \sum_{i=1}^{n} t_iu_i$ where $t_1, \ldots, t_n \in k^*$ and $u_1, \ldots u_n$ are pairwise distinct paths in Q. With this notation, **the support** of r is the set $supp(r) := \{u_1, \ldots, u_n\}$ (with the convention $supp(0) = \emptyset$). A **subexpression of** r is a linear combination $\sum_{i\in E} t_iu_i$ with $E \subseteq \{1, \ldots, n\}$. Later, we will need the following fact: if $r = r_1 + \ldots + r_n \in {}_ykQ_x$ is such that $supp(r_1), \ldots, supp(r_n)$ are pairwise disjoint, then $r_{i_1} + \ldots + r_{i_t}$ is a subsexpression of r, for any indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$. An ideal I of kQ is called **admissible** provided that: 1) any morphism in I is a linear combination of paths of length at least 2, 2) for any vertex x, there exists an integer n such that any path starting or arriving at x and of length greater than n lies in I. A morphism in I is called a **relation** (of I). In particular, **a minimal relation of** I (see [11]) is a non zero relation r of I such that 0 and r are the only subexpressions of r which are relations. With this definition, any relation of I is the sum of minimal relations with pairwise disjoint supports. If $u \in I$ is a path, then u called a **monomial relation**. In particular, I is called monomial if it is generated by a set of monomial relations. A pair (Q, I) where Q is a locally finite and I is an admissible ideal of kQ is called a **bound quiver**. In such a case, kQ/I is a locally bounded k-category and is connected if and only if Q is connected (i.e. the underlying graph of Q is connected). Conversely,

if $\mathcal{C}$ is a locally bounded k-category, then there exists an isomorphism $kQ/I \xrightarrow{\sim} \mathcal{C}$ where (Q, I) is a bound quiver such that Q is the ordinary quiver of $\mathcal{C}$. Such an isomorphism is called **admissible presentation of** $\mathcal{C}$. If the ideal I is monomial, the admissible presentation and $\mathcal{C}$ are called monomial. Notice that $\mathcal{C}$ may have different admissible presentations.

Fundamental group of a presentation (see [11]). Let (Q, I) be a bound quiver and let $x_0 \in Q_0$. For every arrow $x \xrightarrow{a} y \in Q_1$ we define its formal inverse a^{-1} with source $s(a^{-1}) = y$ and target $t(a^{-1}) = x$. With these notations, **a walk in** Q **with source** x **and target** y is a sequence $a_1, \ldots, a_n$ (with $n \geqslant 0$) of arrows and formal inverses of arrows such that $s(a_i) = t(a_{i-1})$ for any i. If $n > 0$, this walk is denoted by $a_n \ldots a_1$. Hence paths are particular cases of walks and the concatenation of paths extends naturally to walks. The **homotopy relation** of (Q, I) is the equivalence relation on the set of walks in Q, denoted by $\sim_I$ and generated by the following properties:

1. $\alpha\alpha^{-1} \sim_I e_y$ and $\alpha^{-1}\alpha \sim_I e_x$ for any arrow $x \xrightarrow{\alpha} y$ in Q,
2. $u \sim_I v$ for any $u, v \in supp(r)$ where r is a minimal relation of I,
3. $wvu \sim_I wv'u$ for any walks w, v, v', u such that $v \sim_I v'$ and such that the concatenations wvu and $wv'u$ are well-defined (i.e. $\sim_I$ is compatible with the concatenation).

The $\sim_I$-equivalence class of a walk γ will be denoted by $[\gamma]_I$. Let $\pi_1(Q, I, x_0)$ be the set of equivalence classes of walks in Q with source and target equal to x_0. The concatenation of walks endows this set with a group structure (with unit e_{x_0}) and this group is called the **fundamental group of** (Q, I). If Q is connected, the isomorphism class of this group does not depend on $x_0 \in Q_0$ and $\pi_1(Q, I, x_0)$ is denoted by $\pi_1(Q, I)$. If $\mathcal{C}$ is a connected locally bounded k-category and if $kQ/I \simeq \mathcal{C}$ is an admissible presentation, the fundamental group $\pi_1(Q, I)$ is called the fundamental group of this presentation.

Dilatations, transvections (see [9]). Let Q be a quiver. A **dilatation** of kQ is an automorphism $D \in Aut_0(kQ)$ such that $D(\alpha) \in k^*\alpha$ for any $\alpha \in Q_1$. The dilatations of kQ form a subgroup $\mathcal{D}$ of $Aut_0(kQ)$. Let (α, u) be a bypass in Q and let $\tau \in k$. This defines $\varphi_{\alpha,u,\tau} \in Aut_0(kQ)$ as follows: $\varphi_{\alpha,u,\tau}(\alpha) = \alpha + \tau u$ and $\varphi_{\alpha,u,\tau}(\beta) = \beta$ for any arrow $\beta \neq \alpha$. The automorphism $\varphi_{\alpha,u,\tau}$ is called a **transvection**. The composition of transvections is ruled as follows. Let $\varphi_{\alpha,u,\tau}$ and $\varphi_{\alpha,u,\tau'}$, then $\varphi_{\alpha,u,\tau}\varphi_{\alpha,u,\tau'} = \varphi_{\alpha,u,\tau+\tau'}$ and $\varphi_{\alpha,u,\tau}^{-1} = \varphi_{\alpha,u,-\tau}$. If (α, u, β, v) and (β, v, α, u) are not a double bypasses, then $\varphi_{\alpha,u,\tau}\varphi_{\beta,v,\nu} = \varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}$,. If (α, u, β, v) is a double bypass and if Q has no oriented cycle, then $\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau} = \varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$, where w is the path obtained from u after replacing β by v. The subgroup of $Aut_0(kQ)$ generated by all the transvections is denoted by $\mathcal{T}$. Dilatations and transvections are useful to compare different admissible presentations of the same locally bounded k-category because of the following proposition:

Proposition 1.1. *(see [9, Prop. 2.1, Prop. 2.2]) Let $kQ/I \simeq A$ and $kQ/J \simeq A$ be admissible presentations of the basic finite dimensional algebra A. If Q has no oriented cycle, then there exists $\psi \in Aut_0(kQ)$ such that $\psi(I) = J$. Moreover, $\mathcal{T}$ is a normal subgroup of $Aut_0(kQ)$ and $Aut_0(kQ) = \mathcal{T}\mathcal{D} = \mathcal{D}\mathcal{T}$.*

The dilatations and the transvections were introduced because they allow comparisons between the fundamental groups of presentations of the same locally bounded k-category:

Proposition 1.2. *(see [9, Prop. 2.5]) Let I be an admissible ideal of kQ, let $\varphi \in Aut_0(kQ)$ and set $J = \varphi(I)$. If φ is a dilatation, then $\sim_I$ and $\sim_J$ coincide. If $\varphi = \varphi_{\alpha,u,\tau}$ is a transvection, then:*

1. *if $\alpha \sim_I u$ and $\alpha \sim_J u$ then $\sim_I$ and $\sim_J$ coincide.*
2. *if $\alpha \not\sim_I u$ and $\alpha \sim_J u$ then $\sim_J$ is generated by $\sim_I$ and $\alpha \sim_J u$.*
3. *if $\alpha \not\sim_I u$ and $\alpha \not\sim_J u$ then $I = J$ and $\sim_I$ and $\sim_J$ coincide.*

If there exists a transvection φ such that $\varphi(I) = J$ and such the second point above occurs, then we shall say that $\sim_J$ ***is a direct successor of*** *$\sim_I$.*

Here the expression "$\sim_I$ is generated by $\sim_J$ and $\alpha \sim_I u$" means that $\sim_I$ is the equivalence relation on the set of walks in Q, compatible with the concatenation and generated by the two following properties: 1) $\gamma \sim_J \gamma' \Rightarrow \gamma \sim_I \gamma'$, 2) $\alpha \sim_I u$. Following [9, Def. 2.7], if A is a basic connected finite dimensional algebra with ordinary quiver Q without oriented cycle, we define **the quiver** Γ **of the homotopy relations of** A to be the quiver such that $\Gamma_0 = \{\sim_I \mid kQ/I \simeq A\}$ and such that there exists arrow $\sim_I \to \sim_J$ if and only if $\sim_J$ is a direct successor of $\sim_I$. Recall ([9, Rem. 5, Prop. 2.8]) that Γ is finite, connected, without oriented cycle and such that for any oriented path with source $\sim_I$ and target $\sim_J$, the identity map on the walks in Q induces a surjective group morphism $\pi_1(Q, I) \twoheadrightarrow \pi_1(Q, J)$.

Gröbner bases Let E be a k-vector space with an ordered basis $(e_1, \ldots, e_n)$, let $(e_1^*, \ldots, e_n^*)$ be the associated dual basis of E^*, and let F be a subspace of E. A Gröbner basis (see [1] for the ususal definition) of F is a basis $(r_1, \ldots, r_d)$ such that:

1. $r_j \in e_{i_j} + Span(e_l \ ; \ l < i_j)$ for some i_j, for any $j \in \{1, \ldots, r\}$,
2. $i_1 < i_2 < \ldots < i_r$,
3. $e_{i_j}^*(r_{j'}) = 0$ for any $j \neq j'$.

It is well known that F admits a unique Gröbner basis. Also, $r \in F$ if and only if: $r = \sum_{j=1}^{d} e_{i_j}^*(r) r_j$. In the sequel, we will use this notion in the following setting: E is the vector space with basis (for some order to be defined) the family of non trivial paths in a finite quiver Q without oriented cycles and F is the underlying subspace of E associated to an admissible ideal I of kQ. Notice that in this setting, if $r \in E$ and if u is a non trivial path, then: $u \in supp(r) \Leftrightarrow u^*(r) \neq 0$.

2 Combinatorics on the paths in a quiver

2.1 Derivation of paths

Definition 2.1. *Let $u = \alpha_n \ldots \alpha_1$ and v be paths in Q. Then v is called derived of u (of order t) if there exist indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$ and with bypasses $(\alpha_{i_1}, v_i), \ldots, (\alpha_{i_t}, u_t)$ such that v is obtained from u after replacing α_{i_l} by v_l for each l:*

$$v = \alpha_n \ldots \alpha_{i_t+1} v_t \alpha_{i_t-1} \ldots \alpha_{i_l+1} v_l \alpha_{i_l-1} \ldots \alpha_{i_1+1} v_1 \alpha_{i_1-1} \ldots \alpha_1$$

Remark 2.2. *If $\alpha \in Q_1$, then u is derived of α if and only if (α, u) is a bypass.*

With the above definition, the following lemma is easily verified using the fact that Q has no multiple arrows.

Lemma 2.3. 1. *If v is derived of u with both orders t and t', then t=t'.*

2. *If v is derived of u of order t then there exists a sequence of paths $u_0 = u, u_1, \ldots, u_t = v$ such that u_i is derived of u_{i-1} of order 1 for any i.*
3. *If v is derived of u of order t, then $|v| \geqslant |u| + t$.*
4. *If v is derived of u of order t and if w is derived of v of order t', then w is derived of u of order at least t.*
5. *Let u, v, w be paths verifying:*
   - *v is derived of u,*
   - *w is derived of v,*
   - *w is derived of u of order 1,*

   then we have:
   $$u = u_2 \alpha u_1, \quad v = u_2 \theta u_1, \quad w = u_2 \theta' u_1$$
   where u_1, u_2 are paths, (α, θ) is a bypass and θ' is derived of θ.
6. *If v (resp. v') is derived of u (resp. of u') of order t (resp. t'), then $v'v$ is derived of $u'u$ of order $t' + t$, whenever these compositions of paths are well defined.*

The following example shows that the inequality in the 4-th point of the preceding lemma may be an equality.

Example 2.4. *Let (α, u, β, v) be a double bypass. Let u_1, u_2 be the paths such that $u = u_2 \beta u_1$. Then u is derived of α of order 1, $w := u_2 v u_1$ is derived of u of order 1 and w is derived of u of order 1.*

2.2 Order between paths, order between bypasses

Now, we construct a total order on the set of non trivial paths in Q. This construction is a particular case of the one introduced in [5], also it depends on an arbitrary order $\lhd$ on Q_1. We assume that this order $\lhd$ is fixed for this subsection. We shall write $\lhd$ for the lexicographical order induced by $\lhd$ on the set of nontrivial paths in Q. For details on the correctness of the following definition we refer the reader to [5].

Definition 2.5. *For $\alpha \in Q_1$, set:*

$$W(\alpha) = Card(B(\alpha)) \quad \textit{where} \quad B(\alpha) = \{(\alpha, u) \mid (\alpha, u) \textit{ is a bypass in } Q\}$$

For $u = \alpha_n \dots \alpha_1$ a path in Q (with $\alpha_i \in Q_1$), let us set:

$$W(u) = W(\alpha_n) + \dots + W(\alpha_1)$$

These data define a total order $<$ on the set of non trivial paths in Q as follows:

$$u < v \Leftrightarrow \begin{cases} & W(u) < W(v) \\ \textit{or} & \\ & W(u) = W(v) \textit{ and } u \lhd v \end{cases}$$

We shall also write $<$ for the lexicographical order induced by $<$ on the set of couples of paths.

Remark 2.6. *If u and v are (non trivial) paths such that vu is well defined, then $W(vu) = W(u) + W(v)$.*

Lemma 2.7. *1. If u, v, u', v' are paths such that $v < u$ and $v' < u'$ then $v'v < u'u$ whenever these compositions are well defined.*

2. If (α, u) is a bypass, then $W(u) < W(\alpha)$. So $u < \alpha$.

3. If v is derived of u, then $v < u$.

4. If (α, u, β, v) is a double bypass and if w is the path obtained from u after replacing β by v, then:

$$(\beta, v) < (\alpha, w) < (\alpha, u)$$

Proof: 1) is a direct consequence of Definition 2.5 and Remark 2.6.

2) Let us write $u = a_n \dots a_1$ with $a_i \in Q_1$ for each i (hence $a_i \neq a_j$ if $i \neq j$ because Q has no oriented cycle). Therefore:

. $B(a_1), \dots, B(a_n)$ are pairwise disjoint,

. $W(u) = W(a_1) + \dots + W(a_n)$

Notice that if $(a_i, v) \in B(a_i)$, then $(\alpha, a_n \dots a_{i+1} v a_{i-1} \dots a_1) \in B(\alpha)$. Thus, we have a well defined mapping:

$$\begin{array}{rccc} \theta: & B(a_1) \sqcup \dots \sqcup B(a_n) & \longrightarrow & B(\alpha) \\ & (a_i, v) & \longmapsto & (\alpha, a_n \dots a_{i+1} v a_{i-1} \dots a_1) \end{array}$$

This mapping is one-to-one, indeed:

. if $\theta(a_i, v) = \theta(a_i, v')$ with $(a_i, v), (a_i, v') \in B(a_i)$ then:

$$a_n \dots a_{i+1} v a_{i-1} \dots a_1 = a_n \dots a_{i+1} v' a_{i-1} \dots a_1$$

and therefore $(a_i, v) = (a_i, v')$,

. if $\theta(a_i, v) = \theta(a_j, v')$ with $(a_i, v) \in B(a_i)$, $(a_j, v') \in B(a_j)$ and $j < i$, then:

$$a_n \dots a_{i+1} v a_{i-1} \dots a_1 = a_n \dots a_{j+1} v' a_{j-1} \dots a_1$$

So:

$$v a_{i-1} \dots a_1 = a_i \dots a_{j+1} v' a_{j-1} \dots a_1$$

Since v and a_i are parallel and since Q has no oriented cycle, we infer that $v = a_i$ which is impossible because $(a_i, v) \in B(a_i)$.

On the other hand, θ is not onto. Indeed, if there exists $(a_i, v) \in B(a_i)$ verifying $\theta(a_i, v) = (\alpha, u)$, then:

$$a_n \dots a_1 = u = a_n \dots a_{i+1} v a_{i-1} \dots a_1$$

which implies $a_i = v$, a contradiction. Since θ is one-to-one and not onto, we deduce that:

$$W(u) = Card(B(\alpha)) > Card(B(a_1) \sqcup \dots \sqcup B(a_n)) = W(u)$$

This proves that $W(u) < W(\alpha)$ and that $u < \alpha$.

3) is a direct consequence of 1) and of 2).

4) Let us write $u = u_2 \beta u_1$ (with u_1, u_2 paths) so that $w = u_2 v u_1$. From 2), we have:

$$W(\alpha) > W(u) = W(u_1) + W(\beta) + W(u_2) \geqslant W(\beta)$$

So $\beta < \alpha$ and therefore $(\beta, v) < (\alpha, w)$. Using 2) again, we also have:

$$W(w) = W(u_2) + W(v) + W(u_1) < W(u_2) + W(\beta) + W(u_1) = W(u)$$

So $w < u$ and therefore $(\alpha, w) < (\alpha, u)$ ■

2.3 Image of a path by a product of transvections

In this paragraph, we apply the previous constructions to find an easy way to compute $\psi(u)$ when $\psi \in \mathcal{T}$ and u is a path in Q. We begin with the following lemma on the description of $\psi(\alpha)$ when $\psi \in \mathcal{T}$ and $\alpha \in Q_1$. Recall that Q has no multiple arrows and no oriented cycles.

Lemma 2.8. *Let $\psi \in \mathcal{T}$ and let $\alpha \in Q_1$. Then $\psi(\alpha) - \alpha$ is a linear combination of paths parallel to α and length greater than or equal to 2. In particular, $\alpha \in supp(\psi(\alpha))$ and $\alpha^*(\psi(\alpha)) = 1$.*

Proof: The conclusion is immediate if ψ is a transvection because Q has no multiple arrows. The conclusion in the general case is obtained using an easy induction on the number of transvections whose product equal ψ. ■

The preceding lemma gives the following description of $\psi(u)$ when $\psi \in \mathcal{T}$ and u is a path. We omit the proof which is immediate thanks to Lemma 2.8 and to point 6) of Lemma 2.3.

Proposition 2.9. *Let $\psi \in \mathcal{T}$ and let $u = \alpha_n \dots \alpha_1$ be a path in Q (with $a_i \in Q_1$ for any i). For each i, let:*

$$\psi(\alpha_i) = \alpha_i + \sum_{j=1}^{m_i} \lambda_{i,j} u_{i_j}$$

be a normal form for $\psi(\alpha_i)$. Then $supp(\psi(u))$ is the set of the paths in Q described as follows. Let $r \in \{0, \dots, n\}$, let $1 \leqslant i_1 < \dots < i_r \leqslant n$ be indices, for each $l \in \{1, \dots, r\}$ let $j_l \in \{1, \dots, m_i\}$, then the following path obtained from u after replacing α_{i_l} by u_{j_l} for each l belongs to $supp(\psi(u))$:

$$\alpha_n \dots \alpha_{i_r+1} u_{j_r} \alpha_{i_r-1} \dots \alpha_{i_l+1} u_{j_l} \alpha_{i_l-1} \dots \alpha_{i_1+1} u_{j_1} \alpha_{i_1-1} \dots \alpha_1$$

Moreover, this path appears in $\psi(u)$ with coefficient:

$$\lambda_{i_1,j_1} \dots \lambda_{i_r,j_r}$$

As a consequence, $\psi(u) - u$ is a linear combination of paths derived of u.

Remark 2.10. *If (α, u) is a bypass and if $v \in supp(\psi(u) - u)$, then (α, v) is also a bypass and $(\alpha, v) < (\alpha, u)$.*

Now we are able to state the main result of this paragraph. It describes $\psi(\alpha)$ ($\alpha \in Q_1$) using a particular writing of ψ as a product of transvections.

Proposition 2.11. *Let $(\alpha_1, u_1) < \dots < (\alpha_n, u_n)$ be an increasing sequence of bypasses, let $\tau_1, \dots, \tau_n \in k^*$ and set $\psi = \varphi_{\alpha_n,u_n,\tau_n} \dots \varphi_{\alpha_1,u_1,\tau_1}$. For any $\alpha \in Q_1$, there is a normal form for $\psi(\alpha)$:*

$$\psi(\alpha) = \alpha + \sum_{i \text{ such that } \alpha=\alpha_i} \tau_i u_i$$

Proof: Let us prove that the conclusion of the proposition is true using an induction on $n \geqslant 1$. By definition of a transvection, the proposition holds of $n = 1$. Assume that $n \geqslant 2$ and that the conclusion of the proposition holds if we replace $\psi = \varphi_{\alpha_n,u_n,\tau_n} \dots \varphi_{\alpha_1,u_1,\tau_1}$ by $\varphi_{\alpha_{n-1},u_{n-1},\tau_{n-1}} \dots \varphi_{\alpha_1,u_1,\tau_1}$. Therefore, for $\alpha \in Q_1$, we have a normal form:

$$\varphi_{\alpha_{n-1},u_{n-1},\tau_{n-1}} \dots \varphi_{\alpha_1,u_1,\tau_1}(\alpha) = \alpha + \sum_{i \leqslant n-1,\ \alpha=\alpha_i} \tau_i u_i$$

So:

$$\psi(\alpha) = \varphi_{\alpha_n,u_n,\tau_n}(\alpha) + \sum_{i \leqslant n-1,\ \alpha=\alpha_i} \tau_i \varphi_{\alpha_n,u_n,\tau_n}(u_i) \qquad (i)$$

Let $i \in \{1, \dots, n-1\}$. Thanks to Lemma 2.7, the inequality $(\alpha_i, u_i) < (\alpha_n, u_n)$ implies that $(\alpha_i, u_i, \alpha_n, u_n)$ is not a double bypass. Thus, α_n does not appear in the path u_i. This proves that:

$$(\forall i \in \{1, \dots, n-1\}) \quad \varphi_{\alpha_n,u_n,\tau_n}(u_i) = u_i \qquad (ii)$$

The definition of $\varphi_{\alpha_n,u_n,\tau_n}$, together with (i) and (ii), imply the equality:

$$\psi(\alpha) = \alpha + \sum_{\alpha=\alpha_i} \tau_i u_i \qquad (iii)$$

It only remains to prove that the equality (iii) is a normal form. Remark that all the scalars which appear in the right-hand side of (iii) are non zero. Moreover, if $i \in \{1, \dots, n\}$ verifies $\alpha = \alpha_i$, then $\alpha \neq u_i$, because

(α, u_i) is a bypass. Finally, if $1 \leqslant i < j \leqslant n$ verify $\alpha = \alpha_i = \alpha_j$, then $(\alpha, u_i) = (\alpha_i, u_i) < (\alpha_j, u_j) = (\alpha, u_j)$ so $u_i \neq u_j$. Therefore, (iii) is a normal form for $\psi(\alpha)$. ■

When $\psi \in \mathcal{T}$ is like in Proposition 2.11, we shall say that ψ is written as a decreasing product of transvections. Later we will prove that any $\psi \in \mathcal{T}$ can be written uniquely as a decreasing product of transvections. The description in Proposition 2.11 will be particularly useful in the sequel. We end this paragraph with two propositions concerning the description of $\psi(r)$ when $\psi \in \mathcal{T}$ and r is a linear combination of paths. The following proposition gives conditions for $\psi^{-1}(r')$ to be a subexpression of r when r' is a subexpression of $\psi(r)$.

Proposition 2.12. *Let $\psi \in \mathcal{T}$, let $r \in {}_ykQ_x$ and let r' be a subexpression of $\psi(r)$. Let $\simeq$ be the equivalence relation on the set of paths in Q generated by:*

$$v \in supp(\psi(u)) \Rightarrow u \simeq v$$

Assume that for any $u, v \in supp(\psi(r))$ verifying $u \simeq v$ we have:

$$u \in supp(r') \Leftrightarrow v \in supp(r')$$

Then $\psi^{-1}(r')$ is a subexpression of r.

Proof: Let $\simeq'$ be the trace of $\simeq$ on $supp(r)$ and let us write $supp(r) = c_1 \sqcup \ldots \sqcup c_n$ as a disjoint union of its $\simeq'$-classes. This partition of $supp(r)$ defines a decomposition of $r = r_1 + \ldots + r_n$ where r_i is the subexpression of r verifying $supp(r_i) = c_i$. For each i, let us fix a normal form:

$$r_i = \sum_{j=1}^{n_i} t_{i,j} u_{i,j}$$

so that we have the following normal form for r:

$$r = \sum_{i=1}^{n} \sum_{j=1}^{n_i} t_{i,j} u_{i,j}$$

Let us set $r'_i := \psi(r_i)$. In order to prove that $\psi^{-1}(r')$ is a subexpression of r, we will prove that there exist indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$ verifying $r' = r'_{i_1} + \ldots + r'_{i_t}$ (so that $\psi^{-1}(r') = r'_{i_1} + \ldots + r'_{i_t}$). In this purpose, we will successively prove the following facts:

1) $u, v \in supp(r'_i) \Rightarrow u \simeq v$, for any i,
2) $supp(r'_1), \ldots, supp(r'_n)$ are pairwise disjoint,
3) for each i, r'_i is a subexpression of $\psi(r)$,
4) if $i \in \{1, \ldots, n\}$ verifies $supp(r') \cap supp(r'_i) \neq \emptyset$, then $supp(r'_i) \subseteq supp(r')$,

1) Let $i \in \{1, \ldots, n\}$ and let $u, v \in supp(r'_i)$. So there exist $u', v' \in supp(r_i)$ such that $u \in supp(\psi(u'))$ and $v \in supp(\psi(v'))$. By definition of $\simeq$ and of r_i, we deduce that:

$$u, v \in supp(r'_i) \Rightarrow u \simeq v \qquad (i)$$

2) Let $i, j \in \{1, \ldots, n\}$ be such that there exists $v \in supp(r'_i) \cap supp(r'_j)$. So there exist $u \in supp(r_i)$ and $u' \in supp(r_j)$ such that $v \in supp(\psi(u))$ and $v \in supp(\psi(u'))$. This implies that $u \simeq v \simeq u'$. Since $u \in c_i = supp(r_i)$ and $u' \in c_j = supp(r_j)$, we deduce that $c_i = c_j$ and therefore $i = j$. So:

$$i \neq j \Rightarrow supp(r'_i) \cap supp(r'_j) = \emptyset \qquad (ii)$$

3) We have $\psi(r) = r'_1 + \ldots + r'_n$ so (ii) implies that:

$$r'_i \text{ is a subexpression of } \psi(r) \text{ for any } i \qquad (iii)$$

4) Let $i \in \{1, \ldots, n\}$ and assume that there exists $u \in supp(r'_i) \cap supp(r')$. If $v \in supp(r'_i)$ then $u \simeq v$ thanks to (i). So, by assumption on r', we have $v \in supp(r')$. This proves that:

$$supp(r'_i) \cap supp(r') \neq \emptyset \Rightarrow supp(r'_i) \subseteq supp(r') \qquad (iv)$$

Now, we can prove that $\psi^{-1}(r')$ is a subexpression of r. Thanks to (iii), the elements $r', r'_1, \ldots, r'_n$ are subexpressions of $\psi(r)$. So (iv) and the equality $\psi(r) = r'_1 + \ldots + r'_n$ imply that there exist indices $1 \leqslant i_1 < \ldots < i_t \leqslant n$ such that $r' = r'_{i_1} + \ldots + r'_{i_t}$. So $\psi^{-1}(r') = r_{i_1} + \ldots + r_{i_n}$. This proves that $\psi^{-1}(r')$ is a subexpression of r. ■

The last proposition of this subsection gives a sufficient condition on $u \in supp(r)$ to verify $u \in supp(\psi(r))$.

Proposition 2.13. *Let $\psi \in \mathcal{T}$, let $r \in {}_ykQ_x$ and let $u \in supp(r)$. Then, at least one of the two following facts is verified:*

- *$u \in supp(\psi(r))$,*
- *there exists $v \in supp(r)$ such that $u \neq v$ and such that $v \in supp(\psi(u))$.*

As a consequence, if u is not derived of v for any $v \in supp(r)$, then:

$$u \in supp(\psi(r)) \quad and \quad u^*(\psi(r)) = u^*(r)$$

Proof: Let us fix a normal form $r = \sum_{i=1}^{n} t_i u_i$ where we may assume that $u = u_1$. Let us assume that $u \notin supp(\psi(r))$, i.e. $u^*(\psi(r)) = 0$. Recall from Proposition 2.9 that $u^*(\psi(u)) = 1$, so:

$$0 = u^*(\psi(r)) = t_1 + \sum_{i=2}^{n} t_i u^*(\psi(u_i)) \qquad (i)$$

Therefore, there exists $i_0 \in \{2, \ldots, n\}$ such that $u^*(\psi(u_{i_0})) \neq 0$. So:

$$u_{i_0} \in supp(r), \quad u_{i_0} \neq u_1 = u \quad \text{and} \quad u_1^*(\psi(u_{i_0})) \neq 0$$

This proves the first assertion of the proposition. Now let us assume that u is not derived of v for any $v \in supp(r)$. Let $i \in \{2, \ldots, n\}$. Since $u = u_1 \neq u_i$, Proposition 2.9 gives the following implications:

$$u \in supp(\psi(u_i)) \Rightarrow u \in supp(\psi(u_i) - u_i) \Rightarrow u \text{ is derived of } u_i$$

By assumption on u, this implies that $u^*(\psi(u_i)) = 0$ for any $i \geqslant 2$. Using (i), we deduce the announced conclusion: $u^*(\psi(r)) = t_1 = u^*(r) \neq 0$ ■

2.4 Ordering products of transvections

In Proposition 2.11 we have seen that $\psi(\alpha)$ may be easily computed when $\psi \in \mathcal{T}$ and $\alpha \in Q_1$ provided that ψ is written as a decreasing product of transvections. In this subsection, we will prove that any $\psi \in \mathcal{T}$ can be uniquely written that way. Recall that $<$ is an order on the set of non trivial paths in Q defined in Definition 2.5. We introduce first some notations.

Definition 2.14. *Let (α, u) be a bypass. We set $\mathcal{T}_{<(\alpha,u)}$ and $\mathcal{T}_{\leqslant(\alpha,u)}$ to be the subgroups of $\mathcal{T}$ generated by the following sets of transvections:*

$$\{\varphi_{\beta,v,\tau} \mid (\beta, v) < (\alpha, u) \text{ and } \tau \in k\} \quad \text{for } \mathcal{T}_{<(\alpha,u)}$$
$$\{\varphi_{\beta,v,\tau} \mid (\beta, v) \leqslant (\alpha, u) \text{ and } \tau \in k\} \quad \text{for } \mathcal{T}_{\leqslant(\alpha,u)}$$

Also, we define $\mathcal{T}_{(\alpha,u)}$ to be the following subgroup of $\mathcal{T}$:

$$\mathcal{T}_{(\alpha,u)} = \{\varphi_{\alpha,u,\tau} \mid \tau \in k\}$$

Remark 2.15.

- *$\mathcal{T}_{(\alpha,u)}$ is indeed a subgroup of $\mathcal{T}$ because $\varphi_{\alpha,u,\tau}\varphi_{\alpha,u,\tau'} = \varphi_{\alpha,u,\tau+\tau'}$ for any $\tau, \tau' \in k$. Actually, the following mapping is an isomorphism of abelian groups:*

$$\begin{array}{ccc} k & \longrightarrow & \mathcal{T}_{(\alpha,u)} \\ \tau & \longmapsto & \varphi_{\alpha,u,\tau} \end{array}$$

- *$\mathcal{T}_{\leqslant(\alpha,u)}$ is generated by $\mathcal{T}_{<(\alpha,u)} \cup \mathcal{T}_{(\alpha,u)}$.*
- *If $(\alpha, u) < (\beta, v)$, then $\mathcal{T}_{\leqslant(\alpha,u)} \subseteq \mathcal{T}_{\leqslant(\beta,v)}$ and $T_{<(\alpha,u)} \subseteq T_{<(\beta,v)}$.*
- *$\mathcal{T} = \bigcup_{(\alpha,u)} \mathcal{T}_{\leqslant(\alpha,u)}$ and if (α_m, u_m) is the greatest bypass in Q, then $\mathcal{T} = \mathcal{T}_{\leqslant(\alpha_m,u_m)}$ (recall that Q has finitely many bypasses because it has no oriented cycle).*

The following lemma proves that any $\psi \in \mathcal{T}$ is a decreasing product of transvections.

Lemma 2.16.

- *$\mathcal{T}_{<(\alpha,u)}$ is a normal subgroup of $\mathcal{T}_{\leqslant(\alpha,u)}$, for any bypass (α, u).*
- *Let $(a_1, v_1) < \ldots < (a_N, v_N)$ be the (finite) increasing sequence of all the bypasses in Q. Then:*
  - *$\mathcal{T}_{\leqslant(a_i,v_i)} = \mathcal{T}_{<(a_{i-1},v_{i-1})}$ if $i \geqslant 1$,*
  - *$\mathcal{T}_{<(a_1,v_1)} = 1$,*

- $\mathcal{T}_{\leqslant(a_i,v_i)} = \mathcal{T}_{(a_i,v_i)}\mathcal{T}_{(a_{i-1},v_{i-1})}\dots\mathcal{T}_{(a_1,v_1)}$.

Proof: Thanks to Remark 2.15, we only need to prove that if $\tau,\nu \in k$ and if $(\beta,v),(\alpha,u)$ are bypasses such that $(\beta,v)<(\alpha,u)$, then:

$$\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau} \in \varphi_{\alpha,u,\tau}\mathcal{T}_{<(\alpha,u)} \qquad (\star)$$

Let us distinguish two cases wether (α,u,β,v) is a double bypass or not. Notice that thanks to Lemma 2.7 and to the inequality $(\beta,v)<(\alpha,u)$ we know that (β,v,α,u) is not a double bypass. If (α,u,β,v) is a double bypass, then we have from Section 1 (recall that Q as no multiple arrows):

$$\varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau} = \varphi_{\alpha,u,\tau}\varphi_{\alpha,w,\tau\nu}\varphi_{\beta,v,\nu}$$

where w is the path obtained from u after replacing β by v. Moreover, Lemma 2.7 implies that $(\beta,v)<(\alpha,w)<(\alpha,u)$. Therefore, $(\star)$ is satisfied when (α,u,β,v) is a double bypass. If (α,u,β,v) is not a double bypass, then we know from Section 1 that:

$$\varphi_{\alpha,u,\tau}\varphi_{\beta,v,\nu} = \varphi_{\beta,v,\nu}\varphi_{\alpha,u,\tau}$$

So $(\star)$ is also satisfied when (α,u,β,v) is not a double bypass. ■

Using the preceding lemma we are able to prove that any $\psi \in \mathcal{T}$ can be uniquelely written as a decreasing product of transvections.

Proposition 2.17. *Let (α,u) be a bypass and let $\psi \in \mathcal{T}_{\leqslant(\alpha,u)}$. Then, there exist a non negative integer n, a sequence of bypasses $(\alpha_1,u_1),\dots,(\alpha_n,u_n)$ and non zero scalars $\tau_1,\dots,\tau_n \in k^*$ verifying:*

(i) $\psi = \varphi_{\alpha_n,u_n,\tau_n}\dots\varphi_{\alpha_1,u_1,\tau_1}$,

(ii) $(\alpha_1,u_1)<\dots<(\alpha_n,u_n)\leqslant(\alpha,u)$.

Moreover, the integer n and the sequence $(\alpha_1,u_1,\tau_1),\dots,(\alpha_n,u_n,\tau_n)$ are unique for these properties.

Proof: Thanks to Lemma 2.16 we know that there exist bypasses $(\alpha_1,u_1),\dots,(\alpha_n,u_n)$ and non zero scalars $\tau_1,\dots,\tau_n \in k^*$ such that (i) and (ii) are satisfied. So we only need to prove the uniqueness property. Let A,B and T be the following sets:

$$\begin{aligned} A &:= \{\alpha \in Q_1 \mid \psi(\alpha)\neq\alpha\} \\ B &:= \{(\alpha,u) \mid (\alpha,u) \text{ is a bypass, } \alpha\in A \text{ and } u\in supp(\psi(\alpha))\} \\ T &:= \{(\alpha,u,\tau) \mid (\alpha,u)\in B \text{ and } \tau = u^*(\psi(\alpha))\} \end{aligned}$$

Notice that the definition of A,B,T depend on ψ only (and not on the triples (α_i,u_i,τ_i)). Let $\beta\in Q_1$. Then Proposition 2.11 gives a normal form:

$$\psi(\beta) = \beta + \sum_{i \text{ such that } \beta=\alpha_i} \tau_i u_i$$

By definition of a normal form, we deduce the following equalities:

$$\begin{aligned} A &= \{\alpha_1,\dots,\alpha_n\} \\ B &= \{(\alpha_1,u_1),\dots,(\alpha_n,u_n)\} \\ T &= \{(\alpha_1,u_1,\tau_1),\dots,(\alpha_n,u_n,\tau_1)\} \end{aligned}$$

This proves that n and $(\alpha_1,u_1,\tau_1),\dots,(\alpha_n,u_n,\tau_n)$ are uniquely determined by the sets A,B,T (which depend on ψ only) and by the total order $<$. ■

3 Comparison of the presentations of a monomial algebra

Let $kQ/I \simeq A$ be an admissible presentation of A. Thanks to Proposition 1.1, we know that there exists ψ equal to a product of transvections and of a dilatation and such that $\psi(I_0)=I$. The aim of this section is to exhibit ψ_I the "simplest" possible among all the ψ's verifying $\psi(I_0)=I$. We shall see that ψ_I verifies a property which makes it unique. In order to find ψ_I we will use specific properties of the Groebner basis of I (due to the fact that I_0 is monomial). So, throughout the section, $<$ will denote a total order on the set of non trivial paths in Q, as in Definition 2.5. We begin by giving some useful properties on the automorphisms ψ verifying $\psi(I_0)=I_0$.

Lemma 3.1. *Let $D \in \mathcal{D}$ be a dilatation. Then $D(I_0) = I_0$. As a consequence, if $kQ/I \simeq A$ is an admissible presentation, then there exists $\psi \in \mathcal{T}$ such that $\psi(I_0) = I$.*

Proof: The first assertion is due to the fact that $D(u) \in k^*u$ for any path u and to the fact that I_0 is monomial. The second one is a consequence of the first one and of Proposition 1.1. ■

Lemma 3.2. *Let (α, u) be a bypass in Q. Then exactly one of the two following assertions is satisfied:*

- *$\varphi_{\alpha,u,\tau}(I_0) = I_0$ for any $\tau \in k$.*
- *$\varphi_{\alpha,u,\tau}(I_0) \neq I_0$ for any $\tau \in k^*$.*

Proof: Assume that $\tau \in k^*$ verifies $\varphi_{\alpha,u,\tau}(I_0) = I_0$ and let $\mu \in k$. Let $v \in I_0$ be a path. If α does not appear in v, then $\varphi_{\alpha,u,\nu}(v) = v \in I_0$. Assume that α appears in v, i.e. $v = v_2\alpha v_1$ with v_1, v_2 paths in which α does not appear (because Q has no oriented cycle). Therefore, $\varphi_{\alpha,u,\tau}(v) = v + \tau v_2 u v_1$. Since $\varphi_{\alpha,u,\tau}(I_0) = I_0$, we deduce that $v_2uv_1 \in I_0$. This implies that $\varphi_{\alpha,u,\nu}(v) = v + \nu v_2 u v_1 \in I_0$. Since I_0 is monomial, we infer that $\varphi_{\alpha,u,\nu}(I_0) = I_0$. ■

Lemma 3.3. *Let $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ be an increasing sequence of bypasses, let $\tau_1, \ldots, \tau_n \in k^*$ and set $\psi = \varphi_{\alpha_n,u_n,\tau_n} \ldots \varphi_{\alpha_1,u_1,\tau_1}$. Then:*

$$\psi(I_0) = I_0 \quad \Leftrightarrow \quad \varphi_{\alpha_i,u_i,\tau_i}(I_0) = I_0 \text{ for any } i$$

Proof: Let us assume that $\psi(I_0) = I_0$. Let $i \in \{1, \ldots, n\}$, let $u = a_r \ldots a_1 \in I_0$ be a path (with $a_i \in Q_1$) and fix $i \in \{1, \ldots, n\}$. We aim at proving that $\varphi_{\alpha_i,u_i,\tau_i}(u) \in I_0$. If $a_j \neq \alpha_i$ for any $j \in \{1, \ldots, r\}$ then $\varphi_{\alpha_i,u_i,\tau_i}(u) = u \in I_0$. Now assume that there exists $j \in \{1, \ldots, r\}$ such that $a_j = \alpha_i$ (j is necessarily unique because Q has no oriented cycle). Therefore:

$$\varphi_{\alpha_i,u_i,\tau_i}(u) = u + \tau_i a_r \ldots a_{j+1} u_i a_{j-1} \ldots a_1 \tag{i}$$

On the other hand, Proposition 2.9 and Proposition 2.11 imply that $a_r \ldots a_{j+1} u_i a_{j-1} \ldots a_1 \in supp(\psi(u))$. Thus, we have (recall that $\psi(u) \in I_0$ and that I_0 is monomial):

$$a_r \ldots a_{j+1} u_i a_{j-1} \ldots a_1 \in I_0 \tag{ii}$$

From (i) and (ii) we deduce that $\varphi_{\alpha_i,u_i,\tau_i}(u) \in I_0$ for any path $u \in I_0$. So $\varphi_{\alpha_i,u_i,\tau_i}(I_0) = I_0$ for any i. The remaining implication is immediate. ■

Remark 3.4. *The three preceding lemmas imply that the group $Aut_0(kQ, I_0)$ defined as follows:*

$$Aut_0(kQ, I_0) := \{\psi \in Aut(kQ) \mid \psi(x) = x \text{ for any } x \in Q_0, \text{ and } \psi(I_0) = I_0\}$$

is generated by the dilatations and by all the transvections preserving I_0:

$$Aut_0(kQ, I_0) = < \mathcal{D} \cup \{\varphi \mid \varphi \text{ is a transvection such that } \varphi(I_0) = I_0\} >$$

Now we give some properties on the Groebner basis of the admissible ideals I of kQ such that $kQ/I \simeq A$. Recall that for such an I, there exists $\psi \in \mathcal{T}$ such that $\psi(I_0) = I$ (see Lemma 3.1).

Proposition 3.5. *Let $\psi \in \mathcal{T}$ and let let $I = \psi(I_0)$. Let B_0 (resp. B) be the Groebner basis of I_0 (resp. of I). Then B_0 is made of all the paths in Q which belong to I_0. Moreover, the mapping:*

$$\begin{array}{ccc} B & \longrightarrow & B_0 \\ r & \longmapsto & max(supp(r)) \end{array} \tag{$\star$}$$

is well defined and bijective. For $u \in B_0$, let us write $r_u \in B$ for the inverse image of u under $(\star)$. Then $supp(r_u - u)$ is a set of paths derived of u.

Proof: Let $u_1 < \ldots < u_n$ be the increasing sequence of all the non trivial paths in Q. Let $(r_1, \ldots, r_d)$ be the Groebner basis of I and for each $j \in \{1, \ldots, d\}$, let $i_j \in \{1, \ldots, n\}$ be such that:

$$r_j \in u_{i_j} + Span(u_l \ ; \ l < i_j)$$

Since I_0 is monomial, B_0 is made of all the paths in Q belonging to I_0.

Let $j \in \{1, \ldots, d\}$. Since $u_{i_j} = max(supp(r_j))$, the path u_{i_j} is not derived of u for any $u \in supp(r_j)$ (thanks to Lemma 2.7). So Proposition 2.13 implies that $u_{i_j} \in supp(\psi^{-1}(r_j)) \in I_0$. Because I_0 is

monomial, this proves that $u_{i_j} \in I_0$. Therefore, the mapping $(\star)$ is well defined. It is also one-to-one because of the definition of the Groebner basis of I. Now let us prove that $(\star)$ is onto. Let $u \in B_0$. Proposition 2.9 implies that $u = max(supp(\psi(u))$. Since $\psi(u) \in I$, we deduce that there exists $j \in \{1, \dots, d\}$ such that $u = u_{i_j} = max(supp(r_j))$. This proves that $(\star)$ is onto and therefore bijective.

Let us prove the last assertion of the proposition. To do this, we will prove by induction on $j \in \{1, \dots, d\}$ that the following assertion is true:

$$H_j : "supp(r_j - u_{i_j}) \text{ is a set of paths derived of } u_{i_j}"$$

Proposition 2.9 implies that for any j:

$$u_{i_j} = max(supp(\psi(u_{i_j}))) \text{ and } u^*_{i_j}(\psi(u_{i_j})) = 1 \qquad (i)$$

Moreover, $\psi(u_{i_j}) \in I$ because $(\star)$ is well defined and because $\psi(I_0) = I$. Let us prove that H_1 is true. Both r_1 and $\psi(u_{i_1})$ lie in I. Moreover, $u_{i_1} = max(supp(r_1))$ by definition of u_{i_1} and $u_{i_1} = max(supp(\psi(u_{i_1})))$ because of Proposition 2.9. So H_1 is true. Assume that $j \geqslant 2$ and that $H_1, \dots, H_{j-1}$ are true. Since $\psi(u_{i_j}) \in I$ and because of (i), we have:

$$\psi(u_{i_j}) = r_j + \sum_{\substack{j' < j, \\ u_{i_{j'}} \in supp(\psi(u_{i_j}))}} u^*_{i_{j'}}(\psi(u_{i_j}))r_{j'}$$

So:

$$r_j - u_{i_j} = \psi(u_{i_j}) - u_{i_j} - \sum_{\substack{j' < j, \\ u_{i_{j'}} \in supp(\psi(u_{i_j}))}} u^*_{i_{j'}}(\psi(u_{i_j}))\left[(r_{j'} - u_{i_{j'}}) + u_{i'_j}\right] \qquad (ii)$$

Notice that in the above equality:

(iii) $supp(\psi(u_{i_j}) - u_{i_j})$ is a set of paths derived of u_{i_j} (thanks to Proposition 2.9),

(iv) if $j' < j$ verifies $u_{i_{j'}} \in supp(\psi(u_{i_j}))$, then:

- (v) $u_{i_{j'}}$ is derived of u_{i_j} (see (iii) above),
- (vi) $supp(r_{j'} - u_{i_{j'}})$ is a set of paths derived of $u_{i_{j'}}$ (because $H_{j'}$ is true) and therefore derived of u_{i_j} (thanks to (v) and to Lemma 2.3).

The points $(ii) - (vi)$ prove that H_j is true. This proves that H_j is true for any $j \in \{1, \dots, d\}$ and finishes the proof of the proposition. ■

Now we can state and prove the proposition giving the existence and the uniqueness of the automorphism ψ_I mentionned at the beginning of the section.

Proposition 3.6. *Let $kQ/I \simeq A$ be an admissible presentation. Then there exists a unique $\psi_I \in \mathcal{T}$ verifying the following conditions:*

1) $\psi_I(I_0) = I$,
2) *if (α, u) is a bypass such that $u \in supp(\psi_I(\alpha))$ then $\varphi_{\alpha,u,\tau}(I_0) \neq I_0$ for any $\tau \in k^*$ (see Lemma 3.2).*

Proof: • Let us prove the existence of ψ_I. Thanks to Lemma 3.1 we know that there exists $\psi \in \mathcal{T}$ verifying 1). Let us set:

$$\mathcal{A} := \{\psi \in \mathcal{T} \mid \psi(I_0) = I\}$$

and let us assume that for any $\psi \in \mathcal{A}$, the condition 2) is not verified. So, for any $\psi \in \mathcal{A}$, we have a finite (recall that Q has no oriented cycle) and non empty set of bypasses (see Lemma 3.2):

$$B_\psi = \left\{ (\alpha, u) \;\middle|\; \begin{array}{l} (\alpha, u) \text{ is a bypass} \\ u \in supp(\psi(\alpha)) \\ \varphi_{\alpha,u,\tau}(I_0) = I_0 \text{ for any } \tau \in k \end{array} \right\}$$

For each $\psi \in \mathcal{A}$, let $(\alpha_\psi, u_\psi) = max\ B_\psi$ and let us fix $\psi \in A$ such that:

$$(\alpha_\psi, u_\psi) = min\ \{(\alpha_{\psi'}, u_{\psi'}) \mid \psi' \in \mathcal{A}\}$$

For simplicity we shall use the following notations:

$$(\alpha, u) := (\alpha_\psi, u_\psi), \quad \tau := u^*(\psi(\alpha)) \text{ and } \psi' := \psi\varphi_{\alpha,u,-\tau}$$

Notice that $\psi' \in \mathcal{A}$ because $(\alpha, u) \in B_\psi$. In order to get a contradiction, we are going to prove that $(\alpha_{\psi'}, u_{\psi'}) < (\alpha, u)$. To do this, let us prove first that $(\alpha, u) \notin B_{\psi'}$. Thanks to Proposition 2.17, we have the equality:

$$\psi = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_1, u_1, \tau_1}$$

where $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ and where $\tau_1, \ldots, \tau_n \in k^*$. On the other hand, since since $u^*(\psi(\alpha)) = \tau \neq 0$, we know from Proposition 2.11 that:

$$(\exists i \in \{1, \ldots, n\}) \;\; (\alpha_i, u_i, \tau_i) = (\alpha, u, \tau)$$

Let us set:

$$\psi_1 := \varphi_{\alpha_{i-1}, u_{i-1}, \tau_{i-1}} \cdots \varphi_{\alpha_1, u, \tau_1} \in \mathcal{T}_{<(\alpha, u)}$$

Hence, the following equality holds:

$$\psi' = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_{i+1}, u_{i+1}, \tau_{i+1}} \varphi_{\alpha, u, \tau} \psi_1 \varphi_{\alpha, u, \tau}^{-1}$$

Since $\psi_1 \in \mathcal{T}_{<(\alpha, u)}$, Lemma 2.16 implies that $\varphi_{\alpha, u, \tau} \psi_1 \varphi_{\alpha, u, \tau}^{-1} \in \mathcal{T}_{<(\alpha, u)}$. Therefore, Proposition 2.17 gives the equality:

$$\varphi_{\alpha, u, \tau} \psi_1 \varphi_{\alpha, u, \tau}^{-1} = \varphi_{\beta_m, v_m, \nu_m} \cdots \varphi_{\beta_1, v_1, \nu_1}$$

where $(\beta_1, v_1) < \ldots < (\beta_m, v_m) < (\alpha, u)$ and $\nu_1, \ldots, \nu_m \in k^*$. As a consequence, we have:

$$\psi' = \varphi_{\alpha_n, u_n, \tau_n} \cdots \varphi_{\alpha_{i+1}, u_{i+1}, \tau_{i+1}} \varphi_{\beta_m, v_m, \nu_m} \cdots \varphi_{\beta_1, v_1, \nu_1}$$

where $(\beta_1, v_1) < \ldots < (\beta_m, v_m) < (\alpha, u) < (\alpha_{i+1}, u_{i+1}) < \ldots < (\alpha_n, u_n)$ and where $\tau_{i+1}, \ldots, \tau_n, \nu_1, \ldots, \nu_m \in k^*$. In particular, Proposition 2.11 implies that $u \notin supp(\psi'(\alpha))$. Therefore:

$$(\alpha, u) \notin B_{\psi'} \tag{i}$$

Hence:

$$(\alpha, u) \neq (\alpha_{\psi'}, u_{\psi'}) = max \; B_{\psi'} \tag{ii}$$

Now let us prove that $(\alpha_{\psi'}, u_{\psi'}) \leqslant (\alpha, u)$. To do this, it suffices to pove that the following implication holds for any bypass (β, v):

$$v \in supp(\psi'(\beta)) \text{ and } (\alpha, u) < (\beta, v) \;\Rightarrow \varphi_{\beta, v, t}(I_0) \neq I_0 \text{ for any } \tau \in k^* \tag{iii}$$

Let (β, v) be a bypass such that $v \in supp(\psi'(\beta))$ and such that $(\alpha, u) < (\beta, v)$. Since $\psi' = \psi \varphi_{\alpha, u, -\tau}$, we have:

$$\psi'(\beta) = \begin{cases} \psi(\beta) & \text{if } \beta \neq \alpha \\ \psi(\beta) - \tau\psi(u) & \text{if } \beta = \alpha \end{cases}$$

Therefore, $v \in supp(\psi'(\beta)) \subseteq supp(\psi(\beta)) \cup supp(\psi(u))$. Remark that if $v \in supp(\psi(u))$, then Proposition 2.9 implies that v is derived of u (we have $u \neq v$ because $\beta = \alpha$ and $(\alpha, u) < (\beta, v)$) and therefore $(\alpha, u) > (\alpha, v) = (\beta, v)$ whereas we assumed that $(\alpha, u) < (\beta, v)$. This proves that $v \in supp(\psi(\beta))$. Since $(\beta, v) > (\alpha, u) = (\alpha_\psi, u_\psi) = max \; B_\psi$ we deduce that $\varphi_{\beta, v, \tau}(I_0) = I_0$ for any $\tau \in k$. This proves that the implication (iii) is satisfied, thus:

$$(\alpha_{\psi'}, u_{\psi'}) \leqslant (\alpha, u) \tag{iv}$$

From (ii) and (iv) we deduce that:

$$(\alpha_{\psi'}, u_{\psi'}) < (\alpha_\psi, u_\psi) = (\alpha, u)$$

This contradicts the minimality of (α_ψ, u_ψ) and proves the existence of ψ.

• Now let us prove the uniqueness of ψ_I. Assume that $\psi, \psi' \in \mathcal{T}$ verify the conditions 1) and 2). In order to prove that $\psi = \psi'$ we only need to prove that $\theta^*(\psi(\alpha)) = \theta^*(\psi'(\alpha))$ for any bypass (α, θ). Let $\alpha \in Q_1$ and assume that there exists a minimal path θ such that (α, θ) is bypass and such that $\theta^*(\psi(\alpha)) \neq \theta^*(\psi'(\alpha))$. We may assume that $\theta^*(\psi(\alpha)) \neq 0$, i.e. $\theta \in supp(\psi(\alpha))$. Since ψ verifies 2), we deduce that there exist paths u and v such that:

$$u \in I_0, \;\; v \notin I_0 \text{ and } \varphi_{\alpha, \theta, 1}(u) = u + v \notin I_0$$

Notice that Proposition 2.9 gives:

$$\begin{cases} v^*(\psi(u)) = \theta^*(\psi(\alpha)) \text{ and } u^*(\psi(u)) = 1 \\ v^*(\psi'(u)) = \theta^*(\psi'(\alpha)) \text{ and } u^*(\psi'(u)) = 1 \end{cases} \tag{i}$$

Since $u \in I$, we have $\psi(u), \psi'(u) \in I_0$. Therefore, using Proposition 3.5 and with the same notations concerning the Groebner bases, we have:

$$\begin{cases} \psi(u) = r_u + \sum\limits_{w \in \mathcal{A}_\psi} w^*(\psi(u))r_w \\ \psi'(u) = r_u + \sum\limits_{w \in \mathcal{A}_{\psi'}} w^*(\psi'(u))r_w \end{cases}$$

where $\mathcal{A}_\psi$ is equal to:

$$\mathcal{A}_\psi := \{w \in supp(\psi(u)) \mid w \neq u \text{ and } w \in I_0\}$$

So:

$$\begin{cases} v^*(\psi(u)) = v^*(r_u) + \sum\limits_{w \in \mathcal{A}_\psi} w^*(\psi(u))v^*(r_w) \\ v^*(\psi'(u)) = v^*(r_u) + \sum\limits_{w \in \mathcal{A}_{\psi'}} w^*(\psi'(u))v^*(r_w) \end{cases} \quad (ii)$$

Let $w \in \mathcal{A}_\psi$ be such that $v^*(r_w) \neq 0$, i.e. $v \in supp(r_w)$. Since $v \notin I_0$ and since $w \in I_0$ we deduce that $v \in supp(r_w - w)$. So:

. v is derived of w (thanks to Proposition 3.5 and because $v \in supp(r_w - w)$).

. v is derived of u of order 1 (because $\varphi_{\alpha,\theta,1}(u) = u + v$).

. w is derived of u (because $w \in \mathcal{A}_\psi$ and thanks to Proposition 2.9).

Using Lemma 2.3, these three facts imply that:

$$u = u_2\alpha u_1,\ v = u_2\theta u_1 \text{ and } w = u_2\theta' u_1 \quad (iii)$$

where u_1, u_2 are paths and where θ' is a path derived of θ. In particular, (α, θ') is a bypass such that $\theta' < \theta$ (see Lemma 2.7). Therefore, the minimality of θ forces $\theta'^*(\psi(\alpha)) = \theta'^*(\psi'(\alpha)))$. Moreover, (iii) and Proposition 2.9 imply that

$$w^*(\psi(u)) = \theta'^*(\psi(\alpha)) = \theta'^*(\psi'(\alpha)) = w^*(\psi'(u))$$

Therefore we have proved the following implication:

$$w \in \mathcal{A}_\psi \text{ and } v^*(r_w) \neq 0 \ \Rightarrow\ w^*(\psi(u))v^*(r_w) = w^*(\psi'(u))v^*(r_w) \quad (iv)$$

After exchangeing the roles of ψ and ψ', the arguments used to prove (iv) also give the following implication:

$$w \in \mathcal{A}_{\psi'} \text{ and } v^*(r_w) \neq 0 \ \Rightarrow\ w^*(\psi(u))v^*(r_w) = w^*(\psi'(u))v^*(r_w) \quad (v)$$

From (ii), (iv) and (v) we deduce that $v^*(\psi(u)) = v^*(\psi'(u))$, and from (i) we infer that $\theta^*(\psi(\alpha)) = \theta^*(\psi'(\alpha))$ whereas we assumed the contrary. This proves that $\psi = \psi'$. ■

4 Proof of the main theorem

The aim of this section is to prove that the quiver Γ of the homotopy relations of the admissible presentations of A has $\sim_{I_0}$ as unique source. This fact will be used in order to exhibit the universal cover of A. Notice that $\sim_{I_0}$ is a source of Γ. Indeed, all minimal relations in I_0 are monomial relations so, for any $\sim_I \in \Gamma_0$ we have $\gamma \sim_{I_0} \gamma' \Rightarrow \gamma \sim_I \gamma'$. In order to prove that $\sim_{I_0}$ is the unique source in Γ we will prove that for any admissible presentation $kQ/I \simeq A$, the decomposition of ψ_I (given by Proposition 3.6) into a decreasing product of transvections (see Proposition 2.17) defines a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_I$. In this purpose, we begin with the following proposition.

Proposition 4.1. *Let $kQ/I \simeq A$ be an admissible presentation. Then, for any bypass (α, u) we have:*

$$u \in supp(\psi_I(\alpha)) \Rightarrow u \sim_I \alpha$$

Proof: For simplicity we shall write ψ for ψ_I. Thanks to Proposition 2.17 we have:

$$\psi = \varphi_{\alpha_n,u_n,\tau_n} \cdots \varphi_{\alpha_1,u_1,\tau_1}$$

with $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ and $\tau_1, \ldots, \tau_n \in k^*$. Notice that Proposition 2.11 implies that we only need to prove that $\alpha_i \sim_I u_i$ for any i. We will prove this fact by a decreasing induction on $m \in \{1, \ldots, n\}$. Let us set:

$$H_m : "\alpha_i \sim_I u_i \text{ for any } i \in \{m, m+1, \ldots, n\}"$$

H_{n+1} is true because $\{i \mid n+1 \leqslant i \leqslant n\}$ is empty, so let us assume that $m \in \{1, \ldots, n\}$ is such that H_{m+1} is true and let us prove that so is H_m. From Proposition 2.11 we have $u_m \in supp(\psi(\alpha_m))$. So Proposition 3.6 provides a path $u \in I_0$ such that $\varphi_{\alpha_m, u_m, 1}(u) \notin I_0$. Therefore, there exist paths v_1, v_2 such that:

$$u = v_2 \alpha_m v_1, \quad v := v_2 u_m v_1 \notin I_0 \text{ and } \varphi_{\alpha_m, u_m, 1}(u) = u + v \tag{i}$$

Since $\psi(u) \in I$ we have:

$$\psi(u) = r_1 + \ldots + r_N$$

where $r_1, \ldots, r_N$ are minimal relations in I with pairwise disjoint supports. Remark that $u, v \in supp(\psi(u))$ thanks to Proposition 2.9 and to Proposition 2.11. Without loss of generality, we may assume that $v \in supp(r_1)$. Let $i \in \{1, \ldots, N\}$ be such that $u \in supp(r_i)$. If $i = 1$ then $u \sim_I v$ hence (i) gives $\alpha_m \sim_I u_m$. So we can assume that $i \neq 1$. Since $r_1 \in I$ we have $\psi^{-1}(r_1) \in I_0$. Because I_0 is monomial we deduce that $v \notin supp(\psi^{-1}(r_1))$. Thanks to Proposition 2.13, this proves that:

$$\text{there exists } w \in supp(r_1) \text{ such that } v \text{ is derived from } w \tag{ii}$$

Therefore:

- w is derived of u since $w \in supp(r_1) \subseteq supp(\psi(u))$ (see Proposition 2.9, $u \neq w$ because $u \notin supp(r_1)$),
- v is derived of w (see (ii)),
- v is derived of u of order 1 (because of (i)).

Thanks to Lemma 2.3, these three points imply that:

$$w = v_2 \theta v_1 \text{ and } u_m \text{ is derived of } \theta \tag{iii}$$

Since $w \in supp(\psi(u))$, the equalities $w = v_2 \theta v_1$, $u = v_2 \alpha_m v_1$ and Proposition 2.9 imply that $\theta \in supp(\psi(\alpha_m))$. Hence, there exists $j \in \{1, \ldots, n\}$ such that:

$$(\alpha_m, \theta) = (\alpha_j, u_j)$$

Since u_m is derived of θ (see (iii)), we get $u_j = \theta > u_m$ (see Lemma 2.7) and therefore $j > m$. Since H_{m+1} is true, we also have:

$$\alpha_m = \alpha_j \sim_I u_j = \theta \tag{iv}$$

Finally, since r_1 is a minimal relation in I such that $v, w \in supp(r_1)$, we have $v \sim_I w$. Using (i), (iii) and (iv) we deduce that:

$$\alpha_m \sim_I u_m$$

This proves that H_m is true. So H_1 is true, i.e. $\alpha_i \sim_I u_i$ for any i. ■

Remark 4.2. *In the preceding proposition we have proved that $\alpha \sim_I u$ for any $u \in supp(\psi(\alpha))$. On the other hand, $\sim_{I_0}$ is weaker than $\sim_I$ (i.e. $\gamma \sim_{I_0} \gamma' \Rightarrow \gamma \sim_I \gamma'$). These two properties are linked in general. Indeed, in [10, Prop. 4.2.35, Prop. 42.36] the author has proved that if I is an admissible ideal (non necessarily monomial) of kQ and if $\psi \in \mathcal{T}$ is such that $\alpha \sim_{\psi(I)} u$ for any bypass (α, u) such that $u \in supp(\psi(\alpha))$, then $\sim_I$ is weaker than $\sim_{\psi(I)}$.*

Now we can exhibit a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_I$, whenever $kQ/I \simeq A$.

Proposition 4.3. *Let $kQ/I \simeq A$ be an admissible presentation. Let $(\alpha_1, u_1) < \ldots < (\alpha_n, u_n)$ be the bypasses and let $\tau_1, \ldots, \tau_n \in k^*$ be the scalars such that $\psi_I = \varphi_{\alpha_n, u_n, \tau_n} \ldots \varphi_{\alpha_1, u_1, \tau_1}$ (see Proposition 2.17). For each $i \in \{1, \ldots, n\}$, set:*

$$I_i := \varphi_{\alpha_i, u_i, \tau_i} \ldots \varphi_{\alpha_1, u_1, \tau_1}(I_0)$$

then, for each i, one of the two following situations occurs:

- *$\sim_{I_{i-1}}$ and $\sim_{I_i}$ coincide,*
- *$\varphi_{\alpha_i, u_i, \tau_i}$ induces an arrow $\sim_{I_{i-1}} \rightarrow \sim_{I_i}$ in Γ.*

In particular, there exists a path in Γ starting at $\sim_{I_0}$ and ending at $\sim_{I_n} = \sim_I$.

Proof: Let $i \in \{1, \ldots, n\}$ and set $\psi_i := \varphi_{\alpha_i, u_i, \tau_i} \ldots \varphi_{\alpha_1, u_1, \tau_1}$. Thus $I_i = \psi_i(I_0)$. Using Proposition 2.11 and Proposition 3.6 it is easily verified that:

$$\psi_i = \psi_{I_i}$$

Therefore, Proposition 4.1 applied to I_i gives:

$$\alpha_i \sim_{I_i} u_i$$

Since $I_i = \varphi_{\alpha_i,u_i,\tau_i}(I_{i-1})$, this proves that (see 1.2) either $\sim_{I_{i-1}}$ and $\sim_{I_i}$ coincide or $\varphi_{\alpha_i,u_i,\tau_i}$ induces an arrow $\sim_{I_{i-1}} \rightarrow \sim_{I_i}$ in Γ. Thus, the following vertices of Γ:

$$\sim_{I_0}, \sim_{I_1}, \ldots, \sim_{I_n} = \sim_I$$

are the vertices of a path in Γ (maybe with repetitions) starting at $\sim_{I_0}$ and ending at $\sim_I$. ■

The preceding proposition and the fact that Γ has no oriented cycle gives immediately the following corollary which was proved by the author in [9] in the case of triangular and without double bypass algebras over an algebraically closed field of characteristic zero.

Corollary 4.4. *Let Q be a quiver without oriented cycle and without multiple arrows, let I_0 be an admissible and monomial ideal of kQ and let $A = kQ/I_0$. Then the quiver Γ of the homotopy relations of the admissible presentations of A admits $\sim_{I_0}$ as unique source.*

Proposition 4.3 also allows us to prove the main theorem of this text. It extends [9, Thm. 2] to monomial triangular algebras without multiple arrows. Notice that in the following theorem we make no assumption on the characteristic of k. Also recall that $\pi_1(Q, I_0) = \pi_1(Q)$.

Theorem 2. *Let A be a monomial, triangular, basic and connected k-algebra. Assume that the quiver Q of A has no multiple arrows. Fix $\varphi\colon kQ/I_0 \xrightarrow{\sim} A$ a monomial presentation of A. Let $p\colon (\widehat{Q}, \widehat{I_0}) \rightarrow (Q, I_0)$ be the universal cover of quiver with relations (see [11]) and let $\overline{p}\colon k\widehat{Q}/\widehat{I_0} \rightarrow kQ/I_0$ be induced by p. Then, for any Galois covering $F\colon \mathcal{C} \rightarrow A$ with group G and with $\mathcal{C}$ connected and locally bounded, there exists a commutative diagram:*

$$\begin{array}{ccc} k\widehat{Q}/\widehat{I_0} & \xrightarrow{F'} & \mathcal{C} \\ {\scriptstyle \varphi\circ\overline{p}}\downarrow & & \downarrow{\scriptstyle F} \\ A & \xrightarrow{\sim} & A \end{array}$$

where the bottom horizontal arrow is an isomorphism which restricts to the identity map on the set of objects of A. Moreover, $F'\colon k\widehat{Q}/\widehat{I_0} \rightarrow \mathcal{C}$ is a Galois covering with group K a normal subgroup of $\pi_1(Q)$ such that there exists an exact sequence of groups:

$$1 \rightarrow K \rightarrow \pi_1(Q) \rightarrow G \rightarrow 1$$

Proof: The proof of the theorem is almost identical to the proof of [8, Thm. 2]: one uses Proposition 4.3 instead of [8, Lem. 4.3]. ■

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