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► **To cite this version:**

Ivan Marin, Jean Michel. ”Galois” automorphisms of complex reflection groups. 2007. hal-00123495v1

HAL Id: hal-00123495

<https://hal.science/hal-00123495v1>

Preprint submitted on 9 Jan 2007 (v1), last revised 12 Mar 2009 (v3)

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“GALOIS” AUTOMORPHISMS OF COMPLEX REFLECTION GROUPS

I. MARIN AND J. MICHEL

ABSTRACT. Let G be a complex reflection group and K its field of definition (the subfield of \mathbb{C} generated by the reflection character). We show that $\text{Gal}(K/\mathbb{Q})$ injects into the group A of outer automorphisms of G which preserve reflections, and that this injection with a few exceptions can be chosen such that it commutes with the Galois action on characters of G ; further, replacing if needed K by an extension of order 2, the injection can be lifted to $\text{Aut}(G)$, and every irreducible representation affords a model which is equivariant with respect to this lifting. Along the way we give the structure of the group A and show that the fundamental invariants of G can be chosen rational.

1. INTRODUCTION

Let G be a finite group generated by (pseudo)reflections in a vector space V of dimension $r < \infty$ over \mathbb{C} . In this paper, we study the automorphisms of G which preserve the set of reflections. It turns out that most of them correspond to the action of a Galois group.

It is well known (see e.g. [Bens, 7.1.1]) that the representation V of G can be already realized over the subfield of K of \mathbb{C} generated by the traces of the elements of G ; the field K is called the *field of definition* of G . Further, it is a well known result of Benard and Bessis ([Bena], [Bes]) that *any* irreducible representation of G can be realized over K .

Let $\Gamma = \text{Gal}(K/\mathbb{Q})$, let $\text{Aut}(G)$ be the group of automorphisms of G , and let $\text{Out}(G) \simeq \text{Aut}(G)/(G/ZG)$ be the group of outer automorphisms. Our main observation may be stated as follows:

Theorem 1.1. *Assume G irreducible. Then for any irreducible representation ρ (with eight exceptions) of G there exists an injective homomorphism: $\Gamma \xrightarrow{\bar{\tau}_\rho} \text{Out}(G)$ such that for any $\gamma \in \Gamma$, the representations $\gamma(\rho)$ and $\rho \circ \bar{\tau}_\rho(\gamma)$ are isomorphic.*

The exceptions are the four 5-dimensional representations of G_{27} , two of the 3 rational representations of dimension 6 of G_{29} , and the two rational representations of dimension 120 of G_{34} . In these cases, the complex conjugation \mathbf{c} is an element of Γ , and its image in any injective morphism $\Gamma \rightarrow \text{Out}(G)$ will exchange in pairs the mentioned characters.

Date: January, 9th, 2007.

The map $\bar{\tau}_\rho$ above can “almost” be lifted to $\text{Aut}(G)$. More precisely we have

Theorem 1.2. *Under the assumptions of theorem 1.1, there exists an extension K' of K such that $[K' : K] \leq 2$ (and such that K'/\mathbb{Q} is still an abelian extension), and an injective homomorphism $\tilde{\tau}_\rho : \text{Gal}(K'/\mathbb{Q}) \rightarrow \text{Aut}(G)$ such that the composition of $\tilde{\tau}_\rho$ with the natural epimorphism $\text{Aut}(G) \rightarrow \text{Out}(G)$ factors through Γ and induces $\bar{\tau}_\rho$.*

One may take $K' = K$ except when G is G_{27} or some of the dihedral groups $G(e, e, 2)$.

The maps $\bar{\tau}_\rho$ and $\tilde{\tau}_\rho$ can be taken independent of ρ except for the groups $G(de, e, r)$ when e and r have a common divisor ≥ 3 .

When G is not one of the groups $G(de, e, r)$ mentioned above, since there is no ambiguity, we will write $\bar{\tau}$ (resp. $\tilde{\tau}$) for the constant value of $\bar{\tau}_\rho$ (resp. $\tilde{\tau}_\rho$) (for the groups G_{27} , G_{29} and G_{34} and an exceptional representation for theorem 1.1, we take $\tilde{\tau}_\rho$ to have the value it takes on non-exceptional representations).

Theorem 1.1 has the following consequence, which can be formulated without exception and without assuming G irreducible.

Corollary 1.3. *Let G be any finite complex reflection group, and let χ be an irreducible character of G . If, for all $a \in \text{Out}(G)$ we have $\chi \circ a = \chi$, then χ takes its values in \mathbb{Q} .*

We turn now to the study of the subgroup $A \subset \text{Out}(G)$ of automorphisms which preserve the set of reflections. Let $N \subset \text{Out}(G)$ be the subgroup induced by $N_{\text{GL}(V)}(G)$. Then we have

Theorem 1.4. *Let G be a finite irreducible complex reflection group. For any $\rho \in \text{Irr}(G)$, we have $A = N \rtimes \text{Im}(\bar{\tau}_\rho)$.*

The main ingredient for proving theorem 1.4 is that (cf. theorem 4.1) any faithful irreducible representation where the reflections of G still act by reflections is a Galois conjugate of V .

Suppose we can find a model over some field L of a faithful representation ρ of G such that $\rho(G)$ as a set is globally invariant by $\text{Gal}(L/\mathbb{Q})$. Then any $\gamma \in \text{Gal}(L/\mathbb{Q})$ induces a permutation of $\rho(G)$ and thus a permutation on G , which is an automorphism. We get thus an homomorphism $\tilde{\tau} : \text{Gal}(L/\mathbb{Q}) \rightarrow \text{Aut}(G)$. If $L = K'$ this $\tilde{\tau}$ is an $\tilde{\tau}_\rho$ suitable for theorem 1.2.

Given an homomorphism $\text{Gal}(L/\mathbb{Q}) \xrightarrow{\tilde{\tau}} \text{Aut}(G)$, and a model of an arbitrary representation ρ over L , we say that the model is $\tilde{\tau}$ -equivariant if for any $\gamma \in \text{Gal}(L/\mathbb{Q})$ and any $g \in G$, we have $\rho(\tilde{\tau}(\gamma)(g)) = \gamma(\rho(g))$. Considerations as above led us to check the following theorem:

Theorem 1.5. *Let G be an irreducible finite complex reflection group which is not G_{22} and let K' and $\tilde{\tau}$ be as in theorem 1.2. Let ρ be any irreducible representation of G . Then there is a model of ρ over K' which is $\tilde{\tau}_\rho$ -equivariant.*

In the case of G_{22} , we have $K = K' = \mathbb{Q}(i, \sqrt{5})$. There is no globally invariant model over K' of the reflection representation. We need to replace K' by the extension $K'' = \mathbb{Q}(e^{\frac{2i\pi}{20}})$ to get a $\tilde{\iota}_\rho$ -equivariant model (then K'' also works for any other representation). This example illustrates the fact that asking for a $\tilde{\iota}_\rho$ -equivariant model is stronger than merely asking as in theorem 1.2 for a model such that $\rho \circ \tilde{\iota}_\rho(\gamma) \simeq \gamma(\rho)$ for any γ .

Theorem 1.5 has the following consequence (which could have been observed from the explicit values of the invariants discovered by various authors, mostly in the 19th century). In this proposition we take K'' as above for G_{22} , and in the other cases we let $K'' = K'$.

Corollary 1.6. *Let $G \subset \mathrm{GL}(V)$ be any irreducible complex reflection group where V is a K'' -vector space, where K'' is as above. There is \mathbb{Q} -form $V = V_0 \otimes_{\mathbb{Q}} K''$ such that the fundamental invariants of G can be taken rational, i.e. in the symmetric algebra of the dual of V_0 .*

Let V^{reg} be the complement in V of the reflecting hyperplanes for G . The fundamental group of the variety V^{reg}/G is the *braid group* of G . We deduce from 1.6 the following

Corollary 1.7. *The variety V^{reg}/G is defined over \mathbb{Q} .*

The techniques we use for checking theorem 1.5 enable us also to give a probably shorter proof of the Benard-Bessis theorem (see remark 7.6).

Acknowledgments. We thank David Harari for teaching us about the Brauer group, and Gunter Malle for a thorough reading of a first version of the paper.

2. BACKGROUND AND NOTATION

We will use the following notation throughout this paper. V is a vector space over the subfield K of \mathbb{C} . G is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections. We assume that K is the *field of definition* of G , that is, it is the subfield of \mathbb{C} generated by the traces of the elements of G . The group G is *irreducible* if the representation V of G is irreducible. It is *imprimitive* if V admits a non-trivial decomposition $V = V_1 \oplus \dots \oplus V_n$ stabilized by the action of G (that is, the V_i are permuted by G); otherwise G is *primitive*.

We will consider only irreducible groups G unless explicitly stated otherwise. Corollary 1.3 is about general reflection groups. Let us show how it can be deduced from theorem 1.1.

Proposition 2.1. *Let G be any finite complex reflection group, let K be its field of definition and let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$. Let \mathcal{S} be the set of irreducible representations of G which cannot be realized over \mathbb{Q} . Then for any $\rho \in \mathcal{S}$ there exists an injection $\Gamma \xrightarrow{\tilde{\iota}_\rho} \mathrm{Out}(G)$ such that for any $\gamma \in \Gamma$ the representations $\gamma(\rho)$ and $\rho \circ \tilde{\iota}_\rho(\gamma)$ are isomorphic.*

Proof of proposition 2.1. Proposition 2.1 is an obvious consequence of 1.1 when G is irreducible. For a general reflection group G , we have a decomposition into irreducible groups: $V = V_1 \oplus \dots \oplus V_n$, $G = G_1 \times \dots \times G_n$ where $G_i \subset \mathrm{GL}(V_i)$. If K_i is the field of definition of G_i then K is the subfield of \mathbb{C} generated by K_1, \dots, K_n , whence if $\Gamma_i = \mathrm{Gal}(K_i/\mathbb{Q})$ we have natural quotient morphisms $\Gamma \rightarrow \Gamma_i$ such that the product map $\Gamma \rightarrow \prod_i \Gamma_i$ is injective. We deduce an injective map $\Gamma \rightarrow \mathrm{Out}(G)$ by composing $\Gamma \rightarrow \prod_i \Gamma_i$ with the individual maps $\Gamma_i \rightarrow \mathrm{Out}(G_i)$ deduced from 1.1 and then with the natural injection $\mathrm{Out}(G_1) \times \dots \times \mathrm{Out}(G_n) \rightarrow \mathrm{Out}(G)$. Since any irreducible representation ρ of G is of the form $\rho_1 \otimes \dots \otimes \rho_n$ where $\rho_i \in \mathrm{Irr}(G_i)$, proposition 2.1 follows readily. \square

Statement 1.3 is an obvious corollary of 2.1.

The Shephard-Todd classification of irreducible finite reflection groups (see *e.g.* [Cohen]) shows that there is one infinite series, identified by three integer parameters d, e and r , and denoted by $G(de, e, r)$; this series consists of imprimitive groups, except for $G(1, 1, r) = \mathfrak{S}_r$ and $G(d, 1, 1)$ which is the cyclic group of order d . In addition there are 34 exceptional primitive ones, denoted G_4 to G_{37} .

3. THE INFINITE SERIES

The group $G(de, e, r)$ is defined as the subgroup of $GL_r(\mathbb{C})$ consisting of n by n monomial matrices such that each matrix has entries in the set μ_{de} of de -th roots of unity, and the product of non-zero entries is in μ_d (for the case of $\mathfrak{S}_r = G(1, 1, r)$ this defines a non-irreducible representation; one has to quotient by the one-dimensional fixed points). In particular, $G(de, e, r)$ is a normal subgroup of index e in $G(de, 1, r)$.

In this text we will denote by ζ_n the primitive root of unity $e^{2i\pi/n}$. The field of definition of $G(de, e, r)$ is the field $K = \mathbb{Q}(\zeta_{de})$, except when $d = 1$ and $r = 2$. In this last case, $G(e, e, 2)$ is the dihedral group of order $2e$, whose field of definition is $K = \mathbb{Q}(\cos(\frac{2\pi}{e})) = \mathbb{Q}(\zeta_e + \zeta_e^{-1})$ and we set $K' = \mathbb{Q}(\zeta_e)$.

It is clear from these definitions that, using the model $V = V_0 \otimes_{\mathbb{Q}} K'$ where $V = K'^r$ which is implicit when we talk about matrices, the set $G(de, e, r)$ is globally invariant under the induced action of $\mathrm{Gal}(K'/\mathbb{Q})$ on $GL_r(K')$. It follows that this action induces a morphism $\eta : \mathrm{Gal}(\mathbb{Q}(\zeta_{de})/\mathbb{Q}) \rightarrow \mathrm{Aut}(G)$. The purpose of this section is to show that a slight variation on η satisfies the properties of theorem 1.2, which implies the weaker theorem 1.1. Actually we will get a more precise version of 1.2, stating the \tilde{l}_ρ equivariance of some models, which gives also theorem 1.5.

For $i \in \mathbb{Z}$ prime to de , we denote by $\mathfrak{s}_i \in \mathrm{Gal}(\mathbb{Q}(\zeta_{de})/\mathbb{Q})$ the element defined by $\mathfrak{s}_i(\zeta_{de}) = \zeta_{de}^i$.

3.1. Dihedral groups. The group $G(e, e, 2)$ is generated by

$$t = \begin{pmatrix} 0 & \zeta_e \\ \zeta_e^{-1} & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with relations $s^2 = t^2 = 1$, $(ts)^e = 1$ (this is the usual presentation of the dihedral group of order $2e$). For i prime to e we have $\eta(\mathfrak{s}_i)(s) = s$ and $\eta(\mathfrak{s}_i)(t) = (ts)^i s$. In particular $\eta(\mathfrak{s}_i)(ts) = (ts)^i$. We may also check that for any $x \in G$ we have $\eta(\mathfrak{s}_{-i})(x) = s\eta(\mathfrak{s}_i)(x)s$.

Recall that $K = \mathbb{Q}(\zeta_e)$ and $K' = \mathbb{Q}(\zeta_e + \zeta_e^{-1})$. It follows that the quotient $\Gamma' = \text{Gal}(K'/\mathbb{Q}) \rightarrow \Gamma = \text{Gal}(K/\mathbb{Q})$ is obtained by identifying \mathfrak{s}_i and \mathfrak{s}_{-i} , thus the above computations show that the composed morphism $\Gamma' \xrightarrow{\eta} \text{Aut}(G) \rightarrow \text{Out}(G)$ factors through Γ , leading to the following commutative diagram where we have written $\tilde{\iota}$ for η .

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\tilde{\iota}} & \text{Aut}(G) \\ \downarrow & \searrow & \downarrow \\ \Gamma & \xrightarrow{\bar{\iota}} & \text{Out}(G) \end{array}$$

The maps $\tilde{\iota}$ and $\bar{\iota}$ above satisfy the hypotheses of theorems 1.1 and 1.2.

We note that the above reflection representation is actually $\tilde{\iota}$ -equivariant, thus we have property 1.5 for the reflection representation.

It is clear that 1.1 and 1.2 hold for the linear characters of G , because they take values in \mathbb{Q} . The remaining irreducible representations of G have a model of the form

$$t \mapsto \begin{pmatrix} 0 & \zeta_e^r \\ \zeta_e^{-r} & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad ts \mapsto \begin{pmatrix} \zeta_e^r & 0 \\ 0 & \zeta_e^{-r} \end{pmatrix}$$

for some r , and are determined up to isomorphism by the value of $\zeta_e^r + \zeta_e^{-r}$, so 1.1, 1.2 and 1.5 also hold for them.

Construction of ι . We will now determine when it is possible to take $K' = K$. It is classical that the short exact sequence

$$1 \rightarrow \text{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q}(\zeta_e + \zeta_e^{-1})) \rightarrow \text{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q}) \rightarrow \text{Gal}(\mathbb{Q}(\zeta_e + \zeta_e^{-1})/\mathbb{Q}) \rightarrow 1$$

splits precisely when $\mathbb{Q}(\zeta_e)$ contains a quadratic imaginary extension of \mathbb{Q} . This is the case if and only if $4|e$ or there exists an odd prime factor of e congruent to 3 modulo 4. In that case one can define injective morphisms $\Gamma \rightarrow \text{Aut}(G)$ by composition.

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\tilde{\iota}} & \text{Aut}(G) \\ \downarrow & \searrow & \downarrow \\ \Gamma & \xrightarrow{\bar{\iota}} & \text{Out}(G) \end{array}$$

We cannot expect to lift this morphism to $\text{Aut}(G)$ in general, as already shows the example $e = 5$. In that case, it is easily checked that $\Gamma = \text{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$ has order two and that all involutive automorphisms of G are inner.

3.2. The case $G(d, 1, r)$. The group $G = G(d, 1, r)$ is generated by the diagonal matrix t with diagonal entries $(\zeta_d, 1, \dots, 1)$ and the permutation matrices $\{s_1, \dots, s_{r-1}\}$ corresponding to the transpositions $(k, k+1)$. We have $\eta(\mathfrak{s}_\alpha)(t) = t^\alpha$ and $\eta(\mathfrak{s}_\alpha)(s_k) = s_k$.

We show an easy proof of 1.1 and 1.2 for these groups starting from the case $r = 2$. Then G has order $2d^2$, and its irreducible representations have dimension 1 and 2. The case of linear characters is an immediate check. The 2-dimensional representations are given, for $\zeta_1, \zeta_2 \in \mu_d, \zeta_1 \neq \zeta_2$, by the model

$$\rho_{\zeta_1, \zeta_2}(t) = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} \quad \rho_{\zeta_1, \zeta_2}(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is readily checked that

$$\rho_{\zeta_1, \zeta_2} \circ \eta(\mathfrak{s}_\alpha) = \mathfrak{s}_\alpha \circ \rho_{\zeta_1, \zeta_2} = \rho_{\zeta_1^\alpha, \zeta_2^\alpha}$$

and this shows theorems 1.1, 1.2 for the groups $G(d, 1, 2)$ by taking $K' = K$ and $\tilde{t} = \eta$. The first equality in the above formula shows also that the model is η -equivariant, giving thus 1.5.

Theorems 1.1 and 1.2 are then easily deduced from this for the groups $G(d, 1, r)$. This comes from the fact that, for $r \geq 3$, each irreducible representation of $G(d, 1, r)$ is determined up to isomorphism by its restriction to the parabolic subgroup $G(d, 1, r-1)$ (see e.g. [ArKo] corollary 3.12).

We will give a proof of 1.5 for $G(d, 1, r)$ in the same spirit in 7.2. But in the next section we will more constructively give explicit models which are \tilde{t}_ρ -equivariant for a suitable \tilde{t}_ρ .

3.3. The general case $G(de, e, r)$. We recall from [Ar] and [ArKo] explicit models for the irreducible representations of $G(de, 1, r)$ and $G(de, e, r)$. Let \mathcal{L} be the set of de -tuples $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_{de-1})$ of partitions with total size r . We denote by $\mathcal{T}(\boldsymbol{\lambda})$ the set of standard tableaux of shape $\boldsymbol{\lambda}$, that is the set of tuples $\mathbf{T} = (T_0, \dots, T_{de-1})$ such that T_i is a filling of the Young diagram of λ_i with numbers in $[1, r]$, with the conditions that each of these numbers appear (exactly once) in one of the filled diagrams, and that they are increasing across the rows and columns of each T_i . For $m \in [1, r]$ we let $\mathbf{T}(m) = i$ if m is placed in T_i .

To $\boldsymbol{\lambda} \in \mathcal{L}$ we associate the \mathbb{Q} -vector space $V^0(\boldsymbol{\lambda})$ of basis $\mathcal{T}(\boldsymbol{\lambda})$, and define $V(\boldsymbol{\lambda}) = V^0(\boldsymbol{\lambda}) \otimes \mathbb{Q}(\zeta_{de})$. Explicit formulas in [ArKo] describe a representation $\rho_{\boldsymbol{\lambda}}$ of $G(de, 1, r)$ over $V(\boldsymbol{\lambda})$, and the $\rho_{\boldsymbol{\lambda}}$ provide a complete set of representatives for the irreducible representations of $G(de, 1, r)$ (see [ArKo] corollary 3.14). We recall from [ArKo] the following facts :

- If $\mathbf{T} = (T_0, \dots, T_{de-1})$ then $\rho_{\boldsymbol{\lambda}}(t)\mathbf{T} = \zeta_{de}^{\mathbf{T}(1)}\mathbf{T}$.
- The matrices $\rho_{\boldsymbol{\lambda}}(s_i)$ for $1 \leq i \leq n-1$ are rational in the basis $\mathcal{T}(\boldsymbol{\lambda})$, thus belong to $\text{GL}(V^0(\boldsymbol{\lambda}))$.

From the formulas $\eta(\mathfrak{s}_\alpha)(t) = t^\alpha$ and $\eta(\mathfrak{s}_\alpha)(s_k) = s_k$ and the above facts, it is clear that these models over $\mathbb{Q}(\zeta_{de})$ satisfy theorem 1.2 by taking $\tilde{t} = \eta$;

actually, these computations show that the model ρ_{λ} is \tilde{t} -equivariant, which gives theorem 1.5.

For analysis of the restriction of ρ_{λ} to $G(de, e, r)$, we need to be more explicit on two points.

- We describe precisely $\rho_{\lambda}(s_i)$: given a tuple of tableaux \mathbf{T} , let $\mathbf{T}_{i \leftrightarrow i+1}$ be the tuple obtained by exchanging the numbers i and $i+1$ in \mathbf{T} when this is still a tuple of standard tableaux, and 0 otherwise (this last case will only occur when the numbers i and $i+1$ occur in the same tableau and on either the same line or the same column). Then, if i and $i+1$ occur in the same tableau, define their axial distance $a(i, i+1)$ to be the distance between the diagonals where they occur (more precisely, if i occurs at coordinates i_0, i_1 and $i+1$ at coordinates j_0, j_1 we set $a(i, i+1) = (i_0 - i_1) - (j_0 - j_1)$). It is clear that for a standard tableau we always have $a(i, i+1) \neq 0$. Finally, take the convention that when i and $i+1$ do not occur in the same tableau, then $a(i, i+1) = \infty$ so that $1/a(i, i+1) = 0$. Then the formula is:

$$(3.1) \quad \rho_{\lambda}(s_i)\mathbf{T} = \frac{1}{a(i, i+1)}\mathbf{T} + \left(1 + \frac{1}{a(i, i+1)}\right)\mathbf{T}_{i \leftrightarrow i+1}$$

- Given λ , we describe the partition tuple $\mathfrak{s}_{\alpha}(\lambda)$ such that $\mathfrak{s}_{\alpha} \circ \rho_{\lambda} \simeq \rho_{\mathfrak{s}_{\alpha}(\lambda)}$. It is given by $\mathfrak{s}_{\alpha}(\lambda) = (\lambda_0, \lambda_{\alpha}, \lambda_{2\alpha}, \dots, \lambda_{(de-1)\alpha})$. If we define the operator $\Sigma_{\alpha} : \mathcal{T}(\lambda) \rightarrow \mathcal{T}(\mathfrak{s}_{\alpha}(\lambda))$ by $\Sigma_{\alpha}(T) = (T_0, T_{\alpha}, T_{2\alpha}, \dots, T_{(de-1)\alpha})$, we have more precisely $\Sigma_{\alpha} \circ \mathfrak{s}_{\alpha} \circ \rho_{\lambda} = \rho_{\mathfrak{s}_{\alpha}(\lambda)} \circ \Sigma_{\alpha}$.

We now look at the group $G(de, e, r)$. It is the normal subgroup of index e generated by $t^e, s_1^t, s_1, \dots, s_r$; the quotient is cyclic, generated by t . Let χ be the generator of the group of linear characters of the quotient which maps t to ζ_e . From the formulae for the action of t and s_i , it is easy to find the tuple $\sigma(\lambda)$ such that $\chi \otimes \rho_{\lambda} \simeq \rho_{\sigma(\lambda)}$; it is given by $\sigma(\lambda) = (\lambda_d, \lambda_{d+1}, \dots, \lambda_{d+de-1})$ (where the indices are taken \pmod{de}). Let C_{λ} be the subgroup of $\langle \sigma \rangle$ which stabilizes λ , and let σ^b be its generator. By Clifford theory, the representation ρ_{λ} restricts to $G(de, e, r)$ as the sum of $\#C_{\lambda}$ distinct irreducible representations.

We can actually define an operator S on $V(\lambda)$ associated to σ^b whose eigenspaces will be the irreducible constituents. For $\mathbf{T} = (T_0, \dots, T_{de-1})$ we set $S(\mathbf{T}) = (T_{bd}, T_{bd+1}, \dots, T_{bd+de-1})$. Since $\sigma^b(\lambda) = \lambda$, this is another tuple of tableaux of shape λ . It is easy to check that S commutes to $\rho_{\lambda}(s_i)$ and to $\rho_{\lambda}(s_1^t)$, and that $S \circ \rho_{\lambda}(t) = \zeta_e^b \rho_{\lambda}(t) \circ S$. It follows that S commutes with the action of $G(de, e, r)$ and its eigenspaces $V(\lambda, \omega) = \text{Ker}(S - \omega)$ for $\omega \in \mu_{\#C_{\lambda}}$ afford irreducible representations of $G(de, e, r)$. We denote by $\rho_{\lambda, \omega}$ the representation afforded by $V(\lambda, \omega)$; it is clear that

$$p_{\omega} := \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \omega^{-i} S^i$$

is the $G(de, e, r)$ -invariant projector on $V(\lambda, \omega)$.

From this we get, if we denote $\chi_{\lambda,\omega}$ the character of $\rho_{\lambda,\omega}$:

$$\chi_{\lambda,\omega}(g) = \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \omega^{-i} \text{Trace}(\rho_{\lambda}(g)S^i).$$

We get then, using that the matrix S is rational in our chosen basis of $V(\lambda)$:

$$\mathfrak{s}_{\alpha}(\chi_{\lambda,\omega}(g)) = \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \mathfrak{s}_{\alpha}(\omega)^{-i} \text{Trace}(\mathfrak{s}_{\alpha}(\rho_{\lambda}(g))S^i).$$

From this we get, if we write S_{λ} for S to keep track where it acts:

$$\mathfrak{s}_{\alpha}(\chi_{\lambda,\omega}(g)) = \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \omega^{-i\alpha} \text{Trace}(\Sigma_{\alpha}^{-1} \rho_{\mathfrak{s}_{\alpha}(\lambda)}(g) \Sigma_{\alpha} S_{\lambda}^i).$$

Now, it is easy to check that $\Sigma_{\alpha} S_{\lambda} \Sigma_{\alpha}^{-1} = S_{\mathfrak{s}_{\alpha}(\lambda)}^{\alpha}$, so

$$\mathfrak{s}_{\alpha}(\chi_{\lambda,\omega}(g)) = \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \omega^{-i\alpha} \text{Trace}(\rho_{\mathfrak{s}_{\alpha}(\lambda)}(g) S_{\mathfrak{s}_{\alpha}(\lambda)}^{i\alpha}) = \chi_{\mathfrak{s}_{\alpha}(\lambda),\omega}(g).$$

While we have by a similar computation

$$\begin{aligned} \chi_{\lambda,\omega}(\tilde{i}(\mathfrak{s}_{\alpha})(g)) &= \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \omega^{-i} \text{Trace}(\mathfrak{s}_{\alpha}(\rho_{\lambda}(g))S_{\lambda}^i) \\ &= \frac{1}{\#C_{\lambda}} \sum_{i=0}^{\#C_{\lambda}-1} \omega^{-i} \text{Trace}(\rho_{\mathfrak{s}_{\alpha}(\lambda)}(g) S_{\mathfrak{s}_{\alpha}(\lambda)}^{i\alpha}) \\ &= \chi_{\mathfrak{s}_{\alpha}(\lambda),\mathfrak{s}_{\alpha}^{-1}(\omega)}(g). \end{aligned}$$

We will now make similar computations at the level of models. We first recall Ariki's models (*cf.* [Ar, section 2]), and then introduce a simpler one more suited to computations. Ariki chooses a basis of $V(\lambda,\omega)$ given by the $p_{\omega}(\mathbf{T})$, where he chooses for \mathbf{T} representatives of the S -orbits on \mathcal{T} given by the subset \mathcal{T}_0 of tuples of tableaux which satisfy $\mathbf{T}(1) < bd$. Setting $\mathbf{T}^{(\omega)} = p_{\omega}(\mathbf{T})$, we then get formulas for the action of the generators of $G(de, e, r)$ on the basis $\{\mathbf{T}^{(\omega)}\}_{\mathbf{T} \in \mathcal{T}_0}$. We will write $\theta = \zeta_e^b$ to simplify notations. Using that $\rho_{\lambda}(t)p_{\omega} = p_{\omega\theta}\rho_{\lambda}(t)$, we get $\rho_{\lambda}(t)\mathbf{T}^{(\omega)} = \zeta_{de}^{\mathbf{T}(1)}\mathbf{T}^{(\omega\theta)}$ from which we get

$$\rho_{\lambda,\omega}(t^e)\mathbf{T}^{(\omega)} = \zeta_d^{\mathbf{T}(1)}\mathbf{T}^{(\omega)}.$$

To write the formula for s_i , first note that if $\mathbf{T} \in \mathcal{T}_0$ then $\mathbf{T}_{i \leftrightarrow i+1}$ is not in general in \mathcal{T}_0 when $i = 1$. We have to take $\mathbf{T}'_{1 \leftrightarrow 2} = S^{\lfloor \frac{\mathbf{T}(2)}{bd} \rfloor} \mathbf{T}_{1 \leftrightarrow 2}$ to get an element of \mathcal{T}_0 . Using that $p_{\omega}(\mathbf{T}'_{1 \leftrightarrow 2}) = \omega^{\lfloor \frac{\mathbf{T}(2)}{bd} \rfloor} p_{\omega}(\mathbf{T}_{1 \leftrightarrow 2})$, we get for $i > 1$

$$\rho_{\lambda,\omega}(s_i)\mathbf{T}^{(\omega)} = \frac{1}{a(i, i+1)}\mathbf{T}^{(\omega)} + \left(1 + \frac{1}{a(i, i+1)}\right)\mathbf{T}_{i \leftrightarrow i+1}^{(\omega)}.$$

and

$$\rho_{\lambda, \omega}(s_1) \mathbf{T}^{(\omega)} = \frac{1}{a(1, 2)} \mathbf{T}^{(\omega)} + \left(1 + \frac{1}{a(1, 2)}\right) \omega^{-\lfloor \frac{\mathbf{T}(2)}{bd} \rfloor} \mathbf{T}'_{1 \leftrightarrow 2}^{(\omega)}.$$

Finally, using the above formulas we get

$$\rho_{\lambda, \omega}(s_1^t) \mathbf{T}^{(\omega)} = \zeta_{de}^{\mathbf{T}(1) - \mathbf{T}(2)} \rho_{\lambda, \omega}(s_1) \mathbf{T}^{(\omega)}.$$

We now introduce a model which does not depend on a choice of representatives \mathcal{T}_0 , using the equality $p_{\omega \theta^{-1}} \circ \rho_{\lambda}(g) = p_{\omega} \rho_{\lambda}(\text{Ad } t(g))$ which implies $\chi_{\lambda, \theta^i}(g) = \chi_{\lambda, 1}(\text{Ad}(t^{-i})g)$. We choose the same model as Ariki of $\rho_{\lambda, 1}$, except that, as the image by p_1 of a tuple \mathbf{T} is the same as that of all elements of the same S -orbit, we take as basis elements the averages of S -orbits. We denote by (\mathbf{T}) the average of the S -orbit of \mathbf{T} . We thus get a basis $\{(\mathbf{T})\}_{(\mathbf{T}) \in \mathcal{T}(\lambda)/S}$ of $V(\lambda, 1)$. We take the same basis for $V(\lambda, \theta^i)$ and define $\rho_{\lambda, \theta^i}(g) := \rho_{\lambda, 1}(\text{Ad}(t)^{-i}g)$. We need to record the following formulas for $\rho_{\lambda, 1}$:

$$\begin{aligned} \rho_{\lambda, 1}(t^e)(\mathbf{T}) &= \zeta_d^{\mathbf{T}(1)}(\mathbf{T}), \\ (3.2) \quad \rho_{\lambda, 1}(s_i)(\mathbf{T}) &= \frac{1}{a(i, i+1)}(\mathbf{T}) + \left(1 + \frac{1}{a(i, i+1)}\right)(\mathbf{T}_{i \leftrightarrow i+1}) \text{ for any } i \\ \rho_{\lambda, 1}(s_1^k)(\mathbf{T}) &= \zeta_{de}^{k(\mathbf{T}(1) - \mathbf{T}(2))} \rho_{\lambda, 1}(s_1)(\mathbf{T}). \end{aligned}$$

(note that the above formulas make sense since both $\zeta_d^{\mathbf{T}(1)}$ and $\zeta_{de}^{\mathbf{T}(1) - \mathbf{T}(2)}$ are constant on a given S -orbit). It becomes then a simple matter to check that

$$\mathfrak{s}_{\alpha} \circ \rho_{\lambda, \theta^i} = \rho_{\lambda, \theta^i} \circ \text{Ad } t^{i(1-\alpha)} \circ \eta(\mathfrak{s}_{\alpha}) = \rho_{\lambda, \theta^i} \circ \text{Ad } t^i \circ \eta(\mathfrak{s}_{\alpha}) \circ \text{Ad } t^{-i}$$

(this formula is obvious for the generators t and s_i where $i \neq 1$. It is thus sufficient to check it for s_1^k ; one uses the above formulas and the fact that $\rho_{\lambda, 1}(s_1)$ is a rational matrix). For any i we define $\tilde{\nu}_{\rho_{\lambda, \theta^i}} : \Gamma \rightarrow \text{Aut}(G)$ by $\tilde{\nu}_{\rho_{\lambda, \theta^i}}(\mathfrak{s}_{\alpha}) = \text{Ad } t^i \circ \eta(\mathfrak{s}_{\alpha}) \circ \text{Ad } t^{-i}$. The formula above becomes $\mathfrak{s}_{\alpha} \circ \rho_{\lambda, \theta^i} = \rho_{\lambda, \theta^i} \circ \tilde{\nu}_{\rho_{\lambda, \theta^i}}(\mathfrak{s}_{\alpha})$, *i.e.* that ρ_{λ, θ^i} is $\tilde{\nu}_{\rho_{\lambda, \theta^i}}$ -equivariant, which proves 1.2 and theorem 1.5.

4. REFLECTION REPRESENTATIONS

Before going on to prove 1.2 for exceptional groups, we use the previous results to show proposition 1.4. By a reflection representation of a reflection group G , we mean a faithful representation such that the image of a reflection of G is a reflection.

Theorem 4.1. *Let $G \subset \text{GL}(V)$ be an irreducible complex reflection group, where V is a vector space over the field of definition K of G . Then the reflection representations of G are the transformed of V by the action of $\text{Gal}(K/\mathbb{Q})$.*

Proof. The theorem is proved by a computer check for the exceptional reflection groups, by looking at the character tables. Both the action of the Galois group, and which representations are faithful and send reflections to reflections are readily seen from the character tables.

We now look at the groups $G(de, e, r)$. We first look at the case $e = 1$. The result is clear if G is cyclic so we assume $r > 1$, and use the notations of section 3.3.

Let us determine when $\rho_{\lambda}(t)$ is a reflection. First, notice that there is only one tableau corresponding to a given young diagram and with a given content if and only if the diagram is a column or a line. For a partition λ we denote by $|\lambda|$ the size of λ (the sum of the lengths of its parts). Since $\mathbf{T} \in \mathcal{T}(\lambda)$ are eigenvectors of $\rho_{\lambda}(t)$ with eigenvalue $\zeta_{de}^{\mathbf{T}(1)}$, there must be exactly one $\mathbf{T} \in \mathcal{T}(\lambda)$ such that $\mathbf{T}(1) \neq 0$. This means that there must be exactly one $i \neq 0$ such that $|\lambda_i| \neq 0$, with a Young diagram a line or a column. We must also have that λ_0 is a line or a column; if $|\lambda_0| = 0$ then the representation is of dimension 1 and cannot be faithful since $r > 1$. Also if $|\lambda_i| > 1$ then $r \geq 3$ and there are at least two \mathbf{T} such that $\mathbf{T}(1) = i$. So we must have $\lambda_i = \{1\}$. Finally, note that i must be prime to de otherwise ρ_{λ} is not faithful (since $\rho_{\lambda}(t)$ does not have order de). To summarize, we have $\lambda_0 = \{r-1\}$ or $\{1^{r-1}\}$ and $\lambda_i = \{1\}$ for some i prime to de .

Let us now analyze when $\rho_{\lambda}(s_i)$ is a reflection (or equivalently all $\rho_{\lambda}(s_i)$ for $i = 1, \dots, r-1$ are reflections, since they are conjugate). The formula 3.1 shows that $\rho_{\lambda}(s_i)$ in the basis $\mathcal{T}(\lambda)$ is block-diagonal, with diagonal blocks indexed by the basis elements \mathbf{T} and $\mathbf{T}_{i \leftrightarrow i+1}$, which are thus of size 1 when $\mathbf{T}_{i \leftrightarrow i+1} = 0$ and of size 2 otherwise.

(i) If $\mathbf{T}_{i \leftrightarrow i+1} \neq 0$ the corresponding block is
$$\begin{pmatrix} \frac{1}{a(i,i+1)} & 1 + \frac{1}{a(i,i+1)} \\ 1 - \frac{1}{a(i,i+1)} & \frac{-1}{a(i,i+1)} \end{pmatrix}$$

whose eigenvalues are 1 and -1 .

- (ii) If $\mathbf{T}_{i \leftrightarrow i+1} = 0$, then i and $i+1$ are in the same tableau, and in the same line or column. We have that \mathbf{T} is an eigenvector of $\rho_{\lambda}(s_i)$, for the eigenvalue $\frac{1}{a(i,i+1)} = 1$ when i and $i+1$ are in the same line and $\frac{1}{a(i,i+1)} = -1$ when i and $i+1$ are in the same column.

For $\rho_{\lambda}(s_i)$ to be a reflection there cannot be more than two j with $|\lambda_j| \neq 0$ otherwise case (i) would occur more than once (with i and $i+1$ in different tableaux). Similarly, if there are two j such that $|\lambda_j| \neq 0$ then the two Young diagrams must be lines or columns otherwise case (i) would occur more than once. Actually, the two diagrams must be lines since if one is a column of height > 1 then case (ii) with eigenvalue -1 has to occur at least once. And for $\rho_{\lambda}(s_{r-1})$ to be a reflection one of the lines has to be of length 1, otherwise by removing the last square of each line we can still fill the rest in at least two different ways which leads to too many instances of (i).

We now look at the case when only one $|\lambda_j| \neq 0$. The diagram must not be a line or a column otherwise ρ is not faithful. For $i = r-1$ case (i) will

occur more than once if the diagram has more than two corners. If there are two corners, when they are removed the rest must be a line or a column; this means the diagram must be a hook; and the height of the hook has to be 2 otherwise case (ii) with eigenvalue -1 will occur. Finally if there is one corner the diagram is a rectangle; this rectangle has to be 2×2 otherwise by removing the corner one would get a diagram with two corners which is not a hook which would lead to too many eigenvalues -1 for s_{r-2} .

To summarize, we must have $i \neq j$ such that $\lambda_i = \{n-1\}$ and $\lambda_j = \{1\}$, or only one i with $|\lambda_i| \neq 0$ with diagram either a hook of height 2 or a 2×2 square (the last for $r = 4$).

The above analysis shows that the reflection representations of $G(de, 1, r)$ are the $\rho_{\lambda^{(i)}}$ for i prime to de , where $\lambda_0^{(i)} = \{r-1\}$ and $\lambda_i^{(i)} = \{1\}$. By the formula just below 3.1 we have $\mathfrak{s}_i \circ \rho_{\lambda^{(1)}} = \rho_{\lambda^{(i)}}$ which proves the theorem.

Let us look now at $G(de, e, r)$. We first notice that a representation which is not the restriction of an irreducible representation of $G(de, 1, r)$ is not faithful. To prove this, we use that the central character of ρ_{λ} is given by $\omega_{\rho_{\lambda}}(z) = \zeta_{de}^{\sum_{i=0}^{d-1} i|\lambda_i|}$ (this can be deduced from the construction of ρ_{λ} as induced from a generalized Young subgroup — see *e.g.* [Ze, p. 93–106]). Here z is the generator of the center of $G(de, 1, r)$, which can be identified to the scalar $\zeta_{de} \in \text{GL}(\mathbb{C}^r)$. If ρ_{λ} restricted to $G(de, e, r)$ is not irreducible, there is a divisor b of e such that for all i we have $\lambda_{i+de/b} = \lambda_i$ (then each λ_i occurs b times thus b divides $\text{gcd}(e, r)$). It follows that $\sum_{i=0}^{d-1} i|\lambda_i|$ is divisible by b thus ρ_{λ} is not faithful on the center. It is not even faithful on the center of $G(de, e, r)$ since that center is generated by $z^{\frac{e}{\text{gcd}(e, r)}}$ and b still divides the order of that element.

Thus the representation we are looking at is the restriction to $G(de, e, r)$ of some ρ_{λ} . We use the analysis of when $\rho_{\lambda}(s_i)$ is a reflection. Notice first that the representations with only one diagram are not faithful, since by formula 3.2 s_1 and s_1^t have same image since for any \mathbf{T} we have $\mathbf{T}(1) = \mathbf{T}(2)$. Thus we are in the case where we have $i \neq j$ such that $\lambda_i = \{n-1\}$ and $\lambda_j = \{1\}$. Actually, for the representation to be faithful we must have $\rho_{\lambda}(s_1^{t^k}) \neq \rho_{\lambda}(s_1)$ for any k prime to de , thus $\mathbf{T}(2) - \mathbf{T}(1)$ must be prime to de , *i.e.* $j - i$ must be prime to de .

Analyzing when $\rho_{\lambda}(t^e)$ is a reflection along the same lines we did for $\rho_{\lambda}(t)$ we find that $j \equiv 0 \pmod{d}$. Since ρ_{λ} and $\rho_{\sigma(\lambda)}$ have same restriction to $G(e, e, r)$ where σ is the “shift” by d as in section 3.3, we may assume $j = 0$. Thus we get the same λ and can finish the proof of the theorem in the same way as in the case $G(de, 1, r)$. \square

We now prove 1.4 whose statement we recall.

Corollary 4.2. *Let $G \subset \text{GL}(V)$ be an irreducible complex reflection group. Let A be the subgroup of $\text{Out}(G)$ which preserves the reflections and let N be the subgroup induced by $N_{\text{GL}(V)}(G)$. For any $\rho \in \text{Irr}(G)$, we have $A = N \rtimes \text{Im}(\bar{\tau}_{\rho})$.*

Proof. First, it is clear that N is normalized by $\text{Im}(\bar{t}_\rho)$, since if $\nu \in N$ is induced by $n \in \text{GL}(V)$ and $a = \bar{t}_\rho(\gamma)$, then $a \circ \nu \circ a^{-1}$ is the element of N induced by $\gamma(n)$.

We then show that N and $\text{Im}(\bar{t}_\rho)$ meet trivially. Let χ be the character of G on V . If $a \in N$, then for any $g \in G$, we have $\chi(a(g)) = \chi(g)$; while if $\gamma \in \Gamma - \{\text{Id}\}$, since K is the field generated by the values of χ , there is $g \in G$ such that $\gamma(\chi(g)) = \chi(\bar{t}_\rho(\gamma)(g)) \neq \chi(g)$.

Let now $a \in A$ be any element. Then if ρ_0 is the representation of G given by V , then $\rho_0 \circ a$ is another reflection representation. Thus by 4.1 there exists $b \in \text{Im}(\bar{t}_\rho)$ such that $\rho_0 \circ a \circ b \simeq \rho_0$. Thus there is $n \in \text{GL}(V)$ such that for any $g \in G$ we have $n\rho_0(a(b(g)))n^{-1} = \rho_0(g)$. Thus $n \in N_{\text{GL}(V)}(G)$ q.e.d. \square

We note that, in accordance with the above corollary, the various morphisms $\bar{t}_{\rho_{\lambda, \theta^i}}$ for the groups $G(de, e, r)$ differ by an element of N . Indeed they are conjugate by a power of $\text{Ad } t$ which is an element of N .

5. EXCEPTIONAL GROUPS

To verify 1.2 for the exceptional reflection group, we have used the GAP package CHEVIE (see [CHEVIE]), which contains information about them, including:

- Their realization via a permutation representation (on a set of “roots” in V).
- Their character table.

The character tables allow us to determine the permutation of $\text{Irr}(G)$ effected by an element of $\text{Gal}(K/\mathbb{Q})$. The method we have used to check 1.1 is to compute the group of outer automorphisms of G which preserve the set of reflections, and then to find a subgroup which induces the same permutation group of $\text{Irr}(G)$ as $\text{Gal}(K/\mathbb{Q})$.

To compute automorphisms, we use a presentation for G , by means of a diagram (see [BMR] and [BM]); if S is the generating set for this presentation, we enumerate up to inner automorphisms all tuples of reflections of G which satisfy the relations for G , and generate G .

To find directly the orbits of G on the tuples of reflections is a too large computation for the larger groups, such as G_{34} which has about $4 \cdot 10^7$ elements; thus we use a recursive process: we choose the image of the first generator s (which amounts to choosing a representative for each orbit of G on the reflections); then we find the $C_G(s)$ -orbits of tuples which satisfy the relations for $S - \{s\}$ (by a recursive process) and then select those that have the required relations with s ; and finally weed out the tuples which do not generate G . This allows us to find the subgroup A of outer automorphisms which preserve the set of reflections in at most a few hours even for the largest groups.

To find the image of Γ in A is rather easy since for the exceptional groups the morphism $\Gamma \xrightarrow{\bar{t}} A$ is either surjective or has an image of index 2 (the

last occurs exactly for G_5 , G_7 and $G_{28} = F_4$, which, in agreement with corollary 4.2, are also the only exceptional groups where $N_{\text{GL}(V)}(G)$ induces non-inner automorphisms; in these cases, $N_{\text{GL}(V)}(G)$ induces respectively the automorphism $(s, t) \mapsto (t, s)$, the automorphism $(s, t, u) \mapsto (s, u^{-1}, t^{-1})$, the diagram automorphism of F_4).

To check 1.2, we must lift $\bar{\iota}$ to the group $\text{Aut}(G)$, that is find representatives of the image of ι which satisfy the relations for Γ . Since Γ is an abelian 2-group, these relations are order relations, plus commutation relations.

Few representatives of a given automorphism turn out to have the right order, so we first compute all representatives which have the right order, and then it is an easy job to pick among them elements which commute (this procedure of course fails for G_{27} for which we must construct $\tilde{\iota}$ rather than ι ; in this case we take $K' = \mathbb{Q}(\zeta_{15})$; the group $\text{Gal}(K'/\mathbb{Q})$ is generated by \mathfrak{c} and \mathfrak{s}_7 , this last element being of order 4 (it is of order 2 in $\text{Gal}(K/\mathbb{Q})$); we can map these to automorphisms of order 2 and 4 respectively commuting to each other — but it is impossible to map \mathfrak{s}_7 to an automorphism of order 2).

The computational problem we find in this procedure is to compute the product of two automorphisms; since an automorphism is represented by the image of S , for this we need an expression as product of the generators of the image of S . For all groups but G_{34} this was solved by enumerating all elements of G by standard permutation group algorithms using a base and a strong generating set. For G_{34} we managed by only considering automorphisms which extend the one for G_{33} ; since the normalizer of G_{33} in G_{34} is G_{33} times the center of G_{34} , such automorphisms map the last generator of G_{34} to a conjugate by G_{33} , which makes the computation feasible.

Table 1 summarizes our results for the exceptional reflection groups; it gives the diagram for G , and then gives the value of K and describes ι in terms of the generators S given by the diagram.

The nodes of the diagrams are labeled by the elements of S ; we give to these elements names in the list s, t, u, v, w, x in that order. A number inside the node gives the order of the corresponding reflection when it is not 2. To express the braid relations, we use the conventions for Coxeter diagrams: two nodes which are not connected correspond to commuting reflections. A single line connecting two nodes s and t corresponds to the relation $sts = tst$, a double line to $stst = tsts$, a triple line to $ststst = tststs$, and a line labeled with the number n corresponds to a braid relation $\underbrace{stst \cdots}_{n \text{ terms}} = \underbrace{tsts \cdots}_{n \text{ terms}}$.

These conventions are extended by additional ones; a circle joining 3 nodes s, t and u , with the number n inside (when there is no number inside, 3 is to be understood) corresponds to a clockwise circular relation $\underbrace{stust \cdots}_{n \text{ terms}} = \underbrace{tustu \cdots}_{n \text{ terms}} = \underbrace{ustus \cdots}_{n \text{ terms}}$; for instance, the relation for G_7, G_{11}, G_{19} and the generators s, t, u of G_{31} is $stu = tus = ust$. Similarly the 6 inside the

triangle for G_{33} and G_{34} means that in addition to length 3 braid relations implied by the sides of the triangle, there is the circular relation $tuvtw = wtvut = vtwtu$.

The double line in G_{29} expresses a length 4 braid relation between v and ut : $vutvut = utvutv$. The triangle symbol for G_{24} and G_{27} corresponds to the relation $s(uts)^2 = (stu)^2t$. Finally, the groups G_{13} and G_{15} have more complicated relations which are spelled out below the diagram.

The groups $G_{28} = F_4$, $G_{35} = E_6$, $G_{36} = E_7$ and $G_{37} = E_8$, which are rational, do not appear in the table since our theorems are trivial for them.

The map ι is specified by giving for generators γ of Γ the tuple $\iota(\gamma)(S)$ (where S is in the order s, t, u, v, w, x). The elements of Γ are denoted by \mathbf{c} for complex conjugation and by \mathfrak{s}_i for the following element: let $\mathbb{Q}(\zeta_n)$ be the smallest cyclotomic field containing K ; then \mathfrak{s}_i is the restriction to K of the Galois automorphism of $\mathbb{Q}(\zeta_n)$ which sends ζ_n to ζ_n^i .

Table 1: The morphism $\tilde{\iota}$

Group	Diagram	Field	$\tilde{\iota}$
G_4		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (s^{-1}, st^{-1})$
G_5		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (s^{-1}, st^{-1})$
G_6		$\mathbb{Q}(\zeta_{12})$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}),$ $\mathfrak{s}_7 \mapsto (st^{-1}st, t)$
G_7		$\mathbb{Q}(\zeta_{12})$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}, (u^{-1})^s),$ $\mathfrak{s}_7 \mapsto (sut, t, u)$
G_8		$\mathbb{Q}(i)$	$\mathbf{c} \mapsto (s^{-1}, t^{-1})$
G_9		$\mathbb{Q}(\zeta_8)$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}),$ $\mathfrak{s}_5 \mapsto (stts, t)$
G_{10}		$\mathbb{Q}(\zeta_{12})$	$\mathbf{c} \mapsto ((s^{-1})^{t^{-1}}, t^{-1})$ $\mathfrak{s}_7 \mapsto (stts, t^{-1})$
G_{11}		$\mathbb{Q}(\zeta_{24})$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}, (u^{-1})^s),$ $\mathfrak{s}_{13} \mapsto (stsut, t, u),$ $\mathfrak{s}_{19} \mapsto (s, t, (u^{-1})^{tu^{-1}s})$
G_{12}		$\mathbb{Q}(\sqrt{-2})$	$\mathbf{c} \mapsto (t, s, u)$

Table 1: The morphism \tilde{t}

Group	Diagram	Field	\tilde{t}
G_{13}	<p>$tust = ust,$ $stust = ustus$</p>	$\mathbb{Q}(\zeta_8)$	$\mathfrak{c} \mapsto (s, u, t^s),$ $\mathfrak{s}_3 \mapsto (s, u^{tu}, u^{su})$
G_{14}		$\mathbb{Q}(\zeta_3, \sqrt{-2})$	$\mathfrak{c} \mapsto (s^{-1}, (t^{-1})^s),$ $\mathfrak{s}_7 \mapsto (s, t^{stst})$
G_{15}	<p>$stu = ust,$ $tusts = ustst$</p>	$\mathbb{Q}(\zeta_{24})$	$\mathfrak{c} \mapsto (s, t^{-1}, u^s),$ $\mathfrak{s}_{13} \mapsto (s^{usu}, t, u),$ $\mathfrak{s}_{19} \mapsto (s, t, u^{sus})$
G_{16}		$\mathbb{Q}(\zeta_5)$	$\mathfrak{s}_2 \mapsto (s^2, (t^2)^{s^2t^{-1}s})$
G_{17}		$\mathbb{Q}(\zeta_{20})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}),$ $\mathfrak{s}_7 \mapsto (s^{t^{-1}st^2}, t^2)$
G_{18}		$\mathbb{Q}(\zeta_{15})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}),$ $\mathfrak{s}_7 \mapsto (s^{t^{-1}s}, t^2)$
G_{19}		$\mathbb{Q}(\zeta_{60})$	$\mathfrak{c} \mapsto (s^t, t^{-1}, u^{-1}),$ $\mathfrak{s}_7 \mapsto (s^{ts}, t^{st^{-1}}, u^2),$ $\mathfrak{s}_{41} \mapsto (s, (t^{-1})(ut)^{su}, u)$
G_{20}		$\mathbb{Q}(\zeta_3, \sqrt{5})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}),$ $\mathfrak{s}_7 \mapsto (s, s^{(ts^{-1})^2})$
G_{21}		$\mathbb{Q}(\zeta_{12}, \sqrt{5})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}),$ $\mathfrak{s}_7 \mapsto (s^{(t^{-1})^{st}}, t),$ $\mathfrak{s}_{13} \mapsto (s^{t^{-1}s^{ts}}, t)$
G_{22}		$\mathbb{Q}(i, \sqrt{5})$	$\mathfrak{c} \mapsto (u, t, s),$ $\mathfrak{s}_7 \mapsto (u^{st}, u^{ts}, t^{us})$
$G_{23} = H_3$		$\mathbb{Q}(\sqrt{5})$	$\mathfrak{s}_2 \mapsto (u^{tsts}, u, t)$

Table 1: The morphism \tilde{t}

Group	Diagram	Field	\tilde{t}
G_{24}		$\mathbb{Q}(\sqrt{-7})$	$\mathbf{c} \mapsto (u, t, s)$
G_{25}		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}, u^{-1})$
G_{26}		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}, u^{-1})$
G_{27}		$\mathbb{Q}(\zeta_3, \sqrt{5})$ $K' = \mathbb{Q}(\zeta_{15})$	$\mathbf{c} \mapsto (s^{tut}, t, u),$ $\mathfrak{s}_7 \mapsto (u^{ts}, t^u, t)$
G_{29}		$\mathbb{Q}(i)$	$\mathbf{c} \mapsto (s, t, u, v^u)$
$G_{30} = H_4$		$\mathbb{Q}(\sqrt{5})$	$\mathfrak{s}_2 \mapsto (u^{tsts}, u, t, v^{utstuststuvuts})$
G_{31}		$\mathbb{Q}(i)$	$\mathbf{c} \mapsto (u, t, s, w, v)$
G_{32}		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (s^{-1}, t^{-1}, u^{-1}, v^{-1})$
G_{33}		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (w, v, u, t, s)$
G_{34}		$\mathbb{Q}(\zeta_3)$	$\mathbf{c} \mapsto (w, v, u, t, s, x^{wvtuvwstwtvs})$

We have some observations to make on table 1. First, the morphism \tilde{t} is not unique; often, there are even many different orbits of possible \tilde{t} up to inner automorphisms of G . Each time we have chosen one which preserves a parabolic subgroup, which will be used to construct nice models of the reflection representation of G in section 7.

This is not always the choice with the simplest formulas. In particular, for Shephard groups (those which have the same diagram, apart from the order of the reflections, as a Coxeter group, thus the same braid group as a Coxeter group), it turns out that it is always possible to find a morphism which maps the complex conjugation \mathbf{c} to the automorphism which sends each element of S to its inverse. When \mathbf{c} does not generate $\text{Gal}(K/\mathbb{Q})$, one

can send \mathfrak{s}_7 to $(s^{ts^{-1}}, t^{-1})$ for G_{10} and \mathfrak{s}_2 to $(s^2, (s^2)^{t^{-2}s^2})$ for G_{16} to get such a morphism (in the case of G_{14} , \mathfrak{s}_7 can keep the same image as in table 1). A general explanation for this phenomenon would be desirable.

6. GALOIS DESCENT

The existence of a model over K globally invariant by $\Gamma = \text{Gal}(K/\mathbb{Q})$ of a faithful representation of G , namely its reflection representation, enabled us in the $G(de, e, r)$ case to construct the morphism $\iota : \Gamma \rightarrow \text{Aut}(G)$.

In this section, we show how by Galois descent, one can conversely try to construct a globally invariant model for any given irreducible representation — more precisely we will try to get a $\tilde{\iota}$ -equivariant model. The possible obstruction to do so is an element of the Brauer group of K' that turns out to be always trivial except for the group G_{22} .

6.1. Non-abelian Galois cohomology. Recall from [Se] that, if H is a group acting on the left on a group Q , then a map $h \mapsto A_h$ from H to Q is a *cocycle* if $A_{st} = A_s s(A_t)$ for all $s, t \in H$. A particular cocycle, called the *zero cocycle*, is given by sending all elements of H to the trivial element of Q . The set of cocycles is denoted $Z^1(H, Q)$. Two cocycles $h \mapsto A_h$ and $h \mapsto B_h$ are said to be cohomologous if there exists $a \in Q$ such that $B_h = a^{-1} A_h h(a)$. This defines an equivalence relation on $Z^1(H, Q)$ whose set of equivalence classes is denoted $H^1(H, Q)$. The set of cocycles cohomologous to the zero cocycle is called the set of coboundaries and is denoted $B^1(H, Q)$. If Q is commutative these definitions coincide with those of ordinary group cohomology.

Application to 1.5. Let $\rho : G \rightarrow \text{GL}(E)$ where E is a K' -vector space be an irreducible representation. Assume we choose a \mathbb{Q} -form $E = E_0 \otimes_{\mathbb{Q}} K'$. This defines an action of $\Gamma' = \text{Gal}(K'/\mathbb{Q})$ on E and by 1.1 for $\gamma \in \Gamma'$ we have $\gamma \circ \rho \simeq \rho \circ \tilde{\iota}(\gamma)$. This means that there exists $A_\gamma \in \text{GL}(E)$ such that for any $g \in G$ we have $A_\gamma \gamma(\rho(g)) A_\gamma^{-1} = \rho(\tilde{\iota}(\gamma)(g))$. Since ρ is absolutely irreducible, by Schur's lemma, A_γ gives a well-defined element of $\text{PGL}(E)$. It is immediate to check that $\gamma \mapsto A_\gamma$ is in a fact a cocycle, that is, an element of $Z^1(\Gamma', \text{PGL}(E))$.

Assume now that ρ has a globally invariant model, of the form $g \mapsto a\rho(g)a^{-1}$; this means that there exists a map $\iota' : \Gamma' \rightarrow \text{Aut}(G)$ such that $\gamma(a\rho(g)a^{-1}) = a\rho(\iota'(\gamma)(g))a^{-1}$, or in other terms that the model is ι' -equivariant for some $\iota' : \Gamma' \rightarrow \text{Aut}(G)$. If $\iota' = \tilde{\iota}$ we get the equality in $\text{PGL}(E)$ that $a^{-1}\gamma(a) = A_\gamma$, that is that $\{\gamma \mapsto A_\gamma\} \in B^1(\Gamma', \text{PGL}(E))$. Thus the obstruction to the existence of $\tilde{\iota}$ -equivariant model is an element of $H^1(\Gamma', \text{PGL}(E))$.

In practice to apply this procedure we encounter another problem which turns out to involve also Galois cohomology: although we know that any ρ has a model over K' (even K), sometimes (*e.g.* when using the CHEVIE data) we are not given such a model but a model $E = E_0 \otimes_{K'} L$ over some

Galois extension L of K' . Again the fact that for $\gamma \in \Pi = \text{Gal}(L/K')$ we have $\gamma \circ \rho \simeq \rho$ means that we can find an intertwiner A_γ such that $A_\gamma \gamma(\rho(g)) A_\gamma^{-1} = \rho(g)$, which is an element of $Z^1(\Pi, \text{PGL}(E))$; and again the existence of a model over K' is equivalent to this cocycle to be a coboundary.

We will now give an algorithm to check the vanishing of a cocycle of a Galois group into the projective linear group.

6.2. Brauer groups. In this subsection and the next one, $K_0 \subset K$ will denote an arbitrary Galois extension of number fields. Let $\Gamma = \text{Gal}(K/K_0)$ and let $\text{Br}(K/K_0) = H^2(\Gamma, K^\times)$ be the Brauer group. Let $E = E_0 \otimes_{K_0} K$ be a K_0 -form of a finite dimensional K -vector space E . The short exact sequence $1 \rightarrow K^\times \rightarrow \text{GL}(E) \rightarrow \text{PGL}(E) \rightarrow 1$ gives rise to a Galois cohomology long exact sequence (of pointed sets; see [Se, prop. 2 p. 133]):

$$1 \rightarrow H^0(\Gamma, K^\times) \rightarrow H^0(\Gamma, \text{GL}(E)) \rightarrow H^0(\Gamma, \text{PGL}(E)) \rightarrow H^1(\Gamma, K^\times) \rightarrow \\ H^1(\Gamma, \text{GL}(E)) \rightarrow H^1(\Gamma, \text{PGL}(E)) \rightarrow H^2(\Gamma, K^\times)$$

and in particular a coboundary operator $H^1(\Gamma, \text{PGL}(E)) \rightarrow H^2(\Gamma, K^\times)$. This map sends $\{\gamma \mapsto A_\gamma\} \in H^1(\Gamma, \text{PGL}(E))$ to the class of $(\gamma, \tau) \mapsto \tilde{A}_\gamma \gamma(\tilde{A}_\tau) \tilde{A}_{\gamma\tau}^{-1}$ in $H^2(\Gamma, K^\times)$, where, for all $\gamma \in \Gamma$, \tilde{A}_γ is a representative of A_γ in $\text{GL}(E)$. This definition does not depend on the choice of the representatives \tilde{A}_γ and this map is known to be injective (this uses the version of Hilbert's theorem 90 which says that $H^1(\Gamma, \text{GL}(E))$ is trivial; see [Se] ch. X proposition 8 and proposition 9).

In the important case where K is a cyclic extension of degree n of K_0 , assume $\Gamma = \langle \gamma \rangle$ and let $N : K \rightarrow K_0$ be the norm map $x \mapsto x\gamma(x) \dots \gamma^{n-1}(x)$. Then an explicit isomorphism from $H^2(\Gamma, K^\times)$ to the 0-th Tate cohomology group $\hat{H}^0(\Gamma, K^\times) = K_0^\times / NK^\times$ is given by the Nakayama map

$$c \mapsto \prod_{k=0}^{n-1} c(\gamma^k, \gamma)$$

hence $\{\gamma \mapsto A_\gamma\}$ is sent to the class of $\tilde{A}_\gamma \gamma(\tilde{A}_\gamma) \dots \gamma^{n-1}(\tilde{A}_\gamma)$ (a scalar matrix identified to an element of K_0^\times) in K_0^\times / NK^\times .

It follows that, when K is a cyclic extension of K_0 , the vanishing of $c \in H^2(\Gamma, K^\times) = \text{Br}(K/K_0)$ boils down to the verification that some element of K_0^\times is a norm of an element of K . When K is an *abelian* extension of K_0 , we can always reduce to this case by induction on $[K : K_0]$, by the inflation-restriction exact sequence for Brauer groups. Indeed, choosing a cyclic subgroup Γ' of Γ defines a Galois sub-extension $K_1 = K^{\Gamma'}$ of K such that $\text{Gal}(K/K_1)$ is cyclic. We then have the exact sequence (see [Se] ch. X §4 proposition 6)

$$0 \rightarrow \text{Br}(K_1/K_0) \rightarrow \text{Br}(K/K_0) \rightarrow \text{Br}(K/K_1)$$

and we may follow the following procedure : first check that the image in $\text{Br}(K/K_1)$ is zero, and then start again with the induced element in $\text{Br}(K_1/K_0)$, until K_1/K_0 itself becomes a cyclic extension.

We finally mention the following general result.

Proposition 6.1. *If $\#\Gamma$ is prime to $\dim(E)$, then $H^1(\Gamma, \text{PGL}(E)) = 0$*

Proof. Let $N = \dim(E)$. Let $c \in H^1(\Gamma, \text{PGL}(E))$ and let A be the central simple algebra associated to it. Let e be its exponent, that is the order of its class $[A]$ in the Brauer group of K_0 . Since $A \otimes K \simeq M_N(K)$ we have $[A][K] = 0$ in the Brauer group hence $e|[K : K_0]$ and $e|\#\Gamma$ (see [Se] X §4 ex. 2). On the other hand, $A \simeq M_r(D)$ with D a skew field whose center is K_0 and $\dim_{K_0} A = N^2$. It follows that $N^2 = [A : K_0] = r^2[D : K_0]$. Now $[D : K_0] = m^2$ for some integer m (called the index of D). It is a classical fact that the exponent divides the index ([Se] X §5 ex. 3a), thus $e|N$, and $e = 1$ since N and $\#\Gamma$ are coprime. The conclusion follows from the fact that $H^1(\Gamma, \text{PGL}(E))$ embeds in the Brauer group of K_0 . \square

Explicit computations. In practice (*e.g.* to get an explicit model over a smaller field of a representation), it is not enough to solve the problem whether a cocycle $\{\gamma \mapsto A_\gamma\} \in Z^1(\Gamma, \text{PGL}(E))$ is a coboundary. We want an explicit element $M \in \text{PGL}(E)$ such that $A_\gamma = M^{-1}\gamma(M)$ for all $\gamma \in \Gamma$. If we can get a preimage (B_γ) of (A_γ) in $Z^1(\Gamma, \text{GL}(E))$, then by Hilbert's Theorem 90 this cocycle is a coboundary, meaning that there exist $\tilde{M} \in \text{GL}(E)$ such that $B_\gamma = \tilde{M}^{-1}\gamma(\tilde{M})$ for all $\gamma \in \Gamma$. A rather constructive proof of this theorem (see *e.g.* [Se] ch. X proposition 3) goes as follows. For all $C \in \text{End}(E)$, the expression

$$X = \sum_{\gamma \in \Gamma} B_\gamma \gamma(C)$$

satisfies $\gamma(X) = B_\gamma^{-1}X$. Because K has characteristic zero, general arguments imply that “many” $C \in K$ exist such that X is invertible. If this is the case, then $B_\gamma = X\gamma(X)^{-1}$ thus $\tilde{M} = X^{-1}$ is the desired solution. For instance, if $\text{Gal}(K/K_0) = \{1, \gamma\}$ has order 2 with generator γ_0 , then the condition on C is that $-C\gamma(C)$ is not an eigenvalue for $B_{\text{Id}}^{-1}B_{\gamma_0}$.

In the case where K is a cyclic extension of K_0 , the study of the previous section shows that we can always lift a coboundary to a cocycle $\{\gamma \mapsto B_\gamma\}$ for $\text{GL}(E)$. Indeed, if $\Gamma = \langle \gamma \rangle$ we have seen that $\tilde{A}_\gamma \gamma(\tilde{A}_\gamma) \dots \gamma^{n-1}(\tilde{A}_\gamma) = N(\lambda)\text{Id}$ for some $\lambda \in K^\times$ so that $B_\gamma = \lambda^{-1}\tilde{A}_\gamma$ belongs to $Z^1(\Gamma, \text{GL}(E))$ (it is easy to check that the equation $\text{Id} = \tilde{A}_\gamma \dots \gamma^{n-1}(\tilde{A}_\gamma)$ is necessary and sufficient for $\tilde{A}_{\gamma^i} = \tilde{A}_\gamma \dots \gamma^{i-1}(\tilde{A}_\gamma)$ to define a cocycle of $\langle \gamma \rangle$).

6.3. An algorithm to check 1.5. Let $\rho : G \rightarrow \text{GL}(E)$ be an irreducible representation of G over the K' -vector space E for which we would like to get a \tilde{t} -equivariant model. When K'/\mathbb{Q} is cyclic, we found an explicit necessary

and sufficient condition for a class in $H^1(\Gamma', \mathrm{PGL}(E))$ to vanish and thus to give an equivariant model.

In general we know that K'/\mathbb{Q} is an abelian extension. We will show how to reduce by induction to the case of a cyclic extension, by making explicit the inflation-restriction exact sequence in this case.

Choose a tower of (Galois) extensions $\mathbb{Q} = K_0 \subset K_1 \subset \dots \subset K_m = K'$ such that K_r/K_{r-1} is a cyclic extension for all $1 \leq r \leq m$. We fix a \mathbb{Q} -form E_0 of E and a basis of E_0 so that we identify E_0 with \mathbb{Q}^n and E with K'^n .

Assume that, for some $r \in [1, m]$ we have a model $\rho_r : G \rightarrow \mathrm{GL}(E)$ of ρ such that $\gamma \circ \rho = \rho \circ \iota(\gamma)$ for all $\gamma \in \mathrm{Gal}(K'/K_r)$. We will show how to decrease r .

Let $\{\tau \mapsto M_\tau\} \in Z^1(\mathrm{Gal}(K'/K_{r-1}), \mathrm{PGL}(E))$ intertwining $\tau \circ \rho_r$ and $\rho_r \circ \iota(\tau)$ for $\tau \in \mathrm{Gal}(K'/K_{r-1})$. For γ in the subgroup $\mathrm{Gal}(K'/K_r)$ we have $M_\gamma = 1$ by the induction hypothesis; we thus have $M_{\tau\sigma} = M_\tau$ for all $\sigma \in \mathrm{Gal}(K'/K_r)$ by the cocycle condition $M_{\tau\gamma} = M_\tau \tau(M_\gamma)$. Hence M_τ only depends on the class of τ in $\mathrm{Gal}(K_r/K_{r-1})$, and lies in $\mathrm{PGL}_n(K_r)$ because $M_\tau = M_{\tau\gamma} = M_{\gamma\tau} = M_\gamma \gamma(M_\tau) = \gamma(M_\tau)$ for all $\gamma \in \mathrm{Gal}(K'/K_r)$. It follows that $\{\tau \mapsto M_\tau\}$ defines an element of $Z^1(\mathrm{Gal}(K_r/K_{r-1}), \mathrm{PGL}_n(K_r))$; we are thus reduced to a cyclic case; assuming we can solve it we find $X \in \mathrm{GL}_n(K_r)$ such that $M_\tau = X\tau(X)^{-1}$ for all $\tau \in \mathrm{Gal}(K'/K_{r-1})$. Then $\rho_{r-1}(g) = X\rho_r(g)X^{-1}$ is a model of ρ such that $\gamma \circ \rho_{r-1} = \rho_{r-1} \circ \iota(\gamma)$ for all $\gamma \in \mathrm{Gal}(K'/K_{r-1})$.

We could successfully carry out this algorithm for all representations of exceptional groups for which we had a model. However, with the exception of the cyclic case, this procedure does not lead to necessary conditions. Note however the following observation:

Proposition 6.2. *Assume G to be an exceptional irreducible complex reflection group. Then any odd-dimensional irreducible representation has a globally invariant model over K' .*

Proof. A case-by-case analysis shows that $[K' : \mathbb{Q}]$ is always a power of 2. The result is then an immediate consequence of 6.1. \square

7. INVARIANT MODELS

We will now prove theorem 1.5 using the results of the previous sections and multiplicity one property of tensor products.

For two representations ρ_1, ρ_2 of a group G , we will denote $(\rho_1 | \rho_2)$ the dimension of the space of intertwiners of ρ_1 and ρ_2 . This coincides with the usual scalar product of the corresponding characters.

We will make repeated use of the following lemma.

Lemma 7.1. *Let G be a finite group and K be a field of characteristic 0. Let ρ' be a finite-dimensional representation of G defined over K and let $\rho \in \mathrm{Irr}(G)$ be such that $(\rho' | \rho) = 1$. Let χ be the character of ρ . Assume that χ takes values in K . Then ρ admits a model over K and :*

- if $\gamma \in \text{Gal}(K/\mathbb{Q})$ and $a \in \text{Aut}(G)$ satisfy $\gamma \circ \rho' = \rho' \circ a$ and $\gamma \circ \chi = \chi \circ a$, then ρ admits a model over K such that $\gamma \circ \rho = \rho \circ a$.
- if there exists $j : \text{Gal}(K/\mathbb{Q}) \rightarrow \text{Aut}(G)$ such that $\gamma \circ \rho' = \rho' \circ j(\gamma)$ and $\gamma \circ \chi = \chi \circ j(\gamma)$, then ρ admits a j -equivariant model over K .

Proof. Since ρ is an isotypic component of a ρ' and has its character over K it is defined over K ; indeed, the G -equivariant projector

$$p = \frac{\chi(1)}{\#G} \sum_{g \in G} \chi(g^{-1}) \rho'(g)$$

on the ρ -isotypic component of ρ' belongs to $\text{End}_K(V)$. Moreover, if $\gamma \circ \chi = \chi \circ a$, we have

$$\gamma(p) = \frac{\chi(1)}{\#G} \sum_{g \in G} \chi(a(g)^{-1}) \rho'(a(g)) = p$$

thus a \mathbb{Q} -form $V' = V'_0 \otimes_{\mathbb{Q}} K$ induces a \mathbb{Q} -form on $V = \text{Im } p$. It follows that $\gamma \circ \rho = \rho \circ a$, which proves the last two points. \square

7.1. Reflection representations. We now prove theorem 1.5 for the reflection representation of exceptional groups, that is the following proposition.

Proposition 7.2. *Let G be an exceptional irreducible complex reflection different from G_{22} given by $\rho : G \hookrightarrow \text{GL}(V)$ where V is a K -vector space and K is the field of definition of G . Let K' be as in theorem 1.2. Then ρ has a $\tilde{\iota}_\rho$ -equivariant model over K' .*

Recall that $K = K'$ except when $G = G_{27}$.

Models for reflection representations of 2-dimensional groups from the braid group. A way to obtain the 2-dimensional representations of the groups G_4 to G_{22} is by using the following matrices:

$$\mathbf{s} \mapsto \begin{pmatrix} x_1 & \frac{y_1+y_2}{y_1 y_2} - \frac{(z_1+z_2)x_2}{r} \\ 0 & x_2 \end{pmatrix}, \mathbf{t} \mapsto \begin{pmatrix} y_1 + y_2 & 1/x_1 \\ -y_1 y_2 x_1 & 0 \end{pmatrix}, \mathbf{u} \mapsto \begin{pmatrix} 0 & \frac{-r}{y_1 y_2 x_1 x_2} \\ r & z_1 + z_2 \end{pmatrix}$$

where $r = \sqrt{x_1 x_2 y_1 y_2 z_1 z_2}$. The braid groups of G_7 , G_{11} and G_{19} are isomorphic to the same group B , with presentation $\langle \mathbf{s}, \mathbf{t}, \mathbf{u} \mid \mathbf{stu} = \mathbf{tus} = \mathbf{ust} \rangle$; the above matrices give the 2-dimensional representation of B where the eigenvalues of \mathbf{s} (resp. \mathbf{t} , \mathbf{u}) are x_1, x_2 (resp. y_1, y_2, z_1, z_2). This representation factors through the Hecke algebra, the quotient of the group algebra of B by the relations $(\mathbf{s} - x_0)(\mathbf{s} - x_1) = 0$, $(\mathbf{t} - y_0)(\mathbf{t} - y_1)(\mathbf{t} - y_2) = 0$ and $\prod_{i=0}^{i=n-1} (\mathbf{u} - z_i) = 0$ where $n = 3$ (resp. 4,5) for G_7 (resp. G_{11}, G_{19}). In turn the group algebra of G_7 (resp. G_{11}, G_{19}) is the specialization of the Hecke algebra for $x_i \mapsto (-1)^i, y_i \mapsto \zeta_3^i, z_i \mapsto \zeta_3^i$ (resp. $z_i \mapsto \zeta_4^i, z_i \mapsto \zeta_5^i$); the Hecke algebras for G_4 to G_6 are subalgebras of partial specializations of that for G_7 (the same holds for G_8 to G_{15} with respect to G_{11} and G_{16} to G_{22} with respect to G_{19}): in each case, these algebras are generated by conjugates of

a part of the generators (or of some power of them); the other generators are specialized to the group algebra.

For G_4, G_8 and G_{16} the Hecke algebra is generated by \mathbf{u} and ${}^s\mathbf{u}$.

For G_5, G_{10} and G_{18} it is generated by \mathbf{t} and \mathbf{u} .

For G_6, G_9 and G_{17} it is generated by \mathbf{s} and \mathbf{u} .

For G_{14} and G_{21} it is generated by \mathbf{s} and \mathbf{t} .

For G_{12} and G_{22} it is generated by \mathbf{s} , ${}^t\mathbf{s}$ and \mathbf{s}^t .

For G_{20} it is generated by \mathbf{t} and ${}^s\mathbf{t}$.

For G_{13} it is generated by \mathbf{u}^2 , \mathbf{s} and \mathbf{s}^t .

Finally for G_{15} it is generated by \mathbf{s} , \mathbf{t} and \mathbf{u}^2 .

Proof of proposition 7.2. The statement is clear if the rank of G is odd, by proposition 6.2. Then we may assume that, either $G = G_{34}$, or that its rank is 2 or 4. Recall that the groups of rank 2 are numbered G_4, \dots, G_{22} .

We started from a model of the reflection representations coming either from the Hecke algebra as described above, or by root diagrams as considered in [Cohen]. Then we apply the algorithm of Galois descent; in order to solve norm equations we used the computer system MAGMA. All equations thus obtained for the reflection representation were solvable except for G_{22} .

Note that, even when the original model is globally invariant, the induced morphism $\eta : \Gamma \rightarrow \text{Aut}(G)$ does not in general coincide with $\tilde{\iota}_\rho$. Furthermore, the induced action of η on the isomorphism classes of irreducible representations of G may also differ from the natural Galois action of Γ , as shows the second example below.

The models given in table 2 were simplified by the following elementary observation. For all the images $\tilde{\iota}(\Gamma)$ chosen here, it turns out that there is a generator s such that the subgroup H generated by s is stable under $\tilde{\iota}(\Gamma)$. Lemma 7.1 thus implies that there is a basis of the underlying \mathbb{Q} -form of K^2 on which the action of s is diagonal.

Example : the case of G_9 . Let us start with the model

$$s \mapsto \begin{pmatrix} 0 & -\zeta_{24}^{11} \\ \zeta_{24} & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \mathbf{i} & -\zeta_3 \\ 0 & 1 \end{pmatrix}$$

which is defined over $\mathbb{Q}(\zeta_{24})$, an extension of degree 2 of $K = \mathbb{Q}(\zeta_8)$. By Galois descent or directly one finds that conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$ yields the following model over K .

$$s \mapsto \begin{pmatrix} 0 & -\zeta_8 \\ \zeta_8^3 & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \mathbf{i} & -1 \\ 0 & 1 \end{pmatrix}$$

We have $\Gamma \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, with three subgroups of order 2 generated by \mathfrak{s}_{-1} , \mathfrak{s}_5 and \mathfrak{s}_3 . To these three subgroups correspond three intermediate extensions between \mathbb{Q} and K . For each of these extensions, we get a cocycle

TABLE 2. Invariant models for G_4 to G_{21}

G_4	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{-3}} \begin{pmatrix} -1 & \zeta_3 \\ 2 & \zeta_3 \end{pmatrix}$
G_5	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{-3}} \begin{pmatrix} \zeta_3 & \zeta_3 \\ 2 & -1 \end{pmatrix}$
G_6	$s \mapsto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$
G_7	$s \mapsto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad u \mapsto \frac{1}{\sqrt{-3}} \begin{pmatrix} \zeta_3 & \zeta_3 \\ 2 & -1 \end{pmatrix}$
G_8	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad t \mapsto \frac{i-1}{2} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}$
G_9	$s \mapsto \frac{1}{2} \begin{pmatrix} \sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
G_{10}	$s \mapsto \frac{1}{\zeta_3(i-1)} \begin{pmatrix} -i & 2i \\ \frac{1}{2} & 1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
G_{11}	$s \mapsto C = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad u \mapsto \frac{\zeta_3}{(i+1)\sqrt{3}} \begin{pmatrix} -2\zeta_3 & \zeta_3 \\ 2 & 2 \end{pmatrix}$
G_{12}	$s \mapsto D = \frac{1}{2} \begin{pmatrix} 1 & 1 + \frac{1}{\sqrt{-2}} \\ 2 + \sqrt{-2} & -1 \end{pmatrix}, \quad t \mapsto \bar{D}, \quad u \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
G_{13}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad u \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$
G_{14}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \mapsto \frac{\zeta_3^2}{2} \begin{pmatrix} -1 + \sqrt{-2} & 1 \\ -1 & -1 - \sqrt{-2} \end{pmatrix}$
G_{15}	$s \mapsto C, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad u \mapsto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \zeta_3^2 \\ 2\zeta_3 & -1 \end{pmatrix}$
G_{16}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{5}} \begin{pmatrix} 1 - \zeta_5^3 & \zeta_5^4 - 1 \\ \zeta_5 - \zeta_5^2 & \zeta_5 - \zeta_5^3 \end{pmatrix}$
G_{17}	$s \mapsto M = \frac{i}{\sqrt{5}} \begin{pmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^3 - \zeta_5^2 \\ \zeta_5^3 - \zeta_5^2 & \zeta_5 - \zeta_5^4 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}$
G_{18}	$s \mapsto N = \frac{\zeta_3^2}{\sqrt{5}} \begin{pmatrix} \zeta_5^2 - \zeta_5^4 & 1 - \zeta_5^4 \\ \zeta_5 - 1 & \zeta_5^3 - \zeta_5 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}$
G_{19}	$s \mapsto N, \quad t \mapsto M, \quad u \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}$
G_{20}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad t \mapsto \frac{\zeta_3^2}{2\sqrt{-15}} \begin{pmatrix} -5 - \sqrt{-15} & 2 \\ 10 & 5 - \sqrt{-15} \end{pmatrix}$
G_{21}	$s \mapsto \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 + \sqrt{5} & -1 + \frac{1}{\sqrt{5}} \\ -5 + \sqrt{5} & -1 - \frac{1}{\sqrt{5}} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$

$\mathfrak{s}_k \mapsto A_k \in GL_2(K)$, given by the following matrices

$$A_{-1} = \begin{pmatrix} -i & 0 \\ 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} -1 & -1-i \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} i & -1+i \\ 0 & 1 \end{pmatrix}$$

It turns out that the last step of the algorithm is easy, because these three cocycles match, so we get in fact an element of $Z^1(\Gamma, GL_2(K))$. In order to apply Hilbert Theorem 90 we need to find $\lambda \in K$ such that $\lambda + \mathfrak{s}_{-1}(\lambda)A_1 + \mathfrak{s}_5(\lambda)A_2 + \mathfrak{s}_3(\lambda)A_3$ is invertible ; it happens that $\lambda = 1 + \zeta_8 + \zeta_8^3$ is a solution that gives the following invariant model

$$s \mapsto \frac{1}{2} \begin{pmatrix} -1 - \zeta_8 + \zeta_8^3 & 1/2 - \zeta_8 + \zeta_8^3 \\ 2 & 1 + \zeta_8 - \zeta_8^3 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} i & \frac{i-1}{2} \\ 0 & 1 \end{pmatrix}$$

Finally, we know that the subgroup generated by t is stable by $\tilde{\iota}(\Gamma)$. It is thus possible to diagonalize it by a rational matrix. We get the still invariant and simpler model

$$s \mapsto \frac{1}{2} \begin{pmatrix} \sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

An example for G_7 . The model ρ of the reflection representation of G_7 given by

$$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \mapsto \frac{\zeta_3^2}{(1+i)} \begin{pmatrix} -1 & -1 \\ i & -i \end{pmatrix}, \quad u \mapsto \frac{\zeta_3^2}{(1+i)} \begin{pmatrix} -1 & i \\ -1 & -i \end{pmatrix}$$

is globally invariant, giving rise to an homomorphism $\eta : \text{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}) \rightarrow \text{Aut}(G_7)$ given by (specifying automorphisms, as in table 1, by the images of s, t, u) $\mathfrak{c} \mapsto (s, u^{-1}, t^{-1})$, $\mathfrak{s}_7 \mapsto (s, s^{us}, u^{ts})$. However, the automorphism $\eta(\mathfrak{c})$ is not compatible with the action of the complex conjugation on other characters of G , contrary to the homomorphism $\tilde{\iota}$ given in table 1, which corresponds to the model given in table 2. The existence of this example is related to the fact that G_7 is the only exceptional group such that the image $\tilde{\iota}(\Gamma)$ is not central in the group A of proposition 1.4

7.2. Proof of 1.5 for exceptional groups. In this section we will deduce the existence of an $\tilde{\iota}$ -equivariant model for every irreducible representation of G from the existence of such a model for the reflection representation ρ_0 , by applying lemma 7.1. We will need the following proposition.

Proposition 7.3. *If G is of type $G(d, 1, r)$ then, for all distinct $\rho_1, \rho_2 \in \text{Irr}(G)$, we have $(\rho_1 \otimes \rho_0 \mid \rho_2) \leq 1$. This also holds for exceptional groups, except for G_{27}, G_{29}, G_{34} , and $G_{36} = E_7$. For these 4 exceptional groups, we have $(\rho_1 \otimes \rho_0 \mid \rho_2) \leq 2$ for distinct $\rho_1, \rho_2 \in \text{Irr}(G)$.*

Proof. The proof for the exceptional groups is a case-by-case computer check. If G is of type $G(d, 1, r)$ with $r \geq 2$, let H denote its natural parabolic subgroup of type $G(d, 1, r-1)$, and $\rho_1 \in \text{Irr}(G)$. Then $(\text{Ind}_H^G \text{Id} \mid \rho_0) = (\text{Id} \mid \text{Res}_H \rho_0) = 1$ because H is a maximal parabolic subgroup. It

follows that $\rho_1 \otimes \rho_0$ embeds in $\rho_1 \otimes \text{Ind}_H^G \text{Id} \simeq \text{Ind}_H^G \text{Res}_H \rho_1$. In particular, for $\rho_2 \in \text{Irr}(G)$ we have

$$(\rho_1 \otimes \rho_0 \mid \rho_2) \leq (\text{Ind}_H^G \text{Res}_H \rho_1 \mid \rho_2) = (\text{Res}_H \rho_1 \mid \text{Res}_H \rho_2)$$

so it is sufficient to check that, if $\rho_1 \neq \rho_2$, then $(\text{Res}_H \rho_1 \mid \text{Res}_H \rho_2) \leq 1$. This is obvious considering the branching rule (see [Ze, p. 104]). \square

Proof of 1.5 for $G(d, 1, r)$ and exceptional groups distinct from G_{27}, G_{29} or G_{34} . We already proved that there exists $\tilde{\iota} : \Gamma \rightarrow \text{Aut}(G)$ such that

- (i) $\forall \rho \in \text{Irr}(G), \gamma \circ \rho \simeq \rho \circ \tilde{\iota}(\gamma)$.
- (ii) ρ_0 admits a model such that $\gamma \circ \rho_0 = \rho_0 \circ \tilde{\iota}(\gamma)$.

We want to show that all $\rho \in \text{Irr}(G)$ admit an $\tilde{\iota}$ -equivariant model over K , that is a model over K such that $\gamma \circ \rho = \rho \circ \tilde{\iota}(\gamma)$ for all $\gamma \in \Gamma$. This obviously holds if G is a Weyl group, so we may assume that G is not $G_{36} = E_7$. First note that every $\rho \in \text{Irr}(G)$ embeds in some $\rho_0^{\otimes n}$ for some n , because ρ_0 is a faithful representation of G (cf. [FH, problem 2.37]). It follows that we can define the *level* $N(\rho) \in \mathbb{N}$ of ρ by

$$N(\rho) = \min\{n \in \mathbb{N} \mid \rho \hookrightarrow \rho_0^{\otimes n}\}.$$

In particular, $N(\rho) = m + 1$ implies that there exists $\rho' \in \text{Irr}(G)$ with $N(\rho') = m$ such that ρ embeds in $\rho' \otimes \rho_0$. By (i) we know that the representations of level at most 1, that is the trivial representation and ρ_0 , admit an equivariant model. We proceed by induction on the level. Assume that all representations of level m admit an equivariant model, and let $\rho \in \text{Irr}(G)$ such that $N(\rho) = m + 1$. Let $\rho' \in \text{Irr}(G)$ of level m such that $\rho \hookrightarrow \rho' \otimes \rho_0$. By proposition 7.3 we have $(\rho \mid \rho' \otimes \rho_0) = 1$. Moreover $\rho' \otimes \rho_0$ admits an equivariant model over K because ρ' and ρ_0 do. Then lemma 7.1 implies that ρ admits an equivariant model over K and we conclude by induction on the level.

Proof of 1.5 for G_{27}, G_{29}, G_{34} . Let $\mathcal{E} \subset \text{Irr}(G)$ denote the set of exceptional (rational) representations described in the introduction. We note that $\text{Id}, \rho_0 \notin \mathcal{E}$. Let us consider the following algorithm.

Algorithm

- (i) $L \leftarrow \{\text{Id}, \rho_0\}$.
- (ii) For all $\rho' \in L, L' \leftarrow L \cup \{\rho \in \text{Irr}(G) \setminus \mathcal{E} \mid (\rho' \otimes \rho_0 \mid \rho) = 1\}$.
- (iii) If $\#L' > \#L$, then $L \leftarrow L'$ and go to (ii).
- (iv) Return L .

The same arguments as above, based on lemma 7.1, show that all representations in the subset L of $\text{Irr}(G) \setminus \mathcal{E}$ returned by this algorithm admit an equivariant model over K . Note that this algorithm only uses the character table of G . It is then enough to check that it returns $\text{Irr}(G) \setminus \mathcal{E}$ for G_{27}, G_{29}

and G_{34} in order to conclude the proof of the theorem. Indeed, this is the case if one applies step (ii) five times for G_{27} and G_{34} , and four for G_{29} .

7.3. The exceptional case of G_{22} . In this case we have $K = K' = \mathbb{Q}(i, \sqrt{5})$. We start from the following model $\rho : G \hookrightarrow \mathrm{GL}(V)$ over the degree 2 extension $K'' = \mathbb{Q}(\zeta_{20})$ of K .

$$s \mapsto \frac{\zeta_5^2 - \zeta_5}{\zeta_{20}\sqrt{5}} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & \zeta_{20}^9 \\ -\zeta_{20} & 0 \end{pmatrix}, \quad u \mapsto \begin{pmatrix} 0 & -\zeta_{20} \\ \zeta_{20}^9 & 0 \end{pmatrix}$$

This model is $\mathrm{Gal}(K''/\mathbb{Q})$ -globally invariant. Note that the projection $\mathrm{Gal}(K''/\mathbb{Q}) \rightarrow \mathrm{Gal}(K/\mathbb{Q})$ is not split. We check this by noting that $\mathrm{Gal}(K''/\mathbb{Q}) \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ admits only one subgroup isomorphic to $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, namely $\mathrm{Gal}(K''/\mathbb{Q}(\sqrt{5}))$, and this subgroup does not surject on $\mathrm{Gal}(K/\mathbb{Q})$, because $\sqrt{5} \in K$.

As predicted by the theorem of Benard and Bessis, we manage to find a model ρ_0 over K by Galois descent. It is possible to get a model over K such that $\mathfrak{s}_{-1} \circ \rho = \rho \circ \tilde{i}(\mathfrak{s}_{-1})$ or such that $\mathfrak{s}_7 \circ \rho = \rho \circ \tilde{i}(\mathfrak{s}_7)$, but not both. The reason is that it is not possible to get a model such that $\mathfrak{s}_{13} \circ \rho = \rho \circ \tilde{i}(\mathfrak{s}_{13})$.

Lemma 7.4. *ρ does not admit any model over K which is globally invariant by $\mathrm{Gal}(K/\mathbb{Q}(i))$.*

Proof. We assume by contradiction that we are given such a model ρ_1 . Since ρ_1 is faithful there exists $a \in \mathrm{Aut}(G)$ of order 2 such that $\rho_1 \circ a = \mathfrak{s}_{13} \circ \rho_1$. We check that all automorphisms of G preserve the set of reflections, thus by theorem 4.1 and theorem 1.4 $\mathrm{Out}(G)$ acts faithfully and transitively on the set of reflection representations. It follows that a and $\tilde{i}(\mathfrak{s}_{13})$ have the same image in $\mathrm{Out}(G)$, namely $\tau(\mathfrak{s}_{13})$. In particular $a = \tilde{i}(\mathfrak{s}_{13}) \circ \mathrm{Ad}(h)$ for some $h \in G$. Exhausting all possibilities, we find using ρ_0 that the image of the cocycle defined by a in $H^1(\mathrm{Gal}(K/\mathbb{Q}(i)), \mathrm{PGL}_2(K)) \simeq K^\times/N(K^\times)$ is always the class of $1/3$ modulo $N(K^\times)$. Using MAGMA we check that this class is non trivial, thus leading to a contradiction. \square

We now investigate which representations of G admit a globally invariant model over K . This group has 18 irreducible representations, including two 1-dimensional ones. The four 2-dimensional (reflection) representations are deduced from each other through Galois action, so none of them admits a globally invariant model over K . In addition to these, there are 2 faithful 4-dimensional representations and 2 faithful 6-dimensional representations.

The center of G_{22} is cyclic of order 4, generated by $Z = (stu)^5$. We have $Z^2 = (ts)^5$. The other representations have for kernel a non-zero subgroup of $Z(G)$.

We show that the 8 odd-dimensional representations admit \tilde{i} -equivariant models over K by proposition 6.1, as in proposition 6.2. A computer check shows that every non-faithful irreducible representation appear with multiplicity 1 in some tensor product $\rho_1 \otimes \rho_2$, where ρ_1 and ρ_2 are odd-dimensional

irreducible representations. It follows that all these admit $\tilde{\iota}$ -equivariant models.

We now study in some detail the fate of the remaining (faithful) irreducible representations. In particular, we prove the following.

Proposition 7.5. *Let ρ be one of the faithful 4-dimensional or 6-dimensional representations of G_{22} . Then ρ admit globally invariant models over K , which induce morphisms $j_\rho : \Gamma \rightarrow \text{Aut}(G)$. These morphisms can be chosen injective, however the induced morphisms $\overline{j}_\rho : \Gamma \rightarrow \text{Out}(G)$ are never injective.*

In both dimensions, one of the representations considered here satisfy $\rho(Z) = -i$. It is sufficient to consider these to prove the proposition, since they are conjugated in pairs by the Galois action. We let $N = \dim \rho \in \{4, 6\}$.

Since ρ is faithful, the existence of a globally invariant model for ρ leads to a morphism $j : \Gamma \rightarrow \text{Aut}(G)$ satisfying $\gamma \circ \rho \simeq \rho \circ j(\gamma)$ for all $\gamma \in \Gamma$, such that the associated cocycle $J = \{\gamma \mapsto J_\gamma\}$ is cohomologically trivial.

We use the following procedure to check the proposition for an arbitrary $j : \Gamma \rightarrow \text{Aut}(G)$ satisfying $\gamma \circ \rho \simeq \rho \circ j(\gamma)$ for all $\gamma \in \Gamma$. Let $\gamma_0 = \mathfrak{s}_{13}$ and note that $K^{\gamma_0} = \mathbb{Q}(i)$. Recall that γ_0 and \mathfrak{c} generate $\Gamma = \text{Gal}(\mathbb{Q}(i, \sqrt{5})/\mathbb{Q})$. Restricting J to $\langle \gamma_0 \rangle$ yields a class in $\mathbb{Q}(i)^\times / NK^\times$ where $N(x) = x\gamma_0(x)$. If this class is non-zero then J cannot be cohomologically trivial. Otherwise there exists $M \in \text{GL}_N(K)$, which can be explicitly determined, such that $J_{\gamma_0} = M^{-1}\gamma_0(M)$ in $\text{PGL}_N(K)$. Then the cocycle $\gamma \mapsto Q_\gamma = MJ_\gamma(M)^{-1}$ is cohomologous to J and $Q_{\gamma_0} = 1$. Since

$$\gamma_0(Q_\mathfrak{c}) = Q_{\gamma_0\gamma_0}(Q_\mathfrak{c}) = Q_{\gamma_0\mathfrak{c}} = Q_{\mathfrak{c}\gamma_0} = Q_{\mathfrak{c}\mathfrak{c}}(Q_{\gamma_0}) = Q_\mathfrak{c}$$

it follows that $\gamma_0(Q_\mathfrak{c}) = Q_\mathfrak{c}$, i.e. $Q_\mathfrak{c} \in \text{PGL}_N(K)^{\gamma_0} = \text{PGL}_N(K^{\gamma_0})$, and that $Q_{\gamma_0\mathfrak{c}} = Q_\mathfrak{c}$. In other words, Q belongs to the image of the inflation map $Z^1(\text{Gal}(K^{\gamma_0}/\mathbb{Q}), \text{PGL}_N(K^{\gamma_0})) \rightarrow Z^1(\Gamma, \text{PGL}_N(K))$. Since the inflation map between Brauer groups is injective this map is also injective and we are again reduced to a cyclic case, hence to a norm equation solvable with MAGMA.

Using CHEVIE we get all such morphisms. The list obtained does not depend on whether ρ has dimension 4 or 6. If $N = 4$, the morphisms j such that J is cohomologous to 0 all satisfy $\overline{j(\mathfrak{c})} = \overline{j(\mathfrak{s}_7)} = \overline{\tau(\mathfrak{s}_7)}$ and $\overline{j(\mathfrak{s}_{13})} = \overline{\tau(1)}$. Some of these morphisms j are injective.

In case $N = 6$, the morphisms j such that J is cohomologically trivial are all injective, and again send \mathfrak{s}_{13} to an inner automorphism. Moreover $\overline{j(\mathfrak{c})} \in \{\overline{\iota(\mathfrak{c})}, \overline{\iota(\mathfrak{s}_7)}\}$ and the two possibilities occur.

The choice of any morphism such that J is cohomologically trivial leads to a globally invariant model for the representations. One of these morphisms, such that J is cohomologically trivial for both representations, has the following simple form : $j(\mathfrak{s}_{13}) = \text{Ad}(t)$ and $j(\mathfrak{c})$ sends (s, t, u) to (u^{tu}, t, s^{tst}) .

Remark 7.6.

Using the results proved here, one gets a simpler proof of the theorem of Benard and Bessis. First consider non-dihedral $G(de, e, r)$. Since all irreducible representations of the groups $G(de, e, r)$ appear as multiplicity 1 component of some representation of $G(de, 1, r)$, we are reduced by lemma 7.1 to the case of $G(d, 1, r)$, then to its reflection representation by proposition 7.3, and we know that this representation is defined over K by the very definition of these groups. The case of the dihedral groups is classical. Finally, the algorithm used here for exceptional groups completes the proof, and is more efficient than the one used in [Bes].

8. INVARIANTS

We prove corollary 1.6, whose statement we recall. We denote by V^* the dual of a vector space V .

Proposition 8.1. *Let $G \subset \mathrm{GL}(V)$ be any complex reflection group where V is a K -vector space, where K is the field K'' of theorem 1.5. There is a \mathbb{Q} -form $V = V_0 \otimes_{\mathbb{Q}} K$ such that the fundamental invariants of G can be taken rational, i.e. in the symmetric algebra $S(V_0^*)$.*

Proof. The question can clearly be reduced to the case of an irreducible complex reflection group.

By theorem 1.5 we may assume that $G \subset \mathrm{GL}(V_0 \otimes_{\mathbb{Q}} K)$ is globally invariant by $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$. Let f_1, \dots, f_r (where $r = \dim V$) be fundamental invariants, i.e. algebraically independent polynomials such that $S(V^*)^G = K[f_1, \dots, f_n]$. We want to find algebraically independent $g_1, \dots, g_r \in S(V_0^*)$ such that we still have $S(V^*)^G = K[g_1, \dots, g_n]$. Our strategy will be as follows: we will set $g_i = \sum_{\gamma \in \Gamma} \gamma(\lambda f_i)$ where $\lambda \in K$. Then the g_i are still invariant since for $g \in G$ we have $g(g_i) = \sum_{\gamma \in \Gamma} \gamma(\lambda \gamma^{-1}(g) f_i) = \sum_{\gamma \in \Gamma} \gamma(\lambda f_i)$, the last equality since $\gamma^{-1}(g) \in G$.

It is thus sufficient to show that we may choose λ such that g_i are still algebraically independent. By the Jacobian criterion, it is enough to show that we may choose λ such that, if x_1, \dots, x_r is a basis of V_0^* , so that $S(V^*) \simeq K[x_1, \dots, x_r]$, we have $\det \left(\frac{\partial g_i}{\partial x_j} \right)_{i,j} \neq 0$.

We use the following version of the algebraic independence of automorphisms:

Proposition 8.2. [Bbk, Chap. V, §10, Théorème 4]. *Let Q be an infinite field, let K be a finite Galois extension of Q with Galois group Γ , let Ω be an arbitrary extension of K and let $\{X_\gamma\}_{\gamma \in \Gamma}$ be indeterminates indexed by the elements of Γ . Let $F \in \Omega[X_\gamma]_{\gamma \in \Gamma}$ be a polynomial such that $F((\gamma(x))_{\gamma \in \Gamma}) = 0$ for any $x \in K$. Then $F = 0$.*

We apply the proposition with $Q = \mathbb{Q}$, $\Omega = K$ and $F = \det \left(\sum_{\gamma \in \Gamma} X_\gamma \gamma \left(\frac{\partial f_i}{\partial x_j} \right) \right)_{i,j}$. The polynomial F evaluated at $X_1 = 1$ and $X_\gamma = 0$ for $\gamma \neq 1$ is equal to

$\det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j}$ which is non-zero, so F is non-zero. By the theorem, there exists $\lambda \in K$ such that $F((\gamma(\lambda))_\gamma) \neq 0$. But $F((\gamma(\lambda))_\gamma) = \det \left(\frac{\partial g_i}{\partial x_j} \right)_{i,j}$ \square

We now prove 1.7 whose statement we recall.

Proposition 8.3. *The variety V^{reg}/G is defined over \mathbb{Q} .*

Proof. We use the notations of the previous proof, in particular we denote by K the field K'' of 1.5, and we assume that $G \subset \text{GL}(V_0 \otimes_{\mathbb{Q}} K)$ is globally invariant by $\Gamma = \text{Gal}(K/\mathbb{Q})$.

Let \mathcal{H} be the set of reflecting hyperplanes for G and for each $H \in \mathcal{H}$, let l_H be a linear form defining H . For $H \in \mathcal{H}$ let e_H be the order of the subgroup of G fixing H , and let $\Delta = \prod_{H \in \mathcal{H}} l_H^{e_H}$. It is well known that $\Delta \in S(V^*)^G$ (see e.g. [OT, 6.44]). Thus the variety V^{reg}/G is the open subvariety of $\text{Spec}(K[f_1, \dots, f_n])$ whose function ring is the localization by the principal ideal Δ . It is thus enough to show that we may choose the f_i such that a multiple of Δ belongs to $\mathbb{Q}[f_1, \dots, f_n]$. If, as in the previous proposition, we choose $f_i \in S(V_0^*)^G$, it will be enough to show that we have a multiple of Δ in $S(V_0^*)$ since $S(V_0^*) \cap S(V^*)^G = S(V_0^*)^G$. Since \mathcal{H} is globally invariant by Γ , for any $\gamma \in \Gamma$ there exists $\lambda_\gamma \in K$ such that $\gamma(\Delta) = \lambda_\gamma \Delta$, and it is clear that $\{\gamma \mapsto \lambda_\gamma\} \in Z^1(\Gamma, K^\times)$. By Hilbert's theorem 90 we have that $H^1(\Gamma, K^\times)$ is trivial, thus $\{\gamma \mapsto \lambda_\gamma\}$ is a coboundary, i.e. there exists $\lambda \in K^\times$ such that $\lambda_\gamma = \lambda^{-1} \gamma(\lambda)$. We get then that $\gamma(\lambda^{-1} \Delta) = \lambda^{-1} \Delta$ for any γ , thus $\lambda^{-1} \Delta \in S(V_0^*)$. \square

REFERENCES

- [Ar] S. Ariki, "Representation theory of a Hecke algebra for $G(r, p, n)$ ", *J. Algebra* **177** (1995), 164–185.
- [ArKo] S. Ariki and K. Koike, "A Hecke algebra of $\mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$ and construction of its irreducible representations", *Adv. Math.* **106** (1994), 216–243.
- [Bena] M. Benard, "Schur indices and splitting fields of the Unitary reflection groups" *J. Algebra* **38** (1976), 318–342.
- [Bens] D. Benson "Polynomial invariants of finite groups", LMS Lecture Note Series, 190. Cambridge University Press, Cambridge, 1993.
- [Bbk] N. Bourbaki, "Algèbre", Chap. V, "Corps commutatifs", Hermann, Paris, 1950.
- [Bes] D. Bessis, "Sur le corps de définition d'un groupe de réflexions complexe", *Comm. Algebra* **25** (1997), 2703–2716.
- [BM] D. Bessis, J. Michel, "Explicit presentations for exceptional braid groups", *Experimental mathematics* **13** (2004), 257–266.
- [BMR] M. Broué, G. Malle and R. Rouquier, "Complex reflection groups, braid groups, Hecke algebras", *J. reine angew. Math.* **500** (1998), 127–190.
- [CHEVIE] see www.math.jussieu.fr/~jmichel/chevie.
- [Cohen] A. Cohen, "Finite complex reflection groups", *Annales de l'E.N.S.* **9** (1976), 379–436.
- [FH] W. Fulton and J. Harris, "Representation theory", Springer G.T.M. **129** (1991).

- [OT] P. Orlik and H. Terao, “Arrangements of hyperplanes”, Springer G.M.W. **300** (1991).
- [Se] J.-P. Serre, “Corps locaux”, Hermann, Paris, 1968.
- [Ze] A. V. Zelevinsky “Representations of finite classical groups — a Hopf algebra approach”, Springer SLN **869** (1981).

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