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AUTOMORPHISMS OF COMPLEX REFLECTION GROUPS

I. MARIN AND J. MICHEL

Abstract. Let $G \subset \mathrm{GL}(\mathbb{C}^r)$ be a finite complex reflection group. We show that when G is irreducible, apart from the exception $G = \mathfrak{S}_6$, as well as for a large class of non-irreducible groups, any automorphism of G is the product of a central automorphism and of an automorphism which preserves the reflections. We show further that an automorphism which preserves the reflections is the product of an element of $N_{\mathrm{GL}(\mathbb{C}^r)}(G)$ and of a "Galois" automorphism: we show that $\mathrm{Gal}(K/\mathbb{Q})$, where K is the field of definition of G, injects into the group of outer automorphisms of G, and that this injection can be chosen such that it induces the usual Galois action on characters of G, apart from a few exceptional characters; further, replacing if needed K by an extension of degree 2, the injection can be lifted to $\mathrm{Aut}(G)$, and every irreducible representation admits a model which is equivariant with respect to this lifting. Along the way we show that the fundamental invariants of G can be chosen rational.

1. Introduction

Let G be a finite complex reflection group of rank r, that is a finite group generated by (pseudo)reflections in a vector space V of dimension $r < \infty$ over $\mathbb{C}$. We say that G is *irreducible* when G acts irreducibly on V. In this paper, we determine the group $\mathrm{Aut}(G)$ of automorphisms of G when G is irreducible, as well as for a large class of non-irreducible groups.

An automorphism $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(g) \in gZG$ for all $g \in G$, where ZG denotes the center of G, is called *central*. These automorphisms form a normal subgroup of $\mathrm{Aut}(G)$, denoted C.

We call *reflections* of G all the elements which act as a (pseudo)-reflection on V and denote by A the subgroup of automorphisms of G which preserve the reflections. One of our main results is that

Theorem 1.1. *Let G be an irreducible complex reflection group different from $G(1,1,6) \simeq \mathfrak{S}_6$; then $\mathrm{Aut}(G) = C \cdot A$. If $\dim V > 2$, then $\mathrm{Aut}(G) = C \rtimes A$.*

Further, we extend this result to a large class of non-irreducible groups (see 1.11).

For G of rank 2, it may happen that C intersects non-trivially A, and for G of rank 1 we have $C = A$.

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The main observation of this paper is the interpretation of A as "Galois" automorphisms.

Let χ_V be the character of G afforded by V. It is well known (see *e.g.* [Bens, 7.1.1]) that the representation V of G can be realized over the subfield $K \subset \mathbb{C}$ generated by the values of χ_V ; the field K is called the *field of definition* of G. Further, by a well known result of Benard and Bessis ([Bena], [Bes]) *any* irreducible representation of G can be realized over K.

Let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$, and let $\gamma \in \Gamma$. If a_γ is an automorphism of G such that for any $g \in G$ we have $\chi_V \circ a_\gamma = \gamma(\chi_V)$ we say that a_γ is a *Galois automorphism* corresponding to γ.

Theorem 1.2. *Assume G irreducible. Then any faithful irreducible representation where the reflections of G still act by reflections is a Galois conjugate of V.*

As a corollary, for an irreducible group, A identifies to the group of Galois automorphisms.

Note that a_γ is determined by γ up to an element of the normal subgroup N formed of the Galois automorphisms corresponding to the identity element of Γ. Equivalently, N is the subgroup of automorphisms induced by an element of $N_{\mathrm{GL}(V)}(G)$; they contain the inner automorphisms. We call the induced outer automorphisms *diagram* automorphisms (since they extend the corresponding notion for finite Coxeter groups).

What is remarkable is that any element of the Galois group can be represented by an automorphism. We denote by $\mathrm{Out}(G)$ the group of outer automorphisms, identified with $\mathrm{Aut}(G)/(G/ZG)$. Let $\overline{A}$ (resp. $\overline{N}$) be the image of A (resp. N) in $\mathrm{Out}(G)$.

Theorem 1.3. *Assume G irreducible. The natural map $\overline{A} \to \Gamma$ is surjective and admits a section, thus $\overline{A} \simeq \overline{N} \rtimes \Gamma$.*

Theorem 1.3 can be reformulated as an equivariance property of χ_V: there exists an injection $\overline{\iota}_V : \Gamma \to \mathrm{Out}(G)$ such that $\chi_V \circ \overline{\iota}_V(\gamma) = \gamma(\chi_V)$.

It is actually a special case of an equivariance property that can be proved for almost any character. We call a representation *rational* if it admits a (matrix) model over $\mathbb{Q}$.

Theorem 1.4. *Assume G irreducible. For any irreducible character χ of G but eight exceptions there exists an injective homomorphism: $\Gamma \xrightarrow{\overline{\iota}_\chi} \mathrm{Out}(G)$ such that for any $\gamma \in \Gamma$, we have $\chi \circ \overline{\iota}_\chi(\gamma) = \gamma(\chi)$.*

The exceptions are the characters of the four (rational) 5-dimensional representations of G_{27}, two of the 3 rational representations of dimension 6 of G_{29}, and the two rational representations of dimension 120 of G_{34}. In these cases, the complex conjugation $\mathfrak{c}$ is an element of Γ, and its image in any injective morphism $\Gamma \to \mathrm{Out}(G)$ will exchange in pairs the mentioned characters.

For an exceptional group (and a non-exceptional representation), or an imprimitive group $G(de, e, r)$ where $\gcd(e, r) \leq 2$, there is a unique way to

take the injection $\overline{\iota}_\chi$ independent of χ; we will call $\overline{\iota}$ the common value. In the remaining cases, we can make choices of $\overline{\iota}_\chi$ which are in a single orbit for $\overline{N}$-conjugacy.

The numbers above refer to the Shephard-Todd classification of irreducible complex reflection groups.

Theorem 1.4 has the following consequence, which can be formulated without exception and without assuming G irreducible.

Corollary 1.5. *Let G be any finite complex reflection group, and let χ be an irreducible character of G. If, for all $a \in \mathrm{Out}(G)$ we have $\chi \circ a = \chi$, then χ takes its values in $\mathbb{Q}$.*

The map $\overline{\iota}$ can "almost" be lifted to $\mathrm{Aut}(G)$. More precisely we have

Theorem 1.6. *For G irreducible and χ outside the exceptions of theorem 1.4, there exists an extension K' of K, which is abelian over $\mathbb{Q}$ and at most quadratic over K, and an injective homomorphism $\tilde{\iota}_\chi : \mathrm{Gal}(K'/\mathbb{Q}) \to \mathrm{Aut}(G)$ such that the composition of $\tilde{\iota}_\chi$ with the natural epimorphism $\mathrm{Aut}(G) \to \mathrm{Out}(G)$ factors through Γ and induces $\overline{\iota}_\chi$.*

One may take $K' = K$ except when G is G_{27} or G is a dihedral group $G(e,e,2)$ such that $\mathbb{Q}(\zeta_e)$ contains no quadratic imaginary extension of $\mathbb{Q}$.

The map $\tilde{\iota}_\chi$ can be taken independent of χ in the same circumstances as $\overline{\iota}_\chi$; in such a case we will write $\tilde{\iota}$ for the common value.

Suppose we can find a model over some field L of a faithful representation ρ of G such that $\rho(G)$ as a set is globally invariant by $\mathrm{Gal}(L/\mathbb{Q})$. Then any $\gamma \in \mathrm{Gal}(L/\mathbb{Q})$ induces a permutation of $\rho(G)$ and thus a permutation on G, which is an automorphism. We get thus an homomorphism $\tilde{\iota} : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}(G)$. If $L = K'$ this $\tilde{\iota}$ is equivariant with respect to the character of ρ thus is suitable for the $\tilde{\iota}$ of theorem 1.6.

Given an homomorphism $\mathrm{Gal}(L/\mathbb{Q}) \xrightarrow{\tilde{\iota}} \mathrm{Aut}(G)$, and a model of an arbitrary representation ρ over L, we say that the model is $\tilde{\iota}$-equivariant if for any $\gamma \in \mathrm{Gal}(L/\mathbb{Q})$ and any $g \in G$, we have $\rho(\tilde{\iota}(\gamma)(g)) = \gamma(\rho(g))$. Note that this implies the global invariance of $\rho(G)$ under $\mathrm{Gal}(L/\mathbb{Q})$. Considerations as above led us to check the following theorem :

Theorem 1.7. *Let G be an irreducible finite complex reflection group which is not G_{22} and let K' and $\tilde{\iota}_\chi$ be as in theorem 1.6. Let ρ be any irreducible representation of G, with character χ, which is not an exception in 1.4. Then there is a model of ρ over K' which is $\tilde{\iota}_\chi$-equivariant.*

In the case of G_{22}, we have $K = K' = \mathbb{Q}(\mathrm{i}, \sqrt{5})$. There is no globally invariant model over K' of the reflection representation. We need to replace K' by the extension $K'' = \mathbb{Q}(e^{\frac{2i\pi}{20}})$ to get an invariant model (then K'' also works for any other representation). This example illustrates the fact that asking for a $\tilde{\iota}_\chi$-equivariant model is stronger than merely asking as in theorem 1.6 for a model such that $\chi \circ \tilde{\iota}_\chi(\gamma) = \gamma(\chi)$ where χ is the character of ρ.

Theorem 1.7 has the following consequence (which could have been observed from the explicit values of the invariants discovered by various authors, mostly in the 19th century). In this proposition we take K'' as above for G_{22}, and in the other cases we let $K'' = K'$.

Corollary 1.8. *Let $G \subset \mathrm{GL}(V)$ be any irreducible complex reflection group where V is a K''-vector space, where K'' is as above. There is $\mathbb{Q}$-form $V = V_0 \otimes_{\mathbb{Q}} K''$ such that the fundamental invariants of G can be taken rational, i.e. in the symmetric algebra of the dual of V_0.*

Let V^{reg} be the complement in V of the reflecting hyperplanes for G. The fundamental group of the variety V^{reg}/G is the *braid group* of G. Since the variety V/G is an affine space it is defined over $\mathbb{Q}$. A geometric reformulation of 1.8 is that the morphism $V \to V/G$ is also defined over $\mathbb{Q}$. We deduce from 1.8 the following

Corollary 1.9. *The varieties V^{reg}, V^{reg}/G as well as the quotient morphism $V^{\mathrm{reg}} \to V^{\mathrm{reg}}/G$ are defined over $\mathbb{Q}$.*

The techniques we use for checking theorem 1.7 enable us also to give a probably shorter proof of the Benard-Bessis theorem (see remark 9.6).

Finally we extend our results to a large class of non-irreducible finite complex reflection groups. We say that a group G is *indecomposable* if it admits no non-trivial decomposition as a direct product. We show

Proposition 1.10. *An irreducible complex reflection group has a decomposition $G = Z \times \hat{G}$ where $Z \subseteq ZG$, and where $\hat{G}$ is indecomposable non-abelian or trivial. In this decomposition Z is unique.*

We call $\hat{G}$ the non-abelian factor of G, denoted $\mathrm{Nab}(G)$; it is unique up to isomorphism, since Z is unique. We then show:

Theorem 1.11. *Let $G = G_1 \times \ldots G_n$ be the decomposition in irreducible factors of a complex reflection group $G \subset \mathrm{GL}(V)$. Then the following are equivalent:*

- *Any automorphism of G is the product of a central automorphism and of an automorphism which preserves the reflections.*
- *No G_i is isomorphic to $\mathfrak{S}_6$, and the decomposition has the property that for any i, j, we have $\mathrm{Nab}(G_i) \simeq \mathrm{Nab}(G_j) \Rightarrow G_i \simeq G_j$.*

We give the list of central and non-abelian factors of irreducible groups in table 3.

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2. Background on the infinite series

We will use the following notation throughout this paper. V is a vector space over the subfield K of $\mathbb{C}$. G is a finite subgroup of $\mathrm{GL}(V)$ generated by (pseudo)-reflections. We assume that K is the *field of definition* of G, that is, it is the subfield of $\mathbb{C}$ generated by the traces of the elements of G. The group G is *irreducible* if the representation V of G is irreducible.

We denote by $g \mapsto \mathrm{Ad}\, g$ the map which maps $g \in G$ to the inner automorphism $x \mapsto {}^g x = gxg^{-1}$.

The Shephard-Todd classification of irreducible finite reflection groups (see *e.g.* [Cohen]) shows that there is one infinite series, identified by three positive integer parameters d, e and r, and denoted by $G(de, e, r)$. In addition there are 34 exceptional ones, denoted G_4 to G_{37}.

For the convenience of the reader, and for future reference, we review the definition and some properties of the groups $G(de, e, r)$,

2.1. **The groups $G(de, e, r)$.** We will denote by μ_n the subgroup of n-th roots of unity in $\mathbb{C}^\times$,and by ζ_n the primitive root of unity $e^{2i\pi/n}$.

The group $G(de, e, r)$ is defined as the subgroup of $\mathrm{GL}_r(\mathbb{C})$ consisting of r by r monomial matrices with entries in μ_{de}, and such that the product of non-zero entries is in μ_d; this defines an irreducible reflection group except for the case of $\mathfrak{S}_r = G(1, 1, r)$, where one has to quotient by the one-dimensional fixed points, and for $G(2, 2, 2) \simeq \mathfrak{S}_2^2$.

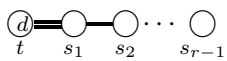
The field of definition K of $G(de, e, r)$ is the field generated by the traces of elements of $G(de, e, r)$. We get $K = \mathbb{Q}(\zeta_{de})$, except when $d = 1$ and $r = 2$. In this last case, $G(e, e, 2)$ is the dihedral group of order $2e$, whose field of definition is $K = \mathbb{Q}(\cos(\frac{2\pi}{e})) = \mathbb{Q}(\zeta_e + \zeta_e^{-1})$.

The group $\mathfrak{S}_r = G(1, 1, r)$ is a subgroup of $G(de, e, r)$; if we denote by $D(de, e, r)$ the subgroup of diagonal matrices in $G(de, e, r)$, we have a semi-direct product decomposition $G(de, e, r) = D(de, e, r) \rtimes \mathfrak{S}_r$.

The group $G(d, 1, r)$ is generated by the set of reflections $\{t, s_1, \ldots, s_{r-1}\}$ where t is the matrix $\mathrm{Diag}(\zeta_d, 1, \ldots, 1) \in D(d, 1, r)$ and $s_k \in \mathfrak{S}_r$ is the permutation matrix corresponding to the transposition $(k, k + 1)$.

The group $G(de, e, r)$ is a normal subgroup of index e in $G(de, 1, r)$; it is generated by the set of reflections $\{t', s_1', s_1, \ldots, s_{r-1}\}$ where $t' = t^e$ and $s_1' = s_1^t$; here t refers to the generator $t = \mathrm{Diag}(\zeta_{de}, 1, \ldots, 1)$ of $G(de, 1, r)$. The generator t' should be omitted (being trivial) if $d = 1$ and s_1' should be omitted (being unnecessary) if $e = 1$. The quotient $G(de, 1, r)/G(de, e, r)$ is cyclic, generated by the image of t.

2.2. **Presentations.** The following diagram gives a presentation of $G(d, 1, r)$:



which means that $t, s_1, \ldots, s_{r-1}$ satisfy the same *braid relations* as the Coxeter group $W(B_r)$ (actually the braid group of $G(d, 1, r)$ is isomorphic to

that of $W(B_r)$, see [BMR]), and that one gets a presentation of $G(d,1,r)$ by adding the *order relations* $t^d = 1$ and $s_i^2 = 1$.

For $G(de,e,r)$, to get a presentation of the braid group one needs to replace the relation $ts_1ts_1 = s_1ts_1t$ implied by the above diagram by the following braid relations involving t' and s_1':

$$\begin{aligned} t's_1's_1 &= s_1's_1t', \\ s_1's_1s_2s_1's_1s_2 &= s_2s_1's_1s_2s_1's_1, \\ \underbrace{s_1t's_1's_1s_1's_1\dots}_{e+1} &= \underbrace{t's_1's_1s_1's_1\dots}_{e+1} \end{aligned} \tag{2.1}$$

and one gets a presentation of $G(de,e,r)$ by adding the order relations $t'^d = 1$ and $s_1'^2 = s_i^2 = 1$.

2.3. **The representations of $G(d,1,r)$.** We recall from [ArKo] explicit models over $K = \mathbb{Q}(\zeta_d)$ for the irreducible representations of $G(d,1,r)$. Let $\mathcal{L}$ be the set of d-tuples $\boldsymbol{\lambda} = (\lambda_0,\dots,\lambda_{d-1})$ of partitions with total size r. We denote by $\mathcal{T}(\boldsymbol{\lambda})$ the set of standard tableaux of shape $\boldsymbol{\lambda}$, that is the set of tuples $\mathbf{T} = (T_0,\dots,T_{d-1})$ such that T_i is a filling of the Young diagram of λ_i with numbers in $[1,r]$, with the conditions that each of these numbers appears (exactly once) in one of the filled diagrams, and that they are increasing across the rows and columns of each T_i. For $m \in [1,r]$ we let $\mathbf{T}(m) = i$ if m is placed in T_i.

To $\boldsymbol{\lambda} \in \mathcal{L}$ we associate the $\mathbb{Q}$-vector space $V^0(\boldsymbol{\lambda})$ of basis $\mathcal{T}(\boldsymbol{\lambda})$, and define $V(\boldsymbol{\lambda}) = V^0(\boldsymbol{\lambda}) \otimes \mathbb{Q}(\zeta_d)$. Explicit formulas in [ArKo] describe a representation $\rho_{\boldsymbol{\lambda}}$ of $G(d,1,r)$ over $V(\boldsymbol{\lambda})$, and the $\rho_{\boldsymbol{\lambda}}$ provide a complete set of representatives for the irreducible representations of $G(d,1,r)$ (see [ArKo] corollary 3.14).

- The matrix $\rho_{\boldsymbol{\lambda}}(t)$ is given by $\rho_{\boldsymbol{\lambda}}(t)\mathbf{T} = \zeta_d^{\mathbf{T}(1)}\mathbf{T}$.
- The matrices $\rho_{\boldsymbol{\lambda}}(s_i)$ for $1 \le i \le n-1$ are rational in the basis $\mathcal{T}(\boldsymbol{\lambda})$, thus belong to $\mathrm{GL}(V^0(\boldsymbol{\lambda}))$. To describe them we introduce some notation: given a tuple of tableaux $\mathbf{T}$, let $\mathbf{T}_{i\leftrightarrow i+1}$ be the tuple obtained by exchanging the numbers i and $i+1$ in $\mathbf{T}$ when this is still a tuple of standard tableaux, and 0 otherwise (this last case will only occur when the numbers i and $i+1$ occur in the same tableau and on either the same line or the same column). Then, if i and $i+1$ occur in the same tableau, define their axial distance $a(i,i+1)$ to be the distance between the diagonals where they occur (more precisely, if i occurs at coordinates i_0, i_1 and $i+1$ at coordinates j_0, j_1 we set $a(i,i+1) = (i_0 - i_1) - (j_0 - j_1)$). It is clear that for a standard tableau we always have $a(i,i+1) \neq 0$. Finally, take the convention that when i and $i+1$ do not occur in the same tableau, then $a(i,i+1) = \infty$ so that $1/a(i,i+1) = 0$. Then the formula is:

$$\rho_{\boldsymbol{\lambda}}(s_i)\mathbf{T} = \frac{1}{a(i,i+1)}\mathbf{T} + (1 + \frac{1}{a(i,i+1)})\mathbf{T}_{i\leftrightarrow i+1} \tag{2.2}$$

2.4. **The representations of $G(de,e,r)$.** As in [Ar], we will get the irreducible representations of $G(de,e,r)$ by Clifford theory, as components of the restriction of $\rho_{\boldsymbol{\lambda}} \in \mathrm{Irr}(G(de,1,r))$ to $G(de,e,r)$.

We recall that $\{t^e, s_1^t, s_1, \dots, s_{r-1}\}$ are the generating reflections for $G(de,e,r)$, and that the quotient $G(de,1,r)/G(de,e,r)$ is cyclic, generated by t. Let χ be the generator of the group of linear characters of the quotient which maps t to ζ_e. From the formulae for the action of t and s_i, we see that there is a permutation $\sigma(\boldsymbol{\lambda})$ of the tuple $\boldsymbol{\lambda}$ such that $\chi \otimes \rho_{\boldsymbol{\lambda}} \simeq \rho_{\sigma(\boldsymbol{\lambda})}$; it is given by $\sigma(\boldsymbol{\lambda}) = (\lambda_d, \lambda_{d+1}, \dots, \lambda_{d+de-1})$ (where the indices are taken $\pmod{de}$). Let $\langle\sigma\rangle$ be the cyclic group generated by σ, and let $C_{\boldsymbol{\lambda}} = \langle\sigma^b\rangle$ be the subgroup which stabilizes $\boldsymbol{\lambda}$. By Clifford theory, the representation $\rho_{\boldsymbol{\lambda}}$ restricts to $G(de,e,r)$ as the sum of $\#C_{\boldsymbol{\lambda}} = e/b$ distinct irreducible representations.

We can actually define an operator S on $V(\boldsymbol{\lambda})$ associated to σ^b whose eigenspaces will be the irreducible constituents. For $\mathbf{T} = (T_0, \dots, T_{de-1})$ we set $S(\mathbf{T}) = (T_{bd}, T_{bd+1}, \dots, T_{bd+de-1})$. Since $\sigma^b(\boldsymbol{\lambda}) = \boldsymbol{\lambda}$, this is another tuple of tableaux of shape $\boldsymbol{\lambda}$. It is easy to check that S commutes to $\rho_{\boldsymbol{\lambda}}(s_i)$ and to $\rho_{\boldsymbol{\lambda}}(s_1^t)$, and that $S \circ \rho_{\boldsymbol{\lambda}}(t) = \zeta_e^b \rho_{\boldsymbol{\lambda}}(t) \circ S$. It follows that S commutes with the action of $G(de,e,r)$ and its eigenspaces $V(\boldsymbol{\lambda},\omega) = \mathrm{Ker}(S-\omega)$ for $\omega \in \mu_{\#C_{\boldsymbol{\lambda}}}$ afford irreducible representations of $G(de,e,r)$. We denote by $\rho_{\boldsymbol{\lambda},\omega}$ the representation afforded by $V(\boldsymbol{\lambda},\omega)$; it is clear that

$$p_\omega := \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \omega^{-i} S^i$$

is the $G(de,e,r)$-invariant projector on $V(\boldsymbol{\lambda},\omega)$.

From this we get, if we denote $\chi_{\boldsymbol{\lambda},\omega}$ the character of $\rho_{\boldsymbol{\lambda},\omega}$:

$$\chi_{\boldsymbol{\lambda},\omega}(g) = \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \omega^{-i} \,\mathrm{Trace}(\rho_{\boldsymbol{\lambda}}(g) S^i).$$

We will now want to describe a model of $\rho_{\boldsymbol{\lambda},\omega}$. We first recall Ariki's model (*cf.* [Ar, section 2]), and then introduce another one, simpler for our purpose. Ariki chooses a basis of $V(\boldsymbol{\lambda},\omega)$ given by the $p_\omega(\mathbf{T})$, where he chooses for $\mathbf{T}$ representatives of the S-orbits on $\mathcal{T}$ given by the subset $\mathcal{T}_0$ of tuples of tableaux which satisfy $\mathbf{T}(1) < bd$. Setting $\mathbf{T}^{(\omega)} = p_\omega(\mathbf{T})$, we then get formulas for the action of the generators of $G(de,e,r)$ on the basis $\{\mathbf{T}^{(\omega)}\}_{\mathbf{T}\in\mathcal{T}_0}$. We will write θ for ζ_e^b to simplify notations. Using that $\rho_{\boldsymbol{\lambda}}(t) p_\omega = p_{\omega\theta} \rho_{\boldsymbol{\lambda}}(t)$, we get $\rho_{\boldsymbol{\lambda}}(t)\mathbf{T}^{(\omega)} = \zeta_{de}^{\mathbf{T}(1)} \mathbf{T}^{(\omega\theta)}$ from which we get

$$\rho_{\boldsymbol{\lambda},\omega}(t^e)\mathbf{T}^{(\omega)} = \zeta_d^{\mathbf{T}(1)} \mathbf{T}^{(\omega)}.$$

To write the formula for s_i, first note that if $\mathbf{T} \in \mathcal{T}_0$ then $\mathbf{T}_{i\leftrightarrow i+1}$ is not in general in $\mathcal{T}_0$ when $i=1$. We have to take $\mathbf{T}'_{1\leftrightarrow 2} = S^{\lfloor \frac{\mathbf{T}(2)}{bd} \rfloor} \mathbf{T}_{1\leftrightarrow 2}$ to get an element of $\mathcal{T}_0$. Using that $p_\omega(\mathbf{T}'_{1\leftrightarrow 2}) = \omega^{\lfloor \frac{\mathbf{T}(2)}{bd} \rfloor} p_\omega(\mathbf{T}_{1\leftrightarrow 2})$, we get for $i > 1$

$$\rho_{\boldsymbol{\lambda},\omega}(s_i)\mathbf{T}^{(\omega)} = \frac{1}{a(i,i+1)} \mathbf{T}^{(\omega)} + (1 + \frac{1}{a(i,i+1)}) \mathbf{T}^{(\omega)}_{i\leftrightarrow i+1}.$$

and

$$\rho_{\boldsymbol{\lambda},\omega}(s_1)\mathbf{T}^{(\omega)} = \frac{1}{a(1,2)}\mathbf{T}^{(\omega)} + (1+\frac{1}{a(1,2)})\omega^{-\lfloor\frac{\mathbf{T}(2)}{bd}\rfloor}\mathbf{T}'^{(\omega)}_{1\leftrightarrow 2}.$$

Finally, using the above formulas we get

$$\rho_{\boldsymbol{\lambda},\omega}(s_1^t)\mathbf{T}^{(\omega)} = \zeta_{de}^{\mathbf{T}(1)-\mathbf{T}(2)}\rho_{\boldsymbol{\lambda},\omega}(s_1)\mathbf{T}^{(\omega)}.$$

We now introduce a model which does not depend on a choice of representatives $\mathcal{T}_0$, using the equality $p_{\omega\theta^{-1}}\circ\rho_{\boldsymbol{\lambda}}(g) = p_\omega\rho_{\boldsymbol{\lambda}}(\mathrm{Ad}\,t(g))$ which implies $\chi_{\lambda,\theta^i}(g) = \chi_{\lambda,1}(\mathrm{Ad}(t^{-i})g)$. We choose the same model as Ariki of $\rho_{\lambda,1}$, except that, as the image by p_1 of a tuple $\mathbf{T}$ is the same as that of all elements of the same S-orbit, we take as basis elements the averages of S-orbits. We denote by $(\mathbf{T})$ the average of the S-orbit of $\mathbf{T}$. We thus get a basis $\{(\mathbf{T})\}_{(\mathbf{T})\in\mathcal{T}(\boldsymbol{\lambda})/S}$ of $V(\boldsymbol{\lambda},1)$. We take the same basis for $V(\boldsymbol{\lambda},\theta^i)$ and define $\rho_{\boldsymbol{\lambda},\theta^i}(g) := \rho_{\boldsymbol{\lambda},1}(\mathrm{Ad}(t)^{-i}g)$. We now have the following formulas for $\rho_{\boldsymbol{\lambda},1}$:

$$\begin{aligned}
&\rho_{\boldsymbol{\lambda},1}(t^e)(\mathbf{T}) = \zeta_d^{\mathbf{T}(1)}(\mathbf{T}),\\
&\rho_{\boldsymbol{\lambda},1}(s_i)(\mathbf{T}) = \frac{1}{a(i,i+1)}(\mathbf{T}) + (1+\frac{1}{a(i,i+1)})(\mathbf{T}_{i\leftrightarrow i+1}) \text{ for any } i\\
&\rho_{\boldsymbol{\lambda},1}(s_1^{t^k})(\mathbf{T}) = \zeta_{de}^{k(\mathbf{T}(1)-\mathbf{T}(2))}\rho_{\boldsymbol{\lambda},1}(s_1)(\mathbf{T}).
\end{aligned} \tag{2.3}$$

(note that the above formulas make sense since both $\zeta_d^{\mathbf{T}(1)}$ and $\zeta_{de}^{\mathbf{T}(1)-\mathbf{T}(2)}$ are constant on a given S-orbit).

3. The group $\overline{N}$

For G irreducible, the subgroup $\overline{N}$ of $\mathrm{Out}(G)$ induced by $N_{\mathrm{GL}(V)}(G)$ has been determined in [BMM, 3.13]. For the convenience of the reader we recall here the result. When K is a real field, G is a finite Coxeter groups and elements of $\overline{N}$ correspond to automorphisms of the Coxeter diagram. Such automorphisms are traditionally denoted by an exponent, for example ${}^3D_4$ denotes an automorphism of order 3 of the group $W(D_4) = G(2,2,4)$. We will use a similar notation for the other cases. With one exception ${}^2F_4$, all such automorphisms result from a normal embedding of G into another reflection group of same rank (the case of ${}^2F_4$ results from the embedding $G_{28} = W(F_4) \subset G_{31}$, which is not normal). We indicate at the same time the relevant embedding.

Automorphism	Embedding
${}^iG(de,e,r)$where $i>1$, $i\vert de$	$G(de,e,r)\lhd G(de,1,r)$
${}^3G(4,2,2)$	$G(4,2,2)\lhd G_6$
${}^4G(3,3,3)$	$G(3,3,3)\lhd G_{26}$
${}^3D_4$	$G(2,2,4) = W(D_4)\lhd G_{28} = W(F_4)$
${}^2G_5$	$G_5\lhd G_{14}$
${}^2G_7$	$G_7\lhd G_{15}$
${}^2F_4$	

This table needs a few comments on how it describes the group $\overline{N}$.

The automorphisms of $G(de,e,r)$ induced by $G(de,1,r)$ are induced by $\operatorname{Ad} t$, where $t = \operatorname{Diag}(\zeta_{de},1,\dots,1)$. Let us determine when the endomorphism $\operatorname{Ad} t^i$ of $G = G(de,e,r)$ is inner. This happens if and only if there exists $g \in G$ such that $t^{-i}g$ is scalar. Since $t^i = \operatorname{Diag}(\zeta_{de}^i,1,\dots,1)$, the element g must be of the form $\operatorname{Diag}(\zeta_{de}^i a,a,\dots,a)$ for some $a \in \mu_{de}$ and the condition that $g \in G$ is $a^r\zeta_{de}^i \in \mu_d$, *i.e.* $\zeta_{de}^i \in \mu_d \cdot \mu_{de}^r = \mu_{de}^e \cdot \mu_{de}^r = \mu_{de}^{\gcd(e,r)}$, *i.e.* we get that $\operatorname{Ad} t^i$ is inner if and only if i is multiple of $\gcd(e,r)$. Thus the image of $\operatorname{Ad} t$ in $\operatorname{Out}(G)$ is of order $\gcd(e,r)$.

For $G = G(2,2,4) = D_4$ the group $\overline{N}$ is generated by ${}^3D_4$ and ${}^2G(2,2,4)$ and is isomorphic to $\mathfrak{S}_3$.

For $G = G(4,2,2)$ the group $\overline{N}$ is generated by ${}^3G(4,2,2)$ and ${}^2G(4,2,2)$ and is isomorphic to $\mathfrak{S}_3$.

For $G = G(3,3,3)$ the group $\overline{N}$ is generated by ${}^3G(3,3,3)$ and the image in $\operatorname{Out}(G)$ of ${}^4G(3,3,3)$, which is of order 2, and is isomorphic to the alternating group $\mathfrak{A}_4$.

The determination of $\overline{N}$ has the following consequence.

Proposition 3.1. *The natural action of* $\operatorname{Out}(G)$ *on* $\operatorname{Irr}(G)$ *is faithful.*

Proof. We note that an element of $\operatorname{Out}(G)$ which acts trivially on $\operatorname{Irr}(G)$ belongs to $\overline{N}$, as it fixes χ_V. Assume that $G = G(de,e,r)$, let $v = \gcd(e,r)$ and consider the de-tuple of partitions $\boldsymbol{\lambda} = (\lambda_0,\dots,\lambda_{de})$ where $\lambda_i = \{r/v\}$ if i is a multiple of de/v and $\lambda_i = \emptyset$ otherwise. With the notations of subsection 2.4, the tuple $\boldsymbol{\lambda}$ is stabilized by $\sigma^{e/v}$ which corresponds to $(\operatorname{Ad} t)^{e/v}$, thus the restriction to G of $\rho_{\boldsymbol{\lambda}} \in \operatorname{Irr} G(de,1,r)$ has an inertia group of cardinality v, hence the subgroup of $\overline{N}$ induced by $\operatorname{Ad} t$ acts faithfully on its set of irreducible components. It thus remains to consider the special cases mentioned above. Inspection of the character tables, verifying that they are not left invariant by the automorphisms in these special cases, concludes the proof. □

4. The structure of $\operatorname{Aut}(G)$

In this section we show that for an irreducible G, every automorphism is the product of a central automorphism and an automorphism which preserves the reflections (except for $G = G(1,1,6) = \mathfrak{S}_6$). Since, for G of rank at least 3, a non-trivial central automorphism does not preserve the reflections, this will prove theorem 1.1.

If $G \subset \operatorname{GL}(V)$ be an irreducible complex reflection group, we may identify ZG to a subgroup of $\mathbb{C}^\times$, thus C is formed of the automorphisms α_χ of the form $\alpha_\chi(g) = g\chi(g)^{-1}$ where $\chi \in \operatorname{Hom}(G,\mathbb{C}^\times)$ is a linear character. This defines in general an endomorphism of G if the values of χ lie in ZG. it is an automorphism (thus a central automorphism) if and only if $z \neq \chi(z)$ for all $z \in ZG - \{1\}$.

4.1. **The infinite series.**

Proposition 4.1. *Any automorphism of a group $G(de, e, r)$ is composed of an automorphism which preserves the reflections with a central automorphism, except for the case of $G(1, 1, 6) = \mathfrak{S}_6$ and for the case of the non-irreducible group $G(2, 2, 2) \simeq \mathfrak{S}_2 \times \mathfrak{S}_2$.*

We first need a lemma.

Lemma 4.2. *The group $D = D(de, e, r)$ of diagonal matrices in $G = G(de, e, r)$ is a characteristic subgroup of G if and only if G is not one of the groups $G(2, 2, 2)$, $G(2, 1, 2)$, $G(4, 2, 2)$, $G(3, 3, 3)$, $G(2, 2, 4)$.*

Proof. We show first that for $r \geq 5$ the group D is the unique maximal normal abelian subgroup of G. Let E be another abelian normal subgroup. Using the semi-direct product decomposition $G = D \rtimes \mathfrak{S}_r$, the image $E/(E \cap D)$ of E in $\mathfrak{S}_r$ is an abelian normal subgroup of $\mathfrak{S}_r$. Since the only such subgroup of $\mathfrak{S}_r$ is trivial if $r \geq 5$, we get that $E/(E \cap D) = 1$ meaning $E \subseteq D$ q.e.d.

We now assume $r \leq 4$. If $d = e = 1$ then $G = \mathfrak{S}_r$ and $D = 1$ is characteristic. We thus assume now $de > 1$. We will show that D is the unique normal abelian subgroup of maximal order, excepted for a few special cases that we handle separately. As before we exclude the non-irreducible and abelian case $G = G(2, 2, 2)$, for which $\mathrm{Out}(G) = \mathrm{GL}_2(\mathbb{F}_2)$ does not preserve $D \simeq \mathbb{Z}/2$. We note for future use that the order of the center ZG is the gcd of the reflection degrees of G, equal to $d \cdot \gcd(e, r)$, and that D is of order $d^r e^{r-1}$ (one chooses arbitrarily $r - 1$ eigenvalues in μ_{de}, and then there are d possibilities for the last one such that the product is in μ_d).

For $r = 1$ we have $G = D$ thus there is nothing to prove. Let E be an abelian normal subgroup of G whose image in $\mathfrak{S}_r$ is non-trivial. For $1 < r \leq 4$, the only non-trivial abelian normal subgroup K of $\mathfrak{S}_r$ is transitive and has order r. It follows that the diagonal matrices D' in E must have equal eigenvalues, that is be in ZG. Thus the maximal such E is the direct product $ZG \times K$, and $\#E = rd \cdot \gcd(e, r)$. It is straightforward to check that $\#E < \#D = d^r e^{r-1}$ except for G one of $G(4, 4, 2)$, $G(2, 1, 2)$, $G(4, 2, 2)$, $G(3, 3, 3)$, $G(2, 2, 4)$. A check shows that the outer automorphism of $G(2, 1, 2)$ does not preserves the diagonal matrices, but that it does preserve them in the incarnation $G(4, 4, 2)$; and that $G(4, 2, 2), G(3, 3, 3), G(2, 2, 4)$ are genuine exceptions. □

We are now in position to prove proposition 4.1, except for the groups $G(2, 1, 2)$, $G(4, 2, 2)$, $G(3, 3, 3)$, $G(2, 2, 4)$, which will be dealt with in the next section. We thus assume that G is not one of these exceptions, and in addition first assume $r \neq 6$.

Let t and $s_1, \ldots, s_{r-1}$ be reflections as in section 2.1, such that G is generated by $t^e, s_1^t, s_1, \ldots s_{r-1}$ (if $d = 1$ we drop t^e and if $e = 1$ we drop s_1^t). Let ϕ be an automorphism of G. Since we are not in one of the exceptions of the lemma, ϕ preserves D thus induces an automorphism $\overline{\phi}$ of $\mathfrak{S}_r$. Since we assumed $r \neq 6$, the automorphism $\overline{\phi}$ is inner, of the form $\mathrm{Ad}\,\sigma$ for some

$\sigma \in \mathfrak{S}_r$, and lifting σ to a permutation matrix in G, we see that up to an inner automorphism we may assume that ϕ induces the identity on $\mathfrak{S}_r$, that is, preserves the shape of a monomial matrix in G.

We will denote by M_σ the permutation matrix corresponding to $\sigma \in \mathfrak{S}_r$. Thus $\phi(s_1)$ is of the form $\mathrm{Diag}(x_1, \dots, x_r)M_{(1,2)}$; the fact that $\phi(s_1)$ is an involution implies that $x_1 = x_2^{-1}$ and that $x_3, \dots, x_n$ are signs. The fact that $\phi(s_1)$ commutes with $\phi(s_3), \dots, \phi(s_{r-1})$ implies that $x_3 = x_4 = \dots = x_r$. Thus $\phi(s_1)$ is of the form $\mathrm{Diag}(\alpha_1^{-1}, \alpha_1, \epsilon, \dots, \epsilon)M_{(1,2)}$ for some sign ϵ and some $\alpha_1 \in \mu_{de}$.

Similarly $\phi(s_i)$ is of the form $\mathrm{Diag}(\epsilon, \dots, \alpha_i^{-1}, \alpha_i, \dots, \epsilon)M_{(i,i+1)}$ where α_i is in ith position (the sign ϵ is the same since s_1 and s_i are conjugate). And $\phi(s_1^t)$ is of the form $\mathrm{Diag}(\alpha'^{-1}, \alpha', \epsilon', \dots, \epsilon')M_{(1,2)}$. We have actually $\epsilon = \epsilon' = 1$ since they occur only when $r > 2$ in which case $s_1^t = (s_1)^{s_2 s_1 s_1^t s_2}$ is conjugate to s_1.

Let $s = \mathrm{Diag}(1, \epsilon\alpha_1^{-1}, \epsilon^2(\alpha_1\alpha_2)^{-1}, \dots, \epsilon^r(\alpha_1 \dots \alpha_r)^{-1}) \in G(de, 1, r)$. Then $\mathrm{Ad}\, s$ induces an automorphism of G which preserves the reflections, and the composed automorphism $\phi' = \mathrm{Ad}\, s \circ \phi$ satisfies $\phi'(s_1) = \epsilon s_1, \dots, \phi'(s_r) = \epsilon s_r$. The element $\phi'(s_1^t)$ is of the form $\mathrm{Diag}(\alpha''^{-1}, \alpha'', \epsilon, \dots, \epsilon)$ for some $\alpha'' \in \mu_{de}$.

The fact that $s_1 s_1^t$ is of order de implies that α'' is a primitive de-th root of unity. Thus there exists an element $\gamma \in \mathrm{Gal}(\mathbb{Q}(\zeta_{de})/\mathbb{Q})$ such that $\gamma(\alpha'') = \zeta_{de}$. Applying γ to the matrices of G gives an automorphism which preserves the reflections, and composing ϕ' with that automorphism, we get an automorphism ϕ'' such that $\phi''(s_i) = \epsilon s_i$ and $\phi''(s_1^t) = \epsilon s_1^t$.

If $d = 1$ we are finished: ϕ'' is the central automorphism given by $s \mapsto \epsilon s$ for any reflection.

Assume now $d > 1$. Then $\phi''(t^e)$ is a diagonal matrix which commutes with $s_2, \dots, s_{r-1}$ thus is of the form $z\,\mathrm{Diag}(\zeta, 1, \dots, 1)$ for some $z \in ZG$ and some $\zeta \in \mu_{de}$. Since it is of order d, z must be of order d and ζ be a primitive d-th root of unity.

Assume first that $e = 1$. Then we do not need s_1^t among the generators, and composing ϕ'' with the automorphism induced by some appropriate element of $\mathrm{Gal}(\mathbb{Q}(\zeta_{de})/\mathbb{Q})$ we may get a ϕ''' such that $\phi'''(s_i) = \epsilon s_i$ and $\phi'''(t) = z't$ for some $z' \in ZG$, thus ϕ''' is a central automorphism.

Finally assume that e and d are not 1. Then we use the third relation $\underbrace{s_1 t^e s_1^t s_1 s_1^t s_1 \dots}_{e+1} = \underbrace{t^e s_1^t s_1 s_1^t s_1 \dots}_{e+1}$ in 2.1 which implies that $\zeta = \zeta_d$, thus ϕ is the central automorphism $s_i \mapsto \epsilon s_i$ and $t^e \mapsto zt^e$.

The same method will apply when $r = 6$ and $de > 1$ if we can exclude the possibility for $\overline{\phi}$ to be a non-inner automorphism of $\mathfrak{S}_6$. Up to conjugation by a permutation matrix, we may assume that $\overline{\phi}$ sends $(1,2)$ to $(1,2)(3,4)(5,6)$. Let $p = M_{(1,2)(3,4)(5,6)}$. We have $\phi(C_G(s_1) \cap D) = C_G(\phi(s_1)) \cap D = C_G(p) \cap D$, the first equality since D is a characteristic subgroup. and the second since $G = D \rtimes \mathfrak{S}_6$ and D is abelian. It follows that $C_G(s_1) \cap D$ and $C_G(p) \cap D$ have the same cardinality. One easily gets $\#(C_G(s_1) \cap D) = (de)^4 d$. On

the other hand, a matrix in $C_G(p) \cap D$ is uniquely determined by (diagonal) coefficients $(a, b, c) \in \mu_{de}^3$ such that $c^2 \in a^{-2}b^{-2}\mu_d$. Since an element in μ_{de} has at most two square roots, it follows that $\#(C_G(p) \cap D) \leq 2d(de)^2$. But $de \geq 2$ implies $2d(de)^2 < (de)^4 d$, a contradiction.

4.2. **The other cases.** We now handle the exceptional groups as well as ones left over from the previous subsection. We first establish a reduction result. We call hyperplanes of G the hyperplanes of V which are invariant subspaces of the reflections in G. Up to a scalar there is a unique G-invariant scalar product on V, to which the orthogonality statement in the next proposition refers.

Proposition 4.3. *Let ϕ be an automorphism of an irreducible reflection group G which maps R inside $R \cdot ZG$, where R is the set of reflections of G. Then if either G is of rank ≥ 3 or G has the property that two orthogonal hyperplanes are conjugate then $\phi \in C \cdot A$.*

Proof. The group G admits a presentation by a set S of distinguished reflections (*i.e*, reflections s with eigenvalue $e^{2i\pi/e_s}$, where e_s is the order of the centralizer in G of the reflecting hyperplane of s), with relations the order relations $s^{e_s} = 1$ and braid relations $w = w'$ where $w, w' \in S^*$ are words of the same length; here the braid relations present the braid group of G (see [Be2, theorem 0.1]). We note that two distinguished reflections are conjugate if and only if their reflecting hyperplanes are conjugate. The words w and w' have the same number of elements in each class of reflections, since the braid relations present the braid group, whose abelianized is the free abelian group on the conjugacy classes of hyperplanes.

For any $s \in S$ we define a reflection $f(s)$ and an element $z(s) \in ZG$ by $\phi(s) = f(s)z(s)$. When $\dim V \geq 3$, $f(s)$ and $z(s)$ are uniquely determined, but when $\dim V = 2$ there might be two choices. We then make an arbitrary such choice, but we ask that if $s, s' \in S$ are conjugate, *i.e.* there exists $g \in G$ such that $s' = {}^g s$ then $z(s') = z(s)$, which is possible since $\phi(s') = {}^{\phi(g)}f(s)z(s)$ and ${}^{\phi(g)}f(s)$ is a reflection. We now extend f and z to the free monoid S^*; we claim that this gives well-defined group homomorphisms from G. Indeed, since (obviously) for any $w \in S^*$ we have $\phi(w) = f(w)z(w)$, it is enough to show that z induces a linear character of G. From the fact that each conjugacy of reflections occurs as many times in w and w' it is clear that $z(w) = z(w')$. And from $\phi(s^{e_s}) = 1 = f(s)^{e_s}z(s)^{e_s}$, since $f(s)^{e_s}$ has at least one eigenvalue 1, we conclude that $f(s)^{e_s} = z(s)^{e_s} = 1$.

If $\dim V \geq 3$, then f is an automorphism: to show that, it is enough to show that f is surjective on R, and for that it is enough to show that f is injective on R. If $s, s' \in R$ and $f(s) = f(s')$ then $\phi(ss'^{-1}) \in ZG$ which implies $ss'^{-1} \in ZG$ which implies $s = s'$ since $\dim V \geq 3$. Thus if $\dim V \geq 3$, we get that ϕ is f composed with a central automorphism (given by $s \mapsto sz(f^{-1}(s))$).

It remains the case $\dim V = 2$ and G has the property that two orthogonal hyperplanes are conjugate for which we give a more complicated argument to prove that f is injective. It is enough to show that some power of f is injective. Since ϕ and f both map ZG into itself, it is easy to show by induction on r that for any r the endomorphisms ϕ^r and f^r still differ by a linear character with values in ZG. Since ϕ is an automorphism, it is of finite order; taking for r this order, and replacing f by f^r, we see that we may assume $\phi = \mathrm{Id}$.

We thus have $f(g) = gz(g)$ for some ZG-valued linear character z, and we want to see that f is injective. We will show that $f^2 = \mathrm{Id}$. It is enough to show that for a reflection r. We have two cases to consider: if $f(r) = r$ then $f^2(r) = r$. Otherwise, we must have $z(r) = \zeta^{-1}\mathrm{Id}$ where ζ is the non-trivial eigenvalue of r, and if $\langle , \rangle$ is a G-invariant scalar product on V, $f(r)$ is a reflection with non-trivial eigenvalue ζ^{-1} and hyperplane orthogonal to that of r. Since two such hyperplanes are conjugate and $f(r)^{-1}$ has the same eigenvalue as r, they are conjugate. Then $z(f(r)) = z(r)^{-1}$ thus $f(f(r)) = f(r)z(f(r)) = rz(r)z(r)^{-1} = r$ and we are done. □

By enumerating the hyperplanes in `CHEVIE`(see [CHEVIE]), it is easy to see that orthogonal hyperplanes are conjugate for all 2-dimensional exceptional groups except G_5 and G_7.

The preceding proposition is sufficient to handle the exceptional groups G_i for $i \geq 12$. Indeed, an easy computer check shows that any conjugacy class c such that there exists a conjugacy class c' satisfying

- $\#c' = \#c$
- c' contains a reflection s'
- c contains an element with the same order as s'

contains the product of a reflection by an element of the center.

Finally, for the groups G_4 to G_{12}, as well as for the exceptions in the infinite series $G(2,1,2), G(4,2,2), G(3,3,3)$ and $G(2,2,4)$ we use the techniques of section 6 to determine all automorphisms and to check that they are the product of an element of C and an automorphism which preserves the reflections.

5. Reflection representations

By a reflection representation of a reflection group G, we mean a faithful representation such that the image of a reflection of G is a reflection. The corresponding characters are called reflection characters. Here we prove theorem 1.2, whose statement we recall.

Theorem 5.1. *Let $G \subset \mathrm{GL}(V)$ be an irreducible complex reflection group, where V is a vector space over the field of definition K of G. Then the reflection characters of G are the transformed of χ_V by the action of* $\mathrm{Gal}(K/\mathbb{Q})$.

Before going on with the proof, we remark that this proves indeed that any automorphism which preserves the reflections is a Galois automorphism attached to some $\gamma \in \mathrm{Gal}(K/\mathbb{Q})$, since it sends χ_V to some $\gamma(\chi_V)$.

of 5.1. The theorem is proved by a computer check for the exceptional reflection groups, by looking at the character tables. Both the action of the Galois group, and which representations are faithful and send reflections to reflections are readily seen from the character tables.

We now look at the groups $G(de, e, r)$. We first look at the case $e = 1$. The result is clear if G is cyclic so we assume $r > 1$, and use the notations of section 2.3.

We first describe the action of $\mathrm{Gal}(K/\mathbb{Q})$ on the representations of $G(d, 1, r)$.

Lemma 5.2. *For $i \in \mathbb{Z}$ prime to d, let $\mathfrak{s}_i \in \mathrm{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ be the element defined by $\mathfrak{s}_i(\zeta_d) = \zeta_d^i$. Then the partition tuple $\mathfrak{s}_i(\boldsymbol{\lambda})$ such that $\mathfrak{s}_i \circ \rho_{\boldsymbol{\lambda}} \simeq \rho_{\mathfrak{s}_i(\boldsymbol{\lambda})}$ is given by $\mathfrak{s}_i(\boldsymbol{\lambda}) = (\lambda_0, \lambda_i, \lambda_{2i}, \ldots, \lambda_{(d-1)i})$ (where the indices are taken* (mod d)*).*

Proof. This is immediate using the model given in 2.3. The action of $\mathfrak{s}_i$ is trivial on $\rho_{\boldsymbol{\lambda}}(s_i)$ and maps $\rho_{\boldsymbol{\lambda}}(t)$ to $\rho_{\boldsymbol{\lambda}}(t)^i$, from which the result follows. □

We now determine which $\rho_{\boldsymbol{\lambda}}$ are faithful reflection representations.

Lemma 5.3. *The faithful reflection representations of $G(d, 1, r)$ are the $\rho_{\boldsymbol{\lambda}^{(i)}}$ for i prime to d, where $\lambda_0^{(i)} = \{r-1\}$ and $\lambda_i^{(i)} = \{1\}$.*

Proof. Let us determine when $\rho_{\boldsymbol{\lambda}}(t)$ is a reflection. First, notice that there is only one standard tableau corresponding to a given young diagram and with a given content if and only if the diagram is a column or a row. For a partition λ we denote by $|\lambda|$ the size of λ (the sum of the lengths of its parts), and call height of λ the number of its parts, namely the number of rows in the corresponding diagram. Since $\mathbf{T} \in \mathcal{T}(\boldsymbol{\lambda})$ are eigenvectors of $\rho_{\boldsymbol{\lambda}}(t)$ with eigenvalue $\zeta_d^{\mathbf{T}(1)}$, there must be exactly one $\mathbf{T} \in \mathcal{T}(\boldsymbol{\lambda})$ such that $\mathbf{T}(1) \neq 0$. This means that there must be exactly one $i \neq 0$ such that $|\lambda_i| \neq 0$, with a Young diagram a line or a column. We must also have that λ_0 is a line or a column; if $|\lambda_0| = 0$ then the representation is of dimension 1 and cannot be faithful since $r > 1$. Also if $|\lambda_i| > 1$ then $r \geq 3$ and there are at least two $\mathbf{T}$ such that $\mathbf{T}(1) = i$. So we must have $\lambda_i = \{1\}$. Finally, note that i must be prime to d otherwise $\rho_{\boldsymbol{\lambda}}$ is not faithful (since $\rho_{\boldsymbol{\lambda}}(t)$ does not have order d). To summarize, we have $\lambda_0 = \{r-1\}$ or $\{1^{r-1}\}$ and $\lambda_i = \{1\}$ for some i prime to d.

Let us now analyze when $\rho_{\boldsymbol{\lambda}}(s_i)$ is a reflection (or equivalently all $\rho_{\boldsymbol{\lambda}}(s_i)$ for $i = 1, \ldots, r-1$ are reflections, since they are conjugate). The formula 2.2 shows that $\rho_{\boldsymbol{\lambda}}(s_i)$ in the basis $\mathcal{T}(\boldsymbol{\lambda})$ is block-diagonal, with diagonal blocks indexed by the basis elements $\mathbf{T}$ and $\mathbf{T}_{i \leftrightarrow i+1}$, which are thus of size 1 when $\mathbf{T}_{i \leftrightarrow i+1} = 0$ and of size 2 otherwise.

(i) If $\mathbf{T}_{i\leftrightarrow i+1} \neq 0$ the corresponding block is $\begin{pmatrix} \frac{1}{a(i,i+1)} & 1+\frac{1}{a(i,i+1)} \\ 1-\frac{1}{a(i,i+1)} & \frac{-1}{a(i,i+1)} \end{pmatrix}$ whose eigenvalues are 1 and -1.

(ii) If $\mathbf{T}_{i\leftrightarrow i+1} = 0$, then i and $i+1$ are in the same tableau, and in the same line or column. We have that $\mathbf{T}$ is an eigenvector of $\rho_{\boldsymbol{\lambda}}(s_i)$, for the eigenvalue $\frac{1}{a(i,i+1)} = 1$ when i and $i+1$ are in the same line and $\frac{1}{a(i,i+1)} = -1$ when i and $i+1$ are in the same column.

For $\rho_{\boldsymbol{\lambda}}(s_i)$ to be a reflection there cannot be more than two j with $|\lambda_j| \neq 0$ otherwise case (i) would occur more than once (with i and $i+1$ in different tableaux). Similarly, if there are two j such that $|\lambda_j| \neq 0$ then the two Young diagrams must be lines or columns otherwise case (i) would occur more than once. Actually, the two diagrams must be lines since if one is a column of height > 1 then case (ii) with eigenvalue -1 has to occur at least once. And for $\rho_{\boldsymbol{\lambda}}(s_{r-1})$ to be a reflection one of the lines has to be of length 1, otherwise by removing the last square of each line we can still fill the rest in at least two different ways which leads to too many instances of (i).

We now look at the case when only one $|\lambda_j| \neq 0$. The diagram must not be a line or a column otherwise ρ is not faithful. For $i = r-1$ case (i) will occur more than once if the diagram has more than two corners. If there are two corners, when they are removed the rest must be a line or a column; this means the diagram must be a hook; and the height of the hook has to be 2 otherwise case (ii) with eigenvalue -1 will occur. Finally if there is one corner the diagram is a rectangle; this rectangle has to be 2×2 otherwise by removing the corner one would get a diagram with two corners which is not a hook which would lead to too many eigenvalues -1 for s_{r-2}.

To summarize, we must have $i \neq j$ such that $\lambda_i = \{r-1\}$ and $\lambda_j = \{1\}$, or only one i with $|\lambda_i| \neq 0$ with diagram either a hook of height 2 or a 2×2 square (the last for $r = 4$).

So we are left with the cases in the statement of the theorem for the images of all reflections to be reflections, and the representation to be faithful. □

The reflection representation of $G(d,1,r)$ is $\rho_{\boldsymbol{\lambda}^{(1)}}$ and by lemma 5.2 we have $\mathfrak{s}_i \circ \rho_{\boldsymbol{\lambda}^{(1)}} = \rho_{\boldsymbol{\lambda}^{(i)}}$ which together with lemma 5.3 proves the theorem for $G(d,1,r)$.

Let us look now at $G(de,e,r)$.

Lemma 5.4. *The faithful irreducible reflection representations of $G(de,e,r)$ are the restriction to $G(de,e,r)$ of the faithful irreducible reflection representations of $G(de,1,r)$.*

Proof. We first notice that a representation which is not the restriction of an irreducible representation of $G(de,1,r)$ is not faithful. To prove this, we use that the central character of $\rho_{\boldsymbol{\lambda}}$ is given by $\omega_{\rho_{\boldsymbol{\lambda}}}(z) = \zeta_{de}^{\sum_{i=0}^{d-1} i|\lambda_i|}$ (this can be deduced from the construction of $\rho_{\boldsymbol{\lambda}}$ as induced from a generalized Young subgroup — see *e.g.* [Ze, p. 93–106]). Here z is the generator of the

center of $G(de,1,r)$, which can be identified to the scalar $\zeta_{de} \in \mathrm{GL}(\mathbb{C}^r)$. If $\rho_{\boldsymbol{\lambda}}$ restricted to $G(de,e,r)$ is not irreducible, there is a divisor b of e such that for all i we have $\lambda_{i+de/b} = \lambda_i$ (then each λ_i occurs b times thus b divides $\gcd(e,r)$). It follows that $\sum_{i=0}^{d-1} i|\lambda_i|$ is divisible by b thus $\rho_{\boldsymbol{\lambda}}$ is not faithful on the center. It is not even faithful on the center of $G(de,e,r)$ since that center is generated by $z^{\frac{e}{\gcd(e,r)}}$ and b still divides the order of that element.

Thus the representation we are looking at is the restriction to $G(de,e,r)$ of some $\rho_{\boldsymbol{\lambda}}$. We use the analysis of when $\rho_{\boldsymbol{\lambda}}(s_i)$ is a reflection. Notice first that the representations with only one diagram are not faithful, since by formula 2.3 s_1 and s_1^t have same image since for any $\mathbf{T}$ we have $\mathbf{T}(1) = \mathbf{T}(2)$. Thus we are in the case where we have $i \neq j$ such that $\lambda_i = \{n-1\}$ and $\lambda_j = \{1\}$. Actually, for the representation to be faithful we must have $\rho_{\boldsymbol{\lambda}}(s_1^{t^k}) \neq \rho_{\boldsymbol{\lambda}}(s_1)$ for any k prime to de, thus $\mathbf{T}(2) - \mathbf{T}(1)$ must be prime to de, *i.e.* $j-i$ must be prime to de.

Analyzing when $\rho_{\boldsymbol{\lambda}}(t^e)$ is a reflection along the same lines we did for $\rho_{\boldsymbol{\lambda}}(t)$ we find that $j \equiv 0 \pmod{d}$. Since $\rho_{\boldsymbol{\lambda}}$ and $\rho_{\sigma(\boldsymbol{\lambda})}$ have same restriction to $G(e,e,r)$ where σ is the "shift" by d as in section 2.4, we may assume $j=0$. Thus we have indeed the same representations as in 5.3, which proves the lemma. □

The theorem for $G(de,e,r)$ follows immediately from the above lemma since the reflection representation of $G(de,e,r)$ is the restriction of the reflection representation of $G(de,1,r)$, and the Galois action commutes with the restriction. □

6. Theorems 1.4 and 1.6 for the exceptional groups

To verify 1.6 for the exceptional reflection group, we have used the `GAP` package `CHEVIE` (see [CHEVIE]), which contains information about them, including:

- Their realization via a permutation representation (on a set of "roots" in V).
- Their character table.

The character tables allow us to determine the permutation of $\mathrm{Irr}(G)$ effected by an element of $\mathrm{Gal}(K/\mathbb{Q})$. The method we have used to check 1.4 is to compute the group of outer automorphisms of G which preserve the set of reflections, and then to find a subgroup which induces the same permutation group on $\mathrm{Irr}(G)$ as $\mathrm{Gal}(K/\mathbb{Q})$. We recall from proposition 3.1 that an element of $\mathrm{Out}(G)$ is completely determined by the permutation it induces on $\mathrm{Irr}(G)$.

To compute automorphisms, we use a presentation for G, by means of a diagram (see [BMR] and [BM]); if S is the generating set for this presentation, to enumerate the automorphisms which preserve the reflections, we enumerate up to inner automorphisms all tuples of reflections of G which satisfy the relations for G, and generate G.

If we want to enumerate all automorphisms, we have to extend the search to other conjugacy classes which have the same size and same order as the conjugacy class of a reflection. We only need to compute the whole of $\mathrm{Aut}(G)$ for 2-dimensional groups, to meet the requirements at the end of section 4, and for these groups the process above is quite fast.

To find directly the orbits of G on the tuples of reflections is a too large computation for the larger groups, such as G_{34} which has about 4.10^7 elements; thus we use a recursive process: we choose the image of the first generator s (which amounts to choosing a representative for each orbit of G on the reflections); then we find the $C_G(s)$-orbits of tuples which satisfy the relations for $S - \{s\}$ (by a recursive process) and then select those that have the required relations with s; and finally weed out the tuples which do not generate G. This allows us compute $\overline{A}$ in at most a few hours even for the largest groups.

To find the image of $\overline{\iota}_\chi(\Gamma)$ in $\overline{A}$ is rather easy since for the exceptional groups the morphism $\Gamma \xrightarrow{\overline{\iota}} \overline{A}$ is either surjective or has an image of index 2 (the last occurs exactly for G_5, G_7 and $G_{28} = F_4$, which, in agreement with theorem 1.3, are also the only exceptional groups where $\overline{N} \neq 1$; in these cases, with the notations of table 1, $\overline{N}$ induces respectively the automorphism $(s,t) \mapsto (t,s)$, the automorphism $(s,t,u) \mapsto (s,u^{-1},t^{-1})$, the diagram automorphism of F_4).

To check 1.6, we must lift $\overline{\iota}$ to the group $\mathrm{Aut}(G)$, that is find representatives in $\mathrm{Aut}(G)$ of elements of the image of $\overline{\iota}$ which satisfy the relations for Γ. Since for exceptional groups Γ is an abelian 2-group, these relations are order relations, plus commutation relations.

Few representatives of a given automorphism turn out to have the right order, so we first compute all representatives which have the right order, and then it is an easy job to pick among them elements which commute (this procedure of course fails for G_{27} for which we must construct $\tilde{\iota}$ rather than ι; in this case we take $K' = \mathbb{Q}(\zeta_{15})$; the group $\mathrm{Gal}(K'/\mathbb{Q})$ is generated by $\mathfrak{c}$ and $\mathfrak{s}_7$, this last element being of order 4 (it is of order 2 in $\mathrm{Gal}(K/\mathbb{Q})$) ; we can map these to automorphisms of order 2 and 4 respectively commuting to each other — but it is impossible to map $\mathfrak{s}_7$ to an automorphism of order 2).

The only computational problem in this procedure is to compute the product of two automorphisms; since an automorphism φ is represented by the image $\varphi(S)$, to do this we need an expression as words in S^* of the elements of $\varphi(S)$. For all groups but G_{34} this was solved by enumerating all elements of G by standard permutation group algorithms using a base and a strong generating set. For G_{34} we managed by only considering automorphisms which extend the one for G_{33}; since the normalizer of G_{33} in G_{34} is G_{33} times the center of G_{34}, such automorphisms map the last generator of G_{34} to a conjugate by G_{33}, which makes the computation feasible.

Table 1 summarizes our results for the exceptional reflection groups; it gives the diagram for G, and then gives the value of K and describes ι in terms of the generators S given by the diagram.

The nodes of the diagrams are labeled by the elements of S; we give to these elements names in the list s, t, u, v, w, x in that order. A number inside the node gives the order of the corresponding reflection when it is not 2. To express the braid relations, we use the conventions for Coxeter diagrams: two nodes which are not connected correspond to commuting reflections. A single line connecting two nodes s and t corresponds to the relation $sts = tst$, a double line to $stst = tsts$, a triple line to $ststst = tststs$, and a line labeled with the number n corresponds to a braid relation $\underbrace{stst\cdots}_{n \text{ terms}} = \underbrace{tsts\cdots}_{n \text{ terms}}$.

These conventions are extended by additional ones; a circle joining 3 nodes s, t and u, with the number n inside (when there is no number inside, 3 is to be understood) corresponds to a clockwise circular relation $\underbrace{stust\cdots}_{n \text{ terms}} = \underbrace{tustu\cdots}_{n \text{ terms}} = \underbrace{ustus\cdots}_{n \text{ terms}}$; for instance, the relation for G_7, G_{11}, G_{19} and the generators s, t, u of G_{31} is $stu = tus = ust$. Similarly the 6 inside the triangle for G_{33} and G_{34} means that in addition to length 3 braid relations implied by the sides of the triangle, there is the circular relation $tuvtuv = uvtuvt = vtuvtu$.

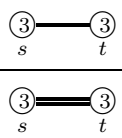
The double line in G_{29} expresses a length 4 braid relation between v and ut: $vutvut = utvutv$. Let us mention that the exceptional characters of G_{29} are the 6-dimensional rational ones which appear in the 10th symmetric power $S^{10}(V)$.

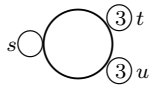
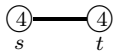
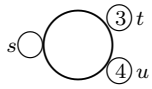
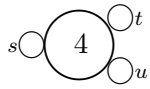
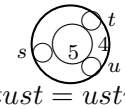
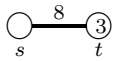
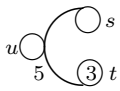
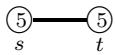
The triangle symbol for G_{24} and G_{27} corresponds to the relation $s(uts)^2 = (stu)^2 t$. Finally, the groups G_{13} and G_{15} have more complicated relations which are spelled out below the diagram.

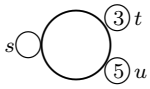
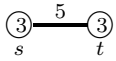
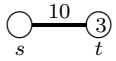
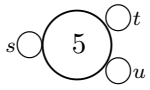
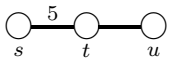
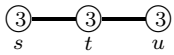
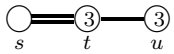
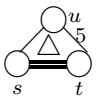
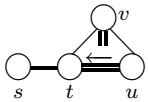
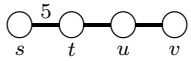
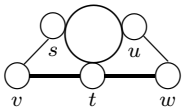
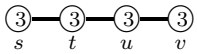
The groups $G_{28} = F_4$, $G_{35} = E_6$, $G_{36} = E_7$ and $G_{37} = E_8$, which are rational, do not appear in the table since our theorems are trivial for them.

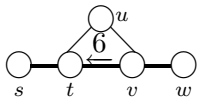
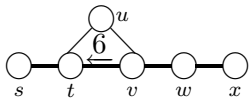
The map ι is specified by giving for generators γ of Γ the tuple $\iota(\gamma)(S)$ (where S is in the order s, t, u, v, w, x). The elements of Γ are denoted by $\mathfrak{c}$ for complex conjugation and by $\mathfrak{s}_i$ for the following element: let $\mathbb{Q}(\zeta_n)$ be the smallest cyclotomic field containing K; then $\mathfrak{s}_i$ is the restriction to K of the Galois automorphism of $\mathbb{Q}(\zeta_n)$ which sends ζ_n to ζ_n^i.

Table 1: The morphism $\tilde{\iota}$

Group	Diagram	Field	$\tilde{\iota}$
G_4		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (s^{-1}, {}^s t^{-1})$
G_5		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (s^{-1}, {}^s t^{-1})$

Group	Diagram	Field	$\tilde{\iota}$
G_6		$\mathbb{Q}(\zeta_{12})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$, $\mathfrak{s}_7 \mapsto (s^{t^{-1}st}, t)$
G_7		$\mathbb{Q}(\zeta_{12})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}, (u^{-1})^s)$, $\mathfrak{s}_7 \mapsto (s^{ut}, t, u)$
G_8		$\mathbb{Q}(\mathrm{i})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$
G_9		$\mathbb{Q}(\zeta_8)$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$, $\mathfrak{s}_5 \mapsto (s^{tts}, t)$
G_{10}		$\mathbb{Q}(\zeta_{12})$	$\mathfrak{c} \mapsto ((s^{-1})^{t^{-1}}, t^{-1})$ $\mathfrak{s}_7 \mapsto (s^{ttss}, t^{-1})$
G_{11}		$\mathbb{Q}(\zeta_{24})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}, (u^{-1})^s)$, $\mathfrak{s}_{13} \mapsto (s^{tsut}, t, u)$, $\mathfrak{s}_{19} \mapsto (s, t, (u^{-1})^{tu^{-1}s})$
G_{12}		$\mathbb{Q}(\sqrt{-2})$	$\mathfrak{c} \mapsto (t, s, u)$
G_{13}	 $tust = ustu$, $stust = ustus$	$\mathbb{Q}(\zeta_8)$	$\mathfrak{c} \mapsto (s, u, t^s)$, $\mathfrak{s}_3 \mapsto (s, u^{tu}, u^{su})$
G_{14}		$\mathbb{Q}(\zeta_3, \sqrt{-2})$	$\mathfrak{c} \mapsto (s^{-1}, (t^{-1})^s)$, $\mathfrak{s}_7 \mapsto (s, t^{sttst})$
G_{15}	 $stu = ust$, $tusts = ustst$	$\mathbb{Q}(\zeta_{24})$	$\mathfrak{c} \mapsto (s, t^{-1}, u^s)$, $\mathfrak{s}_{13} \mapsto (s^{usu}, t, u)$, $\mathfrak{s}_{19} \mapsto (s, t, u^{sus})$
G_{16}		$\mathbb{Q}(\zeta_5)$	$\mathfrak{s}_2 \mapsto (s^2, (t^2)^{s^2t^{-1}s})$
G_{17}		$\mathbb{Q}(\zeta_{20})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$, $\mathfrak{s}_7 \mapsto (s^{t^{-1}st^2}, t^2)$

Group	Diagram	Field	$\tilde{\iota}$
G_{18}		$\mathbb{Q}(\zeta_{15})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$, $\mathfrak{s}_7 \mapsto (s^{t^{-1}s}, t^2)$
G_{19}		$\mathbb{Q}(\zeta_{60})$	$\mathfrak{c} \mapsto (s^t, t^{-1}, u^{-1})$, $\mathfrak{s}_7 \mapsto (s^{ts}, t^{st^{-1}}, u^2)$, $\mathfrak{s}_{41} \mapsto (s, (t^{-1})^{(ut)^{su}}, u)$
G_{20}		$\mathbb{Q}(\zeta_3, \sqrt{5})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$, $\mathfrak{s}_7 \mapsto (s, s^{(ts^{-1})^2})$
G_{21}		$\mathbb{Q}(\zeta_{12}, \sqrt{5})$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1})$, $\mathfrak{s}_7 \mapsto (s^{(t^{-1})^{st}}, t)$, $\mathfrak{s}_{13} \mapsto (s^{t^{-1}s^{ts}}, t)$
G_{22}		$\mathbb{Q}(\mathrm{i}, \sqrt{5})$	$\mathfrak{c} \mapsto (u, t, s)$, $\mathfrak{s}_7 \mapsto (u^{st}, u^{ts}, t^{us})$
$G_{23} = H_3$		$\mathbb{Q}(\sqrt{5})$	$\mathfrak{s}_2 \mapsto (u^{tsts}, u, t)$
G_{24}		$\mathbb{Q}(\sqrt{-7})$	$\mathfrak{c} \mapsto (u, t, s)$
G_{25}		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}, u^{-1})$
G_{26}		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}, u^{-1})$
G_{27}		$\mathbb{Q}(\zeta_3, \sqrt{5})$ $K' = \mathbb{Q}(\zeta_{15})$	$\mathfrak{c} \mapsto (s^{tut}, t, u)$, $\mathfrak{s}_7 \mapsto (u^{ts}, t^u, t)$
G_{29}		$\mathbb{Q}(\mathrm{i})$	$\mathfrak{c} \mapsto (s, t, u, v^u)$
$G_{30} = H_4$		$\mathbb{Q}(\sqrt{5})$	$\mathfrak{s}_2 \mapsto (u^{tsts}, u, t, v^{utstuststuvuts})$
G_{31}		$\mathbb{Q}(\mathrm{i})$	$\mathfrak{c} \mapsto (u, t, s, w, v)$
G_{32}		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (s^{-1}, t^{-1}, u^{-1}, v^{-1})$

Group	Diagram	Field	$\tilde{\iota}$
G_{33}		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (w, v, u, t, s)$
G_{34}		$\mathbb{Q}(\zeta_3)$	$\mathfrak{c} \mapsto (w, v, u, t, s, x^{wvtuvwstuvts})$

We have some observations to make on table 1. First, the morphism $\tilde{\iota}$ is not unique; often, there are even many different orbits of possible $\tilde{\iota}$ up to inner automorphisms of G. Each time we have chosen one which preserves a parabolic subgroup, which will be used to construct nice models of the reflection representation of G in section 9.

This is not always the choice with the simplest formulas. In particular, for Shephard groups (those which have the same diagram, apart from the order of the reflections, as a Coxeter group, thus the same braid group as a Coxeter group), it turns out that it is always possible to find a morphism which maps the complex conjugation $\mathfrak{c}$ to the automorphism which sends each element of S to its inverse. When $\mathfrak{c}$ does not generate $\mathrm{Gal}(K/\mathbb{Q})$, one can send $\mathfrak{s}_7$ to $(s^{ts^{-1}}, t^{-1})$ for G_{10} and $\mathfrak{s}_2$ to $(s^2, (s^2)^{t^{-2}s^2})$ for G_{16} to get such a morphism (in the case of G_{14}, $\mathfrak{s}_7$ can keep the same image as in table 1). A general explanation for this phenomenon would be desirable.

7. Theorems 1.4, 1.6 and 1.7 for the infinite series

The field of definition of $G(de, e, r)$ is $K = \mathbb{Q}(\zeta_{de})$, except when $d = 1$ and $r = 2$. In this last case, $G(e, e, 2)$ is the dihedral group of order $2e$, for which $K = \mathbb{Q}(\cos(\frac{2\pi}{e})) = \mathbb{Q}(\zeta_e + \zeta_e^{-1})$ and we set $K' = \mathbb{Q}(\zeta_e)$.

In any case the natural representation of $G(de, e, r)$ via monomial matrices is over the vector space $V = K'^r$, and writing $V = V_0 \otimes_{\mathbb{Q}} K'$ where V_0 is defined by the standard basis of V, the set $G(de, e, r)$ is globally invariant under the induced action of $\mathrm{Gal}(K'/\mathbb{Q})$ on $\mathrm{GL}_r(K')$. It follows that this action induces a morphism $\eta : \mathrm{Gal}(\mathbb{Q}(\zeta_{de})/\mathbb{Q}) \to \mathrm{Aut}(G)$. The purpose of this section is to show that a slight variation on η satisfies the properties of theorem 1.6, which implies the weaker theorem 1.4. Actually we will get a more precise version of 1.6, stating the $\tilde{\iota}_\chi$ equivariance of some models, which gives also theorem 1.7.

7.1. **Dihedral groups.** In this section we write $s' = s'_1$, $s = s_1$, thus the dihedral group $G(e, e, 2)$ is generated by $S = \{s', s\}$ where

$$s' = \begin{pmatrix} 0 & \zeta_e \\ \zeta_e^{-1} & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with relations $s^2 = s'^2 = 1$, $(s's)^e = 1$. For i prime to e we have $\eta(\mathfrak{s}_i)(s) = s$ and $\eta(\mathfrak{s}_i)(s') = (s's)^i s$. In particular $\eta(\mathfrak{s}_i)(s's) = (s's)^i$. We may also check that for any $x \in G$ we have $\eta(\mathfrak{s}_{-i})(x) = s\eta(\mathfrak{s}_i)(x)s$.

Recall that $K = \mathbb{Q}(\zeta_e + \zeta_e^{-1})$ and let $K' = \mathbb{Q}(\zeta_e)$. It follows that the quotient $\Gamma' = \mathrm{Gal}(K'/\mathbb{Q}) \to \Gamma = \mathrm{Gal}(K/\mathbb{Q})$ is obtained by identifying $\mathfrak{s}_i$ and $\mathfrak{s}_{-i}$, thus the above computations show that the composed morphism $\Gamma' \xrightarrow{\eta} \mathrm{Aut}(G) \to \mathrm{Out}(G)$ factors through Γ, leading to the following commutative diagram where we have written $\tilde{\iota}$ for η.

$$\begin{tikzcd}
\Gamma' \arrow[r, hook, "\tilde{\iota}"'] \arrow[d, two heads] \arrow[dr] & \mathrm{Aut}(G) \arrow[d, two heads] \\
\Gamma \arrow[r, dotted, "\bar{\iota}"'] & \mathrm{Out}(G)
\end{tikzcd}$$

The maps $\tilde{\iota}$ and $\bar{\iota}$ above satisfy the hypotheses of theorems 1.4 and 1.6.

We note that the above reflection representation is actually $\tilde{\iota}$-equivariant, thus we have property 1.7 for the reflection representation.

It is clear that 1.4, 1.6 and 1.7 hold for the linear characters of G, because they take values in $\mathbb{Q}$. The remaining irreducible representations of G have a model of the form

$$s' \mapsto \begin{pmatrix} 0 & \zeta_e^r \\ \zeta_e^{-r} & 0 \end{pmatrix}, \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s's \mapsto \begin{pmatrix} \zeta_e^r & 0 \\ 0 & \zeta_e^{-r} \end{pmatrix}$$

for some r, and are determined up to isomorphism by the value of $\zeta_e^r + \zeta_e^{-r}$, so 1.4, 1.6 and 1.7 also hold for them.

Construction of ι. We will now determine when it is possible to take $K' = K$. It is classical that the short exact sequence

$$1 \to \mathrm{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q}(\zeta_e + \zeta_e^{-1})) \to \mathrm{Gal}(\mathbb{Q}(\zeta_e)/\mathbb{Q}) \to \mathrm{Gal}(\mathbb{Q}(\zeta_e + \zeta_e^{-1})/\mathbb{Q}) \to 1$$

splits precisely when $\mathbb{Q}(\zeta_e)$ contains a quadratic imaginary extension of $\mathbb{Q}$. This is the case if and only if $4|e$ or e has a prime factor congruent to 3 modulo 4. In that case one can define injective morphisms $\Gamma \to \mathrm{Aut}(G)$ by composition.

$$\begin{tikzcd}
\Gamma' \arrow[r, hook, "\tilde{\iota}"'] \arrow[d, two heads] & \mathrm{Aut}(G) \arrow[d, two heads] \\
\Gamma \arrow[r, dotted, "\bar{\iota}"'] \arrow[ur, dotted, "\iota"'] \arrow[u, dotted, bend left] & \mathrm{Out}(G)
\end{tikzcd}$$

We cannot expect to lift this morphism to $\mathrm{Aut}(G)$ in general, as already shows the example $e = 5$. In that case, it is easily checked that $\Gamma = \mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$ has order two and that all involutive automorphisms of G are inner.

If the exact sequence splits, we let $H < \Gamma'$ be a section of Γ. Since H has index two in Γ' we know that H is normal in Γ' and, by elementary Galois theory we deduce from $K = K' \cap \mathbb{R}$ that $H = \mathrm{Gal}(K'/\mathbb{Q}(\mathrm{i}\alpha))$ for

some quadratic number $\alpha \in \mathbb{R}$. Let $\rho : G \to \mathrm{GL}_2(K')$ denote an arbitrary 2-dimensional irreducible representation of G, following the models described above. An intertwiner between ρ and $\mathfrak{c} \circ \rho$ is given by $\rho(s) \in \mathrm{GL}_2(\mathbb{Q})$. Let

$$M = \frac{1}{4\mathrm{i}\alpha} \begin{pmatrix} 1+\mathrm{i}\alpha & -1+\mathrm{i}\alpha \\ -1+\mathrm{i}\alpha & 1+\mathrm{i}\alpha \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}(\mathrm{i}\alpha)).$$

It is easily checked that $M^{-1}\mathfrak{c}(M)$ is equal to $\rho(s)$ up to a scalar multiple (a general way to find such an M will be described in section 8). Let $\rho' : G \to \mathrm{GL}_2(K')$ be defined by $\rho'(g) = M\rho(g)M^{-1}$. We have $\mathfrak{c}(\rho'(g)) = M\rho(s)\mathfrak{c}(\rho(g))\rho(s)M^{-1} = M\rho(g)M^{-1} = \rho'(g)$ hence $\rho'(g) \in \mathrm{GL}_2(K)$. Moreover, for $\gamma \in H$ we have $\gamma(M) = M$ hence $\gamma \circ \rho'(g) = M\gamma \circ \rho(g)M^{-1} = M \circ \rho \circ \tilde{\iota}(\gamma)(g)M^{-1} = \rho' \circ \tilde{\iota}(\gamma)(g)$. It follows that, in these cases, we have morphisms $\iota : \Gamma \hookrightarrow \mathrm{Aut}(G)$ and models over K for all irreducible representations ρ of G, such that $\gamma \circ \rho = \rho \circ \iota(\gamma)$ for all $\gamma \in \Gamma$. Thus, in those cases we can take $K' = K$ in theorems 1.6 and 1.7.

7.2. **The case** $G(d,1,r)$**.** The group $G = G(d,1,r)$ is generated by the set of reflections $S = \{t, s_1, \ldots, s_{r-1}\}$ where $t = \mathrm{Diag}(\zeta_d, 1, \ldots, 1)$ and $s_k = M_{(k,k+1)}$ with the notations of the proof of 5.1. We have $\eta(\mathfrak{s}_\alpha)(t) = t^\alpha$ and $\eta(\mathfrak{s}_\alpha)(s_k) = s_k$.

We provide an easy proof of 1.4 and 1.6 for $G(d,1,r)$ starting from the case $r = 2$. Then G has order $2d^2$, and its irreducible representations have dimension 1 and 2. The case of linear characters is immediate to check. The 2-dimensional representations are given, for $\zeta_1, \zeta_2 \in \mu_d, \zeta_1 \neq \zeta_2$, by the model

$$\rho_{\zeta_1,\zeta_2}(t) = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix} \qquad \rho_{\zeta_1,\zeta_2}(s_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It is readily checked that

$$\rho_{\zeta_1,\zeta_2} \circ \eta(\mathfrak{s}_\alpha) = \mathfrak{s}_\alpha \circ \rho_{\zeta_1,\zeta_2} = \rho_{\zeta_1^\alpha,\zeta_2^\alpha}$$

and this proves theorems 1.4, 1.6 for the groups $G(d,1,2)$ by taking $K' = K$ and $\tilde{\iota} = \eta$. The first equality in the above formula also shows that the model is η-equivariant, thus giving 1.7.

Theorems 1.4 and 1.6 are then easily deduced from this for the groups $G(d,1,r)$, using that, for $r \geq 3$, each irreducible representation of $G(d,1,r)$ is determined up to isomorphism by its restriction to the parabolic subgroup $G(d,1,r-1)$ (see e.g. [ArKo] corollary 3.12).

We will give a proof of 1.7 for $G(d,1,r)$ in the same spirit in 9.2. But in the next subsections we will more constructively show that the models of 2.3 and 2.4 are $\tilde{\iota}_\chi$-equivariant for a suitable $\tilde{\iota}_\chi$.

7.3. **The case of** $G(d,1,r)$**.** From the formulas $\eta(\mathfrak{s}_\alpha)(t) = t^\alpha$ and $\eta(\mathfrak{s}_\alpha)(s_k) = s_k$ and the formulae of subsection 2.3, it is clear that our models over $\mathbb{Q}(\zeta_d)$ satisfy theorem 1.6 by taking $\tilde{\iota} = \eta$; actually, the formulae show that the model $\rho_{\boldsymbol{\lambda}}$ is $\tilde{\iota}$-equivariant, which gives theorem 1.7.

More precisely, if we define the operator $\Sigma_\alpha : \mathcal{T}(\boldsymbol{\lambda}) \to \mathcal{T}(\mathfrak{s}_\alpha(\boldsymbol{\lambda}))$ by $\Sigma_\alpha(T) = (T_0, T_\alpha, T_{2\alpha}, \dots, T_{(d-1)\alpha})$, we have $\Sigma_\alpha \circ \mathfrak{s}_\alpha \circ \rho_{\boldsymbol{\lambda}} = \rho_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})} \circ \Sigma_\alpha$ where $\rho_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}$ is as in 5.2.

7.4. **The general case $G(de, e, r)$.** Using that the matrix S of subsection 2.4 is rational in our chosen basis of $V(\boldsymbol{\lambda})$, we get for $\omega \in \mu_{\#C_{\boldsymbol{\lambda}}}$:

$$\mathfrak{s}_\alpha(\chi_{\boldsymbol{\lambda},\omega}(g)) = \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \mathfrak{s}_\alpha(\omega)^{-i} \operatorname{Trace}(\mathfrak{s}_\alpha(\rho_{\boldsymbol{\lambda}}(g))S^i).$$

From this we get, if we write $S_{\boldsymbol{\lambda}}$ for S to keep track where it acts:

$$\mathfrak{s}_\alpha(\chi_{\boldsymbol{\lambda},\omega}(g)) = \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \omega^{-i\alpha} \operatorname{Trace}(\Sigma_\alpha^{-1}\rho_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}(g)\Sigma_\alpha S_{\boldsymbol{\lambda}}^i).$$

Now, it is easy to check that $\Sigma_\alpha S_{\boldsymbol{\lambda}} \Sigma_\alpha^{-1} = S^\alpha_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}$, so

$$\mathfrak{s}_\alpha(\chi_{\boldsymbol{\lambda},\omega}(g)) = \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \omega^{-i\alpha} \operatorname{Trace}(\rho_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}(g)S^{i\alpha}_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}) = \chi_{\mathfrak{s}_\alpha(\boldsymbol{\lambda}),\omega}(g).$$

While we have by a similar computation

$$\begin{aligned}\chi_{\boldsymbol{\lambda},\omega}(\tilde{\iota}(\mathfrak{s}_\alpha)(g)) &= \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \omega^{-i} \operatorname{Trace}(\mathfrak{s}_\alpha(\rho_{\boldsymbol{\lambda}}(g))S^i_{\boldsymbol{\lambda}}) \\ &= \frac{1}{\#C_{\boldsymbol{\lambda}}} \sum_{i=0}^{\#C_{\boldsymbol{\lambda}}-1} \omega^{-i} \operatorname{Trace}(\rho_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}(g)S^{i\alpha}_{\mathfrak{s}_\alpha(\boldsymbol{\lambda})}) \\ &= \chi_{\mathfrak{s}_\alpha(\boldsymbol{\lambda}),\mathfrak{s}_\alpha{}^{-1}(\omega)}(g).\end{aligned}$$

We need slightly more precise formulas that we will get using our model which simplifies Ariki's one. We claim that if $\theta = \zeta_e^b$ then

$$\mathfrak{s}_\alpha \circ \rho_{\boldsymbol{\lambda},\theta^i} = \rho_{\boldsymbol{\lambda},\theta^i} \circ \operatorname{Ad} t^{i(1-\alpha)} \circ \eta(\mathfrak{s}_\alpha) = \rho_{\boldsymbol{\lambda},\theta^i} \circ \operatorname{Ad} t^i \circ \eta(\mathfrak{s}_\alpha) \circ \operatorname{Ad} t^{-i}$$

indeed this formula is obvious for the generators t and s_i for $i \neq 1$. It is thus sufficient to check it for $s_1^{t^k}$; one uses the formulae 2.3 and the fact that $\rho_{\boldsymbol{\lambda},1}(s_1)$ is a rational matrix.

Let us define $\tilde{\iota}_{\chi_{\boldsymbol{\lambda},\theta^i}} : \Gamma \to \operatorname{Aut}(G)$ by $\tilde{\iota}_{\chi_{\boldsymbol{\lambda},\theta^i}}(\mathfrak{s}_\alpha) = \operatorname{Ad} t^i \circ \eta(\mathfrak{s}_\alpha) \circ \operatorname{Ad} t^{-i}$. The formula above becomes $\mathfrak{s}_\alpha \circ \rho_{\boldsymbol{\lambda},\theta^i} = \rho_{\boldsymbol{\lambda},\theta^i} \circ \tilde{\iota}_{\chi_{\boldsymbol{\lambda},\theta^i}}(\mathfrak{s}_\alpha)$, *i.e.* that $\rho_{\boldsymbol{\lambda},\theta^i}$ is $\tilde{\iota}_{\chi_{\boldsymbol{\lambda},\theta^i}}$-equivariant, which proves theorems 1.6 and 1.7. We note that the various morphisms $\tilde{\iota}_{\chi_{\boldsymbol{\lambda},\theta^i}}$ differ by conjugacy by an element of N. Indeed they are conjugate by a power of $\operatorname{Ad} t$ which is a generator of N. When $\gcd(e, r) = 1$ then $\overline{N}$ is trivial, and when $\gcd(e, r) = 2$ then $\overline{N}$ is of order 2 and it results that $\tilde{\iota}_{\chi_{\boldsymbol{\lambda},\theta^i}}$ does not depend on i: indeed α being prime to de is odd thus $1 - \alpha$ is even and $\operatorname{Ad} t^{i(1-\alpha)}$ is always inner, which takes care of the remarks in 1.4.

8. Galois descent

The existence of a model over K globally invariant by $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$ of a faithful representation of G, namely its reflection representation, enabled us in the $G(de, e, r)$ case to construct the morphism $\tilde{\iota} : \Gamma \to \mathrm{Aut}(G)$.

In this section, we show how by Galois descent, one can conversely try to construct a globally invariant model for any given irreducible representation — more precisely we will try to get a $\tilde{\iota}$-equivariant model. The possible obstruction to do so is an element of the Brauer group of K' that turns out to be always trivial except for the group G_{22}.

8.1. **Non-abelian Galois cohomology.** Recall from [Se] that, if H is a group acting on the left on a group Q, then a map $h \mapsto A_h$ from H to Q is a *cocycle* if $A_{st} = A_s s(A_t)$ for all $s, t \in H$. A particular cocycle, called the *zero cocycle*, is given by sending all elements of H to the trivial element of Q. The set of cocycles is denoted $Z^1(H, Q)$. Two cocycles $h \mapsto A_h$ and $h \mapsto B_h$ are said to be cohomologous if there exists $a \in Q$ such that $B_h = a^{-1} A_h h(a)$. This defines an equivalence relation on $Z^1(H, Q)$ whose set of equivalence classes is denoted $H^1(H, Q)$. The set of cocycles cohomologous to the zero cocycle is called the set of coboundaries and is denoted $B^1(H, Q)$. If Q is commutative these definitions coincide with those of ordinary group cohomology.

Application to 1.7. Let $\rho : G \to \mathrm{GL}(E)$ where E is a K'-vector space be an irreducible representation. Assume we choose a $\mathbb{Q}$-form $E = E_0 \otimes_{\mathbb{Q}} K'$. This defines an action of $\Gamma' = \mathrm{Gal}(K'/\mathbb{Q})$ on E and by 1.4 for $\gamma \in \Gamma'$ we have $\gamma \circ \rho \simeq \rho \circ \tilde{\iota}(\gamma)$. This means that that there exists $A_\gamma \in \mathrm{GL}(E)$ such that for any $g \in G$ we have $A_\gamma \gamma(\rho(g)) A_\gamma^{-1} = \rho(\tilde{\iota}(\gamma)(g))$. Since ρ is absolutely irreducible, by Schur's lemma, A_γ gives a well-defined element of $\mathrm{PGL}(E)$. It is immediate to check that $\gamma \mapsto A_\gamma$ is in a fact a cocycle, that is, an element of $Z^1(\Gamma', \mathrm{PGL}(E))$.

Assume now that ρ has a globally invariant model, of the form $g \mapsto a\rho(g)a^{-1}$; this means that there exists a map $\iota' : \Gamma' \to \mathrm{Aut}(G)$ such that $\gamma(a\rho(g)a^{-1}) = a\rho(\iota'(\gamma)(g))a^{-1}$, or in other terms that the model is ι'-equivariant for some $\iota' : \Gamma' \to \mathrm{Aut}(G)$. If $\iota' = \tilde{\iota}$ we get the equality in $\mathrm{PGL}(E)$ that $a^{-1}\gamma(a) = A_\gamma$, that is that $\{\gamma \mapsto A_\gamma\} \in B^1(\Gamma', \mathrm{PGL}(E))$. Thus the obstruction to the existence of $\tilde{\iota}$-equivariant model is an element of $H^1(\Gamma', \mathrm{PGL}(E))$.

In practice to apply this procedure we encounter another problem which turns out to involve also Galois cohomology: although we know that any ρ has a model over K' (even K), sometimes (*e.g.* when using the `CHEVIE` data) we are not given such a model but a model $E = E_0 \otimes_{K'} L$ over some Galois extension L of K'. Again the fact that for $\gamma \in \Pi = \mathrm{Gal}(L/K')$ we have $\gamma \circ \rho \simeq \rho$ means that we can find an intertwiner A_γ such that $A_\gamma \gamma(\rho(g)) A_\gamma^{-1} = \rho(g)$, which is an element of $Z^1(\Pi, \mathrm{PGL}(E))$; and again the existence of a model over K' is equivalent to this cocycle to be a coboundary.

We will now give an algorithm to check the vanishing of a cocycle of a Galois group into the projective linear group.

8.2. **Brauer groups.** In this subsection and the next one, $K_0 \subseteq K$ will denote an arbitrary Galois extension of number fields. Let $\Gamma = \mathrm{Gal}(K/K_0)$ and let $\mathrm{Br}(K/K_0) = H^2(\Gamma, K^\times)$ be the Brauer group. Let $E = E_0 \otimes_{K_0} K$ be a K_0-form of a finite dimensional K-vector space E. The short exact sequence $1 \to K^\times \to \mathrm{GL}(E) \to \mathrm{PGL}(E) \to 1$ gives rise to a Galois cohomology long exact sequence (of pointed sets; see [Se, prop. 2 p. 133]):

$$1 \to H^0(\Gamma, K^\times) \to H^0(\Gamma, \mathrm{GL}(E)) \to H^0(\Gamma, \mathrm{PGL}(E)) \to H^1(\Gamma, K^\times) \to \\ H^1(\Gamma, \mathrm{GL}(E)) \to H^1(\Gamma, \mathrm{PGL}(E)) \to H^2(\Gamma, K^\times)$$

and in particular a coboundary operator $H^1(\Gamma, \mathrm{PGL}(E)) \to H^2(\Gamma, K^\times)$. This map sends $\{\gamma \mapsto A_\gamma\} \in H^1(\Gamma, \mathrm{PGL}(E))$ to the class of $(\gamma, \tau) \mapsto \tilde{A}_\gamma \gamma(\tilde{A}_\tau)\tilde{A}_{\gamma\tau}^{-1}$ in $H^2(\Gamma, K^\times)$, where, for all $\gamma \in \Gamma$, $\tilde{A}_\gamma$ is a representative of A_γ in $\mathrm{GL}(E)$. This definition does not depend on the choice of the representatives $\tilde{A}_\gamma$ and this map is known to be injective (this uses the version of Hilbert's theorem 90 which says that $H^1(\Gamma, \mathrm{GL}(E))$ is trivial; see [Se] ch. X proposition 8 and proposition 9).

In the important case where K is a cyclic extension of degree n of K_0, assume $\Gamma = \langle\gamma\rangle$ and let $N : K \to K_0$ be the norm map $x \mapsto x\gamma(x)\dots\gamma^{n-1}(x)$. Then an explicit isomorphism from $H^2(\Gamma, K^\times)$ to the 0-th Tate cohomology group $\hat{H}^0(\Gamma, K^\times) = K_0^\times/NK^\times$ is given by the Nakayama map

$$c \mapsto \prod_{k=0}^{n-1} c(\gamma^k, \gamma)$$

hence $\{\gamma \mapsto A_\gamma\}$ is sent to the class of $\tilde{A}_\gamma\gamma(\tilde{A}_\gamma)\dots\gamma^{n-1}(\tilde{A}_\gamma)$ (a scalar matrix identified to an element of $K_0^\times$) in $K_0^\times/NK^\times$.

It follows that, when K is a cyclic extension of K_0, the vanishing of $c \in H^2(\Gamma, K^\times) = \mathrm{Br}(K/K_0)$ boils down to the verification that some element of $K_0^\times$ is a norm of an element of K. When K is an *abelian* extension of K_0, we can always reduce to this case by induction on $[K : K_0]$, by the inflation-restriction exact sequence for Brauer groups. Indeed, choosing a cyclic subgroup Γ' of Γ defines a Galois sub-extension $K_1 = K^{\Gamma'}$ of K such that $\mathrm{Gal}(K/K_1)$ is cyclic. We then have the exact sequence (see [Se] ch. X §4 proposition 6)

$$0 \to \mathrm{Br}(K_1/K_0) \to \mathrm{Br}(K/K_0) \to \mathrm{Br}(K/K_1)$$

and we may follow the following procedure : first check that the image in $\mathrm{Br}(K/K_1)$ is zero, and then start again with the induced element in $\mathrm{Br}(K_1/K_0)$, until K_1/K_0 itself becomes a cyclic extension.

We finally mention the following general result.

Proposition 8.1. *If* $\#\Gamma$ *is prime to* $\dim(E)$*, then* $H^1(\Gamma, \mathrm{PGL}(E)) = 0$

Proof. Let $N = \dim(E)$. Let $c \in H^1(\Gamma, \mathrm{PGL}(E))$ and let A be the central simple algebra associated to it. Let e be its exponent, that is the order of its class $[A]$ in the Brauer group of K_0. Since $A \otimes K \simeq M_N(K)$ we have $[A][K] = 0$ in the Brauer group hence $e|[K : K_0]$ and $e|\#\Gamma$ (see [Se] X §4 ex. 2). On the other hand, $A \simeq M_r(D)$ with D a skew field whose center is K_0 and $\dim_{K_0} A = N^2$. It follows that $N^2 = [A : K_0] = r^2[D : K_0]$. Now $[D : K_0] = m^2$ for some integer m (called the index of D). It is a classical fact that the exponent divides the index ([Se] X §5 ex. 3a), thus $e|N$, and $e = 1$ since N and $\#\Gamma$ are coprime. The conclusion follows from the fact that $H^1(\Gamma, \mathrm{PGL}(E))$ embeds in the Brauer group of K_0. □

Explicit computations. In practice (*e.g.* to get an explicit model over a smaller field of a representation), it is not enough to solve the problem whether a cocycle $\{\gamma \mapsto A_\gamma\} \in Z^1(\Gamma, \mathrm{PGL}(E))$ is a coboundary. We want an explicit element $M \in \mathrm{PGL}(E)$ such that $A_\gamma = M^{-1}\gamma(M)$ for all $\gamma \in \Gamma$. If we can get a preimage (B_γ) of (A_γ) in $Z^1(\Gamma, \mathrm{GL}(E))$, then by Hilbert's Theorem 90 this cocycle is a coboundary, meaning that there exist $\tilde{M} \in \mathrm{GL}(E)$ such that $B_\gamma = \tilde{M}^{-1}\gamma(\tilde{M})$ for all $\gamma \in \Gamma$. A rather constructive proof of this theorem (see e.g. [Se] ch. X proposition 3) goes as follows. For all $C \in \mathrm{End}(E)$, the expression

$$X = \sum_{\gamma \in \Gamma} B_\gamma \gamma(C)$$

satisfies $\gamma(X) = B_\gamma^{-1} X$. Because K has characteristic zero, general arguments imply that "many" $C \in K$ exist such that X is invertible. If this is the case, then $B_\gamma = X\gamma(X)^{-1}$ thus $\tilde{M} = X^{-1}$ is the desired solution. For instance, if $\mathrm{Gal}(K/K_0) = \{1, \gamma\}$ has order 2 with generator γ_0, then the condition on C is that $-C\gamma(C)$ is not an eigenvalue for $B_{\mathrm{Id}}^{-1} B_{\gamma_0}$.

In the case where K is a cyclic extension of K_0, the study of the previous section shows that we can always lift a coboundary to a cocycle $\{\gamma \mapsto B_\gamma\}$ for $\mathrm{GL}(E)$. Indeed, if $\Gamma = \langle\gamma\rangle$ we have seen that $\tilde{A}_\gamma \gamma(\tilde{A}_\gamma) \dots \gamma^{n-1}(\tilde{A}_\gamma) = N(\lambda)\mathrm{Id}$ for some $\lambda \in K^\times$ so that $B_\gamma = \lambda^{-1}\tilde{A}_\gamma$ belongs to $Z^1(\Gamma, \mathrm{GL}(E))$ (it is easy to check that the equation $\mathrm{Id} = \tilde{A}_\gamma \dots \gamma^{n-1}(\tilde{A}_\gamma)$ is necessary and sufficient for $\tilde{A}_{\gamma^i} = \tilde{A}_\gamma \dots \gamma^{i-1}(\tilde{A}_\gamma)$ to define a cocycle of $\langle\gamma\rangle$).

8.3. **An algorithm to check 1.7.** Let $\rho : G \to \mathrm{GL}(E)$ be an irreducible representation of G over the K'-vector space E for which we would like to get a $\tilde{\iota}$-equivariant model. When $K'/\mathbb{Q}$ is cyclic, we found an explicit necessary and sufficient condition for a class in $H^1(\Gamma', \mathrm{PGL}(E))$ to vanish and thus to give an equivariant model.

In general we know that $K'/\mathbb{Q}$ is an abelian extension. We will show how to reduce by induction to the case of a cyclic extension, by making explicit the inflation-restriction exact sequence in this case.

Choose a tower of (Galois) extensions $\mathbb{Q} = K_0 \subset K_1 \subset \cdots \subset K_m = K'$ such that K_r/K_{r-1} is a cyclic extension for all $1 \leq r \leq m$. We fix a $\mathbb{Q}$-form E_0 of E and a basis of E_0 so that we identify E_0 with $\mathbb{Q}^n$ and E with K'^n.

Assume that, for some $r \in [1, m]$ we have a model $\rho_r : G \to \mathrm{GL}(E)$ of ρ such that $\gamma \circ \rho = \rho \circ \iota(\gamma)$ for all $\gamma \in \mathrm{Gal}(K'/K_r)$. We will show how to decrease r.

Let $\{\tau \mapsto M_\tau\} \in Z^1(\mathrm{Gal}(K'/K_{r-1}), \mathrm{PGL}(E))$ intertwining $\tau \circ \rho_r$ and $\rho_r \circ \iota(\tau)$ for $\tau \in \mathrm{Gal}(K'/K_{r-1})$. For γ in the subgroup $\mathrm{Gal}(K'/K_r)$ we have $M_\gamma = 1$ by the induction hypothesis; we thus have $M_{\tau\sigma} = M_\tau$ for all $\sigma \in \mathrm{Gal}(K'/K_r)$ by the cocycle condition $M_{\tau\gamma} = M_\tau \tau(M_\gamma)$. Hence M_τ only depends on the class of τ in $\mathrm{Gal}(K_r/K_{r-1})$, and lies in $\mathrm{PGL}_n(K_r)$ because $M_\tau = M_{\tau\gamma} = M_{\gamma\tau} = M_\gamma \gamma(M_\tau) = \gamma(M_\tau)$ for all $\gamma \in \mathrm{Gal}(K'/K_r)$. It follow that $\{\tau \mapsto M_\tau\}$ defines an element of $Z^1(\mathrm{Gal}(K_r/K_{r-1}), \mathrm{PGL}_n(K_r))$; we are thus reduced to a cyclic case; assuming we can solve it we find $X \in \mathrm{GL}_n(K_r)$ such that $M_\tau = X\tau(X)^{-1}$ for all $\tau \in \mathrm{Gal}(K'/K_{r-1})$. Then $\rho_{r-1}(g) = X\rho_r(g)X^{-1}$ is a model of ρ such that $\gamma \circ \rho_{r-1} = \rho_{r-1} \circ \iota(\gamma)$ for all $\gamma \in \mathrm{Gal}(K'/K_{r-1})$.

We could successfully carry out this algorithm for all representations of exceptional groups for which we had a model. However, with the exception of the cyclic case, this procedure does not lead to necessary conditions. Note however the following observation :

Proposition 8.2. *Assume G to be an exceptional irreducible complex reflection group. Then any odd-dimensional irreducible representation has a globally invariant model over K'.*

Proof. A case-by-case analysis shows that $[K' : \mathbb{Q}]$ is always a power of 2. The result is then an immediate consequence of 8.1. □

9. Invariant models

We will now prove theorem 1.7 using the results of the previous sections and multiplicity one property of tensor products.

For two representations ρ_1, ρ_2 of a group G, we will denote $(\rho_1 \mid \rho_2)$ the dimension of the space of intertwiners of ρ_1 and ρ_2. This coincides with the usual scalar product of the corresponding characters.

We will make repeated use of the following lemma.

Lemma 9.1. *Let G be a finite group and K be a field of characteristic 0. Let ρ' be a finite-dimensional representation of G defined over K and let $\rho \in \mathrm{Irr}(G)$ be such that $(\rho' \mid \rho) = 1$. Let χ be the character of ρ. Assume that χ takes values in K. Then ρ admits a model over K and :*

- *if $\gamma \in \mathrm{Gal}(K/\mathbb{Q})$ and $a \in \mathrm{Aut}(G)$ satisfy $\gamma \circ \rho' = \rho' \circ a$ and $\gamma \circ \chi = \chi \circ a$, then ρ admits a model over K such that $\gamma \circ \rho = \rho \circ a$.*
- *if there exists $j : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Aut}(G)$ such that $\gamma \circ \rho' = \rho' \circ j(\gamma)$ and $\gamma \circ \chi = \chi \circ j(\gamma)$, then ρ admits a j-equivariant model over K.*

Proof. Since ρ is an isotypic component of a ρ' and has its character over K it is defined over K ; indeed, the G-equivariant projector

$$p = \frac{\chi(1)}{\#G} \sum_{g \in G} \chi(g^{-1}) \rho'(g)$$

on the ρ-isotypic component of ρ' belongs to $\mathrm{End}_K(V)$. Moreover, if $\gamma \circ \chi = \chi \circ a$, we have

$$\gamma(p) = \frac{\chi(1)}{\#G} \sum_{g \in G} \chi(a(g)^{-1}) \rho'(a(g)) = p$$

thus a $\mathbb{Q}$-form $V' = V'_0 \otimes_{\mathbb{Q}} K$ induces a $\mathbb{Q}$-form on $V = \mathrm{Im}\, p$. It follows that $\gamma \circ \rho = \rho \circ a$, which proves the last two points. □

9.1. **Reflection representations.** We now prove theorem 1.7 for the reflection representation of exceptional groups, that is the following proposition.

Proposition 9.2. *Let G be an exceptional irreducible complex reflection different from G_{22} given by $\rho : G \hookrightarrow \mathrm{GL}(V)$ where V is a K-vector space and K is the field of definition of G. Let K' be as in theorem 1.6. Then ρ has a $\tilde{\iota}_\rho$-equivariant model over K'.*

Recall that $K = K'$ except when $G = G_{27}$.

Models for reflection representations of 2-dimensional groups from the braid group. A way to obtain the 2-dimensional representations of the groups G_4 to G_{22} is by using the following matrices:

$$\mathbf{s} \mapsto \begin{pmatrix} x_1 & \frac{y_1+y_2}{y_1 y_2} - \frac{(z_1+z_2)x_2}{r} \\ 0 & x_2 \end{pmatrix}, \mathbf{t} \mapsto \begin{pmatrix} y_1+y_2 & 1/x_1 \\ -y_1 y_2 x_1 & 0 \end{pmatrix}, \mathbf{u} \mapsto \begin{pmatrix} 0 & \frac{-r}{y_1 y_2 x_1 x_2} \\ r & z_1+z_2 \end{pmatrix}$$

where $r = \sqrt{x_1 x_2 y_1 y_2 z_1 z_2}$. The braid groups of G_7, G_{11} and G_{19} are isomorphic to the same group B, with presentation $\langle \mathbf{s}, \mathbf{t}, \mathbf{u} \mid \mathbf{stu} = \mathbf{tus} = \mathbf{ust} \rangle$; the above matrices give the 2-dimensional representation of B where the eigenvalues of $\mathbf{s}$ (resp. $\mathbf{t}$, $\mathbf{u}$) are x_1, x_2 (resp. y_1, y_2, z_1, z_2). This representation factors through the Hecke algebra, the quotient of the group algebra of B by the relations $(\mathbf{s} - x_0)(\mathbf{s} - x_1) = 0$, $(\mathbf{t} - y_0)(\mathbf{t} - y_1)(\mathbf{t} - y_2) = 0$ and $\prod_{i=0}^{i=n-1} (\mathbf{u} - z_i) = 0$ where $n = 3$ (resp. 4,5) for G_7 (resp. G_{11}, G_{19}). In turn the group algebra of G_7 (resp. G_{11}, G_{19}) is the specialization of the Hecke algebra for $x_i \mapsto (-1)^i, y_i \mapsto \zeta_3^i, z_i \mapsto \zeta_3^i$ (resp. $z_i \mapsto \zeta_4^i$, $z_i \mapsto \zeta_5^i$); the Hecke algebras for G_4 to G_6 are subalgebras of partial specializations of that for G_7 (the same holds for G_8 to G_{15} with respect to G_{11} and G_{16} to G_{22} with respect to G_{19}): in each case, these algebras are generated by conjugates of a part of the generators (or of some power of them); the other generators are specialized to the group algebra.

For G_4, G_8 and G_{16} the Hecke algebra is generated by $\mathbf{u}$ and ${}^{\mathbf{s}}\mathbf{u}$.

For G_5, G_{10} and G_{18} it is generated by $\mathbf{t}$ and $\mathbf{u}$.

For G_6, G_9 and G_{17} it is generated by $\mathbf{s}$ and $\mathbf{u}$.
For G_{14} and G_{21} it is generated by $\mathbf{s}$ and $\mathbf{t}$.
For G_{12} and G_{22} it is generated by $\mathbf{s}, {}^{\mathbf{t}}\mathbf{s}$ and $\mathbf{s}^{\mathbf{t}}$.
For G_{20} it is generated by $\mathbf{t}$ and ${}^{\mathbf{s}}\mathbf{t}$.
For G_{13} it is generated by $\mathbf{u}^2, \mathbf{s}$ and $\mathbf{s}^{\mathbf{t}}$.
Finally for G_{15} it is generated by $\mathbf{s}, \mathbf{t}$ and $\mathbf{u}^2$.

Proof of proposition 9.2. The statement is clear if the rank of G is odd, by proposition 8.2. Then we may assume that, either $G = G_{34}$, or that its rank is 2 or 4. Recall that the groups of rank 2 are numbered $G_4, \ldots, G_{22}$.

We started from a model of the reflection representations coming either from the Hecke algebra as described above, or by root diagrams as considered in [Cohen]. Then we apply the algorithm of Galois descent; in order to solve norm equations we used the computer system `MAGMA`. All equations thus obtained for the reflection representation were solvable except for G_{22}.

Note that, even when the original model is globally invariant, the induced morphism $\eta : \Gamma \to \mathrm{Aut}(G)$ does not in general coincide with $\tilde{\iota}_\rho$. Furthermore, the induced action of η on the isomorphism classes of irreducible representations of G may also differ from the natural Galois action of Γ, as shows the second example below.

The models given in table 2 were simplified by the following elementary observation. For all the images $\tilde{\iota}(\Gamma)$ chosen here, it turns out that there is a generator s such that the subgroup H generated by s is stable under $\tilde{\iota}(\Gamma)$. Lemma 9.1 thus implies that there is a basis of the underlying $\mathbb{Q}$-form of K^2 on which the action of s is diagonal.

Table 2: Invariant models for G_4 to G_{21}

G_4	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{-3}} \begin{pmatrix} -1 & \zeta_3 \\ 2 & \zeta_3 \end{pmatrix}$
G_5	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{-3}} \begin{pmatrix} \zeta_3 & \zeta_3 \\ 2 & -1 \end{pmatrix}$
G_6	$s \mapsto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$
G_7	$s \mapsto \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad u \mapsto \frac{1}{\sqrt{-3}} \begin{pmatrix} \zeta_3 & \zeta_3 \\ 2 & -1 \end{pmatrix}$
G_8	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad t \mapsto \frac{\mathrm{i}-1}{2} \begin{pmatrix} -\mathrm{i} & 1 \\ 1 & -\mathrm{i} \end{pmatrix}$
G_9	$s \mapsto \frac{1}{2} \begin{pmatrix} \sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$

G_{10}	$s \mapsto \frac{1}{\zeta_3(\mathrm{i}-1)}\begin{pmatrix} -\mathrm{i} & 2\mathrm{i} \\ \frac{1}{2} & 1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$
G_{11}	$s \mapsto C = \frac{1}{\sqrt{6}}\begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad u \mapsto \frac{\zeta_3}{(\mathrm{i}+1)\sqrt{3}}\begin{pmatrix} -2\zeta_3 & \zeta_3 \\ 2 & 2 \end{pmatrix}$
G_{12}	$s \mapsto D = \frac{1}{2}\begin{pmatrix} 1 & 1+\frac{1}{\sqrt{-2}} \\ 2+\sqrt{-2} & -1 \end{pmatrix}, \quad t \mapsto \overline{D}, \quad u \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
G_{13}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad u \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}$
G_{14}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \mapsto \frac{\zeta_3^2}{2}\begin{pmatrix} -1+\sqrt{-2} & 1 \\ -1 & -1-\sqrt{-2} \end{pmatrix}$
G_{15}	$s \mapsto C, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad u \mapsto \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \zeta_3^2 \\ 2\zeta_3 & -1 \end{pmatrix}$
G_{16}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}, \quad t \mapsto \frac{1}{\sqrt{5}}\begin{pmatrix} 1-\zeta_5^3 & \zeta_5^4-1 \\ \zeta_5-\zeta_5^2 & \zeta_5-\zeta_5^3 \end{pmatrix}$
G_{17}	$s \mapsto M = \frac{\mathrm{i}}{\sqrt{5}}\begin{pmatrix} \zeta_5^4-\zeta_5 & \zeta_5^3-\zeta_5^2 \\ \zeta_5^3-\zeta_5^2 & \zeta_5-\zeta_5^4 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}$
G_{18}	$s \mapsto N = \frac{\zeta_3^2}{\sqrt{5}}\begin{pmatrix} \zeta_5^2-\zeta_5^4 & 1-\zeta_5^4 \\ \zeta_5-1 & \zeta_5^3-\zeta_5 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}$
G_{19}	$s \mapsto N, \quad t \mapsto M, \quad u \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_5 \end{pmatrix}$
G_{20}	$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}, \quad t \mapsto \frac{\zeta_3^2}{2\sqrt{-15}}\begin{pmatrix} -5-\sqrt{-15} & 2 \\ 10 & 5-\sqrt{-15} \end{pmatrix}$
G_{21}	$s \mapsto \frac{1}{2\sqrt{3}}\begin{pmatrix} 1+\sqrt{5} & -1+\frac{1}{\sqrt{5}} \\ -5+\sqrt{5} & -1-\sqrt{5} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$

Example : the case of G_9. Let us start with the model

$$s \mapsto \begin{pmatrix} 0 & -\zeta_{24}^{11} \\ \zeta_{24} & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \mathrm{i} & -\zeta_3 \\ 0 & 1 \end{pmatrix}$$

which is defined over $\mathbb{Q}(\zeta_{24})$, an extension of degree 2 of $K = \mathbb{Q}(\zeta_8)$. By Galois descent or directly one finds that conjugating by $\begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix}$ yields the following model over K.

$$s \mapsto \begin{pmatrix} 0 & -\zeta_8 \\ \zeta_8^3 & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \mathrm{i} & -1 \\ 0 & 1 \end{pmatrix}$$

We have $\Gamma \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, with three subgroups of order 2 generated by $\mathfrak{s}_{-1}, \mathfrak{s}_5$ and $\mathfrak{s}_3$. To these three subgroups correspond three intermediate extensions between $\mathbb{Q}$ and K. For each of these extensions, we get a cocycle $\mathfrak{s}_k \mapsto A_k \in GL_2(K)$, given by the following matrices

$$A_{-1} = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} -1 & -1-\mathrm{i} \\ 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \mathrm{i} & -1+\mathrm{i} \\ 0 & 1 \end{pmatrix}$$

It turns out that the last step of the algorithm is easy, because these three cocycles match, so we get in fact an element of $Z^1(\Gamma, \mathrm{GL}_2(K))$. In order to apply Hilbert Theorem 90 we need to find $\lambda \in K$ such that $\lambda + \mathfrak{s}_{-1}(\lambda)A_1 + \mathfrak{s}_5(\lambda)A_2 + \mathfrak{s}_3(\lambda)A_3$ is invertible ; it happens that $\lambda = 1 + \zeta_8 + \zeta_8^3$ is a solution that gives the following invariant model

$$s \mapsto \frac{1}{2}\begin{pmatrix} -1-\zeta_8+\zeta_8^3 & 1/2-\zeta_8+\zeta_8^3 \\ 2 & 1+\zeta_8-\zeta_8^3 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} \mathrm{i} & \frac{\mathrm{i}-1}{2} \\ 0 & 1 \end{pmatrix}$$

Finally, we know that the subgroup generated by t is stable by $\tilde{\iota}(\Gamma)$. It is thus possible to diagonalize it by a rational matrix. We get the still invariant and simpler model

$$s \mapsto \frac{1}{2}\begin{pmatrix} \sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$$

An example for G_7. The model ρ of the reflection representation of G_7 given by

$$s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad t \mapsto \frac{\zeta_3^2}{(1+\mathrm{i})}\begin{pmatrix} -1 & -1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}, \quad u \mapsto \frac{\zeta_3^2}{(1+\mathrm{i})}\begin{pmatrix} -1 & \mathrm{i} \\ -1 & -\mathrm{i} \end{pmatrix}$$

is globally invariant, giving rise to an homomorphism $\eta : \mathrm{Gal}(\mathbb{Q}(\zeta_{12})/\mathbb{Q}) \to \mathrm{Aut}(G_7)$ given by (specifying automorphisms, as in table 1, by the images of s, t, u) $\mathfrak{c} \mapsto (s, u^{-1}, t^{-1}), \mathfrak{s}_7 \mapsto (s, s^{us}, u^{ts})$. However, the automorphism $\eta(\mathfrak{c})$ is not compatible with the action of the complex conjugation on other characters of G, contrary to the homomorphism $\tilde{\iota}$ given in table 1, which corresponds to the model given in table 2. The existence of this example is related to the fact that G_7 is the only exceptional group such that the image $\overline{\iota}(\Gamma)$ does not commute with N.

9.2. **Proof of 1.7 for exceptional groups.** In this section we will deduce the existence of an $\tilde{\iota}$-equivariant model for every irreducible representation of G (outside the exceptional set of theorem 1.4) from the existence of such a model for the reflection representation ρ_0, by applying lemma 9.1. We will need the following proposition.

Proposition 9.3. *If G is of type $G(d,1,r)$ then, for all distinct $\rho_1, \rho_2 \in \mathrm{Irr}(G)$, we have $(\rho_1 \otimes \rho_0 \mid \rho_2) \leq 1$. This also holds for exceptional groups, except for G_{27}, G_{29}, G_{34}, and $G_{36} = E_7$. For these 4 exceptional groups, we have $(\rho_1 \otimes \rho_0 \mid \rho_2) \leq 2$ for distinct $\rho_1, \rho_2 \in \mathrm{Irr}(G)$.*

Proof. The proof for the exceptional groups is a case-by-case computer check. If G is of type $G(d,1,r)$ with $r \geq 2$, let H denote its natural parabolic subgroup of type $G(d,1,r-1)$, and $\rho_1 \in \mathrm{Irr}(G)$. Then $(\mathrm{Ind}_H^G 1_G \mid \rho_0) = (1_G \mid \mathrm{Res}_H \rho_0) = 1$ because H is a maximal parabolic subgroup. It follows that $\rho_1 \otimes \rho_0$ embeds in $\rho_1 \otimes \mathrm{Ind}_H^G 1_G \simeq \mathrm{Ind}_H^G \mathrm{Res}_H \rho_1$. In particular, for $\rho_2 \in \mathrm{Irr}(G)$ we have

$$(\rho_1 \otimes \rho_0 \mid \rho_2) \leqslant (\mathrm{Ind}_H^G \mathrm{Res}_H \rho_1 \mid \rho_2) = (\mathrm{Res}_H \rho_1 \mid \mathrm{Res}_H \rho_2)$$

so it is sufficient to check that, if $\rho_1 \neq \rho_2$, then $(\mathrm{Res}_H \rho_1 \mid \mathrm{Res}_H \rho_2) \leq 1$. This is obvious considering the branching rule (see [Ze, p. 104]). □

Proof of 1.7 for $G(d,1,r)$ and exceptional groups distinct from G_{27}, G_{29} or G_{34}. We already proved that there exists $\tilde{\iota} : \Gamma \to \mathrm{Aut}(G)$ such that

(i) $\forall \rho \in \mathrm{Irr}(G)$, $\gamma \circ \rho \simeq \rho \circ \bar{\iota}(\gamma)$.
(ii) ρ_0 admits a model such that $\gamma \circ \rho_0 = \rho_0 \circ \tilde{\iota}(\gamma)$.

We want to show that all $\rho \in \mathrm{Irr}(G)$ admit an $\tilde{\iota}$-equivariant model over K, that is a model over K such that $\gamma \circ \rho = \rho \circ \tilde{\iota}(\gamma)$ for all $\gamma \in \Gamma$. This obviously holds if G is a Weyl group, so we may assume that G is not $G_{36} = E_7$. First note that every $\rho \in \mathrm{Irr}(G)$ embeds in some $\rho_0^{\otimes n}$ for some n, because ρ_0 is a faithful representation of G (*cf.* [FH, problem 2.37]). It follows that we can define the *level* $N(\rho) \in \mathbb{N}$ of ρ by

$$N(\rho) = \min\{n \in \mathbb{N} \ \mid \ \rho \hookrightarrow \rho_0^{\otimes n}\}.$$

In particular, $N(\rho) = m+1$ implies that there exists $\rho' \in \mathrm{Irr}(G)$ with $N(\rho') = m$ such that ρ embeds in $\rho' \otimes \rho_0$. By (i) we know that the representations of level at most 1, that is the trivial representation and ρ_0, admit an equivariant model. We proceed by induction on the level. Assume that all representations of level m admit an equivariant model, and let $\rho \in \mathrm{Irr}(G)$ such that $N(\rho) = m+1$. Let $\rho' \in \mathrm{Irr}(G)$ of level m such that $\rho \hookrightarrow \rho' \otimes \rho_0$. By proposition 9.3 we have $(\rho \mid \rho' \otimes \rho_0) = 1$. Moreover $\rho' \otimes \rho_0$ admits an equivariant model over K because ρ' and ρ_0 do. Then lemma 9.1 implies that ρ admits an equivariant model over K and we conclude by induction on the level.

Proof of 1.7 for G_{27}, G_{29}, G_{34}. Let $\mathcal{E} \subset \mathrm{Irr}(G)$ denote the set of exceptional (rational) representations described in the introduction. We note that $1_G, \rho_0 \notin \mathcal{E}$. Let us consider the following algorithm.

Algorithm

(i) $L \leftarrow \{1_G, \rho_0\}$.
(ii) For all $\rho' \in L$, $L' \leftarrow L \cup \{\rho \in \mathrm{Irr}(G) \setminus \mathcal{E} \ \mid \ (\rho' \otimes \rho_0 \mid \rho) = 1\}$.
(iii) If $\#L' > \#L$, then $L \leftarrow L'$ and go to (ii).
(iv) Return L.

The same arguments as above, based on lemma 9.1, show that all representations in the subset L of $\mathrm{Irr}(G) \setminus \mathcal{E}$ returned by this algorithm admit an equivariant model over K. Note that this algorithm only uses the character table of G. It is then enough to check that it returns $\mathrm{Irr}(G) \setminus \mathcal{E}$ for G_{27}, G_{29} and G_{34} in order to conclude the proof of the theorem. Indeed, this is the case if one applies step (ii) five times for G_{27} and G_{34}, and four times for G_{29}.

9.3. **The exceptional case of G_{22}.** In this case we have $K = K' = \mathbb{Q}(\mathrm{i}, \sqrt{5})$. We start from the following model $\rho : G \hookrightarrow \mathrm{GL}(V)$ over the degree 2 extension $K'' = \mathbb{Q}(\zeta_{20})$ of K.

$$s \mapsto \frac{\zeta_5^2 - \zeta_5}{\zeta_{20}\sqrt{5}} \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} & -1 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & \zeta_{20}^9 \\ -\zeta_{20} & 0 \end{pmatrix}, \quad u \mapsto \begin{pmatrix} 0 & -\zeta_{20} \\ \zeta_{20}^9 & 0 \end{pmatrix}$$

This model is $\mathrm{Gal}(K''/\mathbb{Q})$-globally invariant. Note that the projection $\mathrm{Gal}(K''/\mathbb{Q}) \to \mathrm{Gal}(K/\mathbb{Q})$ is not split. We check this by noting that $\mathrm{Gal}(K''/\mathbb{Q}) \simeq \mathbb{Z}/4 \times \mathbb{Z}/2$ admits only one subgroup isomorphic to $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$, namely $\mathrm{Gal}(K''/\mathbb{Q}(\sqrt{5}))$, and this subgroup does not surject on $\mathrm{Gal}(K/\mathbb{Q})$, because $\sqrt{5} \in K$.

As predicted by the theorem of Benard and Bessis, we manage to find a model ρ_0 over K by Galois descent. It is possible to get a model over K such that $\mathfrak{s}_{-1} \circ \rho = \rho \circ \tilde{\iota}(\mathfrak{s}_{-1})$ *or* such that $\mathfrak{s}_7 \circ \rho = \rho \circ \tilde{\iota}(\mathfrak{s}_7)$, but not both. The reason is that it is not possible to get a model such that $\mathfrak{s}_{13} \circ \rho = \rho \circ \tilde{\iota}(\mathfrak{s}_{13})$.

Lemma 9.4. *ρ does not admit any model over K which is globally invariant by* $\mathrm{Gal}(K/\mathbb{Q}(\mathrm{i}))$.

Proof. We assume by contradiction that we are given such a model ρ_1. Since ρ_1 is faithful there exists $a \in \mathrm{Aut}(G)$ of order 2 such that $\rho_1 \circ a = \mathfrak{s}_{13} \circ \rho_1$. We will see in table 3 that G has no non-trivial central automorphisms, thus all automorphisms of G preserve the set of reflections, thus by theorem 5.1 $\mathrm{Out}(G)$ acts faithfully and transitively on the set of reflection representations. It follows that a and $\tilde{\iota}(\mathfrak{s}_{13})$ have the same image in $\mathrm{Out}(G)$, namely $\overline{\iota}(\mathfrak{s}_{13})$. In particular $a = \tilde{\iota}(\mathfrak{s}_{13}) \circ \mathrm{Ad}(h)$ for some $h \in G$. Exhausting all possibilities, we find using ρ_0 that the image of the cocycle defined by a in $H^1(\mathrm{Gal}(K/\mathbb{Q}(\mathrm{i})), \mathrm{PGL}_2(K)) \simeq K^\times/N(K^\times)$ is always the class of $1/3$ modulo $N(K^\times)$. Using `MAGMA` we check that this class is non trivial, thus leading to a contradiction. □

We now investigate which representations of G admit a globally invariant model over K. This group has 18 irreducible representations, including two 1-dimensional ones. The four 2-dimensional (reflection) representations are deduced from each other through Galois action, so none of them admits a globally invariant model over K. In addition to these, there are 2 faithful 4-dimensional representations and 2 faithful 6-dimensional representations.

The center of G_{22} is cyclic of order 4, generated by $Z = (stu)^5$. We have $Z^2 = (ts)^5$. The other representations have for kernel a non-zero subgroup of $Z(G)$.

The 8 odd-dimensional representations admit $\tilde{\iota}$-equivariant models over K by proposition 8.1, using the argument of proposition 8.2.

A computer check shows that every non-faithful irreducible representation appear with multiplicity 1 in some tensor product $\rho_1 \otimes \rho_2$, where ρ_1 and ρ_2 are odd-dimensional irreducible representations. It follows that all these admit $\tilde{\iota}$-equivariant models.

We now study in some detail the fate of the remaining (faithful) irreducible representations. In particular, we prove the following.

Proposition 9.5. *Let ρ be one of the faithful 4-dimensional or 6-dimensional representations of G_{22}. Then ρ admit globally invariant models over K, which induce morphisms $j_\rho : \Gamma \to \mathrm{Aut}(G)$. These morphisms can be chosen injective, however the induced morphisms $\overline{j_\rho} : \Gamma \to \mathrm{Out}(G)$ are never injective.*

In both dimensions, one of the representations considered here satisfy $\rho(Z) = -\mathrm{i}$. It is sufficient to consider these to prove the proposition, since they are conjugated in pairs by the Galois action. We let $N = \dim \rho \in \{4, 6\}$.

Since ρ is faithful, the existence of a globally invariant model for ρ leads to a morphism $j : \Gamma \to \mathrm{Aut}(G)$ satisfying $\gamma \circ \rho \simeq \rho \circ j(\gamma)$ for all $\gamma \in \Gamma$, such that the associated cocycle $J = \{\gamma \mapsto J_\gamma\}$ is cohomologically trivial.

We use the following procedure to check the proposition for an arbitrary $j : \Gamma \to \mathrm{Aut}(G)$ satisfying $\gamma \circ \rho \simeq \rho \circ j(\gamma)$ for all $\gamma \in \Gamma$. Let $\gamma_0 = \mathfrak{s}_{13}$ and note that $K^{\gamma_0} = \mathbb{Q}(\mathrm{i})$. Recall that γ_0 and $\mathfrak{c}$ generate $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mathrm{i}, \sqrt{5})/\mathbb{Q})$. Restricting J to $\langle \gamma_0 \rangle$ yields a class in $\mathbb{Q}(\mathrm{i})^\times / NK^\times$ where $N(x) = x\gamma_0(x)$. If this class is non-zero then J cannot be cohomologically trivial; otherwise there exists $M \in \mathrm{GL}_N(K)$, which can be explicitly determined, such that $J_{\gamma_0} = M^{-1}\gamma_0(M)$ in $\mathrm{PGL}_N(K)$. Then the cocycle $\gamma \mapsto Q_\gamma = MJ_\gamma\gamma(M)^{-1}$ is cohomologous to J and $Q_{\gamma_0} = 1$. Since

$$\gamma_0(Q_{\mathfrak{c}}) = Q_{\gamma_0}\gamma_0(Q_{\mathfrak{c}}) = Q_{\gamma_0\mathfrak{c}} = Q_{\mathfrak{c}\gamma_0} = Q_{\mathfrak{c}}\mathfrak{c}(Q_{\gamma_0}) = Q_{\mathfrak{c}}$$

it follows that $\gamma_0(Q_{\mathfrak{c}}) = Q_{\mathfrak{c}}$, i.e. $Q_{\mathfrak{c}} \in \mathrm{PGL}_N(K)^{\gamma_0} = \mathrm{PGL}_N(K^{\gamma_0})$, and that $Q_{\gamma_0\mathfrak{c}} = Q_{\mathfrak{c}}$. In other words, Q belongs to the image of the inflation map $Z^1(\mathrm{Gal}(K^{\gamma_0}/\mathbb{Q}), \mathrm{PGL}_N(K^{\gamma_0})) \to Z^1(\Gamma, PGL_N(K))$. Since the inflation map between Brauer groups is injective this map is also injective and we are again reduced to a cyclic case, hence to a norm equation solvable with `MAGMA`.

Using `CHEVIE` we get all such morphisms. The list obtained does not depend on whether ρ has dimension 4 or 6. If $N = 4$, the morphisms j such that J is cohomologous to 0 all satisfy $\overline{j(\mathfrak{c})} = \overline{j(\mathfrak{s}_7)} = \overline{\iota}(\mathfrak{s}_7)$ and $\overline{j(\mathfrak{s}_{13})} = \overline{\iota}(1)$. Some of these morphisms j are injective.

In case $N = 6$, the morphisms j such that J is cohomologically trivial are all injective, and again send $\mathfrak{s}_{13}$ to an inner automorphism. Moreover $\overline{j(\mathfrak{c})} \in \{\overline{\iota}(\mathfrak{c}), \overline{\iota}(\mathfrak{s}_7)\}$ and the two possibilities occur.

The choice of any morphism such that J is cohomologically trivial leads to a globally invariant model for the representations. One of these morphisms, such that J is cohomologically trivial for both representations, has the following simple form : $j(\mathfrak{s}_{13}) = \mathrm{Ad}(t)$ and $j(\mathfrak{c})$ sends (s, t, u) to (u^{tu}, t, s^{tst}).

Remark 9.6.

Using the results proved here, one gets a simpler proof of the theorem of Benard and Bessis. First consider non-dihedral $G(de, e, r)$. Since all irreducible representations of the groups $G(de, e, r)$ appear as multiplicity 1 component of some representation of $G(de, 1, r)$, we are reduced by lemma 9.1 to the case of $G(d, 1, r)$, then to its reflection representation by proposition 9.3, and we know that this representation is defined over K by the very definition of these groups, or by [Bens, 7.1.1] which apply to all cases. The case of the dihedral groups is classical (their representations are mostly reflection representations). Most exceptional groups are then dealt with by proposition 9.3. Finally, the algorithm used here for the few remaining ones completes the proof, and is more efficient than the one used in [Bes].

10. On the morphisms $\overline{\iota}_\chi$

We first show how to derive corollary 1.5, stated for arbitrary reflection groups, from theorem 1.4. It is an obvious consequence of the following proposition.

Proposition 10.1. *Let G be any finite complex reflection group, let K be its field of definition and let $\Gamma = \mathrm{Gal}(K/\mathbb{Q})$. Let $\mathcal{S}$ be the set of irreducible characters of G which cannot be realized over $\mathbb{Q}$. Then for any $\chi \in \mathcal{S}$ there exists an injection $\Gamma \xrightarrow{\overline{\iota}_\chi} \mathrm{Out}(G)$ such that for any $\gamma \in \Gamma$ $\gamma(\chi) = \chi \circ \overline{\iota}_\chi(\gamma)$.*

Proof of proposition 10.1. Proposition 10.1 is an obvious consequence of 1.4 when G is irreducible. For a general reflection group G, we have a decomposition into irreducible groups: $V = V_1 \oplus \ldots \oplus V_n$, $G = G_1 \times \ldots \times G_n$ where $G_i \subset \mathrm{GL}(V_i)$. If K_i is the field of definition of G_i then K is the subfield of $\mathbb{C}$ generated by $K_1, \ldots, K_n$, whence if $\Gamma_i = \mathrm{Gal}(K_i/\mathbb{Q})$ we have natural quotient morphisms $\Gamma \to \Gamma_i$ such that the product map $\Gamma \to \prod_i \Gamma_i$ is injective. We deduce an injective map $\Gamma \to \mathrm{Out}(G)$ by composing $\Gamma \to \prod_i \Gamma_i$ with the individual maps $\Gamma_i \to \mathrm{Out}(G_i)$ deduced from 1.4 and then with the natural injection $\mathrm{Out}(G_1) \times \ldots \mathrm{Out}(G_n) \to \mathrm{Out}(G)$. Since any irreducible character χ of G is a product of irreducible characters of the G_i, proposition 10.1 follows readily. $\square$

We now study further the structure of $\mathrm{Out}(G)$, w.r.t. the morphisms $\overline{\iota}_V : \Gamma \to \mathrm{Out}(G)$. We now assume that G is irreducible. Recall that, when $G \neq \mathfrak{S}_6$, we have $\mathrm{Out}(G) = \overline{C}.\overline{A}$, as a direct consequence of theorem 1.1,

and $\overline{A} = \overline{N} \rtimes \Gamma$. Moreover it is clear that $\overline{N}$ normalizes $\overline{C}$, and centralizes $\overline{C}$ when $\overline{N}$ acts trivially on the linear characters.

The next lemma is the cornerstone of the interaction between $\overline{\iota}_V(\Gamma)$ and $\overline{C}$.

Lemma 10.2. *Let G be an irreducible complex reflection group. Then $\chi(\overline{\iota}_V(\gamma)(g)) = \gamma(\chi(g))$ for all linear characters χ and all $\gamma \in \Gamma$.*

Proof. The lemma results from the fact that $\overline{\iota}_\chi$ can be chosen to be the same for χ_V and for the linear characters. This is clear from theorem 1.4 since either $\overline{\iota}_\chi$ can be chosen independent of χ, or we are in a $G(de, e, r)$, and the action of $\overline{N}$ is trivial on both V and the linear characters, since they are the restriction from $G(de, 1, r)$ of a character which does not split. □

Proposition 10.3. *$\overline{\iota}_V(\Gamma) \subset \overline{A}$ commutes with $\overline{C}$. In particular $\overline{A}$ commutes with $\overline{C}$ as soon as $\overline{N} = \{1\}$.*

Proof. By theorems 1.6 and 1.7 there exists an extension K'' of K, a $\mathbb{Q}$-form of $V \otimes_K K''$ and a map $\iota : \mathrm{Gal}(K''/\mathbb{Q}) \to \mathrm{Aut}(G)$ such that for $g \in G \subset \mathrm{GL}(V)$ and $\gamma \in \mathrm{Gal}(K''/\mathbb{Q})$ we have $\gamma(g) = \iota(\gamma)(g)$, which induce $\overline{\iota}_V : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Out}(G)$, except possibly for G_{22}. In the case of G_{22}, from the globally invariant model over K'' we get a morphism $\iota : \mathrm{Gal}(K''/\mathbb{Q}) \to \mathrm{Aut}(G)$. The induced morphism j to $\mathrm{Out}(G)$ factors through $\mathrm{Gal}(K/\mathbb{Q})$, as $\gamma \circ \chi_V = \chi_V$ for $\gamma \in \mathrm{Gal}(K''/K)$, and $\overline{N} = 1$ in this case. It then coincides with $\overline{\iota}_V : \mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{Out}(G)$, as $\chi_V \circ j(\gamma) = \chi_V \circ \overline{\iota}_V(\gamma)$ for all $\gamma \in \Gamma$, hence $j(\gamma)\overline{\iota}_V(\gamma)^{-1} \in \overline{N} = \{1\}$.

Let $a_\chi \in C$ a central automorphism defined by $a_\chi(g) = g\chi(g)^{-1}$. It follows that $\iota(\gamma) \circ \alpha_\chi \circ \iota(\gamma)^{-1}(g) = \iota(\gamma)(\iota(\gamma)^{-1}(g)\chi(\iota(\gamma)^{-1}(g))) = g\iota(\gamma)(\chi(\iota(\gamma)^{-1}(g)))$. Now $\chi \circ \iota(\gamma)^{-1} = \chi \circ \overline{\iota}_V(\gamma)^{-1} = \gamma^{-1} \circ \chi$ by lemma 10.2, and $\iota(\gamma)(\chi(\iota(\gamma)^{-1}(g))) = \gamma(\chi(\iota(\gamma)^{-1}(g)))$ as $\chi(\iota(\gamma)^{-1}(g)) \in ZG \subset G$. It follows that $\iota(\gamma)(\chi(\iota(\gamma)^{-1}(g))) = \chi(g)$ hence $\iota(\gamma) \circ \alpha_\chi \circ \iota(\gamma)^{-1}(g) = \alpha_\chi(g)$ whence $\overline{\iota}_V(\Gamma)$ commutes to C. □

11. Invariants

We prove corollary 1.8, whose statement we recall. We denote by V^* the dual of a vector space V.

Proposition 11.1. *Let $G \subset \mathrm{GL}(V)$ be an irreducible complex reflection group where V is a K''-vector space, with K'' as in the introduction. There is a $\mathbb{Q}$-form $V = V_0 \otimes_{\mathbb{Q}} K''$ such that the fundamental invariants of G can be taken rational, i.e. in the symmetric algebra $S(V_0^*)$.*

Proof. By theorem 1.7 and the matrix model over K'' exhibited for G_{22} in section 9.3, we may assume that $G \subset \mathrm{GL}(V_0 \otimes_{\mathbb{Q}} K'')$ is globally invariant by $\Gamma = \mathrm{Gal}(K''/\mathbb{Q})$. Let $f_1, \ldots, f_r$ (where $r = \dim V$) be fundamental invariants, *i.e.* algebraically independent polynomials such that $S(V^*)^G = K''[f_1, \ldots, f_n]$. We want to find algebraically independent $g_1, \ldots, g_r \in S(V_0^*)$ such that we still have $S(V^*)^G = K''[g_1, \ldots, g_n]$. Our strategy will

be as follows: we will set $g_i = \sum_{\gamma\in\Gamma}\gamma(\lambda f_i)$ where $\lambda \in K''$. Then the g_i are still invariant since for $g \in G$ we have $g(g_i) = \sum_{\gamma\in\Gamma}\gamma(\lambda\gamma^{-1}(g)f_i) = \sum_{\gamma\in\Gamma}\gamma(\lambda f_i)$, the last equality since $\gamma^{-1}(g) \in G$.

It is thus sufficient to show that we may choose λ such that g_i are still algebraically independent. By the Jacobian criterion, it is enough to show that we may choose λ such that, if $x_1, \dots, x_r$ is a basis of V_0^*, so that $S(V^*) \simeq K''[x_1, \dots, x_r]$, we have $\det\left(\frac{\partial g_i}{\partial x_j}\right)_{i,j} \neq 0$.

We use the following version of the algebraic independence of automorphisms:

Proposition 11.2. [Bbk1, Chap. V, §10, Théorème 4]. *Let Q be an infinite field, let K be a finite Galois extension of Q with Galois group Γ, let Ω be an arbitrary extension of K and let $\{X_\gamma\}_{\gamma\in\Gamma}$ be indeterminates indexed by the elements of Γ. Let $F \in \Omega[X_\gamma]_{\gamma\in\Gamma}$ be a polynomial such that $F((\gamma(x))_{\gamma\in\Gamma}) = 0$ for any $x \in K$. Then $F = 0$.*

We apply the proposition with $Q = \mathbb{Q}$, $\Omega = K''$ and $F = \det\left(\sum_{\gamma\in\Gamma} X_\gamma\gamma(\frac{\partial f_i}{\partial x_j})\right)_{i,j}$. The polynomial F evaluated at $X_1 = 1$ and $X_\gamma = 0$ for $\gamma \neq 1$ is equal to $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$ which is non-zero, so F is non-zero. By the theorem, there exists $\lambda \in K''$ such that $F((\gamma(\lambda))_\gamma) \neq 0$. But $F((\gamma(\lambda))_\gamma) = \det\left(\frac{\partial g_i}{\partial x_j}\right)_{i,j}$ □

We now prove 1.9 whose statement we recall.

Proposition 11.3. *The morphism $V^{\mathrm{reg}} \to V^{\mathrm{reg}}/G$ is defined over $\mathbb{Q}$.*

Proof. We use the notations of the previous proof, in particular we assume that $G \subset \mathrm{GL}(V_0 \otimes_{\mathbb{Q}} K'')$ is globally invariant by $\Gamma = \mathrm{Gal}(K''/\mathbb{Q})$.

Let $\mathcal{H}$ be the set of reflecting hyperplanes for G and for each $H \in \mathcal{H}$, let l_H be a linear form defining H. For $H \in \mathcal{H}$ let e_H be the order of the subgroup of G fixing H, and let $\Delta = \prod_{H\in\mathcal{H}} l_H^{e_H}$. It is well known that $\Delta \in S(V^*)^G$ (see *e.g.* [OT, 6.44]). Thus the variety V^{reg}/G is the open subvariety of $\mathrm{Spec}(K''[f_1, \dots, f_n])$ whose function ring is the localization by the principal ideal Δ. It is thus enough to show that we may choose the f_i such that a multiple of Δ belongs to $\mathbb{Q}[f_1, \dots, f_n]$. If, as in the previous proposition, we choose $f_i \in S(V_0^*)^G$, it will be enough to show that we have a multiple of Δ in $S(V_0^*)$ since $S(V_0^*) \cap S(V^*)^G = S(V_0^*)^G$. Since $\mathcal{H}$ is globally invariant by Γ, for any $\gamma \in \Gamma$ there exists $\lambda_\gamma \in K''$ such that $\gamma(\Delta) = \lambda_\gamma\Delta$, and it is clear that $\{\gamma \mapsto \lambda_\gamma\} \in Z^1(\Gamma, K''^\times)$. By Hilbert's theorem 90 we have that $H^1(\Gamma, K''^\times)$ is trivial, thus $\{\gamma \mapsto \lambda_\gamma\}$ is a coboundary, *i.e.* there exists $\lambda \in K''^\times$ such that $\lambda_\gamma = \lambda^{-1}\gamma(\lambda)$. We get then that $\gamma(\lambda^{-1}\Delta) = \lambda^{-1}\Delta$ for any γ, thus $\lambda^{-1}\Delta \in S(V_0^*)$. □

12. Quasi-indecomposable groups, and non-irreducible groups

In this chapter, we prove theorems 1.10 and 1.11, and then go on to describe the non-abelian factors and the central automorphisms of irreducible complex reflection groups.

We recall that a group is indecomposable if it has no non-trivial decomposition as a direct product. We recall the statement of the Krull-Remak-Schmidt theorem for finite groups.

Theorem 12.1 (Krull-Remak-Schmidt)**.** *Given two decompositions $G = G_1 \times \ldots \times G_k = H_1 \times \ldots \times H_{k'}$ of the finite group G as a product of indecomposable factors, we have $k = k'$ and there is a permutation σ of $\{1, \ldots, k\}$ and a central automorphism α such that $\alpha(H_i) = G_{\sigma(i)}$.*

Proof. See *e.g.* [Ro, 3.3.8]. □

12.1. **Quasi-indecomposable groups.** We say that a group G is quasi-indecomposable if in any direct product decomposition $G = G_1 \times G_2$, we have either $G_1 \subseteq ZG$ or $G_2 \subseteq ZG$.

Proposition 12.2. *Let G be a finite quasi-indecomposable group. Then G admits a decomposition of the form $G = Z \times \hat{G}$ where $Z \subseteq ZG$ and $\hat{G}$ is either trivial or is a non-abelian indecomposable group. In such a decomposition $\hat{G}$ (called the* non-abelian factor *of G and denoted* $\mathrm{Nab}(G)$*) and Z (called the* central factor *of G) are unique up to isomorphism.*

Proof. First, we observe that G admits such a decomposition. Indeed, being finite, G admits a decomposition $G = G_1 \times \ldots \times G_n$ as a product of indecomposable groups. By assumption, at most one of the G_i is non-abelian; we set $\hat{G}$ equal to that factor and Z equal to the product of the other factors.

Assume now that G admits two such decompositions $G = Z \times \hat{G} = Z_1 \times \hat{G}_1$. If we refine the decompositions to a product of indecomposable groups, by the Krull-Remak-Schmidt theorem there is an automorphism which sends one decomposition to the other, and this automorphism must send the non-abelian factor in one decomposition to the other, whence the isomorphisms between $\hat{G}$ and $\hat{G}_1$ and Z and Z_1. □

We now relate decompositions of quasi-indecomposable groups to central endomorphisms. An endomorphism $\alpha : G \to G$ is *central* if for any $g \in G$, we have $\alpha(g) \in gZG$.

Lemma 12.3. *Let G be a finite quasi-indecomposable group, and let α be a central endomorphism of G with a maximal kernel (that is,* $\ker \alpha$ *is a maximal element for inclusion among the* $\ker \beta$ *where β runs over the central endomorphisms). Then $G = \ker \alpha \times \mathrm{Im}(\alpha)$ is a decomposition of G with* $\ker \alpha \subseteq ZG$ *and* $\mathrm{Im}(\alpha)$ *indecomposable non abelian (or trivial).*

Proof. Since $\ker \alpha \subseteq \ker \alpha^2$ and $\ker \alpha$ is maximal, we have $\ker \alpha^2 = \ker \alpha$, whence $\mathrm{Im}(\alpha^2) = \mathrm{Im}(\alpha)$ since $\#G = \#\ker \alpha \# \mathrm{Im}(\alpha)$. In particular α is

injective on $\mathrm{Im}(\alpha)$, so $\ker\alpha \cap \mathrm{Im}(\alpha) = \{1\}$. Thus $G = \ker\alpha \times \mathrm{Im}(\alpha)$; and $\mathrm{Im}(\alpha)$ is indecomposable, otherwise if $\mathrm{Im}(\alpha) = Z \times \hat{G}$ where Z is central, and if β is the projection $\mathrm{Im}(\alpha) \to \hat{G}$, then $\beta \circ \alpha$ would be central and would have a greater kernel than α, contradicting the maximality of $\ker\alpha$. □

Given a central endomorphism of G, we define an homomorphism $\chi_\alpha : G \to ZG$ by the formula $\chi_\alpha(g) = g\alpha(g^{-1})$. Conversely any homomorphism $\chi : G \to ZG$ gives rise to a central endomorphism by the formula $\alpha_\chi(g) = g\chi(g)^{-1}$. We have $\ker\alpha_\chi = \{z \in ZG \mid \chi(z) = z\}$.

Lemma 12.4. *Let G be a quasi-indecomposable finite group with cyclic center. Then in a decomposition $G = Z \times \hat{G}$ with Z central and $\hat{G}$ indecomposable nonabelian, the factor Z is unique.*

Proof. In view of the previous lemma, it is sufficient to prove that given two central endomorphisms α and β, there is a central automorphism γ such that $\ker\alpha \subseteq \ker\gamma$ and $\ker\beta \subseteq \ker\gamma$. Let us take $\gamma = \alpha \circ \beta$. Then clearly $\ker\beta \subseteq \ker\gamma$, and we will have $\ker\alpha \subseteq \ker\gamma$ if we can show that restricted to ZG the endomorphisms α and β commute. Now $\alpha(\beta(z)) = \beta(z)\chi_\alpha(\beta(z))^{-1} = z\chi_\beta(z^{-1})\chi_\alpha(z^{-1})\chi_\alpha(\chi_\beta(z))$, and this formula is symmetric in α and β since the two endomorphisms χ_α and χ_β of the cyclic group ZG commute with each other. □

Remark 12.5. *Non-unicity of the central factor.*

The group $\mathbb{Z}/2 \times W(B_2)$ does not have a unique central factor. We will see that $W(B_2)$ is indecomposable, so the above is a decomposition into a central and a non-abelian indecomposable factor. If $\{1, \varepsilon\}$ are the two elements of the factor $\mathbb{Z}/2$ and $\{1, w_0\}$ the two elements of $Z(W(B_2))$ then $\{1, \varepsilon w_0\}$ is another direct factor of $W(B_2)$.

Remark 12.6. *Non-unicity of the non-abelian factor.*

Even if the central factor is unique, $\mathrm{Nab}(G)$ may not be. We will see that $W(G_2)$ is quasi-indecomposable. Having a cyclic center, it has a unique central factor which is the center $\{1, w_0\}$. A possible non-abelian factor is $W(A_2) \subset W(G_2)$. However, this is not the only one. If we consider the sign character $x \mapsto \mathrm{sgn}(x)$ of $W(A_2)$ to take its values in $\{1, w_0\}$ then the subgroup formed of $\{x\,\mathrm{sgn}(x) \mid x \in W(A_2)\}$ is another possible non-abelian factor.

We will see however that both factors are unique for most irreducible complex reflection groups. We prove now proposition 1.10.

Proposition 12.7. *Irreducible finite complex reflection groups are quasi-indecomposable.*

Proof. We prove the result case by case, starting with the imprimitive groups $G(de, e, r)$ of rank $r \geq 3$. Recall that such a group is of the form $G = D \rtimes \mathfrak{S}_r$, where D is the subgroup of diagonal matrices in G. Let π be the quotient

map $G \to \mathfrak{S}_r$ and consider a decomposition $G = G_1 \times G_2$. Then we get $\mathfrak{S}_r = \pi(G_1)\pi(G_2)$, a product of two normal subgroups. Since any proper normal subgroup of $\mathfrak{S}_r$ is in $\mathfrak{A}_r$, we must have *e.g.* $\pi(G_1) = \mathfrak{S}_r$. Since $\pi(G_2)$ commutes with $\pi(G_1)$ and the center of $\mathfrak{S}_r$ is trivial (since $r \geq 3$) we have $\pi(G_2) = 1$, *i.e.* $G_2 \subseteq D$. Further, the action of $\pi(G_1) = \mathfrak{S}_r$ on G_2 is trivial, so $G_2 \subseteq ZG$.

To treat the other cases, we will follow the arguments of [M, 6.2 lemme 2]. Assume that $G \subset \mathrm{GL}(V)$, an irreducible group, admits a decomposition $G_1 \times G_2$ where neither of the G_i is abelian. Since an irreducible representation of a product is the tensor product of irreducible representations of the factors, we must have $V = V_1 \otimes V_2$ where V_i affords a faithful irreducible representation of G_i. In particular, we must have $\dim V_i > 1$ since G_i is non-abelian.

These remarks immediately solve the cases of $G(de, e, 2)$ or more generally the irreducible groups of prime rank. It remains to consider the exceptional groups of rank $4, 6$ or 8. For that, we use that if G is not quasi-indecomposable then $V \otimes V \simeq V_1 \otimes V_1 \otimes V_2 \otimes V_2 = (S^2V_1 \oplus \Lambda^2V_1) \otimes (S^2V_2 \oplus \Lambda^2V_2)$ has at least 4 irreducible components (since $\dim V_i > 1$ implies that the decomposition $V_i \otimes V_i = S^2V_i \oplus \Lambda^2V_i$ is non-trivial).

Now it is a theorem of Steinberg [Bbk2, Exercice 3] that for an irreducible complex reflection group, Λ^2V is irreducible. Thus, we see that if S^2V has less than 3 irreducible components, then $V \otimes V = S^2V \oplus \Lambda^2V$ will have less than 4, and thus G will be quasi-indecomposable. When G is real (a Coxeter group) the representation S^2V contains the trivial representation. It turns out that for *any* exceptional irreducible complex reflection group (not only those of rank 4,6 or 8), S^2V is irreducible when G is not real and $S^2V - 1_G$ is irreducible when G is real (this can easily checked by computer from the character table of G). □

We have the following consequence of the Krull-Remak-Schmidt theorem for direct products of quasi-indecomposable groups:

Theorem 12.8. *Let $G = G_1 \times \ldots \times G_n$ be a direct product of finite quasi-indecomposable groups, such that for any i, j we have $\mathrm{Nab}(G_i) \simeq \mathrm{Nab}(G_j) \Rightarrow G_i \simeq G_j$. Then any automorphism of G is of the form $\alpha \circ \beta$, where α is a central automorphism and β satisfies $\beta(G_i) = G_{\sigma(i)}$ for some permutation σ of $\{1, \ldots, n\}$.*

Proof. Let $\gamma \in \mathrm{Aut}(G)$ and let $G = G_1 \times \ldots \times G_k$ be a decomposition of G in indecomposable factors. Let us apply the Krull-Remak-Schmidt theorem to the decomposition given by $H_i = \gamma(G_i)$; we get a central automorphism α of G and $\sigma \in \mathfrak{S}_k$ such that $\alpha(G_i) = H_{\sigma^{-1}(i)}$. Letting $\beta = \alpha^{-1} \circ \gamma$ it follows that γ can be written $\alpha \circ \beta$ with α a central automorphism of G and $\beta(G_i) = G_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_k$.

If we lump together the product of all abelian G_i as a single subgroup Z, we get that any automorphism of a group of the form $Z \times G_1 \times \cdots \times G_n$ where Z is abelian and the G_i are non-abelian indecomposable groups is of

the form $\alpha \circ \beta$ where α is central and $\beta(G_i) = G_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_n$, as $\beta(G_i)$, being non-abelian, cannot be mapped to one of the indecomposable factors of Z. Let $\beta_Z \in \mathrm{Aut}(G)$ being defined by $\beta_Z(g) = \beta(g)$ for $g \in Z$ and $\beta_Z(g) = g$ for $g \in G_i$; similarly, $\beta_G(g) = \beta(g)$ for $g \in G_i$, $\beta_G(g) = g$ for $g \in Z$. We have $\beta = \beta_Z \circ \beta_G$ and $\gamma = (\alpha \circ \beta_Z) \circ \beta_G$. Since β_Z is a central automorphism, replacing α by $\alpha \circ \beta_Z$ and β by β_G we thus can assume that β acts trivially on Z.

Rewriting the group G of the statement as $Z_1 \times \ldots \times Z_n \times \hat{G}_1 \times \ldots \times \hat{G}_n$ where Z_i are the central factors of the G_i and $\hat{G}_i$ are their non-abelian factors, we get that any automorphism of G is of the form $\alpha \circ \beta$ where α is central and β is trivial on $Z_1 \times \ldots \times Z_n$ and effects a permutation σ of the $\hat{G}_i$. The assumption of the statement that $\mathrm{Nab}(G_i) \simeq \mathrm{Nab}(G_j) \Rightarrow G_i \simeq G_j$ now implies that $G_{\sigma(i)} \simeq G_i$, hence by 12.2 there exists isomorphisms $\delta_i : Z_{\sigma(i)} \to Z_i$ between the corresponding central factors. We recall that β acts trivially on each Z_i, and introduce $\delta \in \mathrm{Aut}(G)$ such that $\delta(g) = g$ for $g \in \hat{G}_i$ and $\delta(g) = \delta_i(g)$ for $g \in Z_i$. Letting $\beta' = \delta^{-1} \circ \beta$ we have $\beta'(g) = \beta(g) \in \hat{G}_{\sigma(i)}$ for $g \in \hat{G}_i$ and $\beta'(g) = \delta_i^{-1}(g) \in Z_{\sigma(i)}$ for $g \in G_i$. Since δ is clearly central we get $\gamma = \alpha' \circ \beta'$, with $\alpha' = \alpha \circ \delta$ a central automorphism and $\beta'(G_i) = G_{\sigma(i)}$, which concludes the proof. □

We now prove theorem 1.11.

Theorem 12.9. *Let $G \subset \mathrm{GL}(V)$ be a complex reflection group and let $G = G_1 \times \ldots \times G_n$ be its decomposition in irreducible factors. Then the following are equivalent:*

(i) *No G_i is isomorphic to $\mathfrak{S}_6$, and for any i, j, we have $\mathrm{Nab}(G_i) \simeq \mathrm{Nab}(G_j) \Rightarrow G_i \simeq G_j$.*
(ii) *For any $\rho \in \mathrm{Irr}(G)$, any automorphism of G is of the form $\alpha \circ \beta \circ \gamma$, where α is central, β is induced by $N_{\mathrm{GL}(V)}(G)$, and γ stabilizes the G_i and preserve their reflections.*

Moreover, for every complex reflection group G, any automorphism which preserve the reflections can be written as $\beta \circ \gamma$, with β, γ as above.

Proof. We recall that $N_{\mathrm{GL}(V)}(G)$, mapping reflections to reflections, necessarily permutes the factors G_i. We first observe that $(ii) \Rightarrow (i)$ is clear, since if one of the G_i is isomorphic to $\mathfrak{S}_6$ the exceptional automorphisms of $\mathfrak{S}_6$ are not of the form given in (ii), and, if two non-isomorphic components G_i and G_j have isomorphic non-abelian factors: $G_i = Z_i \times \hat{G}$ and $G_j = Z_j \times \hat{G}$, the automorphism of G which exchanges the isomorphic factors $\hat{G}$ while leaving the other factors invariant is not either of the form given in (ii).

By 12.8 any automorphism is of the form $\alpha_1 \circ \beta_1$ where α_1 is central and β_1 permutes the G_i. Since a permutation of isomorphic G_i can be effected by an element of N, we can write $\beta_1 = \beta_2 \circ \beta_3$ where $\beta_2 \in N$ and $\beta_3(G_i) = G_i$. This proves the last assertion of the theorem. Moreover, since no G_i is isomorphic to $\mathfrak{S}_6$, by 1.1, β_3 itself is of the form $\beta_3 = \alpha_2 \circ \gamma$, where α_2

is central and γ preserves the reflections. Since the subgroup C of central automorphisms is normal, $\alpha_1 \circ \beta_2 \circ \alpha_2 \circ \gamma$ is of the required form. This concludes the proof of the theorem.

□

13. Central endomorphisms, central automorphisms and non-abelian factors of irreducible complex reflection groups

In this section, we describe the central factor and the group C of central automorphisms for irreducible complex reflection groups.

We relate the problem, for complex reflection groups, to linear characters. We recall that the group of linear characters, or equivalently the abelianized of G, is isomorphic to $\prod_H \mathbb{Z}/\#C_G(H)$ where H runs over representative of hyperplane orbits. If S is the generating subset we have taken for G, the hyperplane orbits define a partition σ of S, and a linear character χ is specified by specifying arbitrarily $\chi(s) \in \mu_{e_s}$ for one s in each part of σ, where e_s is the order of s.

Such a character χ will give rise to a central endomorphism if $\chi(s) \in \mu_{\#ZG}$ for any $s \in S$. Let z be a generator of ZG; α_χ will be an automorphism if and only if $z\chi(z)^{-1}$ is still a generator of ZG. Notice that $\ker \alpha_\chi \subseteq ZG$.

We now proceed, case-by-case, to describe σ and give an expression for z in terms of the generators S, from which we will deduce C and the central factor of G as well as a possible non-abelian factor.

For the groups $G(de, e, r)$, we have $S = \{t', s_1', s_1, \dots, s_{r-1}\}$ (where $t' = t^e$ and $s_1' = s_1^t$ as in section 2.1) where t' is omitted if $d = 1$ and s_1' is omitted if $e = 1$. Expressions for a generator of ZG can be found in [BMR]; it is equal to $z = (t's_1 \dots s_{r-1})^r$ if $e = 1$, and $z = t'^{r/\gcd(e,r)}(s_1's_1 \dots s_{r-1})^{\frac{e(r-1)}{\gcd(e,r)}}$ in the general case (this expression is still valid for $d = 1$ by setting $t' = 1$ in that case). We have $ZG \simeq \mu_{d\cdot\gcd(e,r)}$.

When writing the partition σ, we decorate the parts where $e_s \neq 2$ with the value of e_s, separated by a semicolon. Thus σ is $\{t'; d\}, \{s_1', s_1, \dots, s_{r-1}\}$ except when $r = 2$ and e is even, in which case it is $\{t'; d\}, \{s_1'\}, \{s_1\}$. We find that a character χ gives rise to a central endomorphism of $G(de, e, r)$ if and only if $\chi(s_1') = \chi(s_i) = 1$ if $d\gcd(e, r)$ is odd. We assume this condition in the sequel. We have $\chi(z) = \chi(t')^{\frac{r}{\gcd(e,r)}}$.

We first consider the case where $r > 2$ or $r = 2$ and e is odd. If we define j by $\chi(t') = \zeta_d^j$ and $0 \leq j < d$, and if α_χ is the corresponding central endomorphism, we then have that

$$\begin{aligned}\#\ker(\alpha_\chi) = \#\{i \in \{0, \dots, d\gcd(e,r) - 1\} \mid \chi(t')^{\frac{ri}{\gcd(e,r)}} = \zeta_{d\gcd(e,r)}^i\} = \\ \#\{i \in \{0, \dots, d\gcd(e,r) - 1\} \mid i(rj - 1) \equiv 0 \pmod{d\gcd(e,r)}\}\end{aligned}$$

We define $d_{r'}$ to be the largest factor of d which is prime to r; notice that $rj-1$ and $d\gcd(e,r)$ being coprime is equivalent to $rj-1$ and $d_{r'}$ being coprime.

We thus find that

- α_χ is an automorphism if and only if $rj-1$ is prime to $d_{r'}$.
- The maximal kernel for an α_χ is of cardinality $d_{r'}$, obtained when $rj \equiv 1 \pmod{d_{r'}}$. There always exists such a χ as $d_{r'}$ and r are coprime.

If $rj \equiv 1 \pmod{d_{r'}}$ we have $\det(\alpha_\chi(t')) = \det(t')/\chi(t')^r = \zeta_d^{1-rj} \in \mu_{d/d_{r'}}$, thus $\mathrm{Im}(\alpha_\chi) \subseteq G(de, d_{r'}e, r)$ unless it is possible to take $\chi(s_i) = -1$ and $-1 \notin G(de, d_{r'}e, r)$, which happens when d is even and r odd. We finally get that when $r > 2$ or $r = 2$ and e is odd the central factor of $G(de,e,r)$ is $\mathbb{Z}/d_{r'}$ and the non-abelian factor is unique equal to $G(de, d_{r'}e, r)$, unless d is even and r odd when there is another possibility for the non-abelian factor.

In the remaining case of $G(2de', 2e', 2)$, the partition σ is $\{t'; d\}, \{s_1'\}, \{s_1\}$, and $z = t'(s_1's_1)^{e'}$ is of order $2d$.

Define $\epsilon = 0$ if $\chi(s_1's_1) = 1$ or e' is even, and $\epsilon = d$ otherwise, and define j by $\chi(t') = \zeta_d^j$. An analysis similar to the above case shows that $\#\ker\alpha_\chi = \#\{i \in \{0,\dots,2d-1\} \mid i(2j+\epsilon-1) \equiv 0 \pmod{2d}\}$.

We find that χ defines a central automorphism if and only if ϵ is even and $2j+\epsilon-1$ is prime to $d_{2'}$, where $d_{2'}$ is the largest odd divisor of d.

The maximal kernel of α_χ is of order $d_{2'}$ obtained *e.g.* for $\epsilon = 0$ and $2j \equiv 1 \pmod{d_{2'}}$, except when d and e' are both odd, in which case it is of order $2d$, obtained when $\chi(s_1's_1) = -1$ and $j \equiv (1+d)/2 \pmod{d}$. It can be checked that in this last case there are two possibilities for the image of α_χ; if $\chi(s_1') = -1$ it lies in $G(de', de', 2)$, so we get

Proposition 13.1. *Decompositions of $G(de,e,r)$ into a central factor and a non-abelian indecomposable factor are as follows:*

- *If $r = 2$, d is odd and $e = 2e'$ with e' odd, a decomposition is $\mathbb{Z}/2d \times G(de', de', 2)$; there is one other possibility for the non-abelian factor.*
- *In the other cases a decomposition is $\mathbb{Z}/d_{r'} \times G(de, d_{r'}e, r)$; this decomposition is unique unless d is even and r odd, in which case there is one other possibility for the non-abelian factor.*

Table 3 gives for exceptional groups the partition σ (where each part is decorated by the order of the corresponding reflections, when this order is not 2), the order of ZG, an expression in terms of S of a generator of ZG (such expressions can be found in [BMR, table 4] and in [BM]), under the column "automorphism" conditions describing the linear characters giving rise to an element of C (this column is empty when there are no conditions), and under the column "decomposition" the decomposition of G. In this table, $\mathfrak{A}_5.2$ denotes the (unique) nonsplit central extension of $\mathfrak{A}_5$ by $\mathbb{Z}/2$, and $\mathrm{SO}_5(\mathbb{F}_3)'$ is the commutator subgroup of $\mathrm{SO}_5(\mathbb{F}_3)$.

This decomposition turns out to be unique in each case, excepted for G_5 and G_7 where the non-abelian factor given is one of 3 possibilities.

Table 3: Central endomorphisms

G	σ	$\#ZG$	z	automorphism	decomposition
G_4	$\{s,t;3\}$	2	$(st)^3$	$\chi(s)=1$	G_4
G_5	$\{s;3\},\{t;3\}$	6	$(st)^2$	$\chi(st)\neq\zeta_3$	$\mathbb{Z}/3\times G_4$
G_6	$\{s\},\{t;3\}$	4	$(st)^3$	$\chi(t)=1$	G_6
G_7	$\{s\},\{t;3\},\{u;3\}$	12	stu	$\chi(tu)\neq\zeta_3$	$\mathbb{Z}/3\times G_6$
G_8	$\{s,t;4\}$	4	$(st)^3$		G_8
G_9	$\{s\},\{t;4\}$	8	$(st)^3$		G_9
G_{10}	$\{s;3\},\{t;4\}$	12	$(st)^2$	$\chi(s)\neq\zeta_3$	$\mathbb{Z}/3\times G_8$
G_{11}	$\{s\},\{t;3\},\{u;4\}$	24	stu	$\chi(t)\neq\zeta_3$	$\mathbb{Z}/3\times G_9$
G_{12}	$\{s,t,u\}$	2	$(stu)^4$		G_{12}
G_{13}	$\{s\},\{t,u\}$	4	$(stu)^3$		G_{13}
G_{14}	$\{s\},\{t;3\}$	6	$(st)^4$	$\chi(t)\neq\zeta_3$	$\mathbb{Z}/3\times G_{12}$
G_{15}	$\{s\},\{t\},\{u;3\}$	12	$u(st)^2$	$\chi(t)\neq\zeta_3$	$\mathbb{Z}/3\times G_{13}$
G_{16}	$\{s,t;5\}$	10	$(st)^3$	$\chi(s)\neq\zeta_5^2$	$\mathbb{Z}/5\times(\mathfrak{A}_5.2)$
G_{17}	$\{s\},\{t;5\}$	20	$(st)^3$	$\chi(t)\neq\zeta_5^2$	$\mathbb{Z}/5\times G_{22}$
G_{18}	$\{s;3\},\{t;5\}$	30	$(st)^2$	$\chi(s)\neq\zeta_3,\chi(t)\neq\zeta_5^2$	$\mathbb{Z}/15\times(\mathfrak{A}_5.2)$
G_{19}	$\{s\},\{t;3\},\{u;5\}$	60	stu	$\chi(t)\neq\zeta_3,\chi(u)\neq\zeta_5^2$	$\mathbb{Z}/15\times G_{22}$
G_{20}	$\{s,t;3\}$	6	$(st)^5$	$\chi(s)\neq\zeta_3$	$\mathbb{Z}/3\times(\mathfrak{A}_5.2)$
G_{21}	$\{s\},\{t;3\}$	12	$(st)^5$	$\chi(t)\neq\zeta_3$	$\mathbb{Z}/3\times G_{22}$
G_{22}	$\{s,t,u\}$	4	$(stu)^5$		G_{22}
G_{23}	$\{s,t,u\}$	2	$(stu)^5$	$\chi(s)=1$	$\mathbb{Z}/2\times\mathfrak{A}_5$
G_{24}	$\{s,t,u\}$	2	$(stu)^7$	$\chi(s)=1$	$\mathbb{Z}/2\times\mathrm{GL}_3(\mathbb{F}_2)$
G_{25}	$\{s,t,u;3\}$	3	$(stu)^4$		G_{25}
G_{26}	$\{s\},\{t,u;3\}$	6	$(stu)^3$	$\chi(s)=1$	$\mathbb{Z}/2\times G_{25}$
G_{27}	$\{s,t,u\}$	6	$(uts)^5$	$\chi(s)=1$	$\mathbb{Z}/2\times\mathfrak{A}_6$
G_{28}	$\{s,t\},\{u,v\}$	2	$(stuv)^6$		G_{28}
G_{29}	$\{s,t,u,v\}$	4	$(stvu)^5$		G_{29}
G_{30}	$\{s,t,u,v\}$	2	$(stuv)^{15}$		G_{30}
G_{31}	$\{s,t,u,v,w\}$	4	$(stuvw)^6$		G_{31}
G_{32}	$\{s,t,u,v;3\}$	6	$(stuv)^5$	$\chi(s)\neq\zeta_3^2$	$\mathbb{Z}/3\times\mathrm{Sp}_4(\mathbb{F}_3)$
G_{33}	$\{s,t,u,v,w\}$	2	$(stvwu)^9$	$\chi(s)=1$	$\mathbb{Z}/2\times\mathrm{SO}_5(\mathbb{F}_3)'$
G_{34}	$\{s,t,u,v,w,x\}$	6	$(stvwux)^7$		G_{34}
G_{35}	$\{s_1,\dots,s_6\}$	1	1	$\chi(s)=1$	G_{35}
G_{36}	$\{s_1,\dots,s_7\}$	2	$(s_1\dots s_7)^9$	$\chi(s)=1$	$\mathbb{Z}/2\times\mathrm{SO}_7(\mathbb{F}_2)$
G_{37}	$\{s_1,\dots,s_8\}$	2	$(s_1\dots s_8)^{15}$		G_{37}

We remark that for any irreducible complex reflection group, the non-abelian factor of G can always be taken to be a reflection subgroup, unless G is an exceptional group in the list $G_{16}, G_{18}, G_{20}, G_{23}, G_{24}, G_{27}, G_{32}, G_{33}, G_{36}$.

The information we give is sufficient, in rank 2, to determine the intersection of C with the automorphisms which preserve the reflections, thus determining the structure of $\mathrm{Aut}(G)$ in that case. For exceptional groups, the isomorphism type of C is given in table 1 of [Bes] (we note on this table that the group structure of C given by composition of automorphisms is unrelated to the group structure on $\mathrm{Hom}(G, \mathbb{C}^\times)$; in particular C may be non-commutative).

Assume that G is of rank 2 and let H be the group of Galois automorphisms, which is well-determined by theorem 1.6 since $\gcd(e, r) \leq 2$ in our case. It follows from the above table that $C = H$ for $G_{12}, G_{13}, G_{14}, G_{15}$, that H is of index 2 in C for G_8, G_9, G_{10}, G_{11}, that C is of index 2 in H for $G_4, G_6, G_{20}, G_{20}, G_{22}$, and that $C \cap H$ is of index 2 in both C and H for $G_{16}, G_{17}, G_{18}, G_{19}$. Finally, in the two cases G_5 and G_7 where N is not trivial, we have $N \subset C$. In these cases C is non-commutative, isomorphic to $\mathfrak{S}_3$ for G_5 and $\mathbb{Z}/2 \times \mathfrak{S}_3$ for G_7. In both cases $C \cap H$ is the center of C, which is trivial for G_5 and equal to $\mathbb{Z}/2$ for G_7.

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