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# Ribbon tableaux, ribbon rigged configurations and Hall-Littlewood functions at roots of unity

Francois Descouens

**Abstract:** Hall-Littlewood functions indexed by rectangular partitions, specialized at primitive roots of unity, can be expressed as plethysms. We propose a combinatorial proof of this formula using A. Schilling's bijection between ribbon tableaux and ribbon rigged configurations.

## 1 Introduction

In [9, 10], Lascoux, Leclerc and Thibon proved several formulas for Hall-Littlewood functions  $Q'_\lambda(X; q)$  with the parameter  $q$  specialized at primitive roots of unity. This formula implies a combinatorial interpretation of the plethysms  $l_k^{(j)}[h_\lambda]$  and  $l_k^{(j)}[e_\lambda]$ , where  $h_\lambda$  and  $e_\lambda$  are respectively products of complete and elementary symmetric functions, and  $l_k^{(j)}$  the Frobenius characteristics of representations of the symmetric group  $\mathfrak{S}_k$  induced by a transitive cyclic subgroup of  $\mathfrak{S}_k$ . However, the combinatorial interpretation of the plethysms of Schur functions  $l_k^{(j)}[s_\lambda]$  would be far more interesting. This question led the same authors to introduce a new basis  $H_\lambda^{(k)}(X; q)$  of symmetric functions, depending on an integer  $k \geq 1$  and a parameter  $q$ , which interpolate between Schur functions, for  $k = 1$ , and Hall-Littlewood functions  $Q'_\lambda(X; q)$ , for  $k \geq l(\lambda)$ . These were conjectured to behave similarly under specialization at root of unity, and to provide a combinatorial expression of the expansion of the plethysm  $l_k^{(j)}[s_\lambda]$  in the Schur basis for suitable values of the parameters. This conjecture has been proved only for the stable case  $k = l(\lambda)$ , which reduces to the previous result on Hall-Littlewood functions, and  $k = 2$ , which gives the symmetric and antisymmetric squares  $h_2[s_\lambda]$  and  $e_2[s_\lambda]$ . The proof given in [1] relies upon the study of diagonal classes of domino tableaux, i.e. sets of domino tableaux having the same diagonals. Carré and Leclerc proved that the cospin polynomial of such a class has the form  $(1 + q)^a q^b$ , and from this obtained the specialization  $H_{\lambda \cup \lambda}^{(2)}(X; -1)$ .

In [15], Schilling defines ribbon rigged configurations and gives a statistic preserving bijection between this kind of rigged configurations and ribbon tableaux given by product of row partitions. The aim of this note is to provide a similar proof for the stable case, that is, to show that the result on Hall-Littlewood functions at roots of unity follows from an explicit formula for the cospin polynomials of certain diagonal classes of ribbon tableaux, which turn

out to have a very simple characterization through Schilling's bijection.

**Acknowledgment** All the computation on ribbon rigged configurations, ribbon tableaux and Hall-Littlewood functions are made with MuPAD-Combinat (see [4] for more details on this MuPAD package).

## 2 Basic definitions

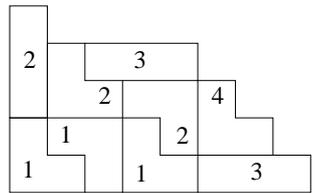
For a partition  $\lambda = (\lambda_1, \dots, \lambda_p)$ , we denote by  $l(\lambda)$  its length  $p$ ,  $|\lambda|$  its weight  $\sum_i \lambda_i$ , and  $\lambda'$  its conjugate. A  $k$ -ribbon is a connected skew diagram of weight  $k$  which does not contain a  $2 \times 2$  square. The first (northwest) cell of a  $k$ -ribbon is called the *head* and the last one (southeast) the *tail*. With  $\lambda$  is associated a  $k$ -core  $\lambda^{(k)}$  and a  $k$ -quotient  $\lambda^{(k)}$ . The  $k$ -core is the unique partition obtained by removing successively  $k$ -ribbons from  $\lambda$ , such that at each step the remaining shape still be a partition. The  $k$ -quotient is a sequence of  $k$  partitions derived from  $\lambda$  (see [5]). Let denote by  $\mathcal{P}$  the set of all the partitions,  $\mathcal{P}_{(k)}$  the set of all the  $k$ -cores (i.e all the partitions from which we can't remove any  $k$ -ribbon) and  $\mathcal{P}^{(k)}$  the cartesian product  $\mathcal{P} \times \dots \times \mathcal{P}$  of length  $k$ . The quotient bijection  $\Phi_k$  is defined by

$$\begin{aligned} \Phi_k : \mathcal{P} &\longrightarrow \mathcal{P}_{(k)} \times \mathcal{P}^{(k)} \\ \lambda &\longmapsto (\lambda^{(k)}, \lambda^{(k)}) . \end{aligned}$$

### 2.1 $k$ -Ribbon tableaux

A  $k$ -ribbon tableau of shape  $\lambda$  and weight  $\mu$  is a tiling of the skew diagram  $\lambda/\lambda^{(k)}$  by labelled  $k$ -ribbons such that the head of a ribbon labelled  $i$  must not be on the right of a ribbon labelled  $j > i$  and its tail must not be on the top of a ribbon labelled  $j \geq i$ . We denote by  $Tab_{\lambda, \mu}^{(k)}$  the set of all  $k$ -ribbon tableaux of shape  $\lambda$  and weight  $\mu$  and by  $Tab_{\lambda}^{(k)}$  the set of all the  $k$ -ribbon tableaux of shape  $\lambda$  and evaluation any composition of  $|\lambda|$ .

**Example 1** A 3-ribbon tableau of shape  $(87651)$  and weight  $(3321)$



The spin of a  $k$ -ribbon  $R$  is defined by  $sp(R) = \frac{h(R)-1}{2}$ , where  $h(R)$  is the height of  $R$ . The spin of a  $k$ -ribbon tableau  $T$  is the sum of the spin of all its ribbons, and the cospin is the associated co-statistic into  $Tab_{\lambda, \mu}^{(k)}$ , i.e

$$cosp(T) = \max \left( sp(T'), T' \in Tab_{\lambda, \mu}^{(k)} \right) - sp(T) .$$

We define the cospin polynomial  $\tilde{G}_{\lambda,\mu}^{(k)}(q)$  as the generating polynomial of  $Tab_{\lambda,\mu}^{(k)}$  with respect to the cospin statistic

$$\tilde{G}_{\lambda,\mu}^{(k)}(q) = \sum_{T \in Tab_{\lambda,\mu}^{(k)}} q^{\text{cosp}(T)} .$$

**Example 2** For  $Tab_{87651,3321}^{(3)}$ , the cospin polynomial is

$$\tilde{G}_{(87651),(3321)}^{(3)}(q) = 3q^5 + 17q^4 + 33q^3 + 31q^2 + 18q + 5 .$$

## 2.2 $k$ -tuples of Young tableaux

Let  $T^\circ = (T^{\circ_1}, \dots, T^{\circ_k})$  be a  $k$ -tuple of semi-standard Young tableaux  $T^{\circ_i}$  of shape  $\lambda^{\circ_i}$  and evaluation  $\mu^{\circ_i}$ . We call shape of  $T^\circ$ , the sequence of shapes  $\lambda^\circ = (\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$  and evaluation of  $T^\circ$ , the vector  $\mu = (\mu_1, \dots, \mu_p)$  defined by  $\mu_i = \sum_{j=1}^k \mu_i^{\circ_j}$ .

**Definition 1** Let  $s$  be a given cell in  $T^\circ$ ,

- $\text{pos}(s)$  is the integer such that the cell  $s$  is in the tableau  $T^{\circ_{\text{pos}(s)}}$ ,
- $T^\circ(s)$  is the label of the cell  $s$ ,
- $\text{row}(s)$  and  $\text{col}(s)$  represent the row and the column of  $s$  in  $T^{\circ_{\text{pos}(s)}}$ ,
- and  $\text{diag}(s)$  is the content of  $s$ , i.e  $\text{diag}(s) = \text{col}(s) - \text{row}(s)$ .

Let denote by  $Tab_{\lambda^\circ, \mu}$  the set of all the  $k$ -tuple of Young tableaux  $T^\circ$  of shape  $\lambda^\circ = (\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$  and evaluation  $\mu$ . In [16], Stanton and White extend the bijection  $\Phi_k$  to a correspondence  $\Psi_k$  between the set of  $k$ -ribbon tableaux of shape  $\lambda$  and evaluation  $\mu$ , and the set of  $k$ -tuples of semi-standard Young tableaux of shape  $\lambda^{(k)}$  and evaluation  $\mu$ .

**Example 3**  $\Psi_3$  send the 3-ribbon tableau of example 1 to the 3-tuple of tableaux

$$\left( \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} , \begin{array}{|c|c|c|} \hline 2 & 3 \\ \hline 1 & 1 & 3 \\ \hline \end{array} , \begin{array}{|c|c|} \hline 1 & 4 \\ \hline \end{array} \right) .$$

**Definition 2** ([14]) Let  $T^\circ = (T^{\circ_1}, \dots, T^{\circ_k})$  be a  $k$ -tuple of semi-standard Young tableaux and  $t$  and  $s$  two cells in  $T^\circ$ . The couple  $(s, t)$  is an inversion if the following conditions hold

- 1-  $\text{diag}(s) = \text{diag}(t)$  and  $\text{pos}(s) < \text{pos}(t)$   
or  
 $\text{diag}(s) = \text{diag}(t) - 1$  and  $\text{pos}(s) > \text{pos}(t)$ ,
- 2-  $\text{row}(s) \leq \text{row}(t)$ ,
- 3-  $T(t) < T(s) < T(t^\uparrow)$ , where  $t^\uparrow$  is the cell directly above  $t$  and  $T(t^\uparrow) = \infty$  if  $t^\uparrow \notin \lambda^\circ$ .

The statistic of inversions on  $T^\circ$ , written  $\text{inv}(T^\circ)$ , is the number of couples which form an inversion in  $T^\circ$ . This statistic permits to extend the correspondence  $\Psi_k$  to a bijection which preserve the statistic (see [14]), i.e

$$\forall T \in \text{Tab}_{\lambda, \mu}^{(k)}, \text{inv}(\Psi_k(T)) = \text{cosp}(T) .$$

Let  $\tilde{I}_{\lambda^\circ, \mu}(q)$  be the generating polynomial of  $\text{Tab}_{\lambda^\circ, \mu}$  with respect to the inversions statistic

$$\tilde{I}_{\lambda^\circ, \mu}(q) = \sum_{T^\circ \in \text{Tab}_{\lambda^\circ, \mu}} q^{\text{inv}(T^\circ)} .$$

**Proposition 1** *The bijection  $\Psi_k$  implies the equality between the two generating polynomials*

$$\tilde{G}_{\lambda, \mu}(q) = \tilde{I}_{\lambda^{(k)}, \mu}(q) . \quad (1)$$

In [3], Haglund, Haiman, Loehr, Remmel and Ulyanov define an other notion of inversions on  $k$ -tuples of tableaux. This statistics is equal to  $\text{inv}(T^\circ)$  up to a constant and gives a combinatorial interpretation for the powers of  $q$  of the Macdonald polynomials  $\tilde{H}_\lambda(X; q, t)$  expanded on the monomials.

### 3 Hall-Littlewood functions at roots of unity

The Hall-Littlewood functions  $Q'_\lambda(X; q)$  are the symmetric functions obtained by

$$Q'_\lambda(X; q) = \prod_{i < j} (1 - qR_{ij})^{-1} s_\lambda(X) ,$$

where  $R_{ij}$  is the raising operator such that  $R_{ij} \cdot s_\lambda = s_{R_{ij} \cdot \lambda}$ , with

$$R_{ij} \cdot \lambda = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots, \lambda_p) .$$

We shall need the  $\tilde{Q}'_\lambda$  version of the Hall-Littlewood functions defined by

$$\tilde{Q}'_\lambda(X; q) = q^{\eta(\lambda)} Q'_\lambda(X; q^{-1}),$$

where  $\eta(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$ .

Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be a partition of length  $k$ , we denote by  $k\lambda$  the partition  $(k\lambda_1, \dots, k\lambda_k)$ . In [11], Lascoux, Leclerc and Thibon showed that Hall-Littlewood functions can be expressed in terms of ribbon tableaux

$$\tilde{Q}'_\lambda(X; q) = \sum_{T \in \text{Tab}_{k\lambda}^{(k)}} q^{\text{cosp}(T)} X^T = \sum_{\mu \vdash |\lambda|} \tilde{G}_{k\lambda, \mu}^{(k)}(q) m_\mu ,$$

where  $X^T$  is  $X_1^{\mu_1} \dots X_p^{\mu_p}$  with  $\mu$  the evaluation of the ribbon tableau  $T$ .

We can remark that in the special case of a partition  $\lambda$  of length  $k$ , the  $k$ -core of  $k\lambda$  is empty and the  $k$ -quotient of  $k\lambda$  is the product of single row partitions

$$(k\lambda)^{(k)} = ((\lambda_k), \dots, (\lambda_1)) .$$

**Proposition 2** *By equation 1, the Hall-Littlewood functions can be expressed as*

$$\tilde{Q}'_{\lambda}(X; q) = \sum_{\mu \vdash |\lambda|} \tilde{I}_{(k\lambda)^{(k)}, \mu}(q) m_{\mu} .$$

This proposition means that Hall-Littlewood functions can be expressed in terms of all the  $k$ -tuples of Young tableaux with shape a product of single row partitions.

**Example 4** *The Hall-Littlewood function  $\tilde{Q}'_{222}$  on the Schur basis is given by*

$$\tilde{Q}'_{222}(X, q) = q^6 s_{222} + (q^5 + q^4) s_{321} + q^3 s_{33} + q^3 s_{411} + (q^4 + q^3 + q^2) s_{42} + (q^2 + q) s_{51} + s_6 .$$

Let denote by  $\Lambda_q$  the vector space of symmetric functions over the field  $\mathbb{C}(q)$ . Let  $k$  be a positive integer and  $\lambda$  a partition, the plethysm of the powersum  $p_{\lambda}$  by the powersum  $p_k$  is defined by

$$p_k \circ p_{\lambda} = p_{k\lambda} . \quad (2)$$

As the powersums form a basis of the vector space  $\Lambda_q$ , the plethysm by a powersum  $p_k$  is defined on any symmetric function  $f$ .

**Theorem 1 ([9])** *Let  $n$  and  $k$  be two positive integers and  $\zeta$  a primitive  $k$ -th root of unity. The specialization of the parameter  $q$  at  $\zeta$  in the Hall-Littlewood function indexed by the rectangular partition  $n^k$ , gives the following formulas*

$$Q'_{n^k}(X; \zeta) = (-1)^{(k-1)n} p_k \circ h_n(X) , \quad (3)$$

$$\tilde{Q}'_{n^k}(X; \zeta) = p_k \circ h_n(X) . \quad (4)$$

In [9], Lascoux, Leclerc and Thibon only give an algebraic proof of this theorem. We shall give a combinatorial proof of this formula using the bijection between ribbon tableaux and ribbon rigged configurations given in [15].

**Example 5** *The Hall-Littlewood function  $\tilde{Q}'_{222}(X; q)$  with  $q$  specialized at  $j = e^{\frac{2i\pi}{3}}$  is*

$$\tilde{Q}'_{222}(X; j) = s_{222} - s_{321} + s_{33} + s_{411} - s_{51} + s_6 \quad (5)$$

$$= \frac{1}{2} (p_{33} + p_6) = p_3 \circ \left( \frac{1}{2} p_{11} + \frac{1}{2} p_2 \right) \quad (6)$$

$$= p_3 \circ h_2(X) . \quad (7)$$

## 4 Ribbon rigged configurations and diagonal classes

Let first recall some basic definitions about  $q$ -analogues of factorials and binomial coefficients. Let  $n$  be a positive integer, we define the  $q$ -factorial  $(q)_n$  by

$$(q)_n = \prod_{i=1}^n (1 + q + \dots + q^{i-1}). \quad (8)$$

Let  $a$  and  $b$  be two positive integers. A  $q$ -analogue of the binomial coefficient  $\binom{a+b}{a,b}$  can be given by

$$\left[ \begin{matrix} a+b \\ a, b \end{matrix} \right] = \frac{(q)_{a+b}}{(q)_a (q)_b}. \quad (9)$$

### 4.1 Ribbon rigged configurations

In this subsection, we briefly recall some basic statements on ribbon rigged configurations introduced by Schilling in [15]. Let  $\nu = (\nu^{(1)}, \dots, \nu^{(p)})$  be an increasing  $p$ -tuple of partitions, i.e the diagram of  $\nu^{(i)}$  is included in the diagram of  $\nu^{(i+1)}$ . Let  $J$  be a sequence of sequences of partitions such that

- 1- for all  $a$  in  $\{1, \dots, p-1\}$ ,  $J^{(a)}$  is a  $l(\nu^{(a)})$ -tuple of partitions  $J^{(a)} = (J_1^{(a)}, \dots, J_{l(\nu^{(a)})}^{(a)})$ ,
- 2- for all  $a$  in  $\{1, \dots, p-1\}$  and for all  $i \leq l(\nu^{(a)})$ , we have  $0 \leq l(J_i^{(a)}) \leq \nu_i^{(a)} - \nu_{i+1}^{(a)}$ ,
- 3- and each part of  $J_i^{(a)}$  (called quantum number) satisfy

$$0 \leq J_{i,j}^{(a)} \leq \nu_i^{(a+1)} - \nu_i^{(a)} \quad (\text{called vacancy number}).$$

**Definition 3** A ribbon rigged configuration of shape  $\nu$  and evaluation  $J$ , written  $(\nu, J)$ , is the filling of the top cells of each column of the partition  $\nu^{(a)}$  which are in the  $i$ -th line with the parts of the partition  $J_i^{(a)}$ .

The graphical representation of a ribbon rigged configuration corresponds to the filling of the top cells of each column of each partition with the numbers of the partition  $J_i^{(a)}$  and the filling of the  $i$ -th row of the frontier of  $\nu^{(a)}$  with the vacancy numbers  $\nu_i^{(a+1)} - \nu_i^{(a)}$ .

**Example 6** This is the graphical representation of the ribbon rigged configuration corresponding to the shape  $((21), (31), (32), (331))$  and the evaluation  $((1), (), ((), (1)), ((1), (2, 1)))$ .



For two partitions  $\delta$  and  $\mu = (\mu_1, \dots, \mu_p)$ , we denote by  $RC(\mu, \delta)$ , the set of all the ribbon rigged configurations  $(\nu, J)$  such that

- 1- the last partition of the sequence  $\nu = (\nu_1, \dots, \nu_p)$  satisfy  $\nu^{(p)} = \delta'$ ,
- 2- and for all  $a$  in  $\{1, \dots, p\}$ , we have  $|\nu^{(a)}| = \mu_1 + \dots + \mu_a$ .

Let denote by  $\Gamma_{\mu, \delta}(\nu)$  the set of all rigged configurations in  $RC(\mu, \delta)$  with shape  $\nu$  and by  $\Gamma_{\mu, \delta}$  the set of all the  $\Gamma_{\mu, \delta}(\nu)$ .

There exists a cocharge statistic on the ribbon rigged configurations defined by

$$\tilde{c}(\nu, J) = \alpha(\nu) + \sum_{\substack{1 \leq a \leq n-1 \\ 1 \leq i \leq \mu_1}} |J_i^{(a)}|, \quad (10)$$

where  $\alpha(\nu)$  is a constant associated to each form of ribbon rigged configurations given by

$$\alpha(\nu) = \sum_{\substack{1 \leq a \leq n-1 \\ 1 \leq i \leq \mu_1}} \nu_{i+1}^{(a)} (\nu_i^{(a+1)} - \nu_i^{(a)}). \quad (11)$$

By enumerating all the elements in  $RC(\mu, \delta)$ , we can express the generating polynomial of ribbon rigged configurations  $\tilde{S}_{\mu, \delta}(q)$  with respect to the cocharge using  $q$ -binomial coefficients

$$\tilde{S}_{\mu, \delta}(q) = \sum_{(\nu, J) \in RC(\mu, \delta)} q^{\tilde{c}(\nu, J)} \quad (12)$$

$$= \sum_{\nu} q^{\alpha(\nu)} \prod_{\substack{1 \leq a \leq n-1 \\ 1 \leq i \leq \mu_1}} \left[ \begin{matrix} \nu_i^{(a+1)} - \nu_{i+1}^{(a)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a)}, \nu_i^{(a+1)} - \nu_i^{(a)} \end{matrix} \right], \quad (13)$$

where  $\nu$  describe the set of all the shape which appear in  $RC(\mu, \delta)$ .

Let denote by  $\tilde{S}_{\mu, \delta}(q, \nu)$  the cocharge polynomial  $\tilde{S}_{\mu, \delta}(q)$  restricted to the subset of ribbon rigged configurations in  $RC(\mu, \delta)$  with shape  $\nu$

$$\tilde{S}_{\mu, \delta}(q, \nu) = q^{\alpha(\nu)} \prod_{\substack{1 \leq a \leq n-1 \\ 1 \leq i \leq \mu_1}} \left[ \begin{matrix} \nu_i^{(a+1)} - \nu_{i+1}^{(a)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a)}, \nu_i^{(a+1)} - \nu_i^{(a)} \end{matrix} \right].$$

**Theorem 2 ([15])** *Let  $\lambda$  be a partition with an empty  $k$ -core and a  $k$ -quotient  $\lambda^{(k)} = (\lambda^{\circ 1}, \dots, \lambda^{\circ k})$ , a sequence of single row partitions. Let  $\delta$  be a partition such that  $\delta_i = |\lambda^{\circ i}|$  and  $\mu$  a partition of weight  $|\lambda|/k$ . In this special case, there exists a bijection  $\Theta_k$ , between  $Tab_{\lambda^{(k)}, \mu}$  and  $RC(\mu, \delta)$  which preserves the statistics, i.e*

$$\forall T^\circ \in Tab_{\lambda^{(k)}, \mu}, \quad \tilde{c}(\Theta_k(T^\circ)) = inv(T^\circ). \quad (14)$$

Let us just recall the steps of the algorithm which permits to compute  $\Theta_k(T^\circ)$ .

**Algorithm 1** ([15])

**Input:**  $T^\circ = (T^{\circ_1}, \dots, T^{\circ_k})$  a  $k$ -tuple of Young tableaux of shape the sequence of single row partitions  $\lambda^\circ = (\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$  and evaluation  $\mu = (\mu_1, \dots, \mu_p)$ .

**Initialization:**  $\nu \leftarrow$  a sequence of  $p$  empty partitions.

For  $i$  from  $k$  down to  $1$  do

  For  $j$  from  $1$  to  $l(\lambda^{\circ_i})$  do

    1- For  $k$  from  $T_j^{\circ_i}$  to  $p$  do

      Add a box in the  $j$ -th row in the partition  $\nu^{(k)}$

    EndFor

    2- Recompute all the vacancy numbers,

    3- Fill the new cells coming from step 1

      with the vacancy number of their row,

    4- Remove a maximal number in the  $(j-1)$ -th row

      of the partitions which have a new box from step 1 in the  $j$ -th row.

  EndFor

EndFor

For  $a$  from  $1$  to  $p-1$  do

  For  $i$  from  $1$  to  $\mu_1$  do

    Replace each number  $\beta$  in the row  $\nu_i^{(a)}$  by  $\nu_i^{(a+1)} - \nu_i^{(a)} - \beta$

  EndFor

EndFor.

This algorithm is implemented in the MuPAD package MuPAD-Combinat.

**Proposition 3** *Let  $k$  be a positive integer and  $\lambda$  and  $\mu$  two partitions such that  $k|\mu| = |\lambda|$ . Let  $\lambda^{(k)} = (\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$  be the  $k$ -quotient of  $\lambda$  and  $\delta$  the partition  $(|\lambda^{\circ_1}|, \dots, |\lambda^{\circ_k}|)$ . Combining the bijection  $\Theta_k$  and  $\Psi_k$ , we have*

$$\tilde{G}_{\lambda, \mu}^{(k)}(q) = \tilde{I}_{\lambda^{(k)}, \mu}(q) = \tilde{S}_{\mu, \delta}(q) = \sum_{\nu} q^{\alpha(\nu)} \prod_{\substack{1 \leq a \leq n-1 \\ 1 \leq i \leq \mu_1}} \left[ \begin{matrix} \nu_i^{(a+1)} - \nu_{i+1}^{(a)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a)}, \nu_i^{(a+1)} - \nu_i^{(a)} \end{matrix} \right]. \quad (15)$$

Using this theorem, we have an explicit formula for the coefficients of the transition matrix between the Hall-Littlewood functions and the monomials.

**Corollary 1** *Let  $\lambda$  be a partition. Let  $\tilde{\lambda}^{l(\lambda)} = (\tilde{\lambda}^{\circ_1}, \dots, \tilde{\lambda}^{\circ_{l(\lambda)}})$  be the  $l(\lambda)$ -quotient of the partition  $l(\lambda)\lambda$  and  $\delta = (\delta_1, \dots, \delta_{l(\lambda)})$  be the partition defined by  $\delta_i = |\tilde{\lambda}^{\circ_i}|$ . Then, the Hall-Littlewood function can be expressed as*

$$\tilde{Q}'_{\lambda}(X; q) = \sum_{\mu=|\lambda|} \tilde{S}_{\mu, \delta}(q) m_{\mu}. \quad (16)$$

The ribbon rigged configurations we considered are different from the rigged configurations defined in [6, 7, 8] which give the expansion of the Hall-Littlewood functions on the Schur basis and correspond to highest weights in crystals of type  $A_n^{(1)}$ .

## 4.2 Diagonal classes of ribbon tableaux

Let  $T^\circ = (T^{\circ_1}, \dots, T^{\circ_k})$  be a sequence of Young tableaux with shape  $\lambda^\circ$  a sequence of single row partitions. Let define two constants  $m_1$  and  $m_2$  by

$$m_1 = \max(l(\lambda^{\circ_i}), i = 1 \dots k) \quad \text{and} \quad m_2 = \max(\lambda_1^{\circ_i}, i = 1 \dots k) .$$

For all  $i \in \{-m_1 + 1, \dots, m_2\}$ , we call  $d_i$  the  $i$ -th diagonal of  $T^\circ$  defined by

$$d_i = \{T^\circ(s) \text{ such that } \text{diag}(s) = i\} ,$$

and we call diagonal vector of  $T^\circ$  the vector  $d_{T^\circ} = (d_{-m_1+1}, \dots, d_{m_2})$ .

Two  $k$ -tuples of Young tableaux  $T^\circ$  and  $T^{\circ'}$  in  $Tab_{\lambda^\circ, \mu}$  are equivalent if and only if for all  $i$  in  $\{-m_1 + 1, \dots, m_2\}$ , the  $i$ -th set in  $d_{T^\circ}$  and  $d_{T^{\circ'}}$  are the same. A diagonal class in  $Tab_{\lambda^\circ, \mu}$  is a set  $D_{\lambda^\circ, \mu}(d)$  of all equivalent  $k$ -tuples of tableaux with a diagonal vector  $d$ . The set of all diagonal classes is denoted by  $D_{\lambda^\circ, \mu}$ . We also define  $\tilde{I}_{\lambda^\circ, \mu}(q, d)$  as the inversion polynomial  $\tilde{I}_{\lambda^\circ, \mu}(q)$  restricted to the diagonal class  $D_{\lambda^\circ, \mu}(d)$

$$\tilde{I}_{\lambda^\circ, \mu}(q, d) = \sum_{T^\circ \in D_{\lambda^\circ, \mu}(d)} q^{\text{inv}(T^\circ)} .$$

Similarly, the same diagonal classes can be defined on  $k$ -ribbon tableaux through the bijection  $\Psi_k^{-1}$ . In [1], Carre and Leclerc give a combinatorial construction of diagonal classes using a notion of labyrinths in the case of domino tableaux (i.e  $k = 2$ ) and proved that cospin polynomials of diagonal classes are of the form  $q^a(1+q)^b$  with  $a$  and  $b$  two positive integers. The combinatorial construction of diagonal classes for  $k > 2$  still an open problem.

## 4.3 A matricial recoding of the bijection

In the following, we shall therefore propose a simpler but similar algorithm for finding the shape of ribbon rigged configurations.

Let  $p$  and  $q$  be two integers and  $\mathcal{M}_{p,q}$  the set of all  $(p \times q)$ -matrices with integer coefficients. We define the operator  $A_E$  on  $\mathcal{M}_{p,q}$  by

$$A_E : \quad \mathcal{M}_{p,q} \quad \longrightarrow \quad \mathcal{M}_{p,q} \\ M = (m_{i,j})_{i,j} \quad \longmapsto \quad N = \begin{cases} n_{i,1} = & m_{i,1} \\ n_{i,j} = & \sum_{1 \leq k \leq j} m_{i,k} , \text{ for } 2 \leq j \leq q . \end{cases} \quad (17)$$

Let  $T^\circ$  be a  $k$ -tuple of Young tableaux of shape  $\lambda^\circ$ , evaluation  $\mu$  and diagonal class  $d$ . Let  $\Theta_k(T^\circ) = (\nu_{T^\circ}, J_{T^\circ})$  be the ribbon rigged configuration corresponding to  $T^\circ$  by  $\Theta_k$ . We construct a  $(m_2 \times l(\mu))$ -matrix  $M^{T^\circ}$  using the following rule

$$M_{i,j}^{T^\circ} = \text{number of cells labelled } j \text{ in } d_{i+1} .$$

**Proposition 4** *The  $j$ -th column of the matrix  $A_E(M^{T^\circ})$  is equal to the  $j$ -th partition of  $\nu_{T^\circ}$ .*

*Proof:* Let  $T^\circ$  be a  $k$ -tuple of single row tableaux and  $\Theta_k(T^\circ) = (\nu_{T^\circ}, J_{T^\circ})$ . In the algorithm 1, we observe that boxes which appear in the  $i$ -th line of a partition  $\nu_{T^\circ}^{(j)}$  only come from elements of the  $i$ -th diagonal of  $T^\circ$  which are smaller than  $j$ . And by definition of the operator  $A_E$ , the entry  $(i, j)$  of the matrix  $A_E(M^{T^\circ})$  corresponds to the number of cells less than  $j$  in the  $i$ -th diagonal.  $\square$

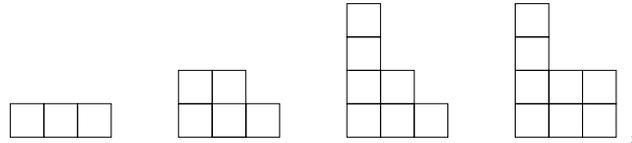
**Example 7** *Let consider the following 3-tuple of single row tableaux  $T^\circ$*

$$\left( \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 3 \\ \hline \end{array} , \begin{array}{|c|c|} \hline 1 & 4 \\ \hline \end{array} , \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array} \right) .$$

*In this case, the matrices  $M^{T^\circ}$  and  $A_E(M^{T^\circ})$  are*

$$M^{T^\circ} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_E(M^{T^\circ}) = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

*The shape of the rigged configuration  $\Theta_k(T^\circ)$  is*



*as can be read from  $A_E(M^{T^\circ})$ .*

#### 4.4 A combinatorial proof of the specialization

**Proposition 5** *Let  $\lambda^\circ$  be a given sequence of single row partitions  $(\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$ ,  $\mu$  a partition of weight  $\sum_i |\lambda^{\circ_i}|$  and  $\delta$  the partition  $(|\lambda^{\circ_1}|, \dots, |\lambda^{\circ_k}|)$ . For all diagonal vectors  $d$  which appears in  $\text{Tab}_{\lambda^\circ, \mu}$ , there exists a sequence of partitions  $\nu$ , such that*

$$\Theta_k(D_{\lambda^\circ, \mu}(d)) = \Gamma_{\mu, \delta}(\nu) .$$

Hence, the explicit expression for the inversion polynomial of a diagonal class  $D_{\lambda^\circ, \mu}(d)$  is

$$\tilde{I}_{\lambda^\circ, \mu}(q, d) = \tilde{S}_{\mu, \delta}(q, \nu) = q^{\alpha(\nu)} \prod_{\substack{1 \leq a \leq n-1 \\ 1 \leq i \leq \mu_1}} \left[ \begin{array}{c} \nu_i^{(a+1)} - \nu_{i+1}^{(a)} \\ \nu_i^{(a)} - \nu_{i+1}^{(a)}, \nu_i^{(a+1)} - \nu_i^{(a)} \end{array} \right], \quad (18)$$

where  $\nu$  is the shape corresponding to the diagonal vector  $d$ .

*Proof:* Let  $d$  be a diagonal vector and  $T^\circ, T^{\circ'}$  two elements in the diagonal class  $D_{\lambda^\circ, \mu}(d)$ . These two  $k$ -tuples of tableaux differ only from a permutation of cells which are in a same diagonal  $d_i$ . By construction, this property implies  $M^{T^\circ} = M^{T^{\circ'}}$  and by proposition 4,  $\Theta_k(T^\circ)$  has the same shape  $\nu$  than  $\Theta_k(T^{\circ'})$ . Consequently, as  $\Theta_k$  is a bijection,  $D_{\lambda^\circ, \mu}(d)$  is embedded into  $\Gamma_{\mu, \delta}(\nu)$ . Conversely, let  $T^\circ$  and  $T^{\circ'}$  be two  $k$ -tuples of tableaux in  $Tab_{\lambda^\circ, \mu}$  which are not in the same diagonal class. This implies that  $M^{T^\circ} \neq M^{T^{\circ'}}$  and consequently the shape of their corresponding ribbon rigged configurations are not the same. Finally, we conclude that

$$\Theta_k(D_{\lambda^\circ, \mu}(d)) = \Gamma_{\mu, \delta}(\nu).$$

The expression of inversion polynomials of diagonal classes in terms of  $q$ -supernomial coefficients follows immediately from the invariance of the statistics under  $\Theta_k$ .  $\square$

In the following, we consider  $k$ -tuples of tableaux of shape  $\lambda^\circ = ((n), \dots, (n))$  for some  $n \geq 1$  and the next corollary characterize in this special case diagonal classes with only one element.

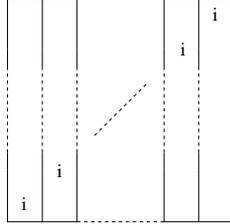
**Corollary 2** *Let  $\lambda^\circ = (\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$  be a sequence of single rows partitions and  $\mu$  a partition of weight  $\sum_{i=1}^k |\lambda^{\circ_i}|$ . Let us describe the diagonal classes with only one element depending on the evaluation  $\mu$ .*

- If each part of  $\mu$  is divisible by  $k$ ,
  - 1- there is an unique diagonal class  $D_{\lambda^\circ, \mu}(d)$  with only one element  $T^\circ = (T^{\circ_1}, \dots, T^{\circ_k})$ ,
  - 2- the  $i$ -th position of each tableau  $T^{\circ_j}$  is filled with the same value,
  - 3-  $\text{inv}(T^\circ) = 0$ , i.e  $\tilde{I}_{\lambda^\circ, \mu}(q, d) = 1$ .
- if one part of  $\mu$  is not divisible by  $k$ , there is no diagonal classes with only one element.

*Proof:* A diagonal class  $D_{\lambda^\circ, \mu}(d)$  has an unique element if and only if there is an unique way to fill  $\lambda^\circ$  according to the vector  $d$ . This implies that, for all  $i$  in  $\{1, \dots, n\}$ , all letters of  $d_{i-1}$  are the same. For filling identically each position in  $d_i$ , the weight  $\mu$  must be of the form  $\mu = (ks_1, \dots, ks_p)$ . On the other hand, if one part of  $\mu$  is not divisible by  $k$ , this implies that for any filling  $T^\circ$  of  $\lambda^\circ$ , there exist two tableaux  $T^{\circ_a}$  and  $T^{\circ_b}$  in  $T^\circ$  which have two different values in one position. By transposition of these two cells, we obtain an other filling  $T^{\circ'}$  which is in the same diagonal class than  $T^\circ$ . Consequently, for this kind of evaluation, all

diagonal classes have more than 2 elements.

Let  $d$  be a diagonal vector such that the diagonal class  $D_{\lambda^\circ, \mu}(d)$  contains only one element  $T^\circ$  and  $\lambda$  be the partition such that  $\lambda^{(k)} = \lambda^\circ$ . The inverse image of  $T^\circ$  by  $\Psi_k$  is a  $k$ -ribbon tableaux in  $Tab_{\lambda, \mu}^{(k)}$ , which only contains  $(k \times k)$ -blocks of the form



Such  $k$ -ribbon tableaux have maximal spin in  $Tab_{\lambda, \mu}^{(k)}$ . Consequently, cospins of such  $k$ -ribbon tableaux are equal to zero. Finally, as  $\Psi_k$  preserve the statistics, this implies that the inversions of the corresponding  $k$ -tuple of tableaux are also equal to zero. By definition of the bijection  $\Psi_k$ , the length of the partition  $\lambda$  is  $k$  because  $\lambda^\circ$  is a  $k$ -tuple of single row partitions. Consequently, due to the increasing condition in the definition of a  $k$ -ribbon tableau, there is a unique way to build a  $k$ -ribbon tableau of shape  $\lambda$  with the previous blocks. This implies the uniqueness of the diagonal classes with only one element for a given evaluation.  $\square$

**Corollary 3** *Let  $\lambda^\circ$  a  $k$ -tuple of partitions and  $\mu$  a partition of weight  $\sum_{i=1}^k |\lambda^{\circ_i}|$  and satisfying the condition  $\mu = (ks_1, \dots, ks_p)$  for some positive integers  $s_1, \dots, s_p$ . Let  $D_{\lambda^\circ, \mu}(d)$  be a diagonal class with only one element  $T^\circ$  and  $\Theta_k(T^\circ) = (\nu_{T^\circ}, J_{T^\circ})$  the corresponding ribbon rigged configuration. The  $i$ -th partition of the shape  $\nu_{T^\circ}$  is the rectangular shape  $(k^{s_1 + \dots + s_i})$ .*

*Proof:* Let  $T^\circ$  be a  $k$ -tuple of tableaux with the same values at the same positions of each tableau. The corresponding matrix  $M^{T^\circ}$  is

$$M^{T^\circ} = \begin{pmatrix} k & 0 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & & & & \vdots \\ k & 0 & \dots & \dots & \dots & 0 \\ 0 & k & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & k & 0 & \dots & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \\ \vdots & & & & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \end{pmatrix},$$

where  $k$  occurs  $s_i$  times in the  $i$ -th column. This implies that the shape is given by the

matrix

$$A_E(M^{T^\circ}) = \begin{pmatrix} k & k & \dots & \dots & \dots & k \\ \vdots & \vdots & & & & \vdots \\ k & k & \dots & \dots & \dots & k \\ 0 & k & k & \dots & \dots & k \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & k & k & \dots & \dots & k \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \\ \vdots & & & & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 0 & k \end{pmatrix} .$$

Then, the  $i$ -th partition in the shape  $\nu_{T^\circ}$  is the rectangular partition  $(k^{s_1+\dots+s_i})$ .  $\square$

**Proposition 6** *All  $k$ -th primitive roots of unity are roots of the inversions polynomials of diagonal classes with strictly more than one element.*

*Proof:* Let  $T^\circ$  be a  $k$ -tuple of tableaux in a diagonal class  $D_{\lambda^\circ, \mu}(d)$  with strictly more than one element and let  $\nu = (\nu^{(1)}, \dots, \nu^{(p)})$  be the shape of the ribbon rigged configuration  $\Theta_k(T^\circ)$ , which is the same for all the  $k$ -tuples of tableaux in this diagonal class (see proposition 5). Let  $h$  be the last position such that the  $h$ -th diagonal  $d_h$  has at least two different elements. Then, the  $(h+1)$ -th partition in  $\nu_{T^\circ}$  is a rectangular partition of width  $k$  and height  $s \leq r$ . As the last part of  $\nu^{(h)} = (\nu_1^{(h)}, \dots, \nu_l^{(h)})$  is equal to  $a$  with  $a < h$ , the following coefficient appears in the inversions polynomial  $\tilde{I}_{\lambda^\circ, \mu}(q, d)$

$$\begin{bmatrix} \nu_l^{(h+1)} - \nu_{l+1}^{(h)} \\ \nu_l^{(h)} - \nu_{l+1}^{(h)}, \nu_l^{(h+1)} - \nu_l^{(h)} \end{bmatrix} = [a-0, k-a] .$$

By definition, all  $k$ -th primitive roots of unity are roots of the  $q$ -binomial coefficient  $[a, k-a]$ . Finally for all  $\zeta$ ,  $k$ -th primitive root of unity,

$$\tilde{I}_{\lambda^\circ, \mu}(\zeta, d) = 0 .$$

$\square$

**Corollary 4** *Let  $\lambda^\circ = (\lambda^{\circ_1}, \dots, \lambda^{\circ_k})$  be a  $k$ -tuple of single row partitions and  $\mu$  a partition of weight  $\sum_{i=1}^k |\lambda^{\circ_i}|$  and  $\zeta$  a  $k$ -th primitive root of unity. We have the following specializations for the inversions polynomials*

- if all the parts of  $\mu$  are divisible by  $k$ ,

$$\tilde{I}_{\lambda^\circ, \mu}(\zeta) = 1 ,$$

- if there exists a part of  $\mu$  which is not divisible by  $k$ ,

$$\tilde{I}_{\lambda^\circ, \mu}(\zeta) = 0 .$$

*Proof:* Using the partition of  $Tab_{\lambda^\circ, \mu}$  into diagonal classes, the inversions polynomial  $\tilde{I}_{\lambda^\circ, \mu}(q)$  can be expressed as

$$\tilde{I}_{\lambda^\circ, \mu}(q) = \sum_d \tilde{I}_{\lambda^\circ, \mu}(q, d) ,$$

where  $d$  describe the set of all the diagonal vectors which appears in  $Tab_{\lambda^\circ, \mu}$ . By splitting the index set with respect to the cardinality of the diagonal classes and specializing  $q$  at  $\zeta$ , we obtain

$$\tilde{I}_{\lambda^\circ, \mu}(q) = \sum_{d/\#D_{\lambda^\circ, \mu}(d) = 1} \tilde{I}_{\lambda^\circ, \mu}(q, d) + \sum_{d/\#D_{\lambda^\circ, \mu}(d) > 1} \tilde{I}_{\lambda^\circ, \mu}(q, d) , \quad (19)$$

$$\tilde{I}_{\lambda^\circ, \mu}(\zeta) = \sum_{d/\#D_{\lambda^\circ, \mu}(d) = 1} \tilde{I}_{\lambda^\circ, \mu}(\zeta, d) + \sum_{d/\#D_{\lambda^\circ, \mu}(d) > 1} \tilde{I}_{\lambda^\circ, \mu}(\zeta, d) . \quad (20)$$

Using the result of corollary 2, we conclude that the first factor of the equation 20 gives 1 if all the parts of  $\mu$  are divisible by  $k$  and 0 otherwise. By proposition 6, the second factor always gives 0 and we obtain the result.  $\square$

Now, we are able to give a combinatorial proof of the specialization of Hall-Littlewood functions given in theorem 1. Let  $n$  and  $k$  be two positive integers. Let denote by  $\Lambda_p^k$  the set of all partitions of weight  $p$  with all parts divisible by  $k$ . Using results of corollary 4,

$$\tilde{Q}'_{n^k}(X; \zeta) = \sum_{\mu \vdash nk} \tilde{I}_{((n), \dots, (n)), \mu}(\zeta) m_\mu \quad (21)$$

$$= \sum_{\mu \in \Lambda_{nk}^k} m_\mu = \sum_{\mu \in \Lambda_{nk}^k} p_k \circ m_{\mu/k} , \quad (22)$$

where  $\mu/k$  denote the partition  $(\frac{\mu_1}{k}, \dots, \frac{\mu_p}{k})$ . By the linearity of the plethysm, we obtain

$$\tilde{Q}'_{n^k}(X; \zeta) = p_k \circ \left( \sum_{\mu \in \Lambda_{nk}^k} m_{\mu/k} \right) , \quad (23)$$

and by definition of the complete function, we conclude that

$$\tilde{Q}'_{n^k}(X; \zeta) = p_k \circ h_n(X) .$$

In the rectangular case, the constant  $\eta(n^k)$  is equal to

$$\eta(n^k) = n \sum_{i=1}^k (i-1) = \frac{nk(k-1)}{2} .$$

This implies that  $\zeta^{n(n^k)} = (-1)^{(k-1)n}$  and the specialization for  $Q'_{n^k}(X; \zeta)$ .

**Remark** In the case where  $\lambda^\circ$  is a  $k$ -tuple of single column partitions  $((1^n), \dots, (1^n))$ , there is a similar bijection between  $k$ -tuples of Young tableaux of shape  $\lambda^\circ$  and ribbon rigged configurations (see [15]). That permits to prove, with the same method, the following specialisation

$$Q'_{k^{nk}}(X; \zeta) = (-1)^{(k-1)n} p_k \circ e_n(X) .$$

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