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QUANTUM FIELD THEORY IN THE LARGE N LIMIT: A REVIEW

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ABSTRACT

We review the solutions of $O(N)$ and $U(N)$ quantum field theories in the large N limit and as $1/N$ expansions, in the case of vector representations. Since invariant composite fields have small fluctuations for large N, the method relies on constructing effective field theories for composite fields after integration over the original degrees of freedom. We first solve a general scalar $U(\phi^2)$ field theory for N large and discuss various non-perturbative physical issues such as critical behaviour. We show how large N results can also be obtained from variational calculations.We illustrate these ideas by showing that the large N expansion allows to relate the $(\phi^2)^2$ theory and the non-linear σ-model, models which are renormalizable in different dimensions. Similarly, a relation between $CP(N-1)$ and abelian Higgs models is exhibited. Large N techniques also allow solving self-interacting fermion models. A relation between the Gross–Neveu, a theory with a four-fermi self-interaction, and a Yukawa-type theory renormalizable in four dimensions then follows. We discuss dissipative dynamics, which is relevant to the

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approach to equilibrium, and which in some formulation exhibits quantum mechanics supersymmetry. This also serves as an introduction to the study of the 3D supersymmetric quantum field theory. Large N methods are useful in problems that involve a crossover between different dimensions. We thus briefly discuss finite size effects, finite temperature scalar and supersymmetric field theories. We also use large N methods to investigate the weakly interacting Bose gas. The solution of the general scalar $U(\phi^2)$ field theory is then applied to other issues like tricritical behaviour and double scaling limit.

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1 Introduction

Studies of the non-perturbative features of quantum field theories are at the forefront of theoretical physics research. Though remarkable progress has been achieved in recent years, still, some of the more fundamental questions have only a descriptive answer, whereas non-perturbative calculable schemes are seldom at hand. The absence of calculable dynamics in realistic models is often supplemented by simpler models in which the essence of the dynamics is revealed. Such a calculable framework for exploring theoretical ideas is given by large N quantum field theories.

Very early quantum field theorists have looked for methods to solve field theory beyond perturbation theory and obtain confirmation of perturbative results. Moreover, some important physical questions are often intrinsically non-perturbative. Let us mention, for illustration, the problem of fermion-pair condensation. A number of similar schemes were proposed, all of mean-field theory nature, variational methods, self-consistent approximations, all reducing the interacting theory to a free fermion theory with self-consistently determined parameters. For example, the quartic fermion self-interaction $(\bar{\psi}\psi)^2$ would be replaced by a term proportional to $\langle\bar{\psi}\psi\rangle\bar{\psi}\psi$, where $\langle\bar{\psi}\psi\rangle$ is the free field average. However, all these methods have several drawbacks: it is unclear how to improve the results systematically, the domain of validity of the approximations are often unknown, in fact there is no obvious small parameter. To return to the fermion example, one realizes that the approximation would be justified if for some reasons the fluctuations of the composite field $\bar{\psi}\psi$ were much smaller than the fluctuations of the fermion field itself.

Large N techniques solve this problem in the spirit of the central limit theorem of the theory of probabilities. If the field has N components, in the large N limit scalar (in the group sense) composite fields are sums of many terms and therefore may have small fluctuations (at least if the different terms are sufficiently uncorrelated). Therefore, if we are able to construct an effective field theory for the scalars, integrating out the original degrees of freedom, we can solve the field theory not only in the large N limit, but also in a systematic $1/N$ expansion. On the technical level, one notes that in vector representations the number of independent scalars is finite and independent of N, unlike what happens for matrix representations. This explains why vector models have been solved much more generally than matrix models.

In this review we describe a few applications of large N techniques to quantum field theories (QFT) with $O(N)$ or $U(N)$ symmetries, where the fields are in the *vector* representation [1]. A summary of results are presented here in the study of the phase structure of quantum field theories. It is demonstrated that large N results nicely complement results obtained from more conventional perturbative renormalization group (RG) [2]. Indeed, the shortcoming of the latter method is that it mainly applies to gaussian or near gaussian fixed points. This re-

stricts space dimension to dimensions in which the corresponding effective QFT is renormalizable, or after dimensional continuation, to the neighbourhood of such dimensions. In some cases, large N techniques allow a study in generic dimensions. In this review we will, in particular, stress two points: first, it is always necessary to check that the $1/N$ expansion is both IR finite and renormalizable. This is essential for the stability of the large N results and the existence of a $1/N$ expansion. Second, the large N expansion is just a technique, with its own (often unknown) limitations and it should not be discussed in isolation. Instead, as we shall do in the following examples, it should be combined, when possible, with other perturbative techniques and the reliability of the $1/N$ expansion should be inferred from the general consistency of all results.

Second-order phase transitions in classical statistical physics provide a first illustration of the usefulness of the large N expansion. Due to the divergence of the correlation length at the critical temperature, one finds that near T_c, system share universal properties which can be described by effective continuum quantum field theories. The N-vector model that we discuss in **Sections 2** and **3** is the simplest example but it has many applications since it allows to describe the critical properties of systems like vapour–liquid, binary mixtures, superfluid Helium or ferromagnetic transitions as well as the statistical properties of polymers. Before showing what kind of information can be provided by large N techniques, we will first shortly recall what can be learned from perturbative RG methods. Long distance properties can be described by a $(\phi^2)^2$ field theory in which analytic calculations can be performed only in an $\varepsilon = 4 - d$ expansion. From lattice model considerations, one expects that the same properties can also be derived from a different QFT, the $O(N)$ non-linear σ model, which, however, can be solved only as an $\varepsilon = d - 2$ expansion. It is somewhat surprising that the same statistical model can be described by two different theories. Since the results derived in this way are valid *a priori* only for ε small, there is no overlap to test the consistency. The large N expansion enables one to discuss generic dimensions and thus to understand the relation between both field theories.

Similar large N techniques can also be applied to other non-linear models and we briefly examine the example of the $CP(N-1)$ model.

Four-fermi interactions have been proposed to generate a composite Higgs particle in four dimensions, as an alternative to a Yukawa-type theory, as one finds in the Standard Model. Again, in the specific example of the Gross–Neveu model, in **Section 4** we will use large N techniques to clarify the relations between these two approaches. We will finally briefly investigate other models with chiral properties, like massless QED or the Thirring model.

Preceding the discussion of supersymmetric models, we study in **Section 5** critical dynamics of purely dissipative systems, which are known to provide a simple field theory extension of supersymmetric quantum mechanics. We then study in **Section 6** two SUSY models in two and three dimensions: the supersymmetric ϕ^4 field theory, which is shown to have a peculiar phase structure in

the large N limit, and a supersymmetric non-linear σ model.

Other applications of the large N expansion include finite size effects (**Section 3.4**) and finite temperature field theory to which we devote **Section 7** for non-supersymmetric theories and **Section 8** for the supersymmetric theories of **Section 6**. In these situations a dimensional crossover occurs between the large size or zero temperature situation, where the infinite volume theory is relevant, to a dimensionally reduced theory in the small volume or high temperature limit. Both effective field theories being renormalizable in different dimensions, perturbative RG cannot describe correctly both situations. Again, large N techniques help understanding the crossover. We often compare in this review the large N results with those obtained by variational methods, since it is often possible to set up variational calculations which parallel the large N limit calculations.

The effect of weak interactions on Bose gases at the Bose–Einstein condensation temperature are analyzed in **Section 9** where large N techniques are employed for the non-perturbative calculations of physical quantities.

We then return in **Section 10** to general scalar boson field theories, and examine multi-critical points (where the large N technique will show some obvious limitations), and the double scaling limit, a toy model for discussing problems encountered in matrix models of 2D quantum gravity.

In this review, we also discuss the breaking of scale invariance. In most quantum field theories spontaneous and explicit breaking of scale invariance occur simultaneously and thus the breaking of scale symmetry is not normally accompanied by the appearance of a massless Nambu–Goldstone boson. Spontaneous breaking of scale invariance, unaccompanied by explicit breaking, associated with a non-zero fixed point and the creation of a massless bound state is demonstrated in **Section A1**. In some cases the dynamics by which scale invariance is broken in a theory which has no trace anomalies in perturbation theory is directly related to the breaking of internal symmetry. This appears in the phase structure of $O(N) \times O(N)$ symmetric models for N large, where the breaking of scale invariance is directly related to the breaking of the internal symmetry. The spontaneous breaking of scale invariance in a supersymmetric, $O(N)$ symmetric vector model in three dimensions was also studied in **Section 6**, were one finds the creation of a massless fermionic bound state as the supersymmetric partner of the massless boson in the supersymmetric ground state.

2 Scalar field theory for N large: general formalism and applications

In this section we present a general formalism that allows studying $O(N)$ symmetric scalar field theories in the large N limit and, more generally, order by order in a large N-expansion.

Of particular interest is the $(\phi^2)^2$ statistical field theory that describes the universal properties of a number of phase transitions. The study of phase transitions and critical phenomena in statistical physics has actually been one of the early applications of large N techniques. It was realized that an N-component spin-model (the spherical model) could be solved exactly in the large N limit [3,4], and the solutions revealed scaling laws and non-trivial (i.e. non-gaussian or mean-field like) critical behaviour . Later, following Wilson (and Wilson–Fisher) [5–7] it was discovered that universal properties of critical systems could be derived from the $(\phi^2)^2$ field theory within the framework of the formal $\varepsilon = 4 - d$ expansion, by a combination of perturbation theory and renormalization group (RG). The peculiarity of this scheme, whose reliability in the physical dimension $d = 3$, and thus $\varepsilon = 1$, could only be guessed, demanded some independent confirmation. This was provided in particular by developing a scheme to solve the N-component $(\phi^2)^2$ field theory in the form of an $1/N$ expansion, whose leading order yields the results of the spherical spin-model [8–18].

Here, we first solve more general $O(N)$ symmetric field theories in the large N limit, reducing the problem to a steepest descent calculation [19,20]. The $(\phi^2)^2$ field theory is then discussed more thoroughly from the point of view of phase transitions and critical phenomena. We also stress the relation between the large N limit and variational principles [21]. Moreover, some other issues relevant to particle physics like triviality, renormalons or Higgs mass are examined in the large N limit.

A remarkable implication of the large N analysis is that two different field theories, the $(\phi^2)^2$ theory and the non-linear σ model, describe the same critical phenomena, a results that holds to all orders in the $1/N$ expansion [22–24]. This result has several generalizations, leading for instance to a relation between the $CP(N-1)$ [25–29] and the abelian Higgs models.

Finally, large N techniques are well adapted to the analysis of finite size effects in critical systems [30], a question we investigate in the more convenient formalism of the non-linear σ model in section 3.4.

2.1 Scalar field theory: the large N formalism

We consider an $O(N)$ symmetric euclidean action (or classical hamiltonian) for an N-component scalar field ϕ:

$$\mathcal{S}(\phi) = \int \left[\tfrac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + NU\big(\phi^2(x)/N\big) \right] \mathrm{d}^d x \,, \tag{2.1}$$

where $U(\rho)$ is a general polynomial, and the explicit N dependence has been chosen to lead to a large N limit. The corresponding partition function is given by a functional integral:

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{S}(\phi)\right]. \tag{2.2}$$

To render the perturbative expansion finite, a cut-off Λ consistent with the symmetry is implied.

The solution of the model in the large N limit is based on an idea of mean field type: it can be expected that, for N large, $O(N)$ invariant quantities like

$$\phi^2(x) = \sum_{i=1}^{N} \phi_i^2(x)$$

self-average and therefore have small fluctuations (as for the central limit theorem this relies on the assumption that the components ϕ_i are somehow weakly correlated). Thus, for example,

$$\left\langle \phi^2(x)\phi^2(y)\right\rangle \underset{N\to\infty}{\sim} \left\langle \phi^2(x)\right\rangle \left\langle \phi^2(y)\right\rangle.$$

This observation suggests to take $\phi^2(x)$ as a dynamical variable, rather than $\phi(x)$ itself. For this purpose, we introduce two additional fields λ and ρ and impose the constraint $\rho(x) = \phi^2(x)/N$ by an integral over λ. For each point of space x, we use the identity

$$1 = N \int \mathrm{d}\rho\, \delta(\phi^2 - N\rho) = \frac{N}{4i\pi} \int \mathrm{d}\rho \mathrm{d}\lambda\, \mathrm{e}^{\lambda(\phi^2 - N\rho)/2}, \tag{2.3}$$

where the λ integration contour runs parallel to the imaginary axis. The insertion of the identity into the integral (2.2) yields a new representation of the partition function:

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}\rho][\mathrm{d}\lambda] \exp\left[-\mathcal{S}(\phi,\rho,\lambda)\right] \tag{2.4}$$

with

$$\mathcal{S}(\phi,\rho,\lambda) = \int \left[\tfrac{1}{2}\left[\partial_\mu\phi(x)\right]^2 + NU\big(\rho(x)\big) + \tfrac{1}{2}\lambda(x)\big(\phi^2(x) - N\rho(x)\big)\right] \mathrm{d}^d x\,. \tag{2.5}$$

The functional integral (2.4) is then gaussian in ϕ, the integral over the field ϕ can be performed and the dependence on N of the partition function becomes explicit. Actually, it is convenient to separate the components of ϕ into one component σ, and $N-1$ components π, and integrate over π only (for $T < T_c$ it may even be convenient to integrate over only $N-2$ components). This does not

affect the large N limit but only the $1/N$ corrections. To generate σ-correlation functions, we add also a source $H(x)$ (a space-dependent magnetic field in the ferromagnetic language) to the action. The partition function then becomes

$$\mathcal{Z}(H) = \int [\mathrm{d}\sigma][\mathrm{d}\rho][\mathrm{d}\lambda] \exp\left[-\mathcal{S}_N(\sigma,\rho,\lambda) + \int \mathrm{d}^d x\, H(x)\sigma(x)\right] \tag{2.6}$$

with

$$\mathcal{S}_N(\sigma,\rho,\lambda) = \int \left[\tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 + NU(\rho) + \tfrac{1}{2}\lambda(x)(\sigma^2(x) - N\rho(x))\right] \mathrm{d}^d x \\ + \tfrac{1}{2}(N-1)\,\mathrm{tr}\ln\left[-\nabla^2 + \lambda\right]. \tag{2.7}$$

ϕ^2-field correlation functions. In this formalism it is natural to consider also correlation functions involving the ρ-field which by construction is proportional to the ϕ^2 composite field. In the framework of phase transitions, near the critical temperature, the ϕ^2 field plays the role of the energy operator.

Remark. One can wonder how much one can still generalize this formalism (with only one vector field). Actually, one can solve also the most general $O(N)$ symmetric field theory with two derivatives. Indeed, this involves adding the two terms

$$Z(\phi^2/N)(\partial_\mu\phi)^2, \quad V(\phi^2/N)(\partial_\mu\phi\cdot\phi)^2/N,$$

where Z and V are two arbitrary functions. These terms can be rewritten

$$Z(\rho)(\partial_\mu\phi)^2, \quad NV(\rho)(\partial_\mu\rho)^2,$$

in such a way that the ϕ integral remains gaussian and can be performed. This reduces again the study of the large N limit to the steepest descent method.

2.2 Large N limit: saddle points and phase transitions

We now study the large N limit, the function $U(\rho)$ being considered as N independent. If we take $\sigma = O(N^{1/2})$, $\rho = O(1)$, $\lambda = O(1)$ all terms in $\mathcal{S}_N$ are of order N and the functional integral can be calculated for N large by the steepest descent method [19].

Saddle points. We look for a uniform saddle point ($\sigma(x), \rho(x), \lambda(x)$ are space-independent because we look for the ground state, thus excluding instantons or solitons)

$$\sigma(x) = \sigma\,, \quad \rho(x) = \rho \quad \text{and} \quad \lambda(x) = m^2 \tag{2.8}$$

because the λ saddle point value must be positive.

The action density $\mathcal{E}$ in zero field H then becomes

$$\mathcal{E} = NU(\rho) + \tfrac{1}{2}m^2(\sigma^2 - N\rho) + \frac{N}{2}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\,\ln[(k^2+m^2)/k^2]. \tag{2.9}$$

Differentiating then $\mathcal{E}$ with respect to σ, ρ and m^2, we obtain the saddle point equations

$$m^2\sigma = 0\,, \tag{2.10a}$$

$$\tfrac{1}{2}m^2 = U'(\rho), \tag{2.10b}$$

$$\sigma^2/N - \rho + \frac{1}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d k}{k^2+m^2} = 0\,. \tag{2.10c}$$

Regularization and large cut-off expansion. In the last equation we have now introduced a cut-off Λ explicitly. This means, more precisely, that we have replaced the propagator by some regularized form

$$\frac{1}{k^2+m^2} \mapsto \tilde{\Delta}_\Lambda(k) = \frac{1}{k^2 D(k^2/\Lambda^2)+m^2} \quad \text{with } D(z) = 1 + O(z), \tag{2.11}$$

where the function $D(z)$ is a function strictly positive for $z > 0$, analytic in the neighbourhood of the real positive semi-axis, and increasing faster than $z^{(d-2)/2}$ for $z \to +\infty$.

We set

$$\frac{1}{(2\pi)^d}\int^{\Lambda}\frac{\mathrm{d}^d k}{k^2+m^2} \equiv \frac{1}{(2\pi)^d}\int \mathrm{d}^d k\, \tilde{\Delta}_\Lambda(k) \equiv \Omega_d(m) \equiv \Lambda^{d-2}\omega_d(m/\Lambda). \tag{2.12}$$

Below we need the first terms of the expansion of $\Omega_d(m)$ for $m^2 \to 0$. One finds for $z \to 0$ and $d > 2$ an expansion which we parametrize as (for details see appendix A2.1)

$$\omega_d(z) = \omega_d(0) - K(d)z^{d-2} + a(d)z^2 + O\left(z^4, z^d\right). \tag{2.13}$$

The constant $K(d)$ is universal, that is independent of the cut-off procedure:

$$K(d) = -\frac{1}{(4\pi)^{d/2}}\Gamma(1-d/2) = -\frac{\pi}{2\sin(\pi d/2)}N_d\,, \tag{2.14a}$$

$$N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\,, \tag{2.14b}$$

where we have introduced for later purpose the usual loop factor N_d.

The constant $a(d)$, in contrast, depends explicitly on the regularization, that is on the way large momenta are cut,

$$a(d) = N_d \times \begin{cases} \displaystyle\int_0^\infty k^{d-5}\mathrm{d}k\left(1 - \frac{1}{D^2(k^2)}\right) & \text{for } d < 4, \\ \displaystyle -\int_0^\infty \frac{k^{d-5}\mathrm{d}k}{D^2(k^2)} & \text{for } d > 4, \end{cases} \tag{2.15}$$

but for $\varepsilon = 4 - d \to 0$ satisfies

$$a(d) \underset{\varepsilon=4-d\to 0}{\sim} 1/(8\pi^2\varepsilon). \tag{2.16}$$

Integrating $\Omega_d(m)$ over m^2, we then obtain a finite expression for the regularized integral arising from the ϕ integration and given in (2.9):

$$\frac{1}{(2\pi)^d}\int^{\Lambda} \mathrm{d}^d k\, \ln[(k^2+m^2)/k^2] = \int_0^m 2s\mathrm{d}s\, \Omega_d(s). \tag{2.17}$$

¿From the expansion (2.13), we infer

$$\begin{aligned}\int_0^m 2s\mathrm{d}s\, \Omega_d(s) = -2\frac{K(d)}{d}m^d + \Omega_d(0)m^2 + \frac{a(d)}{2}m^4\Lambda^{d-4} \\ + O(m^6\Lambda^{d-6}, m^{d+2}\Lambda^{-2}).\end{aligned} \tag{2.18}$$

Finally, for $d = 4$ these expressions have to be modified because a logarithmic contribution appears:

$$\omega_d(z) - \omega_d(0) \sim \frac{1}{8\pi^2} z^2 \ln z\,. \tag{2.19}$$

Phase transitions. Eq. (2.10a) implies either $\sigma = 0$ or $m = 0$. We see from the $\mathrm{tr}\ln$ term in expression (2.7) that m, at this order, is also the mass of the π field. When $\sigma \neq 0$, the $O(N)$ symmetry is spontaneously broken, m vanishes and the massless π-field corresponds to the expected $N-1$ Goldstone modes. If, instead, $\sigma = 0$ the $O(N)$ symmetry is unbroken and the N ϕ-field components have the same mass m.

We then note from Eq. (2.10c) that the solution $m = 0$ can exist only for $d > 2$, because at $d = 2$ the integral is IR divergent. This result is consistent with the Mermin–Wagner–Coleman theorem: in a system with only short range forces a continuous symmetry cannot be broken for $d \leq 2$, in the sense that the expectation value σ of the order parameter must necessarily vanish. The potential Goldstone modes are responsible for this property: being massless, as we expect from general arguments and verify here, they induce an IR instability for $d \leq 2$. Therefore, we discuss below only the dimensions $d > 2$; the dimension $d = 2$ will be examined separately in the more appropriate formalism of the non-linear σ model in section 3.1.

Moreover, we assume now that the polynomial $U(\rho)$ has for $\rho \geq 0$ a unique minimum at a strictly positive value of ρ where $U''(\rho)$ does not vanish, otherwise the critical point would turn out to be a multicritical point, a situation that will be studied in section 10.

(i) *Broken phase.* When $m = 0$, the saddle point Eqs. (2.10) reduce to

$$U'(\rho) = 0\,, \quad \sigma^2/N - \rho + \frac{1}{(2\pi)^d}\int^{\Lambda} \frac{\mathrm{d}^d k}{k^2} = 0\,.$$

The first equation implies that ρ is given by the minimum of U and the second equation then determines the field expectation value. Clearly a solution can be found only if

$$\rho > \rho_c = \frac{1}{(2\pi)^d}\int^{\Lambda} \frac{\mathrm{d}^d k}{k^2} = \Omega_d(0), \tag{2.20}$$

where the definition (2.12) has been used, and then

$$\sigma = \sqrt{N(\rho - \rho_c)}. \tag{2.21}$$

(ii) *The symmetric phase.* When $\sigma = 0$, the saddle point equations (2.10) can be written as

$$\rho - \rho_c = \Omega_d(m) - \Omega_d(0), \quad m^2 = 2U'(\rho)\,. \tag{2.22}$$

The first equation (2.22) implies $\rho \leq \rho_c$. At the value $\rho = \rho_c$ a transition takes place between an ordered phase $\rho > \rho_c$ and a symmetric phase $\rho \leq \rho_c$. The condition

$$U'(\rho_c) = 0 \tag{2.23}$$

determines critical potentials.

In expression (2.7) we see that the σ-propagator then becomes [13]

$$\Delta_\sigma \underset{|p|,m\ll\Lambda}{\sim} \frac{1}{p^2 + m^2}\,. \tag{2.24}$$

Therefore, m is at this order the physical mass or the inverse of the correlation length ξ of the field σ (and thus of all components of the ϕ-field).

The condition $m \ll \Lambda$, or equivalently $\xi \gg 1/\Lambda$ defines the critical domain. The first equation (2.22) then implies that $\rho - \rho_c$ is small in the critical domain. From the second equation (2.22) follows that $U'(\rho)$ is small and thus ρ is close to the minimum of $U(\rho)$. We can then expand $U(\rho)$ around ρ_c:

$$U(\rho) = U'(\rho_c)(\rho - \rho_c) + \tfrac{1}{2}U''_c(\rho - \rho_c)^2 + O\big((\rho - \rho_c)^3\big), \tag{2.25}$$

and it is convenient to set

$$U'(\rho_c) = \tfrac{1}{2}\tau\,, \quad |\tau| \ll \Lambda^2\,.$$

With this parametrization

$$m^2 = 2U''_c(\rho - \rho_c) + \tau + O\big((\rho - \rho_c)^2\big).$$

With our assumptions U_c'' is strictly positive (the sign ensures that the extremum is a minimum). Then τ is positive in the symmetric phase, while $\tau < 0$ corresponds to the broken phase.

At this point we realize that, in the case of a generic critical point, $U(\rho)$ can be approximated by a quadratic polynomial. The problem then reduces to a discussion of the $(\phi^2)^2$ field theory that indeed, in the framework of the $\varepsilon = 4-d$ expansion, describes critical phenomena. Therefore, we postpone a more detailed analysis of the solutions of the saddle point equations and first summarize a few properties of the ϕ^4 field theory from the point of view of perturbative RG.

2.3 $(\phi^2)^2$ field theory, renormalization group, universality and large N limit

¿From now on, the discussion in this section will be specific to the $(\phi^2)^2$ field theory. In terms of the initial N-component scalar field ϕ, we write the action as

$$\mathcal{S}(\phi) = \int \left\{ \frac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r \phi^2(x) + \frac{1}{4!} \frac{u}{N} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x \,. \tag{2.26}$$

¿From the point of view of classical statistical physics, this model has the interpretation of an effective field theory that encodes large distance properties of various statistical models near a second order phase transition. In this framework r is a regular function of the temperature T near the critical temperature T_c. To the critical temperature corresponds a value r_c of the parameter r at which the correlation length ξ (the inverse of the physical mass in field theory language) diverges. For r close to its critical value r_c, $\xi(r)\Lambda \gg 1$ and a continuum limit can be defined.

We denote by $\Gamma^{(\ell,n)}$ the vertex or 1PI functions of ℓ ϕ^2 and n ϕ fields (the coefficients of the expansion of the thermodynamic potential) in Fourier representation. We set

$$r = r_c + \tau \tag{2.27}$$

and τ thus characterizes the deviation from the critical value r_c. In the symmetric phase ($\tau \geq 0$) in zero field, the ϕ and ϕ^2 correlation functions then satisfy, as functions of Λ, the dimensionless coupling constant $g = u\Lambda^{4-d}/N$ and the deviation $\tau \ll \Lambda^2$, RG equations:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \frac{n}{2}\eta(g) - \left(\tau \frac{\partial}{\partial \tau} + \ell\right) \eta_2(g)\right] \Gamma^{(\ell,n)} = \delta_{n0}\delta_{\ell 2} \Lambda^{d-4} B(g), \tag{2.28}$$

where $\beta(g)$ is associated with the flow of the coupling constant g, $\eta(g)$ and $\eta_2(g)$ to the anomalous dimensions of the fields ϕ and ϕ^2, and $B(g)$ is associated to the additive renormalization of the ϕ^2 two-point function.

The RG β-function in dimension $d = 4 - \varepsilon$,

$$\beta(g, \varepsilon) = -\varepsilon g + \frac{N+8}{48\pi^2} g^2 + O(g^3),$$

has for $d < 4$ a non-trivial zero

$$g^* = \frac{48\pi^2\varepsilon}{N+8} + O(\varepsilon^2),$$

which IR attractive since

$$\beta'(g^*) \equiv \omega = \varepsilon + O(\varepsilon^2) > 0\,. \tag{2.29}$$

This property is the starting point of the determination of the universal critical properties of the model within the framework of the so-called ε-expansion.

We explain here, instead, how the model can be solved in the large N limit. In this way we will be able to verify at fixed dimension, in some limit, many results obtained by perturbative methods.

Large N limit. To the action (2.26) corresponds the function

$$U(\rho) = \frac{1}{2} r\rho + \frac{u}{4!}\rho^2. \tag{2.30}$$

The large N limit is taken at $U(\rho)$ fixed and this implies with our conventions that u, the coefficient of ϕ^4/N, is fixed.

The integral over ρ in (2.6) is then gaussian. The integration results in simply replacing $\rho(x)$ by the solution of

$$\tfrac{1}{6} u\rho(x) + r = \lambda(x). \tag{2.31}$$

and one obtains the action

$$\begin{aligned}\mathcal{S}_N(\sigma,\lambda) = \frac{1}{2}\int \mathrm{d}^d x \left[(\partial_\mu\sigma)^2 + \lambda\sigma^2 - \frac{3N}{u}\lambda^2 + \frac{6Nr}{u}\lambda\right] \\ + \frac{(N-1)}{2}\,\mathrm{tr}\ln\left[-\nabla^2 + \lambda(\bullet)\right].\end{aligned} \tag{2.32}$$

Note, however, that the field ρ has a more direct physical interpretation than the field $\lambda(x)$.

Diagrammatic interpretation. In the $(\phi^2)^2$ field theory, the leading perturbative contributions in the large N limit come from chains of "bubble" diagrams of the form displayed in figure 1. These diagrams asymptotically form a geometric series, which the algebraic techniques explained in this section allow to sum.

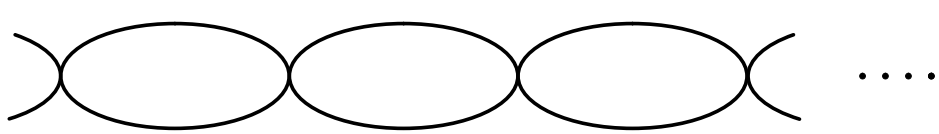

Fig. 1 The dominant diagrams in the large N limit.

The low temperature phase. We first assume that σ, the expectation value of the field, does not vanish, and thus the $O(N)$ symmetry is spontaneously broken. The constant ρ is then given by Eq. (2.10b) which reduces to $U'(\rho) = 0$. The solution must satisfy $\rho > \rho_c$ and Eq. (2.21) then yields $\sigma = \sqrt{N(\rho - \rho_c)}$ (we recall $\rho_c = \Omega_d(0)$, Eq. (2.20)).

The condition (2.23) determines the critical potential U:

$$U'(\rho_c) = 0 \;\Rightarrow\; r = r_c = -u\rho_c/6\,.$$

The expectation value of the field vanishes for $r = r_c$, which thus corresponds to the critical temperature T_c. Then $(r - r_c = \tau)$,

$$U'(\rho) = 0 \;\Rightarrow\; \rho - \rho_c = -(6/u)\tau\,. \tag{2.33}$$

The $O(N)$ symmetry is broken for $\tau < 0$, that is at low temperature, and we can rewrite Eq. (2.21) as

$$\sigma^2 = -(6/u)\tau \propto (-\tau)^{2\beta} \quad \text{with } \beta = \tfrac{1}{2}\,. \tag{2.34}$$

We find that, for N large, the exponent β remains mean-field like or quasi-gaussian in all dimensions.

The high temperature phase. For $\tau > 0$, that is above T_c, σ vanishes. Using Eqs. (2.20,2.27) in Eqs. (2.10b) and (2.10c), we then find

$$m^2 = (u/6)(\rho - \rho_c) + \tau\,, \tag{2.35a}$$

$$\rho - \rho_c = \Omega_d(m) - \Omega_d(0). \tag{2.35b}$$

(i) For $d > 4$, the expansion (2.13) implies that the leading contribution to $\rho - \rho_c$ is proportional to m^2, as the l.h.s. of Eq. (2.35a), and thus, at leading order,

$$m^2 = \xi^{-2} \sim \tau^{2\nu} \quad \text{with} \quad \nu = \tfrac{1}{2}\,, \tag{2.36}$$

which is the mean-field or gaussian result for the correlation exponent ν.

(ii) For $2 < d < 4$, the leading term is now of order m^{d-2}:

$$\rho - \rho_c \sim -K(d)m^{d-2}.$$

In Eq. (2.35a) the leading m-dependent contribution for $m \to 0$ now comes from $\rho - \rho_c$. Keeping only the leading term in (2.13), we obtain ($\varepsilon = 4 - d$)

$$m = \xi^{-1} \sim \tau^{1/(2-\varepsilon)}, \tag{2.37}$$

which shows that the exponent ν is no longer gaussian (or mean-field like):

$$\nu = \frac{1}{2-\varepsilon} = \frac{1}{d-2}\,\cdot \tag{2.38}$$

(iii) For $d = 4$, the leading m-dependent contribution in Eq. (2.35a) still comes from $\rho - \rho_c$:

$$m^2 \sim \frac{48\pi^2}{u} \frac{\tau}{\ln(\Lambda/m)}. \tag{2.39}$$

The correlation length has no longer a power law behaviour but, instead, the behaviour of the gaussian model modified by a logarithm. This is typical of a situation where the gaussian fixed point is stable, in the presence of a marginal operator.

(iv) Examining Eq. (2.10c) for $\sigma = 0$ and $d = 2$, we find that the correlation length becomes large only for $r \to -\infty$. This peculiar situation will be discussed in the framework of the non-linear σ-model.

Critical limit $\tau = 0$. At $\tau = 0$, m vanishes and from the form (2.24) of the σ-propagator, we find that the critical exponent η remains gaussian for all d:

$$\eta = 0 \;\Rightarrow\; d_\phi = \tfrac{1}{2}(d-2)\,. \tag{2.40}$$

We verify that for $d \leq 4$, the exponents β, ν, η satisfy the scaling relation proven within the framework of the ε-expansion:

$$\beta = \nu d_\phi = \tfrac{1}{2}\nu(d - 2 + \eta).$$

Singular free energy and scaling equation of state. In a constant magnetic field H in the σ direction, the free energy density $W(H)$ (defined here as the opposite of the action density $\mathcal{E}$ when the saddle point equations are used) is given by [16]

$$\begin{aligned} W(H) &= \ln \mathcal{Z}/\Omega = -\mathcal{E} \\ &= N\left[\frac{3}{2u}m^4 - \frac{3r}{u}m^2 - \frac{1}{2}m^2\sigma^2/N + H\sigma/N - \int_0^m s\mathrm{d}s\,\Omega_d(s)\right], \end{aligned}$$

where Ω is the d dimensional space volume and ρ has been eliminated using Eq. (2.10c). The saddle point values m^2, σ are given by Eq. (2.10b) and the modified saddle point Eq. (2.10a):

$$m^2\sigma = H\,. \tag{2.41}$$

The magnetization M, expectation value of ϕ, is

$$M = \frac{\partial W}{\partial H} = \sigma\,, \tag{2.42}$$

because partial derivatives of W with respect to m^2 and σ vanish as a consequence of the saddle point equations. The thermodynamic potential density $\mathcal{G}(M)$, Legendre transform of $W(H)$, follows:

$$\begin{aligned} \mathcal{G}(M) &= HM - W(H) \\ &= N\left[-\frac{3}{2u}m^4 + \frac{3r}{u}m^2 + \frac{1}{2}m^2M^2/N + \int_0^m s\mathrm{d}s\,\Omega_d(s)\right]. \end{aligned} \tag{2.43}$$

As a property of the Legendre transformation, the saddle point equation for m^2 is now obtained by expressing that the derivative of $\mathcal{G}$ with respect to m^2 vanishes.

The expansion for large Λ of the tr ln has been given in Eq. (2.18). Introducing r_c, one obtains

$$\mathcal{G}(M)/N = \frac{3}{2}\left(\frac{1}{u^*} - \frac{1}{u}\right) m^4 + \frac{3(r - r_c)}{u} m^2 + \frac{1}{2} m^2 M^2/N - \frac{K(d)}{d} m^d, \quad (2.44)$$

where we have defined

$$u^* = \frac{6}{a(d)} \Lambda^\varepsilon. \quad (2.45)$$

Note that for $d < 4$ the term proportional to m^4 is negligible for m small with respect to the singular term m^d. Thus, at leading order in the critical domain,

$$\mathcal{G}(M)/N = \frac{3}{u} \tau m^2 + \frac{1}{2} m^2 M^2/N - \frac{K(d)}{d} m^d, \quad (2.46)$$

where τ has been defined in (2.27).

Expressing that the derivative with respect to m^2 vanishes,

$$(6/u)\tau + M^2/N - K(d) m^{d-2} = 0\,,$$

we obtain

$$m = \left[\frac{1}{K(d)}\left((6/u)\tau + M^2/N\right)\right]^{1/(d-2)}.$$

It follows that the leading

contribution to the thermodynamic potential, in the critical domain, is given by

$$\mathcal{G}(M)/N \sim \frac{(d-2)}{2d} \frac{1}{\left(K(d)\right)^{2/(d-2)}} \left[(6/u)\tau + M^2/N\right]^{d/(d-2)}. \quad (2.47)$$

¿From $\mathcal{G}(M)$ can be derived various other quantities like the equation of state, which is obtained by differentiating with respect to M. It can be cast into the scaling form

$$H = \frac{\partial \mathcal{G}}{\partial M} = h_0 M^\delta f\left(a_0 \tau / M^2\right), \quad (2.48)$$

where h_0 and a_0 are normalization constants. The exponent δ is given by

$$\delta = \frac{d+2}{d-2}\,, \quad (2.49)$$

in agreement with the general scaling relation $\delta = d/d_\phi - 1$, and the function $f(x)$ by

$$f(x) = (1+x)^{2/(d-2)}. \quad (2.50)$$

The asymptotic form of $f(x)$ for x large implies $\gamma = 2/(d-2)$ again in agreement with the scaling relation $\gamma = \nu(2-\eta)$. Taking into account the values of the critical exponents γ and β, it is then easy to verify that the function f satisfies all required properties like for example Griffith's analyticity. In particular, the equation of state can be cast into the parametric form [34]

$$\begin{aligned} M &= (a_0)^{1/2} R^{1/2} \theta \,, \\ \tau &= 3R\left(1-\theta^2\right), \\ H &= h_0 R^{\delta/2} \theta \left(3-2\theta^2\right)^{2/(d-2)} . \end{aligned}$$

Leading corrections to scaling. The m^4 term yields the leading corrections to scaling. It is subleading by a power of τ:

$$m^4/m^d = O(\tau^{(4-d)/(d-2)}).$$

The exponent governing the leading corrections to scaling in the temperature variable is $\omega\nu$ (ω is defined in Eq. (2.29)) and thus [17]

$$\omega\nu = (4-d)/(d-2) \;\Rightarrow\; \omega = 4-d\,. \tag{2.51}$$

Note that for the special value $u = u^*$ this correction vanishes.

Specific heat exponent. Amplitude ratios. Differentiating twice $\mathcal{G}(M)$ with respect to τ, we obtain the specific heat at fixed magnetization

$$C_H \propto \left[(6/u)\tau + M^2/N\right]^{(4-d)/(d-2)} . \tag{2.52}$$

For $M = 0$, we identify the specific exponent

$$\alpha = \frac{4-d}{d-2}, \tag{2.53}$$

which indeed is equal to $2 - d\nu$, as predicted by scaling relations. Among the universal ratios of amplitudes, one can calculate for example R_ξ^+ and R_c (for definitions, see chapter 29 of ref. [2])

$$(R_\xi^+)^d = \frac{4N}{(d-2)^3}\frac{\Gamma(3-d/2)}{(4\pi)^{d/2}}, \quad R_c = \frac{4-d}{(d-2)^2}. \tag{2.54}$$

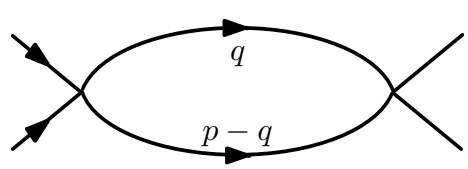

Fig. 2 The "bubble" diagram $B_\Lambda(p,m)$.

The λ and ϕ^2 two-point functions. In the high temperature phase, differentiating twice the action (2.7) with respect to $\lambda(x), \rho(x)$ and replacing the field $\lambda(x)$ by its expectation value m^2, we find the λ-propagator

$$\Delta_\lambda(p) = -\frac{2}{N}\left[\frac{6}{u} + B_\Lambda(p,m)\right]^{-1}, \tag{2.55}$$

where $B_\Lambda(p,m)$ is the bubble diagram of figure 2:

$$B_\Lambda(p,m) = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{(q^2+m^2)\left[(p-q)^2+m^2\right]}, \tag{2.56}$$

and the cut-off symbol Λ means calculated with a regularized propagator as in (2.11).

The λ-propagator is negative because the λ-field is imaginary. Using the relation (2.31), we obtain the ρ two-point function (in the Fourier representation the constant shift only produces a δ-function at $p=0$) and thus as noted in 2.3, the ϕ^2 two-point function

$$\left\langle \phi^2\phi^2 \right\rangle = N^2 \left\langle \rho\rho \right\rangle = -\frac{12N/u}{1+(u/6)B_\Lambda(p,m)}. \tag{2.57}$$

At zero momentum we recover the specific heat. The small m expansion of $B_\Lambda(0,m)$ can be derived from the expansion (2.13). One finds

$$\begin{aligned} B_\Lambda(0,m) &= \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{(q^2+m^2)^2} \\ &= -\frac{\partial}{\partial m^2}\Omega_d(m) \underset{m\ll\Lambda}{=} (d/2-1)K(d)m^{-\varepsilon} - a(d)\Lambda^{-\varepsilon} + \cdots \end{aligned} \tag{2.58}$$

The singular part of the specific heat thus vanishes as m^ε, in agreement with Eq. (2.52) for $M=0$.

In the critical theory ($m=0$ at this order) for $2 \le d < 4$, the denominator is also dominated at low momentum by the integral

$$B_\Lambda(p,0) = \frac{1}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2(p-q)^2} \underset{2<d<4}{=} b(d)p^{-\varepsilon} - a(d)\Lambda^{-\varepsilon} + O\left(\Lambda^{d-6}p^2\right), \tag{2.59}$$

where

$$b(d) = -\frac{\pi}{\sin(\pi d/2)}\frac{\Gamma^2(d/2)}{\Gamma(d-1)}N_d\,, \tag{2.60}$$

and thus

$$\Delta_\lambda(p) \underset{p\to 0}{\sim} -\frac{2}{Nb(d)}p^\varepsilon. \tag{2.61}$$

We again verify consistency with scaling relations. In particular, we note that in the large N limit the *dimension of the field* λ is

$$[\lambda] = [\rho] = [\phi^2] = d - 1/\nu = \tfrac{1}{2}(d+\varepsilon) = 2\,, \tag{2.62}$$

a result important for the $1/N$ perturbation theory.

Remarks.

(i) For $d = 4$, the integral has a logarithmic behaviour:

$$B_\Lambda(p,0) \underset{p\ll\Lambda}{\sim} \frac{1}{8\pi^2}\ln(\Lambda/p) + \text{ const.}\,, \tag{2.63}$$

and still gives *the leading contribution* to the inverse propagator $\Delta_\lambda \propto 1/\ln(\Lambda/p)$.

(ii) Note, therefore, that for $d \le 4$ the contributions generated by the term proportional to $\lambda^2(x)$ in (2.7) are always negligible in the critical domain.

The $\langle\sigma\sigma\rangle$ two-point function at low temperature. In the phase of broken symmetry the action, after translation of expectation values, includes a term proportional to $\sigma\lambda$ and thus the propagators of the fields σ and λ are elements of a 2×2 matrix $\mathbf{M}$:

$$\mathbf{M^{-1}}(p) = \begin{pmatrix} p^2 & \sigma \\ \sigma & -3N/u - \frac{1}{2}NB_\Lambda(p,0) \end{pmatrix}\,, \tag{2.64}$$

where $\sigma = \langle\sigma(x)\rangle$ and B_Λ is given by Eq. (2.59). For $d < 4$ at leading order for $|p| \ll \Lambda$, the determinant is given by

$$1/\det\mathbf{M}(p) \sim -N\left[b(d)p^{d-2} + 6\tau/u\right],$$

where the relation (2.34) has been used. For $|r - r_c| \ll \Lambda^2$, this expression defines a crossover mass scale

$$m_{\rm cr} = (-\tau/u)^{1/(d-2)} \propto \Lambda\big((r_c - r)/\Lambda^2\big)^{1/(d-2)} = \Lambda\big((r_c - r)/\Lambda^2\big)^{\nu}, \tag{2.65}$$

at which a crossover between Goldstone behaviour ($N-1$ massless free particles) and critical behaviour (N massless interacting particles) occurs.

At $d = 4$, the form (2.63) becomes relevant and

$$m_{\rm cr}^2 \propto \frac{r_c - r}{\ln[\Lambda^2/(r_c - r)]}. \tag{2.66}$$

Finally, for $d > 4$, $B_\Lambda(p,0)$ has a limit for $p = 0$ and therefore

$$m_{\rm cr} \propto \sqrt{r_c - r}\,. \tag{2.67}$$

In all dimensions $m_{\rm cr}$ scales near r_c as the physical mass above r_c.

2.4 RG functions and leading corrections to scaling

The RG functions. For a more detailed verification of the consistency between the large N results and RG predictions, we now calculate RG functions at leading order for $N \to \infty$. We set (Eq. (2.45))

$$u = Ng\Lambda^{\varepsilon}, \quad g^* = u^*\Lambda^{-\varepsilon}/N = 6/(Na)\,, \tag{2.68}$$

where the constant $a(d)$ has been defined in (2.13) and behaves for $\varepsilon = 4-d \to 0$ like $a(d) \sim 1/(8\pi^2\varepsilon)$ (Eq. (2.16)).

One then verifies that m solution of Eqs. (2.35) satisfies asymptotically for Λ large an equation that expresses that it is RG invariant:

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(g)\frac{\partial}{\partial g} - \eta_2(g)\tau\frac{\partial}{\partial\tau}\right) m(\tau, g, \Lambda) = 0\,, \tag{2.69}$$

where in the r.h.s. contributions of order $1/\Lambda^2$ have been neglected. The RG functions $\beta(g)$ and $\eta_2(g)$ are then given by

$$\beta(g) = -\varepsilon g(1 - g/g^*), \tag{2.70}$$

$$\nu^{-1}(g) = 2 + \eta_2(g) = 2 - \varepsilon g/g^*. \tag{2.71}$$

When $a(d)$ is positive (but this not true for all regularizations, see the discussion below), one finds an IR fixed point g^*, as well as exponents $\omega = \varepsilon$, and $\nu^{-1} = d-2$, in agreement with Eqs. (2.51,2.38). In the framework of the ε-expansion, ω is associated with the leading corrections to scaling. In the large N limit ω remains smaller than 2 for $\varepsilon < 2$, and this extends the property established near $d = 4$ to all dimensions $2 \le d \le 4$.

Finally, applying the RG equations to the propagator (2.24), one finds

$$\eta(g) = 0\ , \tag{2.72}$$

a result consistent with the value (2.40) found for $\eta = \eta(g^*)$.

Leading corrections to scaling. From the general RG analysis, we expect the leading corrections to scaling to vanish for $u = u^*$. This property has already been verified for the free energy. Let us now consider the correlation length or mass m given by Eq. (2.35). If we keep the leading correction to the integral for m small (Eq. (2.13)), we find

$$1 - \frac{u}{u^*} + (u/6)K(d)m^{-\varepsilon} + O\left(m^{2-\varepsilon}\Lambda^{-2}\right) = \frac{\tau}{m^2}\,, \tag{2.73}$$

where Eq. (2.68) has been used. We see that the leading correction again vanishes for $u = u^*$. Actually, all correction terms suppressed by powers of order ε for $d \to 4$ vanish simultaneously as expected from the RG analysis of the ϕ^4 field

theory. Moreover, one verifies that the leading correction is proportional to $(u-u^*)\tau^{\varepsilon/(2-\varepsilon)}$, which leads to $\omega\nu = \varepsilon/(2-\varepsilon)$, in agreement with Eqs. (2.51,2.38).

In the same way, if we keep the leading correction to the λ-propagator in the critical theory (equation (2.59)), we find

$$\Delta_\lambda(p) = -\frac{2}{N}\left[\frac{6}{u} - \frac{6}{u^*} + b(d)p^{-\varepsilon}\right]^{-1}, \tag{2.74}$$

where terms of order Λ^{-2} and $1/N$ have been neglected. The leading corrections to scaling again cancel for $u = u^*$ exactly, as expected.

Discussion.

(i) One can show that a perturbation due to irrelevant operators is equivalent, at leading order in the critical region, to a modification of the $(\phi^2)^2$ coupling. This can be explicitly verified here. The amplitude of the leading correction to scaling has been found to be proportional to $6/u - a(d)\Lambda^{-\varepsilon}$, where the value of $a(d)$ depends on the cut-off procedure and thus on contributions of irrelevant operators. Let us call u' the $(\phi^2)^2$ coupling constant in another scheme where a is replaced by a'. Identifying the leading correction to scaling, we find the relation

$$\frac{6\Lambda^\varepsilon}{u} - a(d) = \frac{6\Lambda^\varepsilon}{u'} - a'(d),$$

a homographic relation that is consistent with the special form (2.70) of the β-function.

(ii) *The sign of* $a(d)$. It is generally assumed that $a(d)$ is positive for $2 < d < 4$. This is indeed what one finds in the simplest regularization schemes, for example when the function $D(k^2)$ in (2.11) is an increasing function of k^2. Moreover, $a(d)$ is always positive near four dimensions where it diverges like

$$a(d) \underset{d\to 4}{\sim} \frac{1}{8\pi^2\varepsilon}.$$

Then, for $2 < d < 4$ there exists an IR fixed point, corresponding to a non-trivial zero u^* of the β-function. For the value $u = u^*$ the leading corrections to scaling vanish.

However, for $d < 4$ fixed that positivity of $a(d)$ is not assured. For example, in the case of simple lattice regularizations it has been shown that in $d = 3$ the sign is arbitrary.

When $a(d)$ is negative, the RG method for large N (at least in the perturbative framework) is confronted with a serious difficulty. Indeed, the coupling flows in the IR limit to large values where the large N expansion is no longer reliable. It is not known whether this signals a real pathology of the model in the RG sense, or is just an artifact of the large N limit.

Another way of viewing the problem is to examine directly the relation between bare and renormalized coupling constant. Calling $g_{\rm r} m^{4-d}$ the renormalized four-point function at zero momentum, we find

$$m^{4-d} g_{\rm r} = \frac{\Lambda^{4-d} g}{1 + \Lambda^{4-d} g N B_\Lambda(0,m)/6}. \tag{2.75}$$

In the limit $m \ll \Lambda$, the relation can be written as

$$\frac{1}{g_{\rm r}} = \frac{1}{g_{\rm r}^*} + \left(\frac{m}{\Lambda}\right)^{4-d} \left(\frac{1}{g} - \frac{N a(d)}{6}\right), \quad \frac{1}{g_{\rm r}^*} = \frac{(d-2) N K(d)}{12}. \tag{2.76}$$

We see that when $a(d) < 0$, the limiting value $g_{\rm r} = g_{\rm r}^*$ for $m/\Lambda = 0$ cannot be reached by varying g when m/Λ is small but finite (since $g > 0$). In the same way, leading corrections to scaling can no longer be cancelled.

2.5 Small coupling constant and large momentum expansions for $d < 4$

Section 3.1 is devoted to a systematic discussion of the $1/N$ expansion. However, the $1/N$ correction to the two-point function will help us to investigate immediately the following problem: the perturbative expansion of the massless ϕ^4 field theory has IR divergences for any dimension $d < 4$, although we believe the critical theory to exist. In the framework of the $1/N$ expansion, instead, the critical theory ($T = T_c, m^2 = 0$) is defined for any dimension $d < 4$. This implies that the coefficients of the $1/N$ expansion cannot be expanded in a Taylor series of the coupling constant.

To understand the phenomenon, we consider the $\langle \sigma\sigma \rangle$ correlation function at order $1/N$. At this order only one diagram contributes (figure 3), containing two $\lambda^2 \sigma$ vertices. After mass renormalization, in the large cut-off limit, we find

$$\Gamma^{(2)}_{\sigma\sigma}(p) = p^2 + \frac{2}{N(2\pi)^d} \int \frac{{\rm d}^d q}{(6/u) + b(d) q^{-\varepsilon}} \left(\frac{1}{(p+q)^2} - \frac{1}{q^2}\right) + O\left(\frac{1}{N^2}\right). \tag{2.77}$$

We now expand $\Gamma^{(2)}_{\sigma\sigma}$ for $u \to 0$. Note that since the gaussian fixed point is an UV fixed point, the small coupling expansion is also a large momentum expansion.

An analytic study then reveals that the integral has an expansion of the form

$$\sum_{k \geq 1} \alpha_k u^k p^{2-k\varepsilon} + \beta_k u^{(2+2k)/\varepsilon} p^{-2k}. \tag{2.78}$$

The coefficients α_k, β_k can be obtained by performing a Mellin transformation over u on the integral. Indeed, if a function $f(u)$ behaves like u^t for u small, then the Mellin transform

$$M(s) = \int_0^\infty {\rm d}u \, u^{-1-s} f(u),$$

has a pole at $s = t$. Applying the transformation to the integral and inverting q and u integrations, we have to calculate the integral

$$\int_0^\infty \mathrm{d}u \, \frac{u^{-1-s}}{(6/u) + b(d)q^{-\varepsilon}} = \frac{1}{6} \left(\frac{b(d)q^{-\varepsilon}}{6} \right)^{1-s} \frac{\pi}{\sin \pi s} \, .$$

Then, the value of the remaining q integral follows from the generic result (3.27).

The terms with integer powers of u correspond to the formal perturbative expansion where each integral is calculated for ε small enough. α_k has poles at $\varepsilon = (2l+2)/k$ for which the corresponding power of p^2 is $-l$, that is an integer. One verifies that β_l has a pole at the same value of ε and that the singular contributions cancel in the sum [36]. For these dimensions logarithms of u appear in the small u expansion.

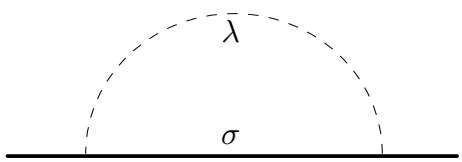

Fig. 3 The diagram contributing to $\Gamma^{(2)}_{\sigma\sigma}$ at order $1/N$.

2.6 Dimension four: triviality, renormalons, Higgs mass

A number of issues concerning the physics of the $(\phi^2)^2$ theory in four dimensions can be addressed within the framework of the large N expansion. For simplicity reasons, we consider here only the critical (i.e. massless) theory.

Triviality and UV renormalons. One verifies that the renormalized coupling constant g_{r}, defined as the value of the vertex $\langle \sigma\sigma\sigma\sigma \rangle$ at momenta of order $\mu \ll \Lambda$, is given by

$$g_{\mathrm{r}} = \frac{g}{1 + \frac{1}{6} N g B_\Lambda(\mu, 0)} \, , \tag{2.79}$$

where $B_\Lambda(p,0)$, which corresponds to the bubble diagram (figure 2), is given by Eq. (2.63):

$$B_\Lambda(p,0) \underset{p \ll \Lambda}{\sim} \frac{1}{8\pi^2} \ln(\Lambda/p) + \text{ const.} \, .$$

We see that when the ratio μ/Λ goes to zero, the renormalized coupling constant vanishes, independently of the value of g (here g is physical, that is $g > 0$). This is the so-called *triviality* property. In the traditional presentation of quantum field field, one usually insists in taking the infinite cut-off Λ limit. Here, one finds then only a free field theory. Another way of formulating the problem is the following: it seems impossible to construct in four dimensions a ϕ^4 field theory consistent (in the sense of satisfying all usual physical requirements) on all scales for non zero coupling. Of course, in the logic of *effective* field theories this

is no longer an issue. The triviality property just implies that the renormalized or effective charge is logarithmically small as indicated by Eqs. (2.79,2.63). Note that if g is generic (not too small) and Λ/μ large, g_r is essentially independent of the initial coupling constant. The renormalized coupling remains an adjustable, though bounded, quantity only when the bare coupling is small enough and the RG flow is thus very slow.

If we proceed formally and, ignoring the problem, express the leading contribution to the four-point function in terms of the renormalized constant,

$$\frac{g}{1+\frac{N}{48\pi^2}g\ln(\Lambda/p)} = \frac{g_{\rm r}}{1+\frac{N}{48\pi^2}g_{\rm r}\ln(\mu/p)}\,,$$

we then find that the function has a pole for

$$p = \mu\,{\rm e}^{48\pi^2/(Ng_{\rm r})}\,.$$

This unphysical pole (called sometimes Landau's ghost) is generated because $g=0$ is an IR fixed point. If we calculate contributions of higher orders, for example to the two-point function, this pole makes the loop integrals diverge. In an expansion in powers of $g_{\rm r}$, each term is instead calculable but one finds, after renormalization, UV contributions of the type

$$\int^{\infty}\frac{{\rm d}^4q}{q^6}\left(-\frac{Ng_{\rm r}}{48\pi^2}\ln(\mu/q)\right)^k \underset{k\to\infty}{\propto} \left(\frac{Ng_{\rm r}}{96\pi^2}\right)^k k!\,.$$

The perturbative manifestation of Landau's ghost is the appearance of contributions to the perturbation series which are not Borel summable. This effect is called UV renormalon effect [40]. By contrast the contributions due to the finite momentum region, which can be evaluated by a semi-classical analysis, are Borel summable, but invisible for N large. Note, finally, that this UV problem is independent of the mass of the field ϕ, which we have taken zero only for simplicity.

IR renormalons. We now illustrate the problem of IR renormalons with the same example of the massless $(\phi^2)^2$ theory (but now zero mass is essential), in four dimensions, in the large N limit [44]. We calculate the contribution of the small momentum region to the mass renormalization, at cut-off Λ fixed. In the large N limit, the mass renormalization then is proportional to (see Eq. (2.77))

$$\int^{\Lambda}\frac{{\rm d}^4q}{q^2\big(1+\frac{1}{6}NgB_\Lambda(q)\big)} \sim \int\frac{{\rm d}^4q}{q^2\big(1+\frac{N}{48\pi^2}g\ln(\Lambda/q)\big)}\,.$$

It is easy to expand this expression in powers of the coupling constant g. The term of order k in the limit $k\to\infty$ behaves as $(-1)^k(N/96\pi^2)^k k!$. This contribution has the alternating sign of the semi-classical contribution. Note that

more generally for N finite on finds $(-\beta_2/2)^k k!$. IR singularities are responsible for additional, Borel summable, contributions to the large order behaviour.

In a theory asymptotically free for large momentum, clearly the roles of IR and UV singularities are interchanged.

The mass of the σ field in the phase of broken symmetry. In four dimensions the ϕ^4 theory is an ingredient of the Standard Model, and the field σ then represents the Higgs field. With some reasonable assumptions, it is possible to establish for finite N a semi-quantitative bound on the Higgs mass. Let us examine here this question for N large.

In the phase of broken symmetry the action, after translation of expectation values, includes a term proportional to $\sigma\lambda$ and thus the propagators of the fields σ and λ are elements of the 2×2 matrix $\mathbf{M}$ defined in (2.64), where $B_\Lambda(p, 0)$ is given by Eq. (2.63).

It is convenient to introduce a RG invariant mass scale M, that we define by

$$\frac{48\pi^2}{Ng} + 8\pi^2 B_\Lambda(p, 0) = \ln(M/p).$$

Then,

$$M \propto \mathrm{e}^{48\pi^2/Ng}\, \Lambda\,.$$

Poles of the propagator correspond to zeros of $\det \mathbf{M}$. Solving the equation $\det \mathbf{M} = 0$, one finds for the mass m_σ of the field σ at this order

$$p^2 \ln(M/p) = -(16\pi^2/N)\sigma^2 \;\Rightarrow\; m_\sigma^2 \ln(iM/m_\sigma) = (16\pi^2/N)\sigma^2.$$

The solution of the equation is complex, because the particle σ can decay into massless Goldstone bosons. At σ fixed, the mass decreases when the cut-off increases or when the coupling constant goes to zero. Expressing that the mass must be smaller than the cut-off, one obtains an upper-bound on m_σ (but which depends somewhat on the precise regularization) [45].

2.7 Other methods. General vector field theories

The large N limit can be obtained by several other algebraic methods. Without being exhaustive, let us list a few. Schwinger–Dyson equations for N large lead to a self-consistent equation for the two-point function [48]. Some versions of the Hartree–Fock approximation or variational methods also yield the large N result as we show in section 2.8. The functional (also called exact) RG takes the form of partial differential equations in the large N limit [49].

¿From the point of view of stochastic quantization or critical dynamics the Langevin equation also becomes linear and self-consistent for N large, because the fluctuations of $\phi^2(x, t)$ are small. As a byproduct the large N expansion of the equilibrium equal-time correlation functions is recovered. This is a topic we study in section 5.

General vector field theories. We have shown how the large N expansion can be generated for a general function $NU(\phi^2/N)$. This method will be applied in section 10 to the study of multicritical points and double scaling limit.

We now briefly explain how the algebraic method of section 2.3 can be generalized to $O(N)$ symmetric actions that depend on several vector fields. Again, the composite fields which are expected to have small fluctuations, are the $O(N)$ scalars constructed from all $O(N)$ vectors. One thus introduces pairs of fields and Lagrange multipliers for all independent $O(N)$ invariant scalar products constructed from the many-component fields [52].

Let us illustrate the idea with the example of two fields ϕ_1 and ϕ_2, and a symmetric interaction arbitrary function of the three scalar invariants $\rho_{12} = \phi_1 \cdot \phi_2/N$, $\rho_{11} = \phi_1^2/N$ and $\rho_{22} = \phi_2^2/N$:

$$\mathcal{S}(\phi_1, \phi_2) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\partial_\mu \phi_1(x)\right]^2 + \tfrac{1}{2} \left[\partial_\mu \phi_2(x)\right]^2 + NU(\rho_{11}, \rho_{12}, \rho_{22}) \right\}. \quad (2.80)$$

We then introduce three pairs of fields $\rho_{ij}(x)$ and $\lambda_{ij}(x)$ and use the identity

$$\exp\left[- \int \mathrm{d}^d x \, NU(\phi_1^2/N, \phi_1 \cdot \phi_2/N, \phi_2^2/N) \right] \propto \int \left[\mathrm{d}\rho_{ij}(x) \, \mathrm{d}\lambda_{ij}(x)\right]$$
$$\times \exp\left\{ - \int \mathrm{d}^d x \left[\sum_{ij} \tfrac{1}{2} \lambda_{ij} \left(\phi_i \cdot \phi_j - N\rho_{ij}\right) + NU(\rho_{11}, \rho_{12}, \rho_{22}) \right] \right\}. \quad (2.81)$$

The identity (2.81) transforms the action into a quadratic form in ϕ_i, the integration over ϕ can thus be performed. The large N limit is then again obtained by the steepest descent method. In the special case in which $U(\rho)$ is a quadratic function, the integral over all ρ's can also be performed. If the action is a general $O(N)$ invariant function of p fields ϕ_i, it is necessary to introduce $p(p-1)/2$ pairs of ρ and λ fields.

2.8 Variational calculations in large N quantum field theory

A possible extra insight into the large N limit and its non-perturbative nature is revealed when variational methods are employed [21]. The gaussian variational wave functional reproduces for N large the saddle point or gap equations of the $1/N$ expansion but also yields a clear picture of the end-point contribution in the variational parameter phase space. It is applicable for finite values of N though it simplifies for N large. However, as usual with variational methods, it is difficult to improve results systematically, while the large N limit is the leading term in an $1/N$ expansion.

To explain the variational method, we use again the concrete example of the $O(N)$ symmetric scalar field theory (2.2) with the euclidean action (2.1):

$$\mathcal{S}(\phi) = \int \left[\tfrac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + NU\big(\phi^2(x)/N\big) \right] \mathrm{d}^d x \,.$$

In section 7 the more general situation of a finite euclidean time interval β with periodic boundary conditions will be discussed. The functional integral (2.2) then represents the quantum partition function

$$\mathcal{Z}(\beta) = \mathrm{tr}\, \mathrm{e}^{-\beta H},$$

where H is the quantum hamiltonian corresponding to the euclidean action, and $\beta = 1/T$ the inverse temperature.

For any action $\mathcal{S}$ and auxiliary action $\mathcal{S}_0$, we can write the identity

$$\mathcal{Z} = \int [\mathrm{d}\phi]\, \mathrm{e}^{-\mathcal{S}(\phi)} = \mathcal{Z}_0 \left\langle \mathrm{e}^{-(\mathcal{S}-\mathcal{S}_0)} \right\rangle_0 , \tag{2.82}$$

where $\langle \bullet \rangle_0$ means expectation value with respect to $\mathrm{e}^{-\mathcal{S}_0}$. The variational principle then relies on the general (convexity) inequality

$$\left\langle \mathrm{e}^{-(\mathcal{S}-\mathcal{S}_0)} \right\rangle_0 \quad \geq \quad \mathrm{e}^{-\left\langle \mathcal{S}-\mathcal{S}_0 \right\rangle_0} . \tag{2.83}$$

The trial action $\mathcal{S}_0$ then is chosen to maximize the r.h.s., a general strategy that also leads to mean field approximations.

In the limit of infinite d-dimensional volume, or equivalently $d-1$-dimensional volume V_{d-1} and zero-temperature ($\beta \to \infty$)

$$\lim_{\beta\to\infty} -\frac{1}{\beta V_{d-1}} \ln \mathcal{Z} = \mathcal{E} , \quad \lim_{\beta\to\infty} -\frac{1}{\beta V_{d-1}} \ln \mathcal{Z}_0 = \mathcal{E}_0 ,$$

where $\mathcal{E}$ and $\mathcal{E}_0$ are the ground state energy (or action) densities corresponding to $\mathcal{S}$ and $\mathcal{S}_0$ respectively. As a consequence one obtains the inequality

$$\mathcal{E} \leq \mathcal{E}_{\mathrm{var.}} = \mathcal{E}_0 + \frac{1}{V_{d-1}\beta} \langle \mathcal{S} - \mathcal{S}_0 \rangle_0 ,$$

One verifies that $V_{d-1}\mathcal{E}_{\mathrm{var.}}$ is also the expectation value of the hamiltonian corresponding to $\mathcal{S}$ in the ground state of the hamiltonian corresponding to $\mathcal{S}_0$. In the zero temperature limit the choice of a trial action $\mathcal{S}_0$ thus becomes equivalent to the choice of a trial wave functional (the ground state) in a Schrödinger representation.

For trial action $\mathcal{S}_0$, we take a free field action with a mass m that is a variational parameter:

$$\mathcal{S}_0 = \frac{1}{2} \int \mathrm{d}^d x \, \left((\partial_\mu \phi)^2 + m^2 \phi^2 \right) .$$

This corresponds, in the Schrödinger representation, to choosing a gaussian wave functional as a variational state:

$$\Psi(\phi) = \exp \left[-\frac{1}{2} \int \mathrm{d}^{d-1} k \, \tilde{\phi}(k) \sqrt{k^2 + m^2} \tilde{\phi}(-k) \right] ,$$

where $\tilde{\phi}(k)$ are the Fourier components of $\phi(x)$ or more generally,

$$\Psi(\phi) = \exp\left[-\frac{1}{2}\int \mathrm{d}^{d-1}x \int \mathrm{d}^{d-1}y\, \phi(x) K_m(x-y)\phi(y)\right],$$

where K_m is determined by minimizing $\mathcal{E}_{\text{var.}}$ [54].

The corresponding energy density is

$$\mathcal{E}_0 = \frac{N}{2(2\pi)^d}\int \mathrm{d}^d k\, \ln[(k^2+m^2)/k^2] = \frac{N}{(2\pi)^{d-1}}\int \mathrm{d}^{d-1}k\, \left[\tfrac{1}{2}\omega(k)\right],$$

where $\omega(k) = \sqrt{k^2+m^2}$.

The variational energy then becomes

$$\mathcal{E}_{\text{var.}} = \frac{N}{2(2\pi)^d}\int \mathrm{d}^d k\, \ln[(k^2+m^2)/k^2] + N\left\langle U\big(\phi^2(x)/N\big) - \tfrac{1}{2}m^2\phi^2(x)/N\right\rangle_0 . \tag{2.84}$$

The r.h.s. contains gaussian expectation values that can be calculated explicitly. We introduce the parameter

$$\rho = \left\langle \phi^2(x)/N\right\rangle_0 = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{k^2+m^2} = \Omega_d(m), \tag{2.85}$$

where $\Omega_d(m)$ is defined by Eq. (2.12). The form of $\mathcal{S}_0$ implies

$$\frac{\partial \mathcal{E}_0}{\partial m^2} = \tfrac{1}{2}\left\langle \phi^2\right\rangle = \tfrac{1}{2}N\rho\,.$$

The important remark, which explains the role of the large N limit, is (see also appendix A3) that for $N\to\infty$ the r.h.s. in Eq. (2.84) simplifies:

$$\left\langle U(\phi^2/N)\right\rangle = U(\left\langle \phi^2/N\right\rangle)\,[1+O(1/N)]\,. \tag{2.86}$$

We then find

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{\longrightarrow} \frac{1}{2(2\pi)^d}\int \mathrm{d}^d k\, \ln[(k^2+m^2)/k^2] + U(\rho) - \tfrac{1}{2}m^2\rho\,,$$

an expression identical (for $\sigma = 0$) to (2.9). Note, however, that here Eq. (2.10c) is automatically satisfied, since it defines the parameter ρ (Eq. (2.85)). The quantity $\mathcal{E}$ can now be minimized with respect to the free parameter m^2, while without the constraint (2.85) it has no minimum.

In the broken phase, previous equations have no solution and one must take as a trial action a free action with a shifted field:

$$\mathcal{S}_0 = \frac{1}{2}\int \mathrm{d}^d x\, \Big((\partial_\mu\phi(x))^2 + m^2(\phi(x)-\phi_0)^2\Big),$$

where $\phi_0^2 = \sigma^2$ is an additional variational parameter. Eq. (2.85) is replaced by

$$\rho = (1/N)\left\langle \phi^2(x)\right\rangle_0 = \Omega_d(m) + \sigma^2/N\,. \tag{2.87}$$

Then,

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{=} \frac{1}{2(2\pi)^d}\int \mathrm{d}^d k\, \ln[(k^2+m^2)/k^2] + U(\rho) - \tfrac{1}{2}m^2(\rho - \sigma^2/N).$$

We note that again $\mathcal{E}_{\text{var}}$ is identical to (2.9) when Eq. (2.10c) is used.

Differentiating the variational energy with respect to m^2, we first notice

$$\frac{\partial}{\partial m^2}\mathcal{E}_0 = \tfrac{1}{2}\left\langle(\phi-\phi_0)^2\right\rangle_0 = \tfrac{1}{2}(N\rho - \sigma^2)$$

and thus find

$$\begin{aligned}\frac{\partial \mathcal{E}_{\text{var.}}}{\partial m^2} &= \tfrac{1}{2}N(\rho-\sigma^2/N) + N\frac{\partial}{\partial m^2}\left[U(\rho) - \tfrac{1}{2}m^2(\rho-\sigma^2/N)\right]\\ &= N\frac{\partial\rho}{\partial m^2}\left[U'(\rho) - \tfrac{1}{2}m^2\right].\end{aligned}$$

Since $\partial\rho/\partial m^2$ is strictly negative, the derivative vanishes only when

$$U'(\rho) - \tfrac{1}{2}m^2 = 0\,,$$

which, combined with Eq. (2.87), yields the set of equations (2.10b) and (2.10c). Differentiating with respect to σ, we recover Eq. (2.10a).

Therefore, in the large N limit the saddle point equations and the variational equations coincide. It should, however, be remembered that since we perform a variational calculation the lowest energy eigenstate is not necessarily the state determined by the solution of Eqs. (2.10). The end-points in the range of variation of m^2 and ${\phi_0}^2$ have to be considered as well and the values of $\mathcal{E}_{\text{var}}$ at these points have to be examined and compared to the extremum values [21].

The $(\phi^2)^2$ field theory at negative coupling for $d = 4$. We again consider the large N self interacting scalar field in $d = 4$ dimensions with ($\rho \equiv \phi^2/N$)

$$U(\rho) = \frac{1}{2}r\rho + \frac{u}{4!}\rho^2. \tag{2.88}$$

The large N limit is defined as $N \to \infty$ holding u, r and the necessary ultraviolet cutoff, Λ, fixed. We have already discussed the triviality of the $(\phi^2)^2$ field theory in $d = 4$ for positive coupling in section 2.6. Here, we add some comments about the situation for negative coupling. Even though the initial functional integral

is defined only for $u > 0$, the large N expansion seems to have a well-defined continuation.

The only divergence appears in

$$\rho(m^2) = \left\langle \phi^2 \right\rangle / N = \rho_c - \frac{m^2}{8\pi^2} \ln(\Lambda/m) + \sigma^2/N \,, \quad \rho_c = \Omega_4(0) \,, \tag{2.89}$$

where a numerical, regularization dependent, constant has been cancelled by adjusting the definition of the cut-off Λ.

We then obtain the energy density

$$\frac{1}{N}\mathcal{E}_{\rm var}(\sigma, m^2) = U(\rho) - \tfrac{1}{2}\int_0^{m^2} s\frac{\partial\rho(s)}{\partial s}{\rm d}s = U(\rho_c) + \frac{1}{32\pi^2}m^4(\ln(\Lambda/m) - \tfrac{1}{4})$$
$$+ \frac{u}{24}\left\{\sigma^2/N + 6\frac{r - r_c}{u} - \frac{1}{8\pi^2}m^2\ln(\Lambda/m)\right\}^2 - \frac{3}{2}\frac{(r - r_c)^2}{u} \,. \tag{2.90}$$

The gap equation can be expressed in terms of the renormalized parameters (see Eq. (2.79)),

$$\frac{1}{u_{\rm r}} = \frac{1}{u} + \frac{1}{48\pi^2}\ln(\Lambda/\mu) \,, \quad \frac{M^2}{u_{\rm r}} = \frac{r - r_c}{u} \,, \tag{2.91}$$

where μ is the renormalization scale and M a renormalized mass parameter, as

$$\frac{m^2}{u_{\rm r}} = -\frac{m^2\ln(\mu/m)}{48\pi^2} + \frac{M^2}{u_{\rm r}} + \frac{\sigma^2}{6N} \,.$$

In the case $u < 0$ the renormalized coupling constant $u_{\rm r}$ does not necessarily approach zero as the cutoff is removed (if $u \to 0^-$ as $\Lambda \to \infty$). One also finds that the solutions of the gap equation reproduce the past results in the literature. However, the ground state energy is lower at the end point $m^2 = 0$ value in which the $O(N)$ symmetry is broken down to $O(N-1)$. One finds in Eq. (2.90) that with $u < 0$, the ground state energy density, $\mathcal{E}_{\rm var} \to -\infty$ as σ is becoming larger. Since $\mathcal{E}_{\rm var}$, in this case, is the upper limit to the exact ground state energy density, one concludes that the theory is inconsistent for $u < 0$.

One finds, however, that when $u < 0$ the $m \neq 0$ solution of (2.10) corresponds to an $O(N)$ symmetric metastable state whose life-time can be calculated. One finds that this metastable state is stabilized by a large tunnelling barrier; its decay rate is proportional to ${\rm e}^{-N}$ and thus, in our large N limit, is an acceptable ground state for the theory. However, this metastable state is not a good candidate for the vacuum of a realistic model because of its peculiar behaviour at finite temperature (see section 7.4). At first, it may sound strange that an $O(N)$ symmetric state will become unstable as the temperature is raised. Indeed, one may expect that the usual behaviour will appear here, in which, as the temperature increases the $O(N)$ symmetric phase is stabilized, whereas the $O(N)$ broken

phase, (which here is the end point $m^2 = 0$ phase) will be destabilized. This expectation is based on the fact that at finite temperature, thermal fluctuations add up to the quantum fluctuations of the field operator ϕ^2 . Thus, one usually expects that a possible negative "mass" at low temperature will turn positive as the temperature increases, and the possible broken symmetry will be restored. This, however, is true as long as one discusses a stable theory $u > 0$. The effect of the temperature is reversed in a metastable situation (here, $u < 0$) since thermal fluctuations help to overcome potential barriers.

The triviality of $(\phi^2)^2$ for $u > 0$ and its inconsistency for $u < 0$ have been derived here for the large N limit of the theory. It seems very likely that these results persist also at finite N. Interesting suggestions came from different view points that the positively coupled theory may have non-trivial implications on practical physics issues. In the standard Weak-Electromagnetic theory, bounds on the Higgs mass can be derived (see for example section 2.6).

3 Models on symmetric spaces in the large N limit

A whole class of geometric models involving only scalar fields, for which the non-linear σ model is the simplest example, shares various classical and quantum properties: The field belongs to a symmetric space, associated to a coset G/H, where G is a Lie group and H a compact maximal subgroup. The action is unique, up to a multiplicative factor which is the coupling constant, and takes the general form

$$\mathcal{S}(\varphi) = \frac{1}{2T} \int \mathrm{d}^d x \, \partial_\mu \varphi^i(x) g_{ij}(\varphi) \partial_\mu \varphi^j(x),$$

where g_{ij} is the metric tensor on the corresponding manifold G/H. This action leads to an infinite number of classical conservation laws in two dimensions. The corresponding quantum models are renormalizable in two dimensions, and then UV asymptotically free. In the classical limit the fields are massless, and correspond to the Goldstone bosons of the G symmetry broken down to H. Since continuous symmetries cannot be broken in two dimensions, the spectrum of such theories is non-perturbative. In dimension $2+\varepsilon$, one finds a critical coupling constant T_c where a phase transition occurs.

To go beyond the $\varepsilon = d-2$ expansion, one can consider using large N techniques. However, only a subclass can be studied in this way. To this subclass belong the $O(N)$ non-linear σ-model and the $CP(N-1)$ model that are discussed below. A number of other models based on Grassmannian manifolds could also be investigated, like $O(N)/O(N-p)\times O(p)$ or similarly $U(N)/U(N-p)\times U(p)$. More general homogeneous spaces with several coupling constants could also be considered.

3.1 The non-linear σ-model in the large N limit

In reference [22] it was first shown that universal quantities could be determined in the form of an $\varepsilon = d-2$ expansion (at least for $N>2$). Again, as for the $\varepsilon = 4-d$ expansion, it is somewhat reassuring that at least in the limiting case $N\to\infty$, the results obtained in this way remained valid even when ε is no longer infinitesimal [64,65].

Moreover, the $1/N$ expansion allows exhibiting a remarkable relation between the non-linear σ-model and the $(\phi^2)^2$ field theory, a relation expected on physical grounds [22].

Finally, large N techniques are well adapted to the analysis of finite size effects in critical systems, a feature we illustrate with a system in a finite volume with periodic boundary conditions.

Non-linear σ-model and $(\phi^2)^2$ field theory. We have noticed that the term proportional to $\int \mathrm{d}^d x \, \lambda^2(x)$ has dimension $4-d$ for $N\to\infty$. It is thus irrelevant in the critical domain for all dimensions $d<4$ and can be omitted at leading

order (this also applies to $d = 4$ where it is marginal but yields only logarithmic corrections). Actually, the constant part in the inverse propagator as written in equation (2.74) plays the role of a large momentum cut-off. We thus omit the λ^2 term in the action (2.32) after shifting the field $\lambda(x)$ by its expectation value m^2 (Eq. (2.8)), $\lambda(x) \mapsto m^2 + \lambda(x)$:

$$\mathcal{S}_N(\sigma, \lambda) = \frac{1}{2} \int \mathrm{d}^d x \left[(\partial_\mu \sigma)^2 + m^2 \sigma^2 + \lambda \sigma^2 - \frac{3N}{u} \lambda^2 - \frac{6N}{u} \left(m^2 - r \right) \lambda \right] + \frac{(N-1)}{2} \operatorname{tr} \ln \left[-\nabla^2 + m^2 + \lambda(\bullet) \right]. \tag{3.1}$$

If we then work backwards, reintroduce the initial field ϕ and integrate over $\lambda(x)$, we find

$$\mathcal{Z} = \int \left[\mathrm{d}\phi(x) \right] \delta \left[\phi^2(x)/N - 6(m^2 - r)/u \right] \exp \left[-\frac{1}{2} \int \left(\partial_\mu \phi(x) \right)^2 \mathrm{d}^d x \right]. \tag{3.2}$$

Under this form we recognize the partition function of the $O(N)$ symmetric non-linear σ-model in an unconventional normalization. We have, therefore, discovered a remarkable correspondence, to all orders in an $1/N$ expansion, between the non-linear σ-model and the $(\phi^2)^2$ field theory.

Actually, reviewing carefully the arguments, one verifies that the identity between the two models has been derived here under the implicit condition $u_{\phi^4} \propto \Lambda^{4-d} \gg (m_\phi)^{4-d}$, which is the generic situation if the ϕ^4 interaction is an effective long distance interaction generated by some microscopic model. If the condition is not satisfied the situation is less clear.

The large N limit. We now write the partition function of the non-linear σ model, with slightly more usual notation, as

$$\mathcal{Z} = \int \left[\mathrm{d}\phi(x) \mathrm{d}\lambda(x) \right] \exp \left[-\mathcal{S}(\phi, \lambda) \right] \tag{3.3}$$

with

$$\mathcal{S}(\phi, \lambda) = \frac{1}{2T} \int \mathrm{d}^d x \left[(\partial_\mu \phi)^2 + \lambda \left(\phi^2 - N \right) \right]. \tag{3.4}$$

By solving in the large N limit the σ-model directly, we are able to exhibit more explicitly the correspondence between the different set of parameters used in the two models.

The field ϕ represents a classical spin of size $\sqrt{N}$. The coupling constant T, which is a loop expansion parameter, represents the temperature of the classical spin model. Eventually, it will be useful to introduce a dimensionless coupling constant t, setting

$$T = \Lambda^{2-d} N t. \tag{3.5}$$

We now separate the field ϕ into $N-1$ components, which we call π in what follows, and over which we integrate as we did in section 2.1, and a remaining component σ. We obtain

$$\mathcal{Z} = \int [\mathrm{d}\sigma(x)\mathrm{d}\lambda(x)] \exp\left[-\mathcal{S}_N(\sigma,\lambda)\right] \tag{3.6}$$

with, for $N \gg 1$,

$$\mathcal{S}_N(\sigma,\lambda) = \frac{1}{2T}\int \left[(\partial_\mu\sigma)^2 + \left(\sigma^2(x) - N\right)\lambda(x)\right] \mathrm{d}^d x + \frac{N}{2}\,\mathrm{tr}\ln\left[-\nabla^2 + \lambda(\cdot)\right]. \tag{3.7}$$

The large N limit is taken here at T fixed. The saddle point equations, analogous to Eqs. (2.10), are

$$m^2\sigma = 0\,, \tag{3.8a}$$

$$\sigma^2/N = 1 - \Omega_d(m)T\,, \tag{3.8b}$$

where we have set $\langle\lambda(x)\rangle = m^2$ and introduced the function (2.12). At low temperature and $d > 2$, σ is different from zero and thus m, which is the mass of the π-field, vanishes. Eq. (3.8b) yields the spontaneous magnetization:

$$\sigma^2/N = 1 - \Omega_d(0)T. \tag{3.9}$$

Setting

$$T_c = 1/\Omega_d(0), \tag{3.10}$$

we can write Eq. (3.9) as

$$\sigma^2/N = 1 - T/T_c\,. \tag{3.11}$$

Thus, T_c is the critical temperature where σ vanishes.

Above T_c, σ instead vanishes and m, which is now the common mass of the π- and σ-field, is for $d > 2$ given by

$$\frac{1}{T_c} - \frac{1}{T} = \Omega_d(0) - \Omega_d(m). \tag{3.12}$$

The physical mass m is solution of an equation quite similar to (2.35). In particular, for $d < 4$, we recover the scaling form (2.37) of the correlation length:

$$\Omega_d(0) - \Omega_d(m) = K(d)m^{d-2} + O(\Lambda^{d-4}m^2) \;\Rightarrow\; \xi = 1/m \propto (T - T_c)^{-1/(d-2)}.$$

In terms of the cut-off Λ and the dimensionless coupling t (Eq. (3.5)), vertex and correlation functions of the non-linear σ model satisfy for $d < 4$ RG equations:

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(t)\frac{\partial}{\partial t} - \frac{n}{2}\zeta(t)\right)\Gamma^{(n)}(p,\Lambda,t) = 0\,,$$

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta(t)\frac{\partial}{\partial t} + \frac{n}{2}\zeta(t)\right)W^{(n)}(p,\Lambda,t) = 0\,.$$

Applying the second equation with $n = 1$ to Eq. (3.11) and expressing in Eq. (3.12) that m is RG invariant, we determine the RG functions at leading order for N large:

$$\beta(t) = (d-2)t(1-t/t_c)\,, \qquad \zeta(t) = (d-2)t/t_c \quad \text{with } t_c = \Lambda^{d-2}T_c/N\,. \tag{3.13}$$

If we add to the action (3.4) the contribution due to a magnetic field in the σ direction,

$$\mathcal{S}(\phi,\lambda) \mapsto \mathcal{S}(\phi,\lambda) - \frac{H}{T}\int \mathrm{d}^d x\, \sigma(x),$$

we can calculate the free energy density $W(H) = T \ln \mathcal{Z}(H)/\Omega$ (Ω is the volume) and its Legendre transform (Eqs. (2.42,2.43)), the thermodynamic potential density function of the magnetization M:

$$\mathcal{G}(M) = N\frac{d-2}{2d}\frac{T^{2/(2-d)}}{\big(K(d)\big)^{2/(d-2)}}(M^2/N - 1 + T/T_c)^{d/(d-2)}, \tag{3.14}$$

a result that extends the scaling form (2.47) to all temperatures below T_c. The calculation of other physical quantities and the expansion in powers of $1/N$ follow from the considerations of previous sections and section 3.1.

Two dimensions and the question of Borel summability. For $d = 2$, the critical temperature vanishes and the parameter m has the form

$$m \sim \Lambda_0\, \mathrm{e}^{-2\pi/T}, \tag{3.15}$$

where (with the definition (2.11))

$$\ln(\Lambda_0/\Lambda) = \frac{1}{4\pi}\int_0^\infty \frac{\mathrm{d}s}{s}\left(\frac{1}{D(s)} - \theta(1-s)\right),$$

in agreement with RG predictions. Note that the field inverse two-point function in the large N-limit is given by

$$\tilde{\Gamma}^{(2)}_{\sigma\sigma}(p) = p^2 + m^2\,. \tag{3.16}$$

The mass term vanishes to all orders in the expansion in powers of the coupling constant t, preventing a perturbative calculation of the mass of the σ-field. The perturbation series is trivially not Borel summable. Most likely, this property remains true for the model at finite N. On the other hand, if the $O(N)$ symmetry is broken by the addition of a term proportional to $\int \mathrm{d}x\, \sigma(x)$ to the action (a magnetic field), the physical mass becomes calculable in perturbation theory.

Corrections to scaling and the dimension 4. In Eq. (3.12) we have neglected corrections to scaling. If we take into account the leading correction, which becomes increasingly important when d approaches 4 from below, we get instead

$$\frac{1}{T_c} - \frac{1}{T} = K(d)m^{d-2} - a(d)\Lambda^{d-4}m^2 + O\left(\Lambda^{d-6}m^4, \Lambda^{-2}m^d\right),$$

where $a(d)$, as we have already discussed, is a constant that explicitly depends on the cut-off procedure and can thus be varied by changing contributions from irrelevant operators.

We can compare this result with the solution of Eqs. (2.35) expanded at the same order:

$$\frac{6}{u}(r - r_c - m^2) = K(d)m^{d-2} - \tilde{a}(d)\Lambda^{d-4}m^2 + O\left(\Lambda^{d-6}m^4\right),$$

where the constant $\tilde{a}(d)$ has the same formal expression as $a(d)$ but corresponds to a different regularization. Eliminating the leading contribution, we find

$$\frac{1}{T_c} - \frac{1}{T} - \frac{6}{u}(r - r_c) = \Lambda^{d-4}m^2\left(\tilde{a}(d) - \frac{6}{Ng} - a(d)\right).$$

We note that it is thus possible to find a regularization of the non-linear σ-model that reproduces the effect of the ϕ^4 coupling constant.

More generally, by comparing with the results of section 2.4, we discover that, although the non-linear σ-model superficially depends on one parameter less than the corresponding ϕ^4 field theory, actually this parameter is hidden in the cut-off function. This remark becomes important in the four dimensional limit where most leading contributions come from the leading corrections to scaling. For example, for $d = 4$ Eq. (3.12) takes a different form, the dominant term in the r.h.s. is proportional to $m^2 \ln m$. We recognize in the factor $\ln m$ the effective ϕ^4 coupling at mass scale m. However, to describe with perturbation theory and RG the physics of the non-linear σ model beyond the $1/N$ expansion, it is necessary to return to the ϕ^4 field theory. This involves addding to the action the operator $\int \mathrm{d}^d x\, \lambda^2(x)$, which irrelevant for $d < 4$, becomes marginal in four dimensions.

3.2 $1/N$-expansion and renormalization group: an alternative formulation

Preliminary remarks. Power counting. Higher order terms in the steepest descent calculation of the functional integral (2.6) generate a systematic $1/N$ expansion.

We now analyze the terms in the action (3.1) from the point of view of large N power counting. The dimension of the field $\sigma(x)$ is $(d-2)/2$. From the critical behaviour (2.61) of the λ-propagator, we inferred the canonical dimension $[\lambda]$ of the field $\lambda(x)$:

$$2\,[\lambda] - \varepsilon = d\,, \qquad \text{i.e.} \quad [\lambda] = 2\,.$$

As noted above, λ^2 has dimension $4 > d$ and thus is irrelevant below four dimensions. The interaction term $\int \lambda(x)\sigma^2(x)\mathrm{d}^d x$ has dimension zero. It is easy to verify that the non-local interactions involving the λ-field, coming from the expansion of the $\mathrm{tr}\ln$, have all also canonical dimension zero:

$$\left[\mathrm{tr}\left[\lambda(x)\left(-\nabla^2+m^2\right)^{-1}\right]^k\right] = k\,[\lambda] - 2k = 0\,.$$

This power counting has the following implication: in contrast with usual perturbation theory, the $1/N$ expansion is exactly renormalizable and thus generates only logarithmic corrections to the leading long distance behaviour for any fixed dimension d, $2 < d \leq 4$. A similar behaviour is found in the ε-expansion (at the IR fixed point) and, thus, one expects here also to be able to calculate universal quantities like critical exponents for example as power series in $1/N$. However, because the interactions are non-local, it is not obvious that the general results of renormalization theory apply here. Therefore, we now construct an alternative quasi-local field theory, for which the standard RG analysis is valid, and which reduces to the large N field theory in some limit [56].

An alternative field theory. To be able to use the standard results of renormalization theory, we reformulate the critical theory to deal with the non-local interactions. Neglecting corrections to scaling, we start from the non-linear σ-model in the form (3.4):

$$\mathcal{Z} = \int [\mathrm{d}\lambda(x)]\,[\mathrm{d}\phi(x)]\exp\left[-\mathcal{S}(\phi,\lambda)\right], \tag{3.17}$$

$$\mathcal{S}(\phi,\lambda) = \frac{1}{2T}\int \mathrm{d}^d x\left[(\partial_\mu\phi)^2 + \lambda\left(\phi^2 - N\right)\right]. \tag{3.18}$$

The difficulty arises from the λ-propagator, absent in the perturbative formulation, and generated by the large N summation. We thus add to the action (3.18) a term quadratic in λ that, in the tree approximation of standard perturbation theory, generates a λ-propagator of the form (2.61). We thus consider the modified action

$$\mathcal{S}_v(\phi,\lambda) = \frac{1}{2}\int \mathrm{d}^d x\left\{\frac{1}{T}\left[(\partial_\mu\phi)^2 + \lambda\left(\phi^2 - N\right)\right] - \frac{1}{v^2}\lambda(-\partial^2)^{-\varepsilon/2}\lambda\right\}. \tag{3.19}$$

In the limit where the parameter v goes to infinity, the coefficient of the additional term vanishes and the initial action is recovered.

Only the critical theory is discussed below, and thus the couplings of all relevant interactions are set to their critical values. These interactions contain a term linear in λ and a polynomial in ϕ^2 of degree depending on the dimension. Note that in some discrete set of dimensions some monomials become just renormalizable. Therefore, we work in generic dimensions and rely on the property that the quantities we calculate are regular functions of the dimension.

The field theory with the action (3.19) can be studied with standard field theory methods. The peculiar form of the λ quadratic term, which is not strictly local, does not create a problem. Similar terms are encountered in statistical systems with long range forces. The main consequence is that the λ-field is not renormalized because counter-terms are always local.

It is convenient to rescale $\phi \mapsto \phi\sqrt{T}$, $\lambda \mapsto v\lambda$:

$$\mathcal{S}_v(\phi,\lambda) = \frac{1}{2}\int \mathrm{d}^d x \left[(\partial_\mu\phi)^2 + v\lambda\phi^2 - \lambda(-\partial^2)^{-\varepsilon/2}\lambda + \text{relevant terms}\right].$$

The renormalized critical action then reads

$$[\mathcal{S}_v]_{\text{ren}} = \frac{1}{2}\int \mathrm{d}^d x \left[Z_\phi\,(\partial_\mu\phi)^2 + v_{\text{r}} Z_v \lambda\phi^2 - \lambda(-\partial^2)^{-\varepsilon/2}\lambda + \text{relevant terms}\right]. \tag{3.20}$$

It follows that the RG equations for vertex functions of l λ fields and n ϕ fields in the critical theory take the form

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta_{v^2}(v)\frac{\partial}{\partial v^2} - \frac{n}{2}\eta(v)\right]\Gamma^{(l,n)} = 0\,. \tag{3.21}$$

The solution to the RG equations (3.21) can be written as

$$\Gamma^{(l,n)}(\ell p, v, \Lambda) = Z^{-n/2}(\ell)\ell^{d-2l-n(d-2)/2}\Gamma^{(l,n)}(p, v(\ell)\Lambda) \tag{3.22}$$

with the usual definitions

$$\ell\frac{\mathrm{d}v^2}{\mathrm{d}\ell} = \beta(v(\ell))\,, \quad \ell\frac{\mathrm{d}\ln Z}{\mathrm{d}\ell} = \eta(v(\ell))\,.$$

We can then calculate the RG functions as power series in $1/N$. It is easy to verify that v^2 has to be taken of order $1/N$. Therefore, to generate a $1/N$ expansion, one first has to sum the multiple insertions of the one-loop λ two-point function, contributions that form a geometric series. The λ propagator then becomes

$$\Delta_\lambda(p) = -\frac{2p^{4-d}}{b(d)D(v)} \tag{3.23}$$

($b(d)$ is given in Eq. (2.60)), where we have defined

$$D(v) = 2/b(d) + Nv^2.$$

We are interested in the neighbourhood of the fixed point $v^2 = \infty$. One verifies that the RG function $\eta(v)$ approaches the exponent η obtained by direct

calculation, and the RG β-function behaves like v^2. The flow equation for large coupling constant becomes

$$\ell \frac{\mathrm{d}v^2}{\mathrm{d}\ell} \sim \rho v^2 \;\Rightarrow\; v^2(\ell) \sim \ell^{\rho}. \tag{3.24}$$

We then note that to each power of the field λ corresponds a power of v. It follows that

$$\begin{aligned} \Gamma^{(l,n)}(\ell p, v, \Lambda) &\propto v^l(\ell)\ell^{d-2l-n(d-2+\eta)} \\ &\propto \ell^{d-(2-\rho/2)l-n(d-2+\eta)}. \end{aligned} \tag{3.25}$$

To compare with the result obtained from the perturbative RG, one has still to take into account that the functions $\Gamma^{(l,n)}$ defined here are obtained by an additional Legendre transformation with respect to the source of ϕ^2. Therefore,

$$2 - \rho/2 = d_{\phi^2} = d - 1/\nu\,. \tag{3.26}$$

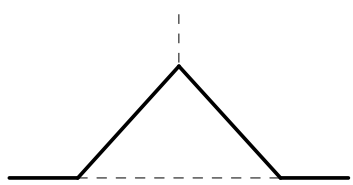

Fig. 4 Diagram contributing to $\Gamma^{(3)}_{\sigma\sigma\lambda}$ at order $1/N$.

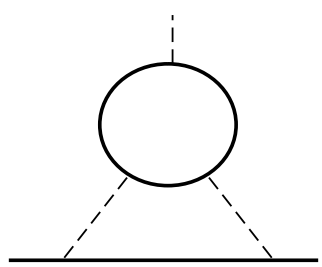

Fig. 5 Diagram contributing to $\Gamma^{(3)}_{\sigma\sigma\lambda}$ at order $1/N$.

RG functions at order $1/N$. Most calculations at order $1/N$ rely on the evaluation of the generic integral

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q}{(p+q)^{2\mu} q^{2\nu}} = p^{d-2\mu-2\nu}\frac{\Gamma(\mu+\nu-d/2)\Gamma(d/2-\mu)\Gamma(d/2-\nu)}{(4\pi)^{d/2}\Gamma(\mu)\Gamma(\nu)\Gamma(d-\mu-\nu)}\,. \tag{3.27}$$

For later purpose, it is convenient to set

$$X_1 = \frac{2N_d}{b(d)} = \frac{4\Gamma(d-2)}{\Gamma(d/2)\Gamma(2-d/2)\Gamma^2(d/2-1)} = \frac{4\sin(\pi\varepsilon/2)\Gamma(2-\varepsilon)}{\pi\Gamma(1-\varepsilon/2)\Gamma(2-\varepsilon/2)}. \quad (3.28)$$

To compare with fixed dimension results, note that $X_1 \sim 2(4-d)$ for $d \to 4$ and $X_1 \sim (d-2)$ for $d \to 2$.

The calculation of the $\langle\phi\phi\rangle$ correlation function at order $1/N$ involves the evaluation of the diagram of figure 3. We want to determine the coefficient of $p^2 \ln\Lambda/p$. Since we work at one-loop order, we can instead replace the λ propagator $q^{-\varepsilon}$ by $q^{2\nu}$ and send the cut-off to infinity. We then use the result (3.27) with $\mu = 1$. In the limit $2\nu \to -\varepsilon$, the integral has a pole. The residue of the pole yields the coefficient of $p^2 \ln\Lambda$ and the finite part contains the $p^2 \ln p$ contribution:

$$\tilde{\Gamma}^{(2)}_{\sigma\sigma}(p) = p^2 + \frac{\varepsilon}{4-\varepsilon}\frac{2N_d}{b(d)D(v)}v^2p^2\ln(\Lambda/p).$$

Expressing that the function satisfies the RG equation, we obtain the function $\eta(v)$.

The second RG function can be deduced from the divergent parts of the $\langle\phi\phi\lambda\rangle$ function:

$$\tilde{\Gamma}^{(3)}_{\sigma\sigma\lambda} = v + A_1 v^3 D^{-1}(v)\ln\Lambda + A_2 v^5 D^{-2}(v)\ln\Lambda + \text{ finite}$$

with

$$A_1 = -\frac{2}{b(d)}N_d = -X_1\,,$$
$$A_2 = -\frac{4N}{b^2(d)}(d-3)b(d)N_d = -2N(d-3)X_1\,,$$

where A_1 and A_2 correspond to the diagrams of figures 4 and 5, respectively.

Applying the RG equation, one finds at order $1/N$ the relation

$$\beta_{v^2}(v) = 2v^2\eta(v) - 2A_1v^4D^{-1}(v) - 2A_2v^6D^{-2}(v). \quad (3.29)$$

One thus obtains

$$\eta(v) = \frac{\varepsilon v^2}{4-\varepsilon}X_1D^{-1}(v), \quad (3.30)$$

$$\beta_{v^2}(v) = \frac{8v^4}{4-\varepsilon}X_1D^{-1}(v) + 4N(1-\varepsilon)v^6X_1D^{-2}(v), \quad (3.31)$$

where the first term in β_{v^2} comes from A_1 and η and the second from A_2.

Extracting the large v^2 behaviour, one infers

$$\eta = \frac{\varepsilon}{N(4-\varepsilon)}X_1 + O(1/N^2), \quad (3.32)$$

$$\rho = \frac{4(3-\varepsilon)(2-\varepsilon)}{N(4-\varepsilon)}X_1 > 0\,,$$

and thus

$$\frac{1}{\nu} = d - 2 + \frac{2(3-\varepsilon)(2-\varepsilon)}{N(4-\varepsilon)} X_1 + O(1/N^2). \tag{3.33}$$

3.3 Some higher order results

The calculations beyond the order $1/N$ are rather technical. The reason is easy to understand: because the effective field theory is renormalizable in all dimensions $2 \leq d \leq 4$, the dimensional regularization, which is so helpful in perturbative calculations, can no longer be used. Therefore, either one keeps a true cut-off or one introduces more sophisticated regularization schemes. For details the reader is referred to the literature [58–66].

Generic dimensions. The exponents γ and η are known up to order $1/N^2$ and $1/N^3$, respectively, in arbitrary dimensions but the expressions are too complicated to be reproduced here. The expansion of γ up to order $1/N$ can be directly deduced from the results of the preceding sections:

$$\gamma = \frac{1}{1-\varepsilon/2}\left(1 - \frac{3}{2N} X_1\right) + O\left(\frac{1}{N^2}\right). \tag{3.34}$$

The exponents ω and $\theta = \omega\nu$, governing the leading corrections to scaling, can also be calculated, for example, from the $\langle \lambda^2 \lambda\lambda \rangle$ function:

$$\omega = \varepsilon\left(1 - \frac{2(3-\varepsilon)^2}{(4-\varepsilon)N} X_1\right) + O\left(\frac{1}{N^2}\right), \tag{3.35}$$

$$\theta = \omega\nu = \frac{\varepsilon}{2-\varepsilon}\left(1 - \frac{2(3-\varepsilon)}{N} X_1\right) + O\left(\frac{1}{N^2}\right). \tag{3.36}$$

Note that the exponents are regular functions of ε up to $\varepsilon = 2$ and free of renormalon singularities at $\varepsilon = 0$.

The equation of state and the spin–spin correlation function in zero field are also known at order $1/N$, but since the expressions are complicated we again refer the reader to the literature for details.

Three dimensional results. Let us give the expansion of η in three dimensions at the order presently available:

$$\eta = \frac{\eta_1}{N} + \frac{\eta_2}{N^2} + \frac{\eta_3}{N^3} + O\left(\frac{1}{N^4}\right)$$

with

$$\eta_1 = \tfrac{8}{3\pi^2}\,, \quad \eta_2 = -\tfrac{8}{3}\eta_1^2\,, \quad \eta_3 = \eta_1^3\big[-\tfrac{797}{18} - \tfrac{61}{24}\pi^2 + \tfrac{27}{8}\psi''(1/2) + \tfrac{9}{2}\pi^2 \ln 2\big],$$

$\psi(x)$ being the logarithmic derivative of the Γ function.

The exponent γ is known only up to order $1/N^2$:

$$\gamma = 2 - \tfrac{24}{N\pi^2} + \tfrac{64}{N^2\pi^4}\left(\tfrac{44}{9} - \pi^2\right) + O\left(\tfrac{1}{N^3}\right).$$

Note that the $1/N$ expansion seems to be rapidly divergent and certainly a direct summation of these terms does not provide precise estimates of critical exponents in three dimensions for relevant values of N.

The nature of the large N expansion. The large order behaviour of the N expansion has been determined explicitly in zero and one dimension (simple integrals and quantum mechanics) [67]. In quantum field theory some estimates are available in ref. [68]. All results seem to indicate that the expansion is divergent in all dimensions, but Borel summable for dimensions $d < 4$.

3.4 Finite size effects: the non-linear σ model

Because finite size effects involve crossover phenomena between different effective dimensions, large N techniques provide convenient tools to study them [30]. It is difficult to discuss systematically finite size effects because the results depend both on the geometry of the system and on boundary conditions. In particular, one must discuss separately boundary conditions whether they break or not translation invariance. In the first case new effects appear, which are surface effects, and that we do not consider here. We study here only periodic conditions, although they are not the only ones preserving translation invariance. For systems that have a symmetry, one can glue the boundaries after having made a group transformation. Thus, here one could also discuss anti-periodic conditions or, more generally, fields differing on both sides by a transformation of the $O(N)$ group.

Even with periodic boundary conditions, the number of different possible situations remains large, the finite sizes in different directions may differ, some sizes may be infinite. Note that QFT at finite temperature, which we begin studying in section 7, can be considered as another example of finite size effects, since the functional integral representing the partition function of a quantum system at finite temperature is also the partition function of a classical system with finite size and periodic boundary conditions in one dimension.

¿From the point of view of the RG, finite size effects, which only affect the IR domain, do not change UV divergences. RG equations remain the same, only the solutions are modified due to the existence of new dimensional parameters. Thus, if finite sizes are characterized by only one length L, solutions will be functions of an additional argument like L/ξ where ξ is the correlation length [69].

Since we want only to demonstrate that large N techniques are useful in the context of finite size effects, we discuss here the non-linear σ-model in a simple geometry with periodic boundary conditions (the geometry of the hypercube or better hypertorus of linear size L).

A characteristic property of a system of finite size is the quantization of momenta, the arguments of field Fourier components. For periodic boundary conditions, if L is the size of the system in a direction μ, we find

$$p_\mu = 2\pi n_\mu / L\,, \quad n_\mu \in \mathbb{Z}\,.$$

In particular, in a massless theory in a finite volume the zero mode $\mathbf{p} = 0$ corresponds to an isolated pole of the propagator. This automatically leads to IR divergences in all dimensions. Therefore, in Eqs. (2.10) the solution $\sigma \neq 0$ no longer exists. This is not surprising: no phase transition is expected in a finite volume.

Finite size scaling in the non-linear σ model. The gap equation (3.8*b*) becomes ($N \gg 1$)

$$G(m, L, \Lambda) \equiv L^{-d} \sum_{n_\mu \in \mathcal{Z}^d} \frac{1}{m^2 + (2\pi \mathbf{n}/L)^2} = \frac{1}{T}\,, \tag{3.37}$$

where the sums are cut by a cut-off Λ.

To write the equation in a more manageable form and be able to define it for continuous dimension, one uses Schwinger's representation

$$\frac{1}{p^2 + m^2} = \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-s(p^2+m^2)}\,.$$

However, in contrast with the infinite volume limit, gaussian integrals over momenta are here replaced by infinite sum over integers which can no longer be calculated exactly. One thus introduces the function

$$\vartheta_0(s) = \sum_{n=-\infty}^{+\infty} \mathrm{e}^{-\pi s n^2}, \tag{3.38}$$

related to Jacobi's elliptic function θ_3 by

$$\vartheta_0(s) = \theta_3(0, \mathrm{e}^{-\pi s}).$$

Poisson's transformation allows to prove the useful identity

$$\vartheta_0(s) = s^{-1/2} \vartheta_0\,(1/s)\,. \tag{3.39}$$

In terms of $\vartheta_0(s)$, the sums can then be written as

$$G(m, L, \Lambda) = L^{-d} \int^\infty \mathrm{d}s\ \mathrm{e}^{-sm^2}\, \vartheta_0^d(4s\pi/L^2),$$

where UV convergence is here ensured by a small s cut-off of order $1/\Lambda^2$.

For $d > 2$, we can introduce the critical temperature T_c, which with the same regularization reads

$$\frac{1}{T_c} = \frac{1}{(4\pi)^{d/2}} \int^{\infty} s^{-d/2} \mathrm{d}s \,.$$

It follows from the identity (3.39) that the difference between both integrals is UV convergent for $d < 4$. After a change of variables $4\pi s/L^2 \mapsto s$ (and keeping only the leading contribution in the critical domain), one finds that the gap equation can be written as

$$L^{d-2} \left(\frac{1}{T} - \frac{1}{T_c} \right) = F(mL) \tag{3.40}$$

with

$$F(z) = \frac{1}{4\pi} \int_0^{\infty} \mathrm{d}s \, \left(\mathrm{e}^{-sz^2/(4\pi)} \, \vartheta_0^d(s) - s^{-d/2} \right) . \tag{3.41}$$

For $|T - T_c| \ll \Lambda^{d-2}$, we thus find a scaling form which is consistent with RG predictions:

$$Lm(T, L) \equiv L/\xi(T, L) = f\big(L/\xi(T, L = \infty)\big),$$

where in addition f is a regular function of T at T_c. More precisely, in the large N limit, using Eq. (3.12) we obtain

$$K(d)\big[Lm(T, L = \infty)\big]^{d-2} = -F\big[Lm(T, L)\big],$$

and we recall that $d-2 = 1/\nu + O(1/N)$. Note that the length ξ has the meaning of a correlation length only for $\xi < L$. Since $\eta = 0$ at this order, m is also directly related to the magnetic susceptibility χ in zero field, $\chi = 1/m^2$.

The function $F(z)$ is decreasing. For $z \to \infty$ the integral is dominated by the small s region, which corresponds to the infinite volume limit,

$$F(z) \sim \Gamma(1 - d/2) \frac{z^{d-2}}{(4\pi)^{d/2}} = -K(d) z^{d-2}.$$

One then verifies that for $T > T_c$ fixed, $L \to \infty$ and thus for $mL \to \infty$, one recovers the infinite volume limit. Alternatively, in the low temperature phase for $T < T_c$ fixed, $L \to \infty$, mL goes to zero. Thus, the contribution of the zero mode $\mathbf{n} = 0$ dominates the sum in equation (3.37). Using the relation (3.39), one then finds

$$F(z) = \frac{1}{z^2} + K_1(d) + O\left(z^2\right),$$

$$K_1(d) = \frac{1}{4\pi} \int_0^{\infty} \mathrm{d}s \, \left(\vartheta_0^d(s) - s^{-d/2} - 1 \right),$$

and thus

$$\chi(L,T)/N = \frac{1}{m^2} = \left(\frac{1}{T} - \frac{1}{T_c}\right) L^d - L^2 K_1(d) + O\left(L^{4-d}/(T-T_c)\right) . \quad (3.42)$$

We see that the susceptibility diverges with the volume, a precursor of the low temperature phase with broken symmetry.

Note, finally, that it is instructive to make a similar analysis for other boundary conditions that have no zero mode.

For $d = 2$, the regime where finite size effects can be seen corresponds to $T \ln(L\Lambda) = O(1)$, that is to a regime of low temperature. The zero mode dominates for $T \ln(L\Lambda) \ll 1$, and the susceptibility then is given by

$$\chi(T,L) \sim L^2 \left[1 + O(T \ln(L\Lambda))\right] .$$

3.5 The $CP(N-1)$ models in the large N limit

We discuss here only one other family of models based on symmetric spaces that can be solved in the large N limit, the $CP(N-1)$ models, because they have been the subject of many studies [25]. In particular, one can show that these models in two dimensions have instanton solutions.

Again one discovers that, within the framework of the large N expansion, $CP(N-1)$ models are related to other models, abelian Higgs models, which are renormalizable in four dimensions.

The models. The field φ_α is an N-component complex vector of unit length:

$$\bar{\varphi} \cdot \varphi = N . \quad (3.43)$$

In addition two vectors φ_α and φ'_α are equivalent if

$$\varphi'_\alpha(x) = \mathrm{e}^{i\Lambda(x)} \varphi_\alpha(x) . \quad (3.44)$$

These conditions characterize the symmetric space $U(N)/[U(N-1) \times U(1)]$, a complex Grassmannian manifold, which is isomorphic to the complex projective manifold $CP(N-1)$.

One form of the unique symmetric classical action is

$$\mathcal{S}(\varphi, A_\mu) = \frac{1}{T} \int \mathrm{d}^2 x \, \overline{\mathrm{D}_\mu \varphi} \cdot \mathrm{D}_\mu \varphi , \quad (3.45)$$

in which T is a coupling constant and D_μ the covariant derivative:

$$\mathrm{D}_\mu = \partial_\mu + i A_\mu . \quad (3.46)$$

The field A_μ is a gauge field for the $U(1)$ transformations

$$\varphi'(x) = \mathrm{e}^{i\Lambda(x)}\,\varphi(x)\,, \quad A'_\mu(x) = A_\mu(x) - \partial_\mu\Lambda(x). \tag{3.47}$$

The $U(N)$ symmetry of the action is obvious and the gauge symmetry implements the equivalence (3.47).

In the tree approximation the fields are massless, and the $2N-2$ independent real components correspond to the Goldstone bosons of the broken symmetry $U(N) \to U(N-1)$, one Goldstone boson being suppressed by the abelian gauge symmetry (the Higgs mechanism).

Since the action contains no kinetic term for A_μ, the gauge field is not a dynamical field but only an auxiliary field that can be integrated out. The action is quadratic in A and the gaussian integration results in replacing in the action A_μ by the solution of the A-field equation

$$NA_\mu = \tfrac{1}{2}i\left(\overline{\varphi}\cdot\partial_\mu\varphi - \overline{\partial_\mu\varphi}\cdot\varphi\right) = i\bar{\varphi}\cdot\partial_\mu\varphi\,, \tag{3.48}$$

where Eq. (3.43) has been used. After this substitution, the composite field $\bar{\varphi}\cdot\partial_\mu\varphi/N$ acts as a gauge field. In the following, however, we find it more convenient to keep A_μ as an independent field.

Note that the $CP(1)$ model is locally isomorphic to the $O(3)$ non-linear σ-model, with the identification

$$\phi^i = \bar{\varphi}_\alpha\sigma^i_{\alpha\beta}\varphi_\beta\,. \tag{3.49}$$

Large N limit. As for the non-linear σ-model, we introduce a Lagrange multiplier $\lambda(x)$ to implement the constraint (3.43), and obtain the action

$$\mathcal{S}(\varphi, A_\mu, \lambda) = \frac{1}{T}\int \mathrm{d}^d x\,\left[\overline{\mathrm{D}_\mu\varphi}\cdot\mathrm{D}_\mu\varphi + \lambda\left(\bar{\varphi}\cdot\varphi - N\right)\right]. \tag{3.50}$$

The integral over φ now is gaussian and can be performed. Integrating over $N-1$ components, one finds (with $\varphi \equiv \varphi_1$),

$$\mathcal{S}_N(\varphi, A_\mu, \lambda) = \frac{1}{T}\int \mathrm{d}^d x\,\left[|\mathrm{D}_\mu\varphi|^2 + \lambda\left(|\varphi|^2 - N\right)\right] + (N-1)\,\mathrm{tr}\ln(-\mathrm{D}_\mu^2 + \lambda). \tag{3.51}$$

Of course, the remaining functional integral over φ, A_μ, λ is well-defined only after a choice of gauge.

The gauge field plays no role in the saddle point equations, which are those of the $O(2N)$ non-linear σ-model. However, the model embodies the physics of the abelian Higgs model and of the Landau–Ginzburg theory of superconductivity. In dimensions $d > 2$, in the broken symmetry phase, the gauge field becomes massive and the number of Goldstone modes is $2N-2$ instead of $2N-1$. In

two dimensions a phase with massless modes is excluded; the model exhibits a symmetric phase with confinement because the Coulomb force is linear.

The abelian Higgs model. The action of the abelian Higgs model can be written as

$$\mathcal{S}(A_\mu, \varphi) = \int \mathrm{d}^d x \left[\frac{N}{4e^2} F_{\mu\nu}^2 + \overline{\mathrm{D}_\mu \varphi} \cdot \mathrm{D}_\mu \varphi + NU(\bar{\varphi} \cdot \varphi / N) \right], \tag{3.52}$$

where the potential is quadratic:

$$U(z) = rz + \tfrac{1}{6} u z^2. \tag{3.53}$$

This model with N charged scalars is renormalizable in $d = 4$ dimensions. In dimension $d = 4 - \varepsilon$, the RG β-functions are

$$\begin{aligned} \beta_u &= -\varepsilon u + \frac{1}{24\pi^2 N} \left[(N+4)u^2 - 18ue^2 + 54e^4 \right], \\ \beta_{e^2} &= -\varepsilon e^2 + \frac{1}{24\pi^2} e^4. \end{aligned}$$

For $d = 4$, the origin $e^2 = g = 0$ is a stable IR fixed point only for $N \geq N_c = 90 + 24\sqrt{15} \approx 183$. Correspondingly, the model has a stable IR fixed point in dimension $4 - \varepsilon$ for $N \geq N_c = N_c(d = 4) + O(\varepsilon)$. This is of course the situation which prevails in the large N limit, and one finds the IR fixed point

$$u^* = e^{2*} = 24\pi^2 \varepsilon + O(\varepsilon^2).$$

We thus expect both effective couplings g and e to run at low momentum to the IR fixed point and the model to depend only on one parameter. As in the case of the non-linear σ model, the linear $|\varphi|^4$ theory and the non-linear $CP(N-1)$ model are equivalent.

This result can be verified by the large N techniques. We introduce the two fields $\rho(x)$ and $\lambda(x)$ as in section 2.1, to implement the constraint $\rho(x) = \bar{\varphi}(x) \cdot \varphi(x)/N$. We then obtain an action of the form

$$\mathcal{S}(\varphi, A_\mu, \lambda, \rho) = \int \mathrm{d}^d x \left[\frac{N}{4e^2} F_{\mu\nu}^2 + \overline{\mathrm{D}_\mu \varphi} \cdot \mathrm{D}_\mu \varphi + \lambda \left(\bar{\varphi} \cdot \varphi - N\rho \right) + NU(\rho) \right]. \tag{3.54}$$

We again integrate over $N-1$ components of the complex field and find ($\varphi \equiv \varphi_1$)

$$\begin{aligned} \mathcal{S}_N(\varphi, A_\mu, \lambda, \rho) = \int \mathrm{d}^d x \left[\frac{N}{4e^2} F_{\mu\nu}^2 + |\mathrm{D}_\mu \varphi|^2 + \lambda \left(|\varphi|^2 - N\rho \right) + NU(\rho) \right] \\ + (N-1) \operatorname{tr} \ln(-\mathrm{D}_\mu^2 + \lambda). \end{aligned} \tag{3.55}$$

For the scalar field, the arguments that show that after translation of the expectation value the four-point interaction is negligible are the same as for the usual

ϕ^4 theory. In the symmetric phase the expansion of the tr ln yields an additional contribution to the gauge field propagator of the form

$$\Delta_{\mu\nu}(p) = \tfrac{1}{2}(p^2\delta_{\mu\nu} - p_\mu p_\nu)B_\Lambda(p,0).$$

We face the same situation as in section 3.1. For all dimensions $d < 4$, $B_\Lambda(p,0)$ behaves for p small like p^{d-4} and, therefore, the contribution to the propagator coming from $F_{\mu\nu}^2$ is negligible. At $d = 4$, it is subleading by $1/\ln p$. Therefore, $1/e^2$ is the coefficient of an irrelevant contribution, which can be omitted in the action. The gauge field then is no longer dynamical and we return to the $CP(N_1)$ model.

4 Fermions in the large N limit

We now illustrate the large N techniques that we have started describing in section 2.1, with the study of models involving fermions in the vector representation of the $U(N)$ group. We first explain how a class of self-interacting fermion models can be solved in the large N limit. When these models have a discrete chiral symmetry they may exhibit two phases, a symmetric phase with massless fermions, and a massive phase where the symmetry is broken. The simplest realization is the Gross–Neveu (GN) model, which we study in more detail [72,73]. The model is renormalizable in two dimensions, and describes in perturbation theory only one phase, the symmetric phase. We also summarize what can be learned from RG equations and $d = 2 - \varepsilon$ expansion.

A different field theory, the Gross–Neveu–Yukawa (GNY) model has the same symmetry, but a different field content since fermions interact through their coupling to a scalar field. The model is renormalizable in four dimensions and, moreover, allows a perturbative analysis of the chiral phase transition. We recall some properties of the model using perturbation theory and RG equations at and near four dimensions. We then show that additional information can be obtained from a large N expansion, and that a relation between the GNY and GN models follows.

Finally, we also briefly examine QED and the massless Thirring model in the large N limit.

In all examples, one of the physical issues we explore is the possibility of spontaneous chiral symmetry breaking and fermion mass generation.

4.1 Large N techniques and fermion self-interactions

We consider a model characterized by a $U(\tilde{N})$ symmetric action for a set of $\tilde{N}$ Dirac fermions $\{\psi^i, \bar{\psi}^i\}$. Since fermions have also a spin, we concentrate on the simple examples where interactions involve only the scalar combination both in the $U(\tilde{N})$ and the spin group sense:

$$\bar{\psi} \cdot \psi \equiv \sum_{\alpha,i} \bar{\psi}^i_\alpha \psi^i_\alpha .$$

The action can then be written as

$$\mathcal{S}(\bar{\psi}, \psi) = -\int \mathrm{d}^d x \left[\bar{\psi}(x) \cdot \partial\!\!\!/ \psi(x) + N U\big(\bar{\psi}(x) \cdot \psi(x)/N\big) \right], \tag{4.1}$$

where U is a general polynomial potential. We have introduced the notation $N = \tilde{N} \operatorname{tr} \mathbf{1}$, the matrix $\mathbf{1}$ being the identity in the space of Dirac γ matrices and N thus the total number of ψ components. Moreover, a chiral invariant regularization (see Eq. (2.11)) is assumed:

$$\partial\!\!\!/ \mapsto \partial\!\!\!/ \sqrt{D(-\nabla^2/\Lambda^2)} \, .$$

From the general discussion of section 2.1, it is now quite clear how to solve such a model in the large N limit and how to generate a systematic large N expansion. We introduce two scalar fields σ and ρ and impose the constraint $\rho(x) = \bar{\psi}(x) \cdot \psi(x)/N$ by an integral over σ. The partition function can then be written as

$$\mathcal{Z} = \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}][\mathrm{d}\rho][\mathrm{d}\sigma] \exp\left[-\mathcal{S}(\bar{\psi}, \psi, \rho, \sigma)\right] \tag{4.2}$$

with

$$\mathcal{S}(\bar{\psi}, \psi, \rho, \sigma) = -\int \left[\bar{\psi} \cdot \partial\!\!\!/ \psi + NU\big(\rho(x)\big) + \sigma(x)\big((\bar{\psi}(x) \cdot \psi(x) - N\rho(x)\big)\right] \mathrm{d}^d x\,. \tag{4.3}$$

In the representation (4.2), the integration over fermion fields is gaussian and can be performed, the N-dependence of the partition function becoming explicit. Again it is convenient to integrate only over $\tilde{N}-1$ components. The action suited for large N calculations then takes the form (now $\psi \equiv \psi_1$)

$$\begin{aligned}\mathcal{S}_N(\bar{\psi}, \psi, \rho, \sigma) = -\int \left[\bar{\psi}\partial\!\!\!/ \psi + NU\big(\rho(x)\big) + \sigma(x)\big((\bar{\psi}(x)\psi(x) - N\rho(x)\big)\right] \mathrm{d}^d x \\ - (\tilde{N}-1)\,\mathrm{tr}\ln\left(\partial\!\!\!/ + \sigma\right).\end{aligned} \tag{4.4}$$

For large N, the action is proportional to N and can be calculated by the steepest descent method. The action density $\mathcal{E}$ for constant fields $\rho, \sigma = M$ reduces to

$$\mathcal{E}(M, \rho)/N = -U(\rho) + M\rho - \frac{1}{2}\int^{\Lambda} \frac{\mathrm{d}^d q}{(2\pi)^d} \ln[(q^2 + M^2)/q^2]. \tag{4.5}$$

At this order M, the σ expectation value, is also the fermion mass. In the continuum limit (or in the critical domain) it must satisfy the physical condition $|M| \ll \Lambda$.

The saddle point equations, expressed in terms of the function (2.12), are

$$M = U'(\rho), \tag{4.6a}$$

$$\rho = \frac{M}{(2\pi)^d}\int^{\Lambda} \frac{\mathrm{d}^d q}{q^2 + M^2} = M\Omega_d(M)\,. \tag{4.6b}$$

In terms of the function Ω_d, the action density then reads

$$\mathcal{E}(M, \rho)/N = -U(\rho) + M\rho - \int_0^M s\mathrm{d}s\,\Omega_d(s). \tag{4.7}$$

From Eq. (4.6b), we infer that ρ/M is positive and that ρ and M vanish simultaneously for $d > 1$. Moreover, the condition $|M| \ll \Lambda$ implies that ρ is small in the natural cut-off scale. We can, therefore, expand U for ρ small:

$$U(\rho) = \mathcal{M}\rho + \tfrac{1}{2}G\rho^2 + O(\rho^3). \tag{4.8}$$

Using Eq. (4.6b) to eliminate ρ, we infer from Eq. (4.6a)

$$M = \mathcal{M} + GM\Omega_d(M) + O(M^2\Omega_d^2).$$

In particular, the fermion mass M vanishes when $\mathcal{M}$ goes to zero except if the equation $G\Omega_d(M) = 1$ has a solution. The latter condition implies $G > 0$, that is attraction between fermions. For a repulsive interaction ($G < 0$), at $\mathcal{M} = 0$ the saddle point is $\rho = M = 0$ and the fermion mass always vanishes.

The value $\mathcal{M} = 0$ is natural if the model has a discrete symmetry $U(\rho) = U(-\rho)$, which prevents the addition of an explicit fermion mass term. Such a symmetry can be realized in the fermion representation, in the form of discrete chiral transformations in even dimensions, and in odd dimensions is a consequence of parity symmetry. In the case of attractive interactions, the issue one can then address concerns the possibility of spontaneous fermion mass generation, consequence of the spontaneous breaking of the symmetry. This is the question we now discuss at large N.

4.2 Discrete chiral symmetry and spontaneous mass generation

We now consider models with a discrete symmetry that prevents the addition of a fermion mass ($\mathcal{M} = 0$). In even dimensions it is a discrete chiral symmetry ($\gamma_S \equiv \gamma_{d+1}$)

$$\psi \mapsto \gamma_S\psi, \quad \bar{\psi} \mapsto -\bar{\psi}\gamma_S\,, \tag{4.9}$$

while in odd dimensions it is simply space reflection. Actually, it is possible to find a unique transformation, which corresponds to space reflection in odd dimensions, and makes sense in all dimensions

$$\mathbf{x} = \{x_1, \ldots, x_\mu, \ldots, x_d\} \mapsto \tilde{\mathbf{x}} = \{x_1, \ldots, -x_\mu, \ldots, x_d\}, \quad \begin{cases} \psi(\mathbf{x}) \mapsto \gamma_\mu\psi(\tilde{\mathbf{x}}), \\ \bar{\psi}(\mathbf{x}) \mapsto -\bar{\psi}(\tilde{\mathbf{x}})\gamma_\mu \end{cases}. \tag{4.10}$$

Then, the potential has the expansion

$$U(\rho) = \tfrac{1}{2}G\rho^2 + O(\rho^4) \;\Rightarrow\; \rho \sim M/G\,, \tag{4.11}$$

where we have assumed an attractive fermion self-interaction that excludes multicritical points ($G > 0$) .

Then, the gap equation (4.6b) has two solutions, a symmetric solution $M = \rho = 0$ and a solution with non-vanishing mass and broken symmetry, $|M| \ll \rho^{1/(d-1)} \ll \Lambda$, which satisfies

$$1 \sim G\Omega_d(M). \tag{4.12}$$

Two dimensions. We first examine dimension 2, which is peculiar. The solution with non-vanishing mass and broken symmetry leads to

$$1 = G\Omega_2(M) \;\Rightarrow\; M \propto \Lambda\,\mathrm{e}^{-2\pi/G},$$

where $|M| \ll \Lambda$ implies that the parameter G has to be small enough. The massive solution has always lower energy density, and is therefore realized (see also the variational analysis at the end of the section).

Higher dimensions. In dimensions $d > 2$, the massive phase can exist only if the ratio ρ/M is smaller than some critical value

$$\rho/M = \Omega_d(0).$$

Since $M/\rho \sim G$, this implies the existence of a critical coupling constant

$$G_c = 1/\Omega_d(0). \tag{4.13}$$

For $G < G_c$ the symmetry is unbroken and fermions are massless: $\rho = M = 0$.

For $G > G_c$, instead, both the trivial symmetric solution $\rho = M = 0$ and a massive solution can be found. In the broken symmetry phase, the gap equation can be written as

$$\frac{1}{G_c} - \frac{1}{G} = \Omega_d(0) - \Omega_d(M).$$

A comparison of energies then shows that the broken phase has a lower energy than the symmetric phase (see also the variational argument at the end of the section). For $G > G_c$, the symmetry is broken and the fermions are massive. At the critical value G_c, a phase transition occurs.

Note that while for $d > 2$ the massless symmetric phase $M = \rho = 0$ can exist, for $d = 2$ instead the symmetric solution does not exist anymore. This phenomenon is associated with the divergence of momentum integral diverges at $q = 0$ in the saddle point equation (4.6*b*). The mechanism is reminiscent of the Goldstone phenomenon, but with one difference: here it is the symmetric phase that is massless and therefore does not exist in low dimensions.

Continuum physics in the broken phase is possible only if $|M| \ll \Lambda$, that is when G is close to its critical value, $G - G_c \ll \Lambda^{2-d}$.

Depending on the position of the dimension with respect to 4, one then finds

$$M \propto \left(\frac{1}{G_c} - \frac{1}{G}\right)^{1/(d-2)} \text{for } d < 4\,,$$

$$M \propto \frac{1}{\sqrt{\ln(\Lambda/M)}} \left(\frac{1}{G_c} - \frac{1}{G}\right)^{1/2} \text{for } d = 4\,,$$

$$M \propto \left(\frac{1}{G_c} - \frac{1}{G}\right)^{1/2} \text{for } d > 4\,.$$

Finally, we note that in the continuum limit the function $U(\rho)$ can always be approximated by the quadratic polynomial (4.11).

The scalar bound state. In the quadratic approximation relevant to the continuum limit, we can integrate over $\rho(x)$. The result of the gaussian integration amounts to replace $\rho(x)$ by the solution of the field equation

$$\rho(x) = \sigma(x)/G\,, \tag{4.14}$$

and the action becomes

$$\mathcal{S}_N(\bar\psi,\psi,\sigma) = -\int \left[\bar\psi\left(\partial\!\!\!/ + \sigma(x)\right)\psi - N\sigma^2(x)/2G\right] \mathrm{d}^d x - (\tilde N - 1)\,\mathrm{tr}\ln\left(\partial\!\!\!/ + \sigma\right). \tag{4.15}$$

It is then instructive to calculate, at leading order, the σ two-point function Δ_σ in the massive phase. Differentiating the action twice with respect to $\sigma(x)$, and then setting $\sigma(x) = M$, one finds after some algebra

$$\begin{aligned}\Delta_\sigma^{-1}(p) &= \frac{N}{G} - \frac{N}{(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q}{q^2+M^2} \\ &\quad + \frac{N}{2(2\pi)^d}\left(p^2+4M^2\right)\int^\Lambda \frac{\mathrm{d}^d q}{(q^2+M^2)\left[(p+q)^2+M^2\right]} \\ &= \tfrac{1}{2}N\left(p^2+4M^2\right)B_\Lambda(p,m),\end{aligned} \tag{4.16}$$

where the definition (2.56), the saddle point equation (4.6*b*) and the relation (4.11) have been used. We see that the inverse propagator vanishes for $ip = 2M$ (we use euclidean conventions), which means that in the broken phase, in the quadratic approximation, one finds a scalar bound state with mass $2M$ (the two-fermion threshold), independently of the dimension d of space.

Variational calculations. Here, like for the scalar fields in section 2.8, it is possible to relate the large N results to results obtained from variational calculations in the large N limit. One takes as a variational action

$$\mathcal{S}_0(\bar\psi,\psi) = -\int \mathrm{d}^d x\, \bar\psi\cdot(\partial\!\!\!/ + M)\psi\,.$$

The arguments follow directly what has been done in the scalar example. We introduce the parameter

$$\rho = \left\langle\bar\psi\cdot\psi\right\rangle_0/N = \frac{\tilde N}{N}\,\mathrm{tr}\int^\Lambda \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{M - ik\!\!\!/}{k^2+M^2} = M\Omega_d(M)\,, \tag{4.17}$$

where $\langle\bullet\rangle$ means expectation value with respect to $\mathrm{e}^{-\mathcal{S}_0}$.

Then, the variational energy density $\mathcal{E}_{\mathrm{var}}$ is given by

$$\mathcal{E}_{\mathrm{var}}/N = -\left\langle U(\bar\psi\cdot\psi)/N\right\rangle_0 + M\left\langle\bar\psi\cdot\psi\right\rangle_0/N - \mathrm{tr}\ln(\partial\!\!\!/ + M)/\,\mathrm{tr}\,\mathbf{1}\,.$$

Again, in the large N limit,

$$\left\langle U(\bar{\psi}\cdot\psi)/N\right\rangle_0 \sim U(\rho)$$

and thus

$$\mathcal{E}_{\rm var}/N = -U(\rho) + M\rho - \int_0^M s{\rm d}s\,\Omega_d(s)\,. \tag{4.18}$$

We recognize Eq. (4.7) but here ρ and M are related by Eq. (4.17) and thus Eq. (4.6b) is automatically satisfied. We have now to look for the minimum of $\mathcal{E}_{\rm var}$ as a function of M. Except for possible end-point solutions, which are not relevant here, and taking into account Eq. (4.7), we recover Eq. (4.6a).

We can now examine the behaviour of $\mathcal{E}_{\rm var}$ near the trivial symmetric solution $M=0$. From Eq. (4.18), one finds

$$\frac{1}{N}\frac{\partial\mathcal{E}_{\rm var}}{\partial M} = M(1-G\Omega_d(M))\frac{\partial\{M\Omega_d(M)\}}{\partial M}\,. \tag{4.19}$$

Thus, for $d>2$,

$$\mathcal{E}_{\rm var}/N \underset{M\to 0}{\sim} \tfrac{1}{2}\Omega_d(0)M^2(1-G/G_c).$$

Therefore, for $G<G_c$ the massless symmetric phase has lower energy, while it is the massive broken phase for $G>G_c$.
For $d=2$,

$$\mathcal{E}_{\rm var}/N \underset{M\to 0}{\sim} -\tfrac{1}{2}GM^2\Omega_2^2(M)$$

and, therefore, the broken phase is always lower.

4.3 The Gross–Neveu model

In the general discussion of the fermion self-interaction in the large N limit, we have seen that interesting physics can be studied by considering only a quartic fermion interaction. Then,

$$\mathcal{S}(\bar{\psi},\psi) = -\int {\rm d}^dx\left[\bar{\psi}\cdot\partial\!\!\!/\psi + \tfrac{1}{2N}G\left(\bar{\psi}\cdot\psi\right)^2\right]. \tag{4.20}$$

This characterizes the Gross–Neveu (GN) model which we now study in more detail. The model illustrates, for $G>0$, the mechanism of spontaneous mass generation and, in even dimensions, chiral symmetry breaking. It is renormalizable and asymptotically free in two dimensions. However, as in the case of the non-linear σ model, the perturbative GN model describes only one phase.

Since in the GN model the symmetry breaking mechanism is non-perturbative, it will eventually be instructive to compare it with a different model with the same symmetries, but where the mechanism is perturbative: the Gross–Neveu–Yukawa model.

RG equations in two and near two dimensions. The GN model is renormalizable in two dimensions, and in perturbation theory describes only the massless symmetric phase. Perturbative calculations in two dimensions can be made with an IR cut-off of the form of a mass term $\mathcal{M}\bar{\psi}\psi$, which breaks softly the chiral symmetry. It is possible to use dimensional regularization in practical calculations.

Note that in two dimensions the symmetry group is really $O(N)$, as one verifies after some relabelling of the fields. Therefore, the $(\bar{\psi}\psi)^2$ interaction is multiplicatively renormalized.

In generic non-integer dimensions $d > 2$, the situation is more complicated because the algebra of γ matrices is infinite-dimensional and an infinite number of four-fermion interactions mix under renormalization. The coupling (4.21) thus has the interpretation of a coupling constant that parametrizes the RG flow that joins the gaussian fixed point to the non-trivial UV fixed point. Its flow equation is obtained by first eliminating all other couplings. This remark is important from the point of view of explicit calculations in a $d-2$ expansion, but because the problem does not appear at leading order, it does not affect the analysis and we disregard here this subtlety.

It is convenient to introduce here a dimensionless coupling constant

$$u = G\Lambda^{2-d}/N. \tag{4.21}$$

As a function of the cut-off Λ, the bare vertex functions satisfy the RG equations [75].

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(u)\frac{\partial}{\partial u} - \frac{n}{2}\eta_\psi(u) - \eta_{\mathcal{M}}(u)\mathcal{M}\frac{\partial}{\partial\mathcal{M}}\right]\Gamma^{(n)}\left(p_i; u, \mathcal{M}, \Lambda\right) = 0\,. \tag{4.22}$$

A direct calculation of the β-function in $d = 2+\varepsilon$ dimension yields [76]

$$\beta(u) = \varepsilon u - (N-2)\frac{u^2}{2\pi} + (N-2)\frac{u^3}{4\pi^2} + \frac{(N-2)(N-7)}{32\pi^3}u^4 + O\left(u^5\right), \tag{4.23}$$

Here $N = \tilde{N}\,\mathrm{tr}\,\mathbf{1}$ is the number of fermion degrees of freedom and thus for $d = 2$ $N = 2\tilde{N}$.

The special case $N = 2$, for which the β-function vanishes identically in two dimensions, corresponds to the Thirring model (since for $N = 2$, $(\bar{\psi}\gamma_\mu\psi)^2 = -2(\bar{\psi}\psi)^2$). The latter model is also equivalent to the sine-Gordon or the $O(2)$ vector model.

Finally, the field and mass RG functions, at the presently available, order are

$$\eta_\psi(u) = \frac{N-1}{8\pi^2}u^2 - \frac{(N-1)(N-2)}{32\pi^3}u^3 + \frac{(N-1)(N^2-7N+7)}{128\pi^4}u^4, \tag{4.24}$$
$$\eta_{\mathcal{M}}(u) = \frac{N-1}{2\pi}u - \frac{N-1}{8\pi^2}u^2 - \frac{(2N-3)(N-2)}{32\pi^3}u^3 + O(u^4).$$

Repulsive interactions. If u is negative, the form of the β-function shows that the model is IR free for any dimension $d \geq 2$ (at least for u small enough), chiral symmetry is never broken and, for $\mathcal{M} = 0$, fermions remain massless. In dimension 2, one finds a gaussian behaviour modified by logarithms. In higher dimensions the theory is gaussian. Therefore, in what follows we discuss only the situation of an attractive self-interaction, that is $u > 0$.

Attractive interactions. As in the case of the non-linear σ model, it is then convenient to express the solutions of the RG equations (4.22) in terms of a RG invariant mass scale $\Lambda(u)$ or its inverse, a length scale ξ of the type of a correlation length,

$$\xi^{-1}(u) \equiv \Lambda(u) \propto \Lambda \exp\left[-\int^u \frac{\mathrm{d}u'}{\beta(u')}\right] . \tag{4.25}$$

We then have to consider separately dimension 2, which is special, and higher dimensions.

Two dimensions [80–85]. For $d = 2$, the model is UV asymptotically free. In the chiral limit ($\mathcal{M} = 0$) the spectrum, then, is non-perturbative, and a number of arguments lead to the conclusion that the chiral symmetry is always broken and a fermion mass generated. From the statistical point of view, this corresponds to a gap in the spectrum of fermion excitations (like in a superconductor). All masses are proportional to the mass parameter $\Lambda(u)$ defined in equation (4.25). For u small

$$\Lambda(u) \propto \Lambda u^{1/(N-2)}\, \mathrm{e}^{-2\pi/(N-2)u}\big(1 + O(u)\big) . \tag{4.26}$$

We see that the continuum limit, which is reached when the masses are small compared to the cut-off, corresponds to $u \to 0$.

S-matrix considerations have then led to the conjecture that, for N finite, the spectrum is

$$m_n = \Lambda(u)\frac{(N-2)}{\pi}\sin\left(\frac{n\pi}{N-2}\right), \quad n = 1, 2 \ldots < N/2\,,\ N > 4\,.$$

The first values of N are special, the model $N = 4$ is conjectured to be equivalent to two decoupled sine-Gordon models.

To each mass value corresponds a representation of the $O(N)$ group. The nature of the representations leads to the conclusion that n odd corresponds to fermions and n even to bosons.

This result is consistent with the spectrum for N large evaluated by semi-classical methods. In particular, the ratio of the masses of the fundamental fermion ψ and the lowest lying boson σ is

$$\frac{m_\sigma}{m_\psi} = 2\cos\left(\frac{\pi}{N-2}\right) = 2 + O(1/N^2). \tag{4.27}$$

Note that the results about breaking of chiral symmetry, the coupling constant dependence of the mass scale, and the ratio of (4.27) are completely consistent with the large N results found in sections 4.1,4.2.

Dimension $d = 2 + \varepsilon$ [75]. As in the case of the σ-model, asymptotic freedom implies the existence of a non-trivial UV fixed point u_c in $2 + \varepsilon$ dimension:

$$u_c = \frac{2\pi}{N-2}\varepsilon\left(1 - \frac{\varepsilon}{N-2}\right) + O\left(\varepsilon^3\right).$$

u_c is also the critical coupling constant for a transition between a phase in which the chiral symmetry is spontaneously broken and a massless small u phase.

Setting $u = u_c$ in the RG functions, one infers the correlation length exponent ν:

$$\nu^{-1} = -\beta'(u_c) = \varepsilon - \frac{\varepsilon^2}{N-2} + O\left(\varepsilon^3\right), \tag{4.28}$$

and the fermion field dimension $[\psi]$:

$$2[\psi] = d - 1 + \eta_\psi(u_c) = 1 + \varepsilon + \frac{N-1}{2(N-2)^2}\varepsilon^2 + O\left(\varepsilon^3\right). \tag{4.29}$$

The dimension of the composite field $\sigma = \bar{\psi}\psi$ is given by

$$[\sigma] = d - 1 - \eta_M(u_c) = 1 - \frac{\varepsilon}{N-2} + O(\varepsilon^2).$$

As for the σ-model, the existence of a non-trivial UV fixed point implies that large momentum behaviour is not given by perturbation theory above two dimensions. This explains why the perturbative result that indicates that the model cannot be renormalized in higher dimensions, cannot be trusted. However, to investigate whether the ε expansion makes sense beyond an infinitesimal neighbourhood of dimension two, other methods are required [86], like the $1/N$ expansion, which is discussed in sections 4.1, 4.2, 4.6.

4.4 The Gross–Neveu–Yukawa model

The Gross–Neveu–Yukawa (GNY) and the GN models have the same chiral and $U(\tilde{N})$ symmetries. The action of the GNY model is (now $\varepsilon = 4 - d$) [87–90]

$$\mathcal{S}(\bar{\psi}, \psi, \sigma) = \int \mathrm{d}^d x \left[-\bar{\psi} \cdot \left(\partial\!\!\!/ + g\Lambda^{\varepsilon/2}\sigma\right)\psi + \tfrac{1}{2}\left(\partial_\mu \sigma\right)^2 + \tfrac{1}{2}m^2\sigma^2 + \frac{\lambda}{4!}\Lambda^\varepsilon\sigma^4\right], \tag{4.30}$$

where σ is an additional scalar field, Λ the momentum cut-off, and g, λ dimensionless "bare", that is effective coupling constants at large momentum scale Λ.

The action still has a reflection symmetry, σ transforming into $-\sigma$ when the fermions transform by (4.9). In contrast with the GN model, however, the chiral transition can be discussed here by perturbative methods. An analogous situation has already been encountered when comparing the $(\phi^2)^2$ field theory with the non-linear σ model. An additional analogy is provided by the property that the GN model is renormalizable in dimension 2 and the GNY model in four dimensions.

The phase transition. Examining the action (4.30), we see that in the tree approximation when m^2 is negative the chiral symmetry is spontaneously broken. The σ expectation value gives a mass to the fermions, a mechanism reminiscent of the Standard Model of weak-electromagnetic interactions,

$$m_\psi = g\Lambda^{\varepsilon/2}\langle\sigma\rangle\,, \tag{4.31}$$

while the σ mass is then

$$m_\sigma^2 = \frac{\lambda}{3g^2}m_\psi^2\,. \tag{4.32}$$

Interactions modify the transition value m_c^2 of the parameter m^2 and thus in what follows we set

$$m^2 = m_c^2 + \tau\,. \tag{4.33}$$

The new parameter τ plays, in the language of phase transitions, the role of the deviation from the critical temperature.

In order to study the model beyond the tree approximation, we discuss now shortly RG equations near four dimensions.

4.5 RG equations near dimension 4

The model (4.30) is trivial above four dimensions, renormalizable in four dimensions and can thus be studied near dimension 4 by RG techniques. Five renormalization constants are required, corresponding to the two field renormalizations, the σ mass, and the two coupling constants. The RG equations thus involve five RG functions. The vertex functions $\Gamma^{(\ell,n)}$, for l ψ and n σ fields, then satisfy

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta_{g^2}\frac{\partial}{\partial g^2} + \beta_\lambda\frac{\partial}{\partial\lambda} - \tfrac{1}{2}\ell\eta_\psi - \tfrac{1}{2}n\eta_\sigma - \eta_m\tau\frac{\partial}{\partial\tau}\right)\Gamma^{(\ell,n)} = 0\,. \tag{4.34}$$

The RG functions at one-loop order are

$$\beta_\lambda = -\varepsilon\lambda + \frac{1}{8\pi^2}\left(\frac{3}{2}\lambda^2 + N\lambda g^2 - 6Ng^4\right), \tag{4.35}$$

$$\beta_{g^2} = -\varepsilon g^2 + \frac{N+6}{16\pi^2}g^4, \tag{4.36}$$

where $N = \tilde{N}\,\mathrm{tr}\,\mathbf{1}$ is the total number of fermion components. In four dimensions $\mathrm{tr}\,\mathbf{1} = 4$ and thus $N = 4\tilde{N}$.

Dimension 4: In dimension 4, the origin $\lambda = g^2 = 0$ is IR stable. Indeed Eq. (4.36) implies that g goes to zero, and Eq. (4.35) then implies that also λ goes to zero. As a consequence, if the bare coupling constants are generic, that is if the effective couplings at cut-off scale are of order 1, the effective couplings at scale $\mu \ll \Lambda$ are small and become asymptotically independent from the initial bare couplings. One finds

$$g^2(\mu) \sim \frac{16\pi^2}{(N+6)\ln(\Lambda/\mu)}\,, \quad \lambda(\mu) \sim \frac{16\pi^2}{\ln(\Lambda/\mu)} R \tag{4.37}$$

where we have defined

$$R = \frac{24N}{(N+6)\left[(N-6)+\sqrt{N^2+132N+36}\right]}. \tag{4.38}$$

This observation allows using renormalized perturbation theory to calculate physical observables. For example, we can evaluate the ratio between the masses of the scalar and fermion fields. To minimize quantum corrections we take for μ a value of order $\langle\sigma\rangle$. A remarkable consequence follows: the ratio (4.32) of scalar and fermion masses is fixed [75,91,92]:

$$\frac{m_\sigma^2}{m_\psi^2} = \frac{\lambda_*}{3g_*^2} = \frac{8N}{(N-6)+\sqrt{N^2+132N+36}}, \tag{4.39}$$

while in the classical limit it is arbitrary. Note that in the large N limit

$$\frac{m_\sigma}{m_\psi} = 2\left(1-\tfrac{15}{N}\right) + O(1/N^2).$$

The ratio has the same limit 2 as in two dimensions (Eq. (4.27)), a result that will be explained by the study of the large N limit in section 4.6.

Of course, if the initial bare couplings are "unnaturally" small, the ratio Λ/μ may not be large enough for the asymptotic regime (4.37) to be reached. The renormalized couplings at scale μ may then be even smaller than in Eq. (4.37) and the ratio will remain arbitrary.

Dimension $d = 4-\varepsilon$. One then finds a non-trivial IR fixed point

$$g_*^2 = \frac{16\pi^2\varepsilon}{N+6}, \quad \lambda_* = 16\pi^2 R\varepsilon\,. \tag{4.40}$$

The matrix of derivatives of the β-functions has two positive eigenvalues ω_1, ω_2,

$$0 < \omega_1 = \varepsilon < \omega_2 = \varepsilon\sqrt{N^2+132N+36}/(N+6), \tag{4.41}$$

and thus the fixed point is IR stable.

The field renormalization RG functions at the same order are

$$\eta_\sigma = \frac{N}{16\pi^2} g^2, \qquad \eta_\psi = \frac{1}{16\pi^2} g^2. \tag{4.42}$$

At the fixed point one finds

$$\eta_\sigma = \frac{N\varepsilon}{N+6}, \quad \eta_\psi = \frac{\varepsilon}{(N+6)}, \tag{4.43}$$

and thus the dimensions d_ψ and d_σ of the fields:

$$d_\psi = \frac{3}{2} - \frac{N+4}{2(N+6)}\varepsilon\,, \qquad d_\sigma = 1 - \frac{3}{N+6}\varepsilon\,. \tag{4.44}$$

The RG function η_m corresponding to the mass operator is at one-loop order:

$$\eta_m = -\frac{\lambda}{16\pi^2} - \eta_\sigma\,,$$

and the correlation length exponent ν is given by

$$\frac{1}{\nu} = 2 + \eta_m = 2 - \varepsilon R - \frac{N\varepsilon}{N+6} = 2 - \varepsilon \frac{5N + 6 + \sqrt{N^2 + 132N + 36}}{6(N+6)}. \tag{4.45}$$

Finally, we can evaluate the ratio of masses (4.32) at the fixed point:

$$\frac{m_\sigma^2}{m_\psi^2} = \frac{\lambda_*}{3g_*^2} = \frac{8N}{(N-6) + \sqrt{N^2 + 132N + 36}}.$$

In $d = 4$ and $d = 4 - \varepsilon$, the existence of an IR fixed point has the same consequence: if we assume that the σ expectation value is much smaller than the cut-off and that the coupling constants are generic at the cut-off scale, then *the ratio of fermion and scalar masses is fixed.*

4.6 GNY and GN models in the large N limit

We now show that the GN model plays with respect to the GNY model (4.30) the role the non-linear σ-model plays with respect to the ϕ^4 field theory [75]. For this purpose we start from the action (4.30) of the GNY model and integrate over $\tilde{N} - 1$ fermion fields. We also rescale for convenience $\Lambda^{(4-d)/2} g\sigma$ into σ, and then get the large N action

$$\begin{aligned}\mathcal{S}_N(\bar\psi, \psi, \sigma) = \int \mathrm{d}^d x \left\{ -\bar\psi\,(\partial\!\!\!/ + \sigma)\,\psi + \Lambda^{d-4} \left[\frac{1}{2g^2}\left((\partial_\mu\sigma)^2 + m^2\sigma^2\right) + \frac{\lambda\sigma^4}{4!g^4} \right] \right\} \\ - (\tilde{N} - 1)\,\mathrm{tr}\ln\,(\partial\!\!\!/ + \sigma)\,.\end{aligned} \tag{4.46}$$

We take the large N limit with Ng^2, $N\lambda$ fixed. When σ is of order one, the action is of order N and can be calculated by the steepest descent method.

We denote by $\mathcal{E}(\sigma)$ the action density for constant field $\sigma(x)$ and vanishing fermion fields:

$$\begin{aligned}\mathcal{E}(\sigma) &= \Lambda^{d-4}\left(\frac{m^2}{2g^2}\sigma^2 + \frac{\lambda}{4!g^4}\sigma^4\right) - \tilde{N}\,\mathrm{tr}\ln\left(\partial\!\!\!/ + \sigma\right) \\ &= \Lambda^{d-4}\left(\frac{m^2}{2g^2}\sigma^2 + \frac{\lambda}{4!g^4}\sigma^4\right) - \frac{N}{2}\int^{\Lambda}\frac{\mathrm{d}^d q}{(2\pi)^d}\ln[(q^2+\sigma^2)/q^2]. \quad (4.47)\end{aligned}$$

The expectation value of σ for N large is a solution to the *gap* equation

$$\mathcal{E}'(\sigma)\Lambda^{4-d} = \frac{m^2}{g^2}\sigma + \frac{\lambda}{6g^4}\sigma^3 - N\Lambda^{4-d}\sigma\,\Omega_d(\sigma) = 0\,, \quad (4.48)$$

where, again, we have introduced the function (2.12). It is also useful to calculate the second derivative to check stability of the extrema:

$$\mathcal{E}''(\sigma)\Lambda^{4-d} = \frac{m^2}{g^2} + \frac{\lambda}{2g^4}\sigma^2 + N\Lambda^{4-d}\int^{\Lambda}\frac{\mathrm{d}^d q}{(2\pi)^d}\frac{\sigma^2 - q^2}{(q^2+\sigma^2)^2}.$$

The solution $\sigma = 0$ is stable provided

$$\mathcal{E}''(0) > 0 \;\Leftrightarrow\; \frac{m^2}{g^2} > N\Lambda^{4-d}\Omega_d(0).$$

Instead, the non-trivial solution to the gap equation exists only for

$$\frac{m^2}{g^2} < N\Lambda^{4-d}\Omega_d(0),$$

but then it is stable. We conclude that the bare mass m_c given by

$$\frac{m_c^2}{g^2} = N\Lambda^{4-d}\Omega_d(0), \quad (4.49)$$

is the critical bare mass (the analogue of the critical temperature for a classical statistical system) where a phase transition occurs. The expression shows that the fermions favour the chiral transition. In particular when d approaches 2, we observe that $m_c^2 \to +\infty$ which implies that the chiral symmetry is always broken in two dimensions. Using Eq. (4.49) and setting

$$\tau = \Lambda^{d-4}(m^2 - m_c^2)/g^2, \quad (4.50)$$

we can write the equation for the non-trivial solution as

$$\tau + \Lambda^{d-4}\frac{\lambda}{6g^4}\sigma^2 + N\big(\Omega_d(0) - \Omega_d(\sigma)\big) = 0\,.$$

We now expand Ω_d for σ small (see Eq. (2.13)):

$$\Omega_d(0) - \Omega_d(\sigma) = K(d)\sigma^{2-\varepsilon} - a(d)\Lambda^{-\varepsilon}\sigma^2 + O\left(\sigma^4/\Lambda^{2+\varepsilon}\right). \tag{4.51}$$

Keeping only the leading terms for $\tau \to 0$, we obtain for $d < 4$ the scaling behaviour

$$\sigma \sim \big(-\tau/NK(d)\big)^{1/(d-2)}. \tag{4.52}$$

Since, at leading order, the fermion mass $m_\psi = \sigma$, it follows immediately that the exponent ν is also given by

$$\nu \sim \beta \sim 1/(d-2) \;\Rightarrow\; \eta_\sigma = 4 - d \;\Leftrightarrow\; d_\sigma = \tfrac{1}{2}(d - 2 + \eta_\sigma) = 1\,. \tag{4.53}$$

At leading order for $N \to \infty$, ν has the same value as in the non-linear σ-model.

At leading order in the scaling limit, the thermodynamic potential density then becomes

$$\mathcal{G}(\sigma) = \tfrac{1}{2}\tau\sigma^2 + (N/d)K(d)|\sigma|^d. \tag{4.54}$$

We note that, although in terms of the σ-field the model has a simple Ising-like symmetry, the scaling equation of state for N large is different. This reflects the property that the fermions become massless at the transition and thus do not decouple.

We also read from the large N action that at this order $\eta_\psi = 0$. Finally, from the large N action we can calculate the σ-propagator at leading order. Quite generally, using the saddle point equation, one finds for the inverse σ-propagator in the massive phase

$$\begin{aligned}\Delta_\sigma^{-1}(p) = \Lambda^{d-4}&\left(\frac{p^2}{g^2} + \frac{\lambda}{3g^4}\sigma^2\right)\\ &+ \frac{N}{2(2\pi)^d}\left(p^2 + 4\sigma^2\right)\int^\Lambda \frac{\mathrm{d}^d q}{(q^2+\sigma^2)\left[(p+q)^2+\sigma^2\right]}.\end{aligned} \tag{4.55}$$

We see that in the scaling limit $p, \sigma \to 0$, the integral yields the leading contribution. Neglecting corrections to scaling, we find that the propagator vanishes for $p^2 = -4\sigma^2$ which is just the $\bar\psi\psi$ threshold. Thus, in this limit, $m_\sigma = 2m_\psi$ in all dimensions, a result consistent with the $d = 2$ and $d = 4$ exact values (Eq. (4.27,4.39)).

At the transition the propagator reduces to

$$\Delta_\sigma \sim \frac{2}{Nb(d)p^{d-2}} \tag{4.56}$$

with (Eq. (2.60))

$$b(d) = -\frac{\pi}{\sin(\pi d/2)} \frac{\Gamma^2(d/2)}{\Gamma(d-1)} N_d \,. \tag{4.57}$$

The result is consistent with the value of η_σ in Eq. (4.53).

Finally, we note that the behaviour of the propagator at the critical point, $\Delta_\sigma(p) \propto p^{2-d}$, implies that the field σ from the point of view of the large N expansion, for $2 \leq d \leq 4$, has the canonical dimension

$$[\sigma] = 1 \,. \tag{4.58}$$

Corrections to scaling and the IR fixed point. The IR fixed point is determined by demanding the cancellation of the leading corrections to scaling. In the example of the action density $\mathcal{E}(\sigma)$, the leading correction to scaling is proportional to

$$\left(\frac{\lambda}{4! g^4} - \frac{N a(d)}{4} \right) \sigma^4,$$

($a(d) \sim 1/8\pi^2\varepsilon$). We now assume $a(d) > 0$, otherwise we are led to problems analogous to those already discussed in section 2.4. Demanding the cancellation of the coefficient of σ^4, we obtain a relation between λ and g^2,

$$g_*^4 = \frac{\lambda_*}{6Na(d)} = \frac{4\lambda_* \varepsilon \pi^2}{3N} + O\left(\varepsilon^2\right).$$

In the same way, it is possible to calculate the leading correction to the σ-propagator (4.55). Demanding the cancellation of the leading correction, we obtain

$$\frac{p^2}{g_*^2} + \frac{\lambda_*}{3g_*^4}\sigma^2 - \tfrac{1}{2}N\left(p^2 + 4\sigma^2\right) a(d) = 0 \,.$$

The coefficient of σ^2 cancels from the previous relation and the cancellation of the coefficient of p^2 implies

$$g_*^2 = \frac{2}{Na(d)} = \frac{16\pi^2\varepsilon}{N} + O\left(\varepsilon^2\right) \;\Rightarrow\; \lambda^* = \frac{192\pi^2\varepsilon}{N} \,,$$

in agreement with the ε-expansion for N large.

The relation to the GN model for dimensions $2 \leq d \leq 4$. In several examples we have observed that the contributions coming from the terms $(\partial_\mu\sigma)^2$ and σ^4 in the large N action could be neglected in the IR critical region for $d \leq 4$. Power counting confirms this property because both terms have a canonical dimension $4 > d$ and are therefore irrelevant. We recognize a situation already encountered

in the $(\phi^2)^2$ field theory in the large N limit. In the scaling region it is possible to omit them and one then finds the action

$$\mathcal{S}_N(\bar{\psi},\psi,\sigma) = \int \mathrm{d}^d x \left[-\bar{\psi}\cdot(\partial\!\!\!/ + \sigma)\,\psi + \Lambda^{d-4}\frac{m^2}{2g^2}\sigma^2 \right]. \tag{4.59}$$

The gaussian integral over the σ field can then be performed explicitly and yields the action of the GN model

$$\mathcal{S}_N(\bar{\psi},\psi) = -\int \mathrm{d}^d x \left[\bar{\psi}\cdot\partial\!\!\!/\psi + \frac{\Lambda^{4-d}}{2m^2} g^2 \left(\bar{\psi}\cdot\psi\right)^2 \right].$$

The GN (with an attractive interaction) and GNY models are thus equivalent for large distance physics, that is in the continuum limit. Again, the arguments above rely on the condition that the effective coupling constants at cut-off scale are generic. In the GN model, in the large N limit, the σ particle, simply appears as a $\bar{\psi}\psi$ bound state at threshold [94,96].

One may then wonder whether the corrections to scaling are different. Indeed superficially it would seem that the GN model depends on a smaller number of parameters than the GNY model. Again this problem is only interesting in four dimensions where corrections to the leading contributions vanish only logarithmically. However, if we examine the divergences of the term $\mathrm{tr}\ln(\partial\!\!\!/ + \sigma)$ in the effective action (4.46) relevant for the large N limit, we find a local polynomial in σ of the form

$$\int \mathrm{d}^4 x \left[A\sigma^2(x) + B\left(\partial_\mu\sigma\right)^2 + C\sigma^4(x) \right].$$

Therefore, the value of the determinant can be modified by a local polynomial of this form by changing the way the cut-off is implemented: additional parameters, as in the case of the non-linear σ-model, are hidden in the cut-off procedure. Near two dimensions these operators can be identified with $(\bar{\psi}\psi)^2, [\partial_\mu(\bar{\psi}\psi)]^2, (\bar{\psi}\psi)^4$. It is clear that by changing the cut-off procedure, we change the amplitude of higher dimension operators. These bare operators in the IR limit have a component on all lower dimensional renormalized operators.

Finally, note that we could have added to the GNY model an explicit breaking term linear in the σ field, which becomes a fermion mass term in the GN model, and which would have played the role of the magnetic field of ferromagnets.

4.7 The large N expansion

Using the large N dimension of fields and power counting arguments, one can then prove that the $1/N$ expansion is renormalizable with arguments quite similar to those presented in section 3.1 [1,2,96].

Alternative theory. To prove that the large N expansion is renormalizable, one proceeds as in the case of the scalar theory in section 3.1. One starts from a

critical action with an additional term quadratic in σ which generates the large N σ-propagator already in perturbation theory:

$$\mathcal{S}(\psi,\bar{\psi},\sigma)=\int \mathrm{d}^d x\left[-\bar{\psi}(\partial\!\!\!/+\sigma)\psi+\frac{1}{2v^2}\sigma(-\partial^2)^{d/2-1}\sigma\right]. \tag{4.60}$$

The initial theory is recovered in the limit $v\to\infty$. One then rescales σ in $v\sigma$. The model is renormalizable without σ field renormalization because divergences generate only local counter-terms. The renormalized action then reads

$$\mathcal{S}_{\mathrm{r}}(\psi,\bar{\psi},\sigma)=\int \mathrm{d}^d x\left[-Z_\psi\bar{\psi}(\partial\!\!\!/+v_{\mathrm{r}}Z_v\sigma)\psi+\frac{1}{2}\sigma(-\partial^2)^{d/2-1}\sigma\right]. \tag{4.61}$$

RG equations follow:

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta_{v^2}(v)\frac{\partial}{\partial v^2}-\frac{n}{2}\eta_\psi(v)\right]\Gamma^{(l,n)}=0\,. \tag{4.62}$$

Again, the large N expansion is obtained by first summing the bubble contributions to the σ-propagator. We define

$$D(v)=\frac{2}{b(d)}+Nv^2.$$

Then, the σ propagator for N large reads

$$\langle\sigma\sigma\rangle=\frac{2}{b(d)D(v)p^{d-2}}. \tag{4.63}$$

The solution to the RG equations can be written as

$$\Gamma^{(l,n)}(\ell p,v,\Lambda)=Z^{-n/2}(\ell)\ell^{d-l-n(d-2)/2}\Gamma^{(l,n)}(p,v(\ell),\Lambda) \tag{4.64}$$

with the usual definitions

$$\ell\frac{\mathrm{d}v^2}{\mathrm{d}\ell}=\beta(v(\ell))\,,\quad \ell\frac{\mathrm{d}\ln Z}{\mathrm{d}\ell}=\eta_\psi(v(\ell))\,.$$

We are interested in the neighbourhood of the fixed point $v^2=\infty$. Then, the RG function $\eta(v)$ approaches the exponent η. The flow equation for the coupling constant becomes

$$\ell\frac{\mathrm{d}v^2}{\mathrm{d}\ell}=\rho v^2,\ \Rightarrow\ v^2(\ell)\sim\ell^\rho.$$

We again note that a correlation function with l σ fields becomes proportional to v^l. Therefore,

$$\Gamma^{(l,n)}(\ell p,v,\Lambda)\propto\ell^{d-(1-\rho/2)l-n(d-2+\eta_\psi)/2}. \tag{4.65}$$

We conclude

$$d_\sigma = \tfrac{1}{2}(d-2+\eta_\sigma) = 1 - \tfrac{1}{2}\rho \;\Leftrightarrow\; \eta_\sigma = 4 - d - \rho\,. \tag{4.66}$$

RG functions at order $1/N$ [97]. A new generic integral is useful here:

$$\frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d q(\not{p}+\not{q})}{(p+q)^{2\mu}q^{2\nu}} = \not{p}p^{d-2\mu-2\nu}\frac{\Gamma(\mu+\nu-d/2)\Gamma(d/2-\mu+1)\Gamma(d/2-\nu)}{(4\pi)^{d/2}\Gamma(\mu)\Gamma(\nu)\Gamma(d-\mu-\nu+1)}\,. \tag{4.67}$$

We first calculate the $1/N$ contribution to the fermion two-point function at the critical point (from a diagram similar to diagram 3)

$$\Gamma^{(2)}_{\bar\psi\psi}(p) = i\not{p} + \frac{2iv^2}{b(d)D(v)(2\pi)^d}\int^\Lambda \frac{\mathrm{d}^d q(\not{p}+\not{q})}{q^{d-2}(p+q)^2}.$$

We need the coefficient of $\not{p}\ln\Lambda/p$. Since we work only at one-loop order, we again replace the σ propagator $1/q^{d-2}$ by $1/q^{2\nu}$ and send the cut-off to infinity. The residue of the pole at $2\nu = d-2$ gives the coefficient of the term $\not{p}\ln\Lambda$ and the finite part the $\not{p}\ln p$ contribution. We find

$$\Gamma^{(2)}_{\bar\psi\psi}(p) = i\not{p} + \frac{2iv^2}{b(d)D(v)}N_d\left(\frac{d-2}{d}\right)\not{p}\ln(\Lambda/p)\,, \tag{4.68}$$

where N_d is the loop factor (2.14*b*). Expressing that the $\langle\bar\psi\psi\rangle$ function satisfies RG equations, we obtain immediately the RG function

$$\eta_\psi(v) = \frac{v^2}{D(v)}\frac{(d-2)}{d}X_1\,, \tag{4.69}$$

where X_1 is given by equation (3.28). We then calculate the function $\langle\sigma\bar\psi\psi\rangle$ at order $1/N$:

$$\Gamma^{(3)}_{\sigma\bar\psi\psi}(p) = v + A_1 D^{-1}(v)v^3\ln\Lambda\,,$$

where A_1 corresponds to the diagram of figure 4:

$$A_1 = -\frac{2}{b(d)}N_d = -X_1\,.$$

The diagram of figure 5 vanishes because the σ three-point function vanishes for symmetry reasons.

The β-function follows:

$$\beta_{v^2}(v) = \frac{4(d-1)v^4}{d}X_1 D^{-1}(v) \tag{4.70}$$

and thus

$$\rho = \frac{8(d-1)N_d}{db(d)N} = \frac{4(d-1)}{dN} X_1 \,.$$

The exponents η_ψ and η_σ at order $1/N$ and, thus, the corresponding dimensions d_ψ, d_σ of the fields, follow:

$$\eta_\psi = \frac{(d-2)}{d}\frac{X_1}{N} = \frac{(d-2)^2}{d}\frac{\Gamma(d-1)}{\Gamma^3(d/2)\Gamma(2-d/2)N}. \tag{4.71}$$

$$2d_\psi = d - 1 - \frac{2(d-2)}{d}\frac{X_1}{N} \,. \tag{4.72}$$

For $d = 4-\varepsilon$, we find $\eta_\psi \sim \varepsilon/N$, result consistent with (4.43) for N large. Whereas for $d = 2+\varepsilon$, one finds $\eta_\psi \sim \varepsilon^2/2N$, consistent with (4.29). The dimension of the field σ is

$$d_\sigma = \tfrac{1}{2}(d-2+\eta_\sigma) = 1 - \frac{2(d-1)}{dN} X_1 + O(1/N^2). \tag{4.73}$$

A similar evaluation of the $\langle \sigma^2 \sigma\sigma \rangle$ function allows to determine the exponent ν to order $1/N$:

$$\frac{1}{\nu} = d - 2 - \frac{2(d-1)(d-2)}{dN} X_1 \,. \tag{4.74}$$

Actually all exponents are known to order $1/N^2$, except η_ψ which is known to order $1/N^3$.

We discuss now shortly two other models with chiral fermions, in which large N techniques can be applied, massless QED and the $U(N)$ massless Thirring model.

4.8 Massless electrodynamics with $U(\tilde{N}) \times U(\tilde{N})$ symmetry.

We first consider $\tilde{N}$ charged massless fermion fields $\psi, \bar{\psi}$, interacting through an abelian gauge field A_μ (massless QED with $\tilde{N}$ flavours) [98]:

$$\mathcal{S}(\bar{\psi}, \psi, A_\mu) = \int \mathrm{d}^d x \left[\tfrac{1}{4e^2} F_{\mu\nu}^2(x) - \bar{\psi}(x) \cdot (\not{\partial} + i\not{A})\, \psi(x) \right] . \tag{4.75}$$

This model possesses, in addition to the $U(1)$ gauge invariance, a chiral $U(\tilde{N}) \times U(\tilde{N})$ symmetry since the fermions are massless. Again, an interesting question is whether the model exhibits in some dimensions $2 \le d \le 4$ a spontaneous breaking of chiral symmetry. As before we use here $N = \tilde{N}\,\mathrm{tr}\,\mathbf{1}$, where N is the total number of fermion components in $\tilde{N}$ differently flavoured Dirac fields.

Dimension $d = 4-\varepsilon$. In terms of the coupling constant standard in dimension 4,

$$\alpha \equiv \frac{e^2}{4\pi} \Lambda^{-\varepsilon} \,, \tag{4.76}$$

the RG β-function reads (taking $\mathrm{tr}\,\mathbf{1} = 4$ in the space of γ matrices) [103]:

$$\beta(\alpha) = -\varepsilon\alpha + \frac{2\tilde{N}}{3\pi}\alpha^2 + \frac{\tilde{N}}{2\pi^2}\alpha^3 - \frac{\tilde{N}(22\tilde{N}+9)}{144\pi^3}\alpha^4 \\ - \frac{1}{64\pi^4}\tilde{N}\left[\tfrac{616}{243}\tilde{N}^2 + \left(\tfrac{416}{9}\zeta(3) - \tfrac{380}{27}\right)\tilde{N} + 23\right]\alpha^5 + O\left(\alpha^6\right). \quad (4.77)$$

The model is IR free in four dimensions. Therefore no phase transition is expected, at least for e^2 small enough. A hypothetical phase transition would rely on the existence on non-trivial fixed points outside of the perturbative regime.

In the perturbative framework, the model provides an example of the famous triviality property. For a generic effective coupling constant at cut-off scale (i.e. bare coupling), the effective coupling constant at scale $\mu \ll \Lambda$ is given by

$$\alpha(\mu) \equiv \frac{e^2(\mu)}{4\pi} \sim \frac{3\pi}{2\tilde{N}\ln(\Lambda/\mu)}.$$

This result can be used to bound the number of charged fields (the number is not huge).

In $4-\varepsilon$ dimension instead, one finds a non-trivial IR fixed point corresponding to a coupling constant

$$e_*^2 = 24\pi^2\varepsilon\Lambda^\varepsilon/N\,,$$

($N = \tilde{N}\,\mathrm{tr}\,\mathbf{1}$) and correlation functions have a scaling behaviour at large distance. As we have discussed in the case of the ϕ^4 field theory, the effective coupling constant at large distance becomes close to the IR fixed point, except when the initial coupling constant is very small.

The RG function associated with the field renormalization is also known up to order α^3:

$$\eta_\psi = \xi\frac{\alpha}{2\pi} - \frac{4\tilde{N}+3}{16\pi^2}\alpha^2 + \frac{40\tilde{N}^2+54\tilde{N}+27}{576\pi^3}\alpha^3 + O\left(\alpha^4\right),$$

where the gauge is specified by a term $(\partial_\mu A_\mu)^2/2\xi$, but it is a non-physical quantity because it is gauge dependent. The simple dependence of η_ψ in ξ reflects the property that if $\bar{\psi}(x)\psi(y)$ is not gauge invariant, $\bar{\psi}(x)\exp[i\oint_y^x A_\mu(s)\mathrm{d}s_\mu]\psi(y)$, instead is.

The large N limit [104]. To solve the model for $N \to \infty$, one first integrates over the fermion fields and one obtains the large N action

$$\mathcal{S}_N(A_\mu) = \int \mathrm{d}^d x\left[\tfrac{1}{4e^2}F_{\mu\nu}^2(x) - \tilde{N}\,\mathrm{tr}\ln\left(\not{\partial} + i\not{A}\right)\right]. \quad (4.78)$$

The large N limit ($N = \tilde{N}\,\mathrm{tr}\,\mathbf{1}$) is taken with $e^2 N$ fixed. Therefore, at leading order, only $\mathcal{S}_{N,\mathrm{F}}(A_\mu)$, the quadratic term in A_μ in the expansion of the fermion determinant, contributes. A short calculation yields

$$\begin{aligned} \mathcal{S}_{N,\mathrm{F}}(A_\mu) &= -N \int \mathrm{d}^d k\, A_\mu(k) \left[k^2 \delta_{\mu\nu} - k_\mu k_\nu\right] K(k) A_\nu(-k), \\ \text{with} \quad K(k) &= \frac{d-2}{4(d-1)} \left[b(d) k^{d-4} - a(d) \Lambda^{d-4}\right] + O\left(\Lambda^{-2}\right), \end{aligned} \tag{4.79}$$

where $b(d)$ is the universal constant (2.60), and $a(d)$ is the constant (2.15) that depends on the regularization.

For $d < 4$, the leading term in the IR region comes from the integral. The behaviour at small momentum of the vector field is modified, which confirms the existence of a non-trivial IR fixed point. The fixed point is found by demanding cancellation of the leading corrections to scaling coming from $F_{\mu\nu}^2$ and the divergent part of the loop integral,

$$e_*^2 = \frac{2(d-1)}{(d-2)a(d)} \frac{\Lambda^{4-d}}{N}.$$

However, there is again no indication of chiral symmetry breaking. Power counting within the $1/N$ expansion confirms that the IR singularities have been eliminated, because the large N vector propagator is less singular than in perturbation theory. Of course, this result is valid only for N large. Since the long range forces generated by the gauge coupling have not been totally eliminated, the problem remains open for d not close to four, or for e^2 not very small and N finite. Some numerical simulations indeed suggest a chiral phase transition for $d = 4$ and $d = 3$, $\tilde{N} \leq N_c \sim 3$.

The exponents corresponding to the IR fixed point have been calculated up to order $1/N^2$. At order $1/N$ (X_1 is defined by Eq. (3.28))

$$\begin{aligned} \eta_\psi(\xi = 0) &= -\frac{(d-1)^2(4-d)}{d(d-2)} \frac{X_1}{N} + O\left(1/N^2\right), \\ \eta_m &= -4 \frac{(d-1)^2}{d(d-2)} \frac{X_1}{N} + O\left(1/N^2\right), \\ \beta'(\alpha^*) &= 4 - d - \frac{(d-3)(d-6)(d-1)^2(4-d)}{d(d-2)} \frac{X_1}{N} + O\left(1/N^2\right). \end{aligned}$$

Finally, note that in the $d = 2$ limit, the integral generates a contribution $Ne^2/\pi k^2$ times the propagator of the free gauge field

$$K(k) \underset{d \to 2}{\sim} \frac{1}{4\pi k^2}.$$

As a direct analysis of the $d = 2$ case confirms, this corresponds to a massive bound state, of mass squared Ne^2/π. However, for generic values of the coupling constant, the mass is of the order of the cut-off Λ. Only when e is "unnaturally" small with respect to the microscopic scale, as one assumes in conventional renormalized perturbation theory, does this mass correspond in the continuum limit to a propagating particle.

Two dimensions. We now assume that the dimensional quantity e^2 is small in the microscopic scale. The model then is a simple extension of the Schwinger model and can be exactly solved by the same method. For $\tilde{N} = 1$, the model exhibits the simplest example of a chiral anomaly, illustrates the property of confinement and spontaneous chiral symmetry breaking. For $\tilde{N} > 1$, the situation is more subtle. The neutral $\bar{\psi}\psi$ two-point function decays algebraically,

$$\left\langle \bar{\psi}(x) \cdot \psi(x) \bar{\psi}(0) \cdot \psi(0) \right\rangle \propto x^{2/\tilde{N}-2},$$

indicating the presence of a massless mode and $\left\langle \bar{\psi}\psi \right\rangle = 0$. Instead, if we calculate the two-point function of the composite operator

$$\mathcal{O}_{\tilde{N}}(x) = \prod_{i=1}^{\tilde{N}} \bar{\psi}_i(x)\psi_i(x),$$

we find

$$\langle \mathcal{O}_{\tilde{N}}(x)\mathcal{O}_{\tilde{N}}(0)\rangle \propto \text{ const. }.$$

We have thus identified an operator which has a non-zero expectation value. As a consequence of the fermion antisymmetry, if we perform a transformation under the group $U(\tilde{N}) \times U(\tilde{N})$ corresponding to matrices U_+, U_-, the operator is multiplied by $\det U_+ / \det U_-$. The operator thus is invariant under the group $SU(\tilde{N}) \times SU(\tilde{N}) \times U(1)$. Its non-vanishing expectation value is the sign of the spontaneous breaking of the remaining $U(1)$ chiral group.

4.9 The $U(\tilde{N})$ Thirring model

We now consider the model [107–109]

$$\mathcal{S}(\bar{\psi}, \psi) = -\int \mathrm{d}^d x \left[\bar{\psi} \left(\not{\partial} + m_0 \right) \psi - \tfrac{1}{2} g J_\mu J_\mu \right], \tag{4.80}$$

where

$$J_\mu = \bar{\psi}\gamma_\mu \cdot \psi \,. \tag{4.81}$$

The special case $\tilde{N} = 1$ corresponds to the simple Thirring model. In two dimensions, it is then equivalent to a free massless boson field theory (with mass term for fermions, one obtains the sine–Gordon model). In order to bosonize the

model in $d = 2$ and to study that large N properties, one introduces an abelian gauge field A_μ coupled to the current J_μ:

$$\tfrac{1}{2} g J_\mu J_\mu \longmapsto A_\mu^2/2g + i A_\mu J_\mu \,. \tag{4.82}$$

One then finds massive QED without a $F_{\mu\nu}^2$ gauge kinetic term:

$$\mathcal{S}(A_\mu, \bar{\psi}, \psi) = -\int \mathrm{d}^2 x \left[\bar{\psi}\left(\partial\!\!\!/ + i A\!\!\!/ + m_0\right)\psi - A_\mu^2/2g\right]. \tag{4.83}$$

If one integrates over the fermions, the fermion determinant generates a kinetic term for the gauge field. For $m_0 = 0$, the situation is thus similar to massless QED, except that the gauge field is now massive.

5 Dissipative dynamics in the large N limit

We now study a dissipative stochastic dynamics described by a Langevin equation, in the large N limit. From the point of critical phenomena, the dissipative Langevin equation describes the simplest time evolution with prescribed equilibrium distribution, and allows calculating relaxation or correlation times or time-dependent correlation functions. We recall that the correlation time diverges at a second order phase transition, a phenomenon called critical slowing-down, and this leads to universal long time evolution [111].

Purely dissipative stochastic dynamics is a problem interesting in its own right, but it will also serve as an introduction to the discussion of supersymmetric models of section 6, since correlation functions associated with the dissipative Langevin equation can be calculated from a functional integral with a supersymmetric action. The corresponding algebraic structure generalizes supersymmetric quantum mechanics.

5.1 Langevin equation in the large N limit

A general dissipative Langevin equation for a scalar field φ is a stochastic differential equation of the form

$$\frac{\partial\varphi(t,x)}{\partial t} = -\frac{\Omega}{2}\frac{\delta\mathcal{A}}{\delta\varphi(t,x)} + \nu(t,x), \tag{5.1}$$

where $\nu(t,x)$ is a gaussian white noise,

$$\langle\nu(t,x)\rangle = 0\,, \qquad \langle\nu(t,x)\nu(t',x')\rangle = \Omega\,\delta(t-t')\delta(x-x'), \tag{5.2}$$

and the constant Ω characterizes the amplitude of the noise. This equation generates a time-dependent field distribution, which converges at large time towards an equilibrium distribution corresponding to the functional measure $\mathrm{e}^{-\mathcal{A}(\varphi)}[\mathrm{d}\varphi]$ if it is normalizable.

$O(N)$ symmetric models in the large N limit. We now consider a model where φ is a N-component field and the static action $\mathcal{A}(\varphi)$ has an $O(N)$ symmetry of the form (2.1):

$$\mathcal{A}(\varphi) = \int \mathrm{d}^d x\, \left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + NU(\varphi^2/N)\right]. \tag{5.3}$$

The corresponding Langevin equation then reads

$$\dot\varphi_i(t,x) = -\tfrac{1}{2}\Omega\left[-\nabla_x^2 + 2U'(\varphi^2/N)\right]\varphi_i(t,x) + \nu_i(t,x) \tag{5.4}$$

with

$$\langle\nu_i(t,x)\rangle = 0\,, \qquad \langle\nu_i(t,x)\nu_j(t',x')\rangle = \Omega\,\delta(t-t')\delta(x-x')\delta_{ij}. \tag{5.5}$$

Here, the components ν_i of the noise are independent variables and in the large N limit the central limit theorem applies to $O(N)$ scalar functions of ν.

As a boundary condition, we choose

$$\varphi(t=-\infty,x)=0\,,$$

which ensures equilibrium at any finite time.

We then set

$$\rho(t,x)=\varphi^2(t,x)/N\,,\quad m^2=2U'(\rho),$$

and assume, as an ansatz, that m^2 goes to a constant in the large N limit. The Langevin equation then becomes linear and can be solved. Introducing the field Fourier components

$$\varphi(t,x)=\int \mathrm{d}^dk\ \mathrm{e}^{ikx}\,\tilde{\varphi}(t,k),\quad \nu(t,x)=\int \mathrm{d}^dk\ \mathrm{e}^{ikx}\,\tilde{\nu}(t,k),$$

one finds

$$\tilde{\varphi}(t,k)=\int_{-\infty}^{t}\mathrm{d}\tau\ \mathrm{e}^{-\Omega(k^2+m^2)(t-\tau)/2}\,\tilde{\nu}(\tau,k),$$

with

$$\langle\tilde{\nu}_i(t,k)\tilde{\nu}_j(t',k')\rangle=\frac{1}{(2\pi)^d}\delta(t-t')\delta(k+k')\delta_{ij}\,.$$

The calculation of $\rho(t,x)$ then involves the quantity with vanishing fluctuations for $N\to\infty$:

$$\sum_i\tilde{\nu}_i(\tau_1,k_1)\tilde{\nu}_i(\tau_2,k_2)\sim\frac{N}{(2\pi)^d}\delta(k_1+k_2)\delta(\tau_1-\tau_2).$$

Therefore,

$$\rho(t,x)=\frac{1}{(2\pi)^d}\int\mathrm{d}^dk\int_{-\infty}^{t}\mathrm{d}\tau\ \mathrm{e}^{-\Omega(k^2+m^2)(t-\tau)}=\frac{1}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{k^2+m^2}=\Omega_d(m),$$

a result that is consistent with the ansatz m constant. One then recognizes the static saddle point equation (2.10c) in the symmetric phase.

For the broken phase the ansatz or boundary conditions have to be slightly modified. One verifies that provided $m=0$, one can impose

$$\varphi(t=-\infty,x)=\sigma\,,$$

where σ is a constant. Then,

$$\tilde{\varphi}(t,k)=\int_{-\infty}^{t}\mathrm{d}\tau\ \mathrm{e}^{-\Omega k^2(t-\tau)/2}\,\tilde{\nu}(\tau,k)+\sigma\delta(k),$$

and

$$\rho(t,x) = \sigma^2/N + \frac{1}{(2\pi)^d}\int \mathrm{d}^d k \int_{-\infty}^{t} \mathrm{d}\tau\ \mathrm{e}^{-\Omega k^2(t-\tau)} = \sigma^2/N + \Omega_d(0),$$

where one recognizes the static saddle point equation (2.10c) in the broken phase.

However, we will not discuss the problem of the large N expansion in this formalism further, because we now introduce an alternative supersymmetric formalism.

5.2 Path integral solution: Supersymmetric formalism

In the case of the purely dissipative Langevin equation (5.1) with gaussian white noise, it can be shown that dynamic correlation functions can also be expressed in terms of a functional integral that generalizes supersymmetric quantum mechanics [112–114].

One introduces a superfield Φ function of two Grassmann coordinates $\bar\theta, \theta$:

$$\Phi(t,x;\bar\theta,\theta) = \varphi(t,x) + \theta\bar\psi(t,x) + \psi(t,x)\bar\theta + \theta\bar\theta\bar\varphi(t,x)\,,$$

and supersymmetric covariant derivatives

$$\bar{\mathrm{D}} = \frac{\partial}{\partial\bar\theta}, \quad \mathrm{D} = \frac{\partial}{\partial\theta} - \bar\theta\frac{\partial}{\partial t}, \tag{5.6}$$

which satisfy the anticommutation relations

$$\mathrm{D}^2 = \bar{\mathrm{D}}^2 = 0\,, \qquad \mathrm{D}\bar{\mathrm{D}} + \bar{\mathrm{D}}\mathrm{D} = -\frac{\partial}{\partial t}\,. \tag{5.7}$$

The generating functional of Φ-field correlation functions then is given by

$$\mathcal{Z}(J) = \int [\mathrm{d}\Phi]\, \mathrm{e}^{-\mathcal{S}(\Phi)+J\cdot\Phi}$$

with

$$\mathcal{S}(\Phi) = \int \mathrm{d}\bar\theta\,\mathrm{d}\theta\,\mathrm{d}t\left[\frac{2}{\Omega}\int \mathrm{d}^d x\,\bar{\mathrm{D}}\Phi\mathrm{D}\Phi + \mathcal{A}(\Phi)\right]. \tag{5.8}$$

Here $J(x,\bar\theta,\theta)$ is a source for Φ field:

$$J\cdot\Phi \equiv \int \mathrm{d}t\,\mathrm{d}^d x\,\mathrm{d}\bar\theta\mathrm{d}\theta\, J(t,x,\bar\theta,\theta)\Phi(t,x,\bar\theta,\theta).$$

Note that with our conventions for the θ integration measure

$$\delta^2(\theta-\theta') = (\theta-\theta')(\bar\theta-\bar\theta')\,.$$

The generalized action $\mathcal{S}(\Phi)$ is supersymmetric. The corresponding supersymmetry generators are

$$\mathrm{Q} = \frac{\partial}{\partial\theta}, \qquad \bar{\mathrm{Q}} = \frac{\partial}{\partial\bar{\theta}} + \theta\frac{\partial}{\partial t}\,. \tag{5.9}$$

Both *anticommute* with D and $\bar{\mathrm{D}}$ and satisfy

$$\mathrm{Q}^2 = \bar{\mathrm{Q}}^2 = 0\,, \qquad \mathrm{Q}\bar{\mathrm{Q}} + \bar{\mathrm{Q}}\mathrm{Q} = \frac{\partial}{\partial t}\,. \tag{5.10}$$

Let us verify, for instance, that $\bar{\mathrm{Q}}$ is the generator of a symmetry. We perform a variation of Φ of the form

$$\delta\Phi(t,\theta,\bar{\theta}) = \bar{\varepsilon}\bar{\mathrm{Q}}\Phi\,, \tag{5.11}$$

which in component form reads

$$\delta\varphi = \psi\bar{\varepsilon}\,, \quad \delta\psi = 0\,, \quad \delta\bar{\psi} = (\bar{\varphi} - \dot{\varphi})\,\bar{\varepsilon}\,, \quad \delta\bar{\varphi} = \dot{\psi}\bar{\varepsilon}\,. \tag{5.12}$$

The term $\mathcal{A}$ is invariant because it does not depend on t and $\bar{\theta}$ explicitly. For the remaining term, the additional property that $\bar{\mathrm{Q}}$ anticommutes with D and $\bar{\mathrm{D}}$ has to be used:

$$\delta\left[\bar{\mathrm{D}}\phi\mathrm{D}\phi\right] = \bar{\mathrm{D}}\left[\bar{\varepsilon}\bar{\mathrm{Q}}\phi\right]\mathrm{D}\phi + \bar{\mathrm{D}}\phi\mathrm{D}\left[\bar{\varepsilon}\bar{\mathrm{Q}}\phi\right] = \bar{\varepsilon}\bar{\mathrm{Q}}\left[\bar{\mathrm{D}}\phi\mathrm{D}\phi\right].$$

The variation of the action density thus is a total derivative. A similar argument applies to Q. This proves that the action is supersymmetric.

This supersymmetry is directly related to the property that the corresponding Fokker–Planck hamiltonian is equivalent to a positive hamiltonian.

Static action. The static action defines the equilibrium distribution. In what follows we have in mind a static action of the form (this includes the action (5.3))

$$\mathcal{A}(\varphi) = \int \mathrm{d}^d x\left[\tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}m^2\varphi^2 + V(\varphi)\right].$$

The propagator Δ is the inverse of the kernel

$$K = -\frac{2}{\Omega}[\bar{\mathrm{D}},\mathrm{D}] - \nabla_x^2 + m^2.$$

Introducing Fourier components, frequency ω corresponding to time and momentum k corresponding to space, we can write this operator more explicitly. From

$$[\bar{\mathrm{D}},\mathrm{D}]\delta^2(\theta-\theta') = 2 + i\omega(\theta'-\theta)(\bar{\theta}+\bar{\theta}')$$

one infers

$$K(\omega,\mathbf{k},\theta',\theta) = \left\{-\frac{2}{\Omega}[\bar{\mathrm{D}},\mathrm{D}] + k^2 + m^2\right\}\delta^2(\theta-\theta')$$
$$= -\frac{4}{\Omega}\left[1+\tfrac{1}{2}i\omega(\theta'-\theta)(\bar{\theta}+\bar{\theta}')\right] + (k^2+m^2)\delta^2(\theta-\theta').$$

To obtain the propagator Δ in superspace, we note

$$\left([\bar{\mathrm{D}},\mathrm{D}]\right)^2 = -\left(2\bar{\mathrm{D}}\mathrm{D}+i\omega\right)\left(2\mathrm{D}\bar{\mathrm{D}}+i\omega\right) = -\omega^2\,.$$

Then,

$$\Delta = \frac{\Omega}{2}\frac{[\bar{\mathrm{D}},\mathrm{D}]+\Omega(k^2+m^2)/2}{\omega^2+\Omega^2(k^2+m^2)^2/4}$$

or, more explicitly,

$$\Delta(\omega,\mathbf{k},\theta',\theta) = \frac{\Omega\left[1+\frac{1}{2}i\omega\left(\theta'-\theta\right)\left(\bar{\theta}+\bar{\theta}'\right)+\frac{1}{4}\Omega\left(k^2+m^2\right)\delta^2(\bar{\theta}'-\bar{\theta})\right]}{\omega^2+\frac{\Omega^2}{4}\left(k^2+m^2\right)^2}. \quad (5.13)$$

5.3 Ward–Takahashi (WT) identities and renormalization

WT identities. The symmetry associated with the Q generator has a simple consequence, correlation functions are invariant under a translation of the coordinate θ. The transformation (5.11) has a slightly more complicated form. Connected correlation functions $W^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i t)$ and proper vertices $\Gamma^{n)}(t_i,x_i,\theta_i,\bar{\theta}_i)$ satisfy the WT identities

$$\bar{\mathrm{Q}}W^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i) = 0\,, \qquad \bar{\mathrm{Q}}\Gamma^{(n)}(t_i,x_i,\theta_i,\bar{\theta}_i) = 0 \quad (5.14)$$

with

$$\bar{\mathrm{Q}} \equiv \sum_{k=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_k}+\theta_k\frac{\partial}{\partial t_k}\right).$$

After Fourier transformation over time, the operator $\bar{\mathrm{Q}}$ takes the form

$$\bar{\mathrm{Q}} = \sum_{k=1}^{n}\left(\frac{\partial}{\partial\bar{\theta}_k}-i\omega_k\theta_k\right). \quad (5.15)$$

One verifies immediately that the propagator (5.13) satisfies the identity (5.14). Actually, the general solution can be written as

$$\widetilde{W}^{(n)}(\omega,\mathbf{k},\theta,\bar{\theta}) = \exp\left[-\frac{i}{4n}\sum_{k,l}(\theta_k-\theta_l)(\omega_k-\omega_l)(\bar{\theta}_k+\bar{\theta}_l)\right]F^{(n)}(\omega,\mathbf{k},\theta,\bar{\theta}),$$

where the function $F^{(n)}(\omega, \mathbf{k}, \theta, \bar{\theta})$ now is invariant under translations of both θ and $\bar{\theta}$.

Example: a two-point function. Let us explore the implications of WT identities for a two-point function. As the relations (5.9,5.10) show, supersymmetry implies translation invariance on time and θ. Therefore, any two-point function $W^{(2)}$ can be written as

$$W^{(2)} = A(t_1 - t_2) + (\theta_1 - \theta_2)\left[(\bar{\theta}_1 + \bar{\theta}_2)B(t_1 - t_2) + (\bar{\theta}_1 - \bar{\theta}_2)C(t_1 - t_2)\right]. \quad (5.16)$$

The WT identity (5.14) then implies

$$2B(t) = \frac{\partial A}{\partial t}. \quad (5.17)$$

The WT identity does not determine the function C. An additional constraint comes from causality. For the two-point function, it implies that the coefficient of $\theta_1\bar{\theta}_2$ vanishes for $t_1 < t_2$ and the coefficient of $\theta_2\bar{\theta}_1$ for $t_2 < t_1$. The last function is thus determined, up to a possible distribution localized at $t_1 = t_2$. One finds

$$2C(t) = -\epsilon(t)\frac{\partial A}{\partial t}, \quad (5.18)$$

where $\epsilon(t)$ is the sign of t, and, therefore,

$$W^{(2)} = \left\{1 + \tfrac{1}{2}(\theta_1 - \theta_2)\left[\bar{\theta}_1 + \bar{\theta}_2 - (\bar{\theta}_1 - \bar{\theta}_2)\epsilon(t_1 - t_2)\right]\frac{\partial}{\partial t_1}\right\} A(t_1 - t_2). \quad (5.19)$$

Renormalization. In the special case of the supersymmetric dynamical action (5.8), a comparison between the two explicit quadratic terms in Φ of the action yields the relation between dimensions [118]

$$[t] - [\bar{\theta}] - [\theta] = 0 \;\Rightarrow [\mathrm{d}t] + [\mathrm{d}\bar{\theta}] + [\mathrm{d}\theta] = 0\,. \quad (5.20)$$

(We recall that since integration and differentiation over anticommuting variables are equivalent operations, the dimension of $\mathrm{d}\theta$ is $-[\theta]$.)

Therefore, the term proportional to $\mathcal{A}(\Phi)$ in the action has the same canonical dimension as in the static case: the power counting is thus the same and the dynamic theory is always renormalizable in the same space dimension as the static theory. Note that Eq. (5.20) also implies

$$2[\Phi] = d + [t]\,,$$

an equation that relates the dimensions of field and time.

One then verifies that supersymmetry is preserved by renormalization and that the most general supersymmetric renormalized action has the form

$$\mathcal{S}_{\mathrm{r}}(\Phi) = \int \mathrm{d}\bar{\theta}\,\mathrm{d}\theta\,\mathrm{d}t\left[\frac{2}{\Omega}Z_\Omega\int \mathrm{d}^d x\,\bar{\mathrm{D}}\Phi\mathrm{D}\Phi + \mathcal{A}_{\mathrm{r}}(\Phi)\right], \quad (5.21)$$

where Z_Ω is the renormalization of the parameter Ω, and thus also of the scale of time.

The renormalized Langevin equation thus remains dissipative; the drift force derives from an action.

5.4 $O(N)$ symmetric models in the large N limit: supersymmetric formalism

We now consider again the $O(N)$ symmetric Langevin equation (5.4) for an N-component field φ, corresponding to the static action (5.3):

$$\mathcal{A}(\varphi) = \int \mathrm{d}^d x \left[\tfrac{1}{2}(\partial_\mu \varphi)^2 + NU(\varphi^2/N)\right].$$

We apply the usual strategy and introduce in the dynamic theory two superfields L and R, which have the form

$$\begin{aligned} L(\theta) &= l + \theta\bar{\ell} + \ell\bar{\theta} + \theta\bar{\theta}\lambda\,, \\ R(\theta) &= \rho + \theta\bar{\sigma} + \sigma\bar{\theta} + \theta\bar{\theta}s\,. \end{aligned}$$

We implement the condition $R = \Phi^2/N$ by an integral over L. The functional integral takes the form

$$\mathcal{Z} = \int [\mathrm{d}\Phi][\mathrm{d}R][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}(\Phi,R,L)}$$

with

$$\mathcal{S}(\Phi,R,L) = \int \mathrm{d}t\mathrm{d}\bar{\theta}\mathrm{d}\theta\mathrm{d}^d x \left[\frac{2}{\Omega}\bar{\mathrm{D}}\Phi\mathrm{D}\Phi + \tfrac{1}{2}(\partial_\mu\Phi)^2 + NU(R) + \tfrac{1}{2}L(\Phi^2 - NR)\right].$$

We integrate over $N-1$ components of the Φ field, keeping one component $\Phi_1 = \phi$ as a test-component. The large N action then reads

$$\begin{aligned} \mathcal{S}_N = &\int \mathrm{d}t\mathrm{d}\bar{\theta}\mathrm{d}\theta\mathrm{d}^d x \left[\frac{2}{\Omega}\bar{\mathrm{D}}\phi\mathrm{D}\phi + \tfrac{1}{2}(\partial_\mu\phi)^2 + NU(R) + \tfrac{1}{2}L(\phi^2 - NR)\right] \\ &+ \tfrac{1}{2}(N-1)\,\mathrm{Str}\,\ln\left\{-2\Omega^{-1}[\bar{\mathrm{D}},\mathrm{D}] - \nabla_x^2 + L\right\}, \end{aligned}$$

where Str means trace in the sense of space, time and Grassmann coordinates.

At leading order at large N, the functional integral can be calculated by the steepest descent method. The two first saddle point equations, obtained by varying the superfields ϕ and R, are

$$\left(-\frac{2}{\Omega}[\bar{\mathrm{D}},\mathrm{D}] + L\right)\phi = 0\,, \tag{5.22a}$$

$$L - 2U'(R) = 0\,. \tag{5.22b}$$

The last saddle point equation, obtained by varying L, involves the ϕ superpropagator Δ:

$$R - \phi^2/N = \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}\omega\, \mathrm{d}^d k\, \Delta(\omega,\mathbf{k},\theta,\theta), \tag{5.22c}$$

where ω and $\mathbf{k}$ refer to the time and space Fourier components, respectively.

The super-propagator can be calculated, for example, by solving

$$\left\{-2\Omega^{-1}[\bar{\mathrm{D}},\mathrm{D}]+k^2+L\right\}\phi=J\,.$$

For Eq. (5.22c), the ϕ-propagator in presence of a L field is needed, but only for L of the form

$$L=m^2+\theta\bar{\theta}\lambda\,,$$

with m^2,λ constants. Then, setting

$$G=k^2+m^2+2i\omega/\Omega$$

one finds

$$\Delta(\omega,\mathbf{k},\theta',\theta)=\frac{4\Omega^{-1}+G\theta'\bar{\theta}'+G^*\theta\bar{\theta}-\lambda\theta'\bar{\theta}'\theta\bar{\theta}}{GG^*+4\lambda/\Omega}-\frac{\theta'\bar{\theta}}{G}-\frac{\theta\bar{\theta}'}{G^*}\,.$$

At coinciding points $\theta=\theta'$ it reduces to

$$\Delta(\omega,\mathbf{k},\theta,\theta)=\frac{4\Omega^{-1}+2(k^2+m^2)\theta\bar{\theta}}{GG^*+4\lambda/\Omega}-\frac{2(k^2+m^2)\theta\bar{\theta}}{GG^*}\,.$$

After integration over ω, one obtains

$$\frac{1}{2\pi}\int\mathrm{d}\omega\,\Delta(\omega,\mathbf{k},\theta,\theta)\equiv\bar{\Delta}(\mathbf{k},\theta)=\frac{1+\Omega(k^2+m^2)\theta\bar{\theta}/2}{\sqrt{(k^2+m^2)^2+4\lambda/\Omega}}-\tfrac{1}{2}\Omega\theta\bar{\theta}\,.$$

Eq. (5.22c) then becomes

$$R-\phi^2/N=\frac{1}{(2\pi)^d}\int\mathrm{d}^dk\,\bar{\Delta}(\mathbf{k},\theta).$$

All saddle point equations reduce to the static equations for $\lambda=0$, and then $F=s=0$, which implies that supersymmetry is preserved, and the ground state energy vanishes. Then, no further analysis is necessary. Of course, we know that supersymmetry is broken when the measure $\mathrm{e}^{-\mathcal{A}(\varphi)}[\mathrm{d}\varphi]$ is not normalizable. But this effect cannot be seen at leading order in perturbation theory nor in the large N limit.

Finally, note that the super-propagator formalism simplifies dynamic $1/N$ calculations.

The action density: saddle point equations in component form. Alternatively, one can start from the action density for constant scalar fields and vanishing Grassmann fields

$$\begin{aligned}\mathcal{E}=&-\frac{2}{\Omega}F^2+NsU'(\rho)+\frac{1}{2}\lambda(\varphi^2-N\rho)+\frac{1}{2}m^2(2F\varphi-Ns)\\&+\frac{N\Omega}{4}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\left(\sqrt{(k^2+m^2)^2+4\lambda/\Omega}-k^2-m^2\right).\end{aligned}\tag{5.23}$$

By differentiating $\mathcal{E}$ with respect to all parameters, one recovers the saddle point equations in component form:

$$F = \Omega m^2 \varphi/4\,, \quad m^2 F + \lambda\varphi = 0\,, \tag{5.24}$$

$$m^2 = 2U'(\rho)\,, \quad \lambda = 2sU''(\rho), \tag{5.25}$$

and

$$\rho - \varphi^2/N = \frac{1}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{(k^2+m^2)^2 + 4\lambda/\Omega}} \tag{5.26a}$$

$$s - 2F\varphi/N = \frac{\Omega}{2}\frac{1}{(2\pi)^d}\int \mathrm{d}^d k \left[\frac{(k^2+m^2)}{\sqrt{(k^2+m^2)^2+4\lambda/\Omega}} - 1\right]. \tag{5.26b}$$

Note that the same action density is obtained in a large N variational calculation, starting from

$$\mathcal{S}_0(\Phi, L) = \int \mathrm{d}t\mathrm{d}\bar\theta\mathrm{d}\theta\mathrm{d}^d x \left[\frac{2}{\Omega}\bar{\mathrm{D}}\Phi\mathrm{D}\Phi + \tfrac{1}{2}(\partial_\mu\Phi)^2 + \tfrac{1}{2}L(\Phi-\Phi_0)^2)\right],$$

but then the two equations (5.26) are constraints, and only Φ_0 and L are variational parameters.

Eliminating F between the two equations (5.24), one finds

$$\varphi(\lambda + \Omega m^4/4) = 0\,. \tag{5.27}$$

This equation has two solutions: $\varphi = 0$ which corresponds to the $O(N)$ symmetric phase, $\lambda + \Omega m^4/4 = 0$ which corresponds to a broken massless phase.

Eliminating F, s, ρ from the action density using the saddle point equations, one obtains the ground state energy density

$$\frac{1}{N}\mathcal{E} = -\frac{1}{2}\lambda\varphi^2 - \frac{\lambda}{2(2\pi)^d}\int \frac{\mathrm{d}^d k}{\sqrt{(k^2+m^2)^2+4\lambda/\Omega}} + \frac{\Omega}{4}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left(\sqrt{(k^2+m^2)^2+4\lambda/\Omega} - k^2 - m^2\right).$$

In the symmetric phase the derivative of $\mathcal{E}$ with respect to λ is

$$\frac{\partial\mathcal{E}}{\partial\lambda} = \frac{N}{\Omega}\frac{\lambda}{(2\pi)^d}\int \frac{\mathrm{d}^d k}{\left[(k^2+m^2)^2+4\lambda/\Omega\right]^{3/2}}.$$

The minimum is at $\lambda = 0$, that is at the supersymmetric point, where $\mathcal{E}$ vanishes.

In the broken symmetry phase λ is non-positive. A short calculation shows that again $\mathcal{E}$ is positive. The minimum is reached for $m^2 = 0$, and therefore $\lambda = 0$. The minimum again is supersymmetric and $\mathcal{E}$ then vanishes at the minimum, independently of the value of φ.

5.5 Quartic potential

We now specialize to the quartic potential

$$U(R) = \tfrac{1}{2} rR + \frac{u}{4!} R^2.$$

Then, the integral over R can be performed, leading to a contribution

$$L = r + \tfrac{1}{6} uR \;\Rightarrow\; \delta\mathcal{S}_N = -3N(L-r)^2/2u.$$

Note that for $u > 0$ the usual static results are recovered, a symmetric phase for $r > r_c$ and a broken symmetry phase otherwise. For $u < 0$ one finds the opposite situation, and there is no sign that the situation is pathological form the static point of view. The absence of an equilibrium state requires higher order calculations.

RG equations. The RG differential operator (2.28) acting on dynamic correlation functions takes at T_c ($r = r_c$) the form

$$\mathrm{D}_{\mathrm{RG}} = \Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} + \eta_\Omega(g) \Omega \frac{\partial}{\partial \Omega} - \frac{n}{2} \eta(g), \tag{5.28}$$

where $g = u\Lambda^{4-d}/N$ and $\eta_\Omega(g)$ is a new independent RG function related to the renormalization constant Z_Ω.

The solution of the RG equation for the two-point function $\widetilde{W}^{(2)}$ leads to the scaling form

$$\widetilde{W}^{(2)}(p, \omega, \theta = 0) \sim p^{-2+\eta-z} G^{(2)}(\omega/p^z). \tag{5.29}$$

The dynamic exponent z which also characterizes, near the critical point, the divergence of the correlation time in the scale of the correlation length, keeps at leading order its classical value $z = 2$.

At order $1/N$ the $\langle LL \rangle$ propagator is needed. In the symmetric phase $\varphi = 0$, it is given by

$$[\Delta_L]^{-1} = -\frac{3}{u} \delta^2(\bar{\theta} - \bar{\theta}') + \frac{1}{(2\pi)^{d+1}} \int \mathrm{d}^d k \, \mathrm{d}\omega'' \, \Delta(\omega'', \mathbf{k}, \theta, \theta') \Delta(\omega - \omega'', \mathbf{p} - \mathbf{k}, \theta, \theta').$$

In the infrared limit $\omega, \mathbf{k} \to 0$, it can be evaluated and used to calculate the ϕ two-point function at order $1/N$. The value of the dynamic exponent z at order $1/N$ follows. It can be written as [119]

$$z = 2 + c\eta\,, \quad c = \left(\frac{4}{4-d} \right) \left\{ \frac{dB(\frac{1}{2}d - 1, \frac{1}{2}d - 1)}{8 \int_0^{1/2} \mathrm{d}x \, [x(2-x)]^{d/2-2}} - 1 \right\},$$

where η has been given in section 3.1 and $B(\alpha, \beta)$ is the mathematical β-function.

The dissipative non-linear σ-model. Within the framework of the large N expansion, we have shown that the results of the static $(\phi^2)^2$ could be reproduced by the non-linear σ-model. This result generalizes to the dynamic theory [120]. In terms of the superfield Φ, the functional integral takes a form

$$\mathcal{Z} = \int [\mathrm{d}\Phi]\delta(\Phi^2 - N) \exp\left[-\frac{1}{T}\int \mathrm{d}t\mathrm{d}\bar{\theta}\mathrm{d}\theta\mathrm{d}^d x \left(\frac{2}{\Omega}\bar{\mathrm{D}}\Phi\mathrm{D}\Phi + \tfrac{1}{2}(\partial_\mu\Phi)^2\right)\right].$$

Therefore, the strategy is the same as in the static case. Since supersymmetry is not broken, the saddle point equations again reduce to the static equations.

6 Supersymmetric models in the large N limit

We have already discussed scalar field theories and self-interacting fermions in the large N limit. We want now to investigate how the results are affected by supersymmetry, and what new properties emerge in this case [121].

Unfortunately, not many supersymmetric models can be constructed which can be studied by large N techniques. We consider here two such models which involve an N-component scalar superfield, in three and two euclidean dimensions. First, we solve at large N a $(\Phi^2)^2$ supersymmetric field theory. We then examine the supersymmetric non-linear σ model, very much as we have done in the non supersymmetric examples.

Both models are the simplest generalization of supersymmetric quantum mechanics as it naturally appears, for example, in the study of stochastic evolution equations of Langevin type (see section 5).

Again the main issue will be the phase structures of these models, and the possibility of spontaneous symmetry breaking [122].

6.1 Supersymmetric scalar field in three dimensions

Apart from the interest in the general phase structure of the supersymmetric $(\Phi^2)^2$ model, it will be of interest here to study the spontaneous breaking of scale invariance that occurs in this model at large N . As we will be seen below, in a certain region of the parameter space, there is only spontaneous breaking of scale invariance and no explicit breaking. Though one may expect this as a result of non-renormalization of the coupling constant, in fact this happens also in the non-supersymmetric case (see appendix A1). When the coupling constant that binds bosonic and fermionic $O(N)$ quanta is tuned to a value at which $O(N)$ singlet massless bound states are created. The resulting massless Goldstone particles appear as a supersymmetric multiplet of a dilaton and dilatino.

Conventions and notation: Supersymmetry and Majorana spinors in $d = 3$. Since the properties of Majorana spinors in three euclidean space dimensions may not be universally known, we briefly recall some of them and explain our notation. In three dimensions the spin group is $SU(2)$. Then, a spinor transforms like

$$\psi_U = U\psi\,, \quad U \in SU(2).$$

The role of Dirac γ matrices is played by the Pauli σ matrices, $\gamma_\mu \equiv \sigma_\mu$. Moreover, σ_2 is antisymmetric while $\sigma_2\sigma_\mu$ is symmetric. This implies

$$\sigma_2\sigma_\mu\sigma_2 = -{}^T\sigma_\mu \;\Rightarrow\; U^* = \sigma_2 U \sigma_2\,.$$

A Majorana spinor corresponds to a neutral fermion and has only two independent components ψ_1, ψ_2. The conjugated spinor is defined by (^{T} means transposed)

$$\bar\psi = {}^T\psi\sigma_2 \;\Leftrightarrow\; \bar\psi_\alpha = i\epsilon_{\alpha\beta}\psi_\beta\,, \tag{6.1}$$

($\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$, $\epsilon_{12} = 1$) and thus $\bar{\psi}$ transforms like

$$[{}^T\psi\sigma_2]_U = [{}^T\psi\sigma_2]U^\dagger\,.$$

In the same way, we define a spinor of Grassmann coordinates

$$\bar{\theta} = {}^T\theta\sigma_2\,.$$

Since the only non-vanishing product is $\theta_1\theta_2$, we have

$$\bar{\theta}_\alpha\theta_\beta = \tfrac{1}{2}\delta_{\alpha\beta}\bar{\theta}\cdot\theta\,.$$

The scalar product of $\bar{\theta}$ and θ is

$$\bar{\theta}\cdot\theta = -2i\theta_1\theta_2 \;\Rightarrow \theta_\alpha\bar{\theta}_\beta = i\delta_{\alpha\beta}\theta_1\theta_2\,.$$

If $\bar{\theta}', \theta'$ is another pair of coordinates, because $\sigma_2\sigma_\mu$ is symmetric, one finds

$$\bar{\theta}\sigma_\mu\theta' = {}^T\theta\sigma_2\sigma_\mu\theta' = -\bar{\theta}'\sigma_\mu\theta\,, \tag{6.2}$$

and for the same reason

$$\bar{\theta}\psi = \bar{\psi}\theta\,.$$

Other useful identities are

$$(\bar{\theta}\psi)^2 = -\tfrac{1}{2}(\bar{\theta}\theta)(\bar{\psi}\psi), \quad (\bar{\theta}\not{p}\psi)^2 = \tfrac{1}{2}p^2(\bar{\psi}\psi)(\bar{\theta}\theta).$$

It is convenient to integrate over θ_1, θ_2 with the measure

$$\mathrm{d}^2\theta \equiv \frac{i}{2}\mathrm{d}\theta_2\mathrm{d}\theta_1\,.$$

Then,

$$\int \mathrm{d}^2\theta\,\bar{\theta}_\alpha\theta_\beta = \tfrac{1}{2}\delta_{\alpha\beta}\,, \quad \int \mathrm{d}^2\theta\,\bar{\theta}\cdot\theta = 1\,.$$

With this convention the identity kernel $\delta^2(\theta' - \theta)$ in θ space is

$$\delta^2(\theta' - \theta) = (\bar{\theta}' - \bar{\theta})\cdot(\theta' - \theta). \tag{6.3}$$

Superfields and covariant derivatives. A superfield $\Phi(\theta)$ can be expanded in θ:

$$\Phi(\theta) = \varphi + \bar{\theta}\psi + \tfrac{1}{2}\bar{\theta}\theta F\,. \tag{6.4}$$

Again, although only two θ variables are independent, we define the covariant derivatives D_α and $\bar{\mathrm{D}}_\alpha$ ($\bar{\mathrm{D}} = \sigma_2 \mathrm{D}$)

$$\mathrm{D}_\alpha \equiv \frac{\partial}{\partial \bar{\theta}_\alpha} - (\partial\!\!\!/ \theta)_\alpha \,, \quad \bar{\mathrm{D}}_\alpha \equiv \frac{\partial}{\partial \theta_\alpha} - (\bar{\theta} \partial\!\!\!/)_\alpha \,.$$

Then the anticommutation relation is

$$\{\mathrm{D}_\alpha, \bar{\mathrm{D}}_\beta\} = -2[\partial\!\!\!/]_{\alpha\beta} \,.$$

Also

$$\bar{\mathrm{D}}_\alpha \mathrm{D}_\alpha = \frac{\partial}{\partial \theta_\alpha} \frac{\partial}{\partial \bar{\theta}_\alpha} - (\bar{\theta} \partial\!\!\!/)_\alpha \frac{\partial}{\partial \bar{\theta}_\alpha} - \frac{\partial}{\partial \theta_\alpha} (\partial\!\!\!/ \theta)_\alpha + \bar{\theta}_\alpha \theta_\alpha \partial^2 \,.$$

Since the σ_μ are traceless, using the identity (6.2) one verifies that $\bar{\mathrm{D}}\mathrm{D}$ can also be written as

$$\bar{\mathrm{D}}_\alpha \mathrm{D}_\alpha = \frac{\partial}{\partial \theta_\alpha} \frac{\partial}{\partial \bar{\theta}_\alpha} - 2(\bar{\theta} \partial\!\!\!/)_\alpha \frac{\partial}{\partial \bar{\theta}_\alpha} + \bar{\theta}_\alpha \theta_\alpha \partial^2 \,,$$

and, therefore, in component form

$$\bar{\mathrm{D}}_\alpha \mathrm{D}_\alpha \Phi = 2F - 2\bar{\theta} \partial\!\!\!/ \psi + \bar{\theta}\theta \partial^2 \varphi \,.$$

Moreover,

$$\left(\bar{\mathrm{D}}_\alpha \mathrm{D}_\alpha\right)^2 = 4\partial^2.$$

Supersymmetry generators and WT identities. Supersymmetry is generated by the operators

$$Q_\alpha = \frac{\partial}{\partial \bar{\theta}_\alpha} + (\partial\!\!\!/ \theta)_\alpha \,, \quad \bar{Q}_\alpha = \frac{\partial}{\partial \theta_\alpha} + (\bar{\theta} \partial\!\!\!/)_\alpha \,,$$

which anticommute with D_α (and thus $\bar{\mathrm{D}}_\alpha$). Then,

$$\{\bar{Q}_\alpha, Q_\beta\} = 2[\partial\!\!\!/]_{\alpha\beta} \,.$$

Supersymmetry implies WT identities for correlation functions. The n-point function $\widetilde{W}^{(n)}(p_k, \theta_k)$ of Fourier components satisfies

$$Q_\alpha \widetilde{W}^{(n)} \equiv \left[\sum_k \frac{\partial}{\bar{\partial} \theta^k_\alpha} - i(p\!\!\!/_k \theta^k)_\alpha \right] \widetilde{W}^{(n)}(p, \theta) = 0 \,.$$

To solve this equation, we set

$$\widetilde{W}^{(n)}(p, \theta) = F^{(n)}(p, \theta) \exp\left[-\frac{i}{2n} \sum_{jk} \bar{\theta}_j (p\!\!\!/_j - p\!\!\!/_k) \theta_k \right] , \tag{6.5}$$

where $F^{(n)}$ is a symmetric function in the exchange $\{p_i, \theta_i\} \leftrightarrow \{p_j, \theta_j\}$. It then satisfies

$$\sum_k \frac{\partial}{\partial \theta^k_\alpha} F^{(n)}(p, \theta) = 0\,,$$

that is, is translation invariant in θ space.

In the case of the two-point function, this leads to the general form

$$\begin{aligned} \widetilde{W}^{(2)}(p, \theta', \theta) &= A(p^2)\left[1 + C(p^2)\delta^2(\theta' - \theta)\right] \mathrm{e}^{i\bar{\theta}\not{p}\theta'}\,. \\ &= A(p^2)\left[1 + C(p^2)(\bar{\theta}' - \bar{\theta})(\theta' - \theta) + i\bar{\theta}\not{p}\theta' - \tfrac{1}{4}p^2\bar{\theta}\theta\bar{\theta}'\theta'\right]. \end{aligned} \tag{6.6}$$

Since the vertex functions $\Gamma^{(n)}$ satisfy the same WT identities, they take the same general form.

General $O(N)$ symmetric action. We now consider the $O(N)$ invariant action

$$\mathcal{S}(\Phi) = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \left[\tfrac{1}{2}\bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + NU(\Phi^2/N)\right], \tag{6.7}$$

where Φ is a N-component vector.

In component notation

$$\int \mathrm{d}^2\theta\, \bar{\mathrm{D}}\Phi\mathrm{D}\Phi = -\int \mathrm{d}^2\theta\, \Phi\bar{\mathrm{D}}_\alpha\mathrm{D}_\alpha\Phi = -\bar{\psi}\not{\partial}\psi + (\partial_\mu\varphi)^2 - F^2. \tag{6.8}$$

Then, since

$$\Phi^2 = \varphi^2 + 2\varphi\bar{\theta}\psi - \tfrac{1}{2}(\bar{\theta}\theta)(\bar{\psi}\psi) + F\varphi\bar{\theta}\theta\,, \tag{6.9}$$

quite generally

$$\int \mathrm{d}^2\theta\, \mathcal{U}(\Phi^2) = \mathcal{U}'(\varphi^2)\left(-\tfrac{1}{2}\bar{\psi}\psi + F\varphi\right) - \mathcal{U}''(\varphi^2)(\bar{\psi}\varphi)(\varphi\psi).$$

In the case of the free theory $U(R) \equiv \mu R$, the action in component form is

$$\mathcal{S} = \int \mathrm{d}^3x\, \left[-\tfrac{1}{2}\bar{\psi}\not{\partial}\psi + \tfrac{1}{2}(\partial_\mu\varphi)^2 - \tfrac{1}{2}F^2 + \mu\left(-\tfrac{1}{2}\bar{\psi}\psi + F\varphi\right)\right].$$

After integration over the auxiliary F field, the action becomes

$$\mathcal{S} = \int \mathrm{d}^3x\, \left[-\tfrac{1}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}\mu\bar{\psi}\psi + \tfrac{1}{2}(\partial_\mu\varphi)^2 + \tfrac{1}{2}\mu^2\varphi^2\right].$$

For later purpose it is also convenient to notice that in momentum representation

$$[-\bar{\mathrm{D}}\mathrm{D} + 2\mu]\delta^2(\theta' - \theta) = 2\left[-2 + \delta^2(\theta' - \theta)\mu\right] \mathrm{e}^{-i\bar{\theta}\not{p}\theta'}\,. \tag{6.10}$$

The free propagator can be written as

$$[-\bar{\mathrm{D}}\mathrm{D}+2\mu]^{-1}=\frac{1}{4(k^2+\mu^2)}\left(\bar{\mathrm{D}}\mathrm{D}+2\mu\right),$$

or more explicitly

$$[-\bar{\mathrm{D}}\mathrm{D}+2\mu]^{-1}\delta^2(\theta'-\theta)=\frac{1}{k^2+\mu^2}\left[1+\tfrac{1}{2}\mu\delta^2(\theta'-\theta)\right]\mathrm{e}^{-i\bar{\theta}\not{k}\theta'}\,. \tag{6.11}$$

For a generic super-potential $U(R)$, we find

$$\begin{aligned}\mathcal{S}=\int\mathrm{d}^3x\,&\left[-\tfrac{1}{2}\bar{\psi}\not{\partial}\psi+\tfrac{1}{2}(\partial_\mu\varphi)^2-\tfrac{1}{2}U'(\varphi^2/N)\bar{\psi}\psi-U''(\varphi^2/N)(\bar{\psi}\varphi)(\varphi\psi)/N\right.\\&\left.+\tfrac{1}{2}\varphi^2U'^2(\varphi^2/N)\right].\end{aligned} \tag{6.12}$$

Note that the theory violates parity symmetry. Actually, a space reflection is equivalent to the change $U\mapsto -U$ (see definition (4.10)). Therefore, theories with $\pm U$ have the same physical properties.

Finally, in the calculations that follow we assume, when necessary, a supersymmetric Pauli–Villars regularization.

6.2 Large N limit: superfield formulation

To study the large N limit, we introduce a constraint on $\Phi^2/N=R$, where R now is a superfield, by integrating over another superfield L:

$$\mathcal{Z}=\int[\mathrm{d}\Phi][\mathrm{d}R][\mathrm{d}L]\,\mathrm{e}^{-\mathcal{S}(\Phi,R,L)},$$

where

$$\mathcal{S}(\Phi,R,L)=\int\mathrm{d}^3x\,\mathrm{d}^2\theta\left\{\tfrac{1}{2}\bar{\mathrm{D}}\Phi\cdot\mathrm{D}\Phi+NU(R)+L(\theta)\left[\Phi^2(\theta)-NR(\theta)\right]\right\}. \tag{6.13}$$

We parameterize the scalar superfields L and R as

$$L(\theta,x)=M+\bar{\theta}\ell+\tfrac{1}{2}\bar{\theta}\theta\lambda\,, \tag{6.14}$$

$$R(\theta,x)=\rho+\bar{\theta}\sigma+\tfrac{1}{2}\bar{\theta}\theta s\,. \tag{6.15}$$

The Φ integral is now gaussian. As usual we integrate over only $N-1$ components and keep a test-component $\Phi_1\equiv\phi$. We find

$$\mathcal{Z}=\int[\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\,\mathrm{e}^{-\mathcal{S}_N(\phi,R,L)} \tag{6.16}$$

with the large N action

$$\mathcal{S}_N = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \left[\tfrac{1}{2}\bar{\mathrm{D}}\phi \mathrm{D}\phi + NU(R) + L\left(\phi^2 - NR\right)\right] + \tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left[-\bar{\mathrm{D}}\mathrm{D} + 2L\right]. \tag{6.17}$$

The two first saddle point equations, obtained by varying ϕ and R, are

$$2L\phi - \bar{\mathrm{D}}\mathrm{D}\phi = 0\,, \tag{6.18a}$$

$$L - U'(R) = 0\,. \tag{6.18b}$$

The last saddle point equation, obtained by varying L, involves the super-propagator Δ of the ϕ-field. For $N \gg 1$, it reads

$$R - \phi^2/N = \langle x, \theta| \left[-\bar{\mathrm{D}}\mathrm{D} + 2L\right]^{-1} |x, \theta\rangle = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\, \Delta(k, \theta, \theta). \tag{6.18c}$$

The super-propagator Δ is solution of the equation

$$\left(-\bar{\mathrm{D}}\mathrm{D} + 2L(\theta)\right)\Delta(k, \theta, \theta') = \delta^2(\theta' - \theta).$$

It is here needed only for $\ell = 0$ and M, λ constants. It can be obtained by solving

$$\left(-\bar{\mathrm{D}}\mathrm{D} + 2L(\theta)\right)\Phi(\theta) = J(\theta).$$

In Fourier representation and in terms of its components, the equation reads

$$2M\varphi - 2F + 2\bar{\theta}(-i\not{k} + M)\psi + \bar{\theta}\theta[(k^2 + \lambda)\varphi + MF] = J(\theta).$$

¿From its solution we infer the form of the propagator

$$\Delta(k, \theta, \theta') = \frac{\left[1 + \frac{1}{2}M(\bar{\theta}\theta + \bar{\theta}'\theta') - \frac{1}{4}(\lambda + k^2)\bar{\theta}\theta\bar{\theta}'\theta'\right]}{k^2 + M^2 + \lambda} - \frac{\bar{\theta}[i\not{k} + M]\theta'}{k^2 + M^2}\,. \tag{6.19}$$

Clearly, one reads in Eq. (6.19) the $\langle\varphi(k)\varphi(-k)\rangle$ propagator $(k^2+M^2+\lambda)^{-1}$ and the $\langle\bar{\psi}(k)\psi(-k)\rangle$ propagator $(i\not{k} + M)/(k^2 + M^2)$. The coefficients of $(\bar{\theta}\theta + \bar{\theta}'\theta')$ and of $(\bar{\theta}\theta\bar{\theta}'\theta')$ are the $\langle\varphi(k)F(-k)\rangle$ and $\langle F(k)F(-k)\rangle$ propagators, respectively.

At coinciding θ arguments, we obtain

$$\Delta(k, \theta, \theta) = \frac{1 + M\bar{\theta}\theta}{k^2 + M^2 + \lambda} - \frac{M\bar{\theta}\theta}{k^2 + M^2}\,. \tag{6.20}$$

Eq. (6.18c) thus is

$$R - \phi^2/N = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k \left[\frac{1 + M\bar{\theta}\theta}{k^2 + M^2 + \lambda} - \frac{M\bar{\theta}\theta}{k^2 + M^2}\right]. \tag{6.21}$$

Saddle point equations in component form. It is now convenient to introduce a notation for the boson mass

$$m_\varphi \equiv m = \sqrt{M^2 + \lambda}\,. \tag{6.22}$$

Eq. (6.18a) implies

$$F - M\varphi = 0\,, \tag{6.23a}$$
$$\lambda\varphi + MF = 0\,. \tag{6.23b}$$

Eliminating F between the two equations, we find

$$\varphi m^2 = 0 \tag{6.24}$$

and, thus, if the $O(N)$ symmetry is broken the boson mass m_φ vanishes.

Then, Eq. (6.18b) yields

$$M = U'(\rho), \tag{6.25a}$$
$$\lambda = m^2 - M^2 = sU''(\rho). \tag{6.25b}$$

Using Eq. (6.9), we write Eq. (6.21) in component form (a cut-off is implied) as

$$\rho - \varphi^2/N = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{p^2+m^2}\,, \tag{6.26a}$$
$$s - 2F\varphi/N = \frac{2M}{(2\pi)^3}\int \mathrm{d}^3p\left(\frac{1}{p^2+m^2} - \frac{1}{p^2+M^2}\right). \tag{6.26b}$$

Introducing the cut-off dependent constant

$$\rho_c = \frac{1}{(2\pi)^3}\int^\Lambda \frac{\mathrm{d}^3p}{p^2} = \Omega_3(0)\,, \tag{6.27}$$

(see Eqs. (2.12–2.15)) we rewrite these equations as

$$\rho - \varphi^2/N = \rho_c - \frac{1}{4\pi}m\,, \tag{6.28a}$$
$$s - 2F\varphi/N = \frac{1}{2\pi}M\left(|M| - m\right). \tag{6.28b}$$

Note that the change $U \mapsto -U$ here corresponds to

$$F \mapsto -F\,, \quad s \mapsto -s\,, \quad M \mapsto -M\,.$$

Action density. Finally, we calculate the action density $\mathcal{E}$ corresponding to the action $\mathcal{S}_N$ (Eq. (6.17)), $\mathcal{E} = \mathcal{S}_N/\text{volume}$, for vanishing fermion fields. We use (see also Eq. (6.8))

$$\frac{1}{2}\int \mathrm{d}^2\theta\, \bar{\mathrm{D}}\phi \mathrm{D}\phi = -\frac{1}{2}F^2\,, \quad \int \mathrm{d}^2\theta\, L\phi^2 = MF\varphi + \frac{1}{2}\lambda\varphi^2 \tag{6.29}$$

and

$$\mathrm{Str}\,\ln\left(-\bar{\mathrm{D}}\mathrm{D} + 2L\right) = \mathrm{tr}\ln(-\partial^2 + M^2 + \lambda) - \mathrm{tr}\ln(\not{\partial} + M). \tag{6.30}$$

Then,

$$\begin{aligned}\mathcal{E}/N = &-\tfrac{1}{2}F^2/N + \tfrac{1}{2}sU'(\rho) + MF\varphi/N + \tfrac{1}{2}\lambda\varphi^2/N - \tfrac{1}{2}Ms - \tfrac{1}{2}\lambda\rho \\ &+ \tfrac{1}{2}\,\mathrm{tr}\ln(-\partial^2 + M^2 + \lambda) - \tfrac{1}{2}\,\mathrm{tr}\ln(\not{\partial} + M).\end{aligned} \tag{6.31}$$

In $d = 3$, the $\mathrm{tr}\ln$ in Eq. (6.31) is given by

$$\tfrac{1}{2}\,\mathrm{tr}\ln(-\partial^2 + M^2 + \lambda) - \tfrac{1}{2}\,\mathrm{tr}\ln(\not{\partial} + M) = \tfrac{1}{2}\rho_c\lambda - \frac{1}{12\pi}\left(m^3 - |M|^3\right). \tag{6.32}$$

The saddle point equations in component form are then recovered from derivatives of $\mathcal{E}$ with respect to the various parameters. Using the saddle point equations (6.18a), (6.28) and (6.25a) to eliminate F, s, ρ, one eliminates also the explicit dependence on the super-potential U and the expression is simplified into

$$\mathcal{E}/N = \tfrac{1}{2}M^2\varphi^2/N + \frac{1}{24\pi}(m - |M|)^2(m + 2|M|). \tag{6.33}$$

In this form we see that $\mathcal{E}$ is positive for all saddle points, and, as a function of m, has an absolute minimum at $m = |M|$, and thus $\lambda = 0$, that is for a supersymmetric ground state.

Moreover, Eq. (6.24) implies $M\varphi = 0$, and thus $\mathcal{E} = 0$. Therefore, if a supersymmetric solution exists, it will have the lowest possible ground state energy and any non-supersymmetric solution will have a higher energy.

Since M, m, and φ are related by the saddle point equations, it remains to verify whether such a solution indeed exists.

In the supersymmetric situation the saddle point equations reduce to $s = 0$, $M\varphi = 0$ and

$$\rho - \rho_c = \varphi^2/N - |M|/4\pi\,, \quad M = U'(\rho).$$

In the $O(N)$ symmetric phase $\varphi = 0$ and $|M| = 4\pi(\rho_c - \rho) = |U'(\rho)|$. In the broken phase $U'(\rho) = 0$ and $\varphi^2/N = \rho - \rho_c$. We will show in the next section that these conditions can be realized by a quadratic function $U(\rho)$ and then in both phases the ground state is supersymmetric and $\mathcal{E}$ vanishes.

6.3 Variational calculations

For completeness, we present here the corresponding variational calculations [125] and apply the arguments of section 2.8 to the action (6.12). In terms of the two parameters

$$\rho = (1/N)\left\langle \varphi^2(x)\right\rangle_0 = \frac{\varphi^2}{N} + \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{p^2+m_\varphi^2}, \tag{6.34a}$$

$$\tilde{\rho} = (1/N)\left\langle \bar{\psi}(x)\psi(x)\right\rangle_0 = \frac{2m_\psi}{(2\pi)^3}\int \frac{\mathrm{d}^3p}{p^2+m_\psi^2}\,, \tag{6.34b}$$

the variational energy density can be written as

$$\begin{aligned}\mathcal{E}_{\mathrm{var}}/N = {}& \tfrac{1}{2}\tilde{\rho}\big(m_\psi - U'(\rho)\big) + \tfrac{1}{2}\rho U'^2(\rho) - \tfrac{1}{2}m_\varphi^2(\rho - \varphi^2/N)\\ &+ \tfrac{1}{2}\operatorname{tr}\ln(-\partial^2 + m_\varphi^2) - \tfrac{1}{2}\operatorname{tr}\ln(\not{\partial} + m_\psi).\end{aligned} \tag{6.35}$$

In the following we choose $m_\psi \geq 0$ and $m_\varphi \geq 0$.

Since the variational energy is larger than or equal to the ground state energy, which in a supersymmetric theory is non-negative, it is sufficient to find values of the three parameters $m_\psi, m_\varphi, \varphi$ such that $\mathcal{E}_{\mathrm{var}}$ vanishes to prove that the ground state is supersymmetric. Choosing $m_\psi = m_\varphi = M$ and using Eqs. (6.34), we find

$$\mathcal{E}_{\mathrm{var}}/N = U'^2(\rho)\frac{\varphi^2}{2N} + \frac{1}{2(2\pi)^3}\int \frac{\mathrm{d}^3p}{p^2+M^2}\left[U'(\rho) - M\right]^2.$$

This expression vanishes whenever $M = U'(\rho)$ together with $M\varphi = 0$. Together with Eq. (6.34a), we recover the three supersymmetric saddle point equations we discussed at the end of section 6.2.

A surprising feature of the variational energy density is the appearance of a divergent contribution

$$\mathcal{E}_{\mathrm{var}}/N = \tfrac{1}{2}\rho_c\big(M - U'(\rho)\big)^2 + \text{ finite}.$$

Therefore, when the equation $M = U'(\rho)$ is not enforced, the variational energy is infinite with the cut-off.

Of course, we can also look for the minimum of $\mathcal{E}_{\mathrm{var}}$ by differentiating with respect to the three parameters. Differentiating $\mathcal{E}_{\mathrm{var}}$ with respect to m_ψ and using the definition of $\tilde{\rho}$, we obtain the fermion mass gap equation

$$m_\psi = U'(\rho). \tag{6.36}$$

If this equation is taken into account the variational energy density simplifies to

$$\frac{1}{N}\mathcal{E}_{\mathrm{var}}(m_\varphi, m_\psi, \varphi) = \frac{m_\psi^2}{2}\frac{\varphi^2}{N} + \frac{1}{24\pi}(m_\varphi - m_\psi)^2(m_\varphi + 2m_\psi). \tag{6.37}$$

which is positive definite (recall that $m_\psi \geq 0$ and $m_\varphi \geq 0$) and vanishes when $m_\psi = m_\varphi$ and $m_\psi \varphi = 0$, that is for a supersymmetric ground state with the $O(N)$ symmetry either broken or unbroken.

When the positive fermion and the boson masses m_ψ, m_φ are identified with their value in terms of the parameters M, m,

$$m_\psi = |M| \quad \text{and} \quad m_\varphi = \sqrt{M^2 + \lambda}, \tag{6.38}$$

one recognizes the expression (6.33). As we have shown this expression has a unique minimum $m_\varphi = m_\psi$, which is supersymmetric. Differentiating then with respect to φ, we find $\varphi m_\psi = 0$, one solution $m_\psi = 0$ corresponding to $O(N)$ symmetry breaking, the other $\varphi = 0$ to an $O(N)$ symmetric phase.

Differentiating Eq. (6.35) with respect to m_φ and using the definition of ρ, we obtain the boson mass gap equation

$$m_\varphi^2 = 2\rho U'(\rho) U''(\rho) + U'^2(\rho) - \tilde{\rho} U''(\rho) \tag{6.39}$$

or, using the value of m_ψ,

$$m_\varphi^2 - m_\psi^2 = U''(\rho)\left(2 m_\psi \rho - \tilde{\rho}\right).$$

Clearly, the supersymmetric solution satisfies both gap equations. In the combination $2m_\psi\rho - \tilde{\rho}$, we recognize the parameter s as in Eq. (6.26b) (F being taken from Eq. (6.23a)), and thus the equation becomes Eq. (6.25b): $\lambda = sU''(\rho)$.

Eqs. (6.36,6.39) have a clear Schwinger–Dyson diagrammatic interpretation for a $U(\rho) = \mu\rho + \frac{1}{2}u\rho^2$ potential (see Eq. (6.12)). Namely,

$$m_\psi = \mu + \frac{u}{N}\left\langle \varphi^2(x)\right\rangle_0 , \tag{6.40a}$$

$$m_\varphi^2 = \mu^2 + 4\frac{\mu u}{N}\left\langle \varphi^2(x)\right\rangle_0 + 3\frac{u^2}{N^2}\left\langle \varphi^2(x)\right\rangle_0^2 - \frac{u}{N}\left\langle \bar{\psi}(x)\psi(x)\right\rangle_0 . \tag{6.40b}$$

6.4 The Φ^4 super-potential in $d = 3$: phase structure

We now consider the special example

$$U(R) = \mu R + \tfrac{1}{2}uR^2 \;\Rightarrow\; U'(R) = \mu + uR\,.$$

The dimensions of the θ variables and the field Φ are

$$[\theta] = -\tfrac{1}{2}\,, \quad [\Phi] = \tfrac{1}{2} \;\Rightarrow\; [u] = 0\,.$$

Power counting thus tells us that the model is renormalizable in three dimensions. Prior to a more refined analysis, one expects coupling constant and field

renormalizations (with logarithmic divergences) and a mass renormalization with linear divergences. Using the solution (6.6) for the two-point function $\Gamma^{(2)}$, one infers that the coefficient $A(p^2)$ has at most a logarithmic divergence, which corresponds to the field renormalization, while the coefficient $C(p^2)$ can have a linear divergence which corresponds to the mass renormalization.

The invariance of physics under the change $U \mapsto -U$ was mentioned above and seen in Eq. (6.12) and the equations that followed. This invariance will be reflected into the phase structure of the model. For the quartic potential the equations (6.25) now are

$$M = \mu + u\rho\,, \quad \lambda = us\,. \tag{6.41}$$

We introduce the critical value of μ,

$$\mu_c = -u\rho_c\,. \tag{6.42}$$

Taking into account equations (6.41), one finds that the equations (6.28) can now be written as

$$M = \mu - \mu_c + u\varphi^2/N - \frac{u}{4\pi}\sqrt{M^2+\lambda}\,, \tag{6.43a}$$

$$\lambda = 2uM\varphi^2/N + \frac{u}{2\pi}M\left(|M| - \sqrt{M^2+\lambda}\right)\,. \tag{6.43b}$$

Eqs. (6.43) relate the fermion mass $m_\psi = |M|$, the boson mass $m_\varphi = \sqrt{M^2+\lambda}$ and the classical field φ. The phase structure of the model is then described by the lowest energy solutions of these equations in the $\{\mu - \mu_c, u\}$ plane. Taking into account the $U \to -U$ symmetry, mentioned above, one can restrict the discussion to $u > 0$.

We find, indeed, that supersymmetry is left unbroken ($\lambda = 0$) and the ground state energy $\mathcal{E} = 0$ in each quadrant in the $\{\mu - \mu_c, u\}$ plane. This is consistent with Eqs. (6.43) having a common solution with $\lambda = 0$ (thus $m_\psi = m_\varphi = |M|$) and $M\varphi = 0$. They then reduce to

$$M = \mu - \mu_c + u\frac{\varphi^2}{N} - \frac{u}{4\pi}|M|\,, \quad M\varphi = 0\,. \tag{6.44}$$

The broken $O(N)$ symmetry phase. The $M = 0$ solution implies a spontaneously broken $O(N)$ symmetry, scalar and fermion $O(N)$ quanta are massless and

$$\varphi^2 = -N(\mu - \mu_c)/u\,, \tag{6.45}$$

which implies that this solution exists only for $\mu < \mu_c$. The solution exists in the fourth (and second) quadrant of the $\{\mu - \mu_c, u\}$ plane. Note that this yields the same exponent $\beta = \frac{1}{2}$ as in the simple $(\varphi^2)^2$ field theory (Eq. (2.34)).

The $O(N)$ symmetric phase. We now choose the solution $\varphi = 0$ of Eqs. (6.44). Then the equation

$$M = \mu - \mu_c - (u/u_c)|M| \tag{6.46}$$

yields the common mass M for the fermions and bosons. In Eq. (6.46) we have introduced the special value of the coupling u,

$$u_c = 4\pi \ . \tag{6.47}$$

This equation splits into two equations, depending on the sign of M. The first solution

$$M = M_+ = (\mu - \mu_c)/(1 + u/u_c) > 0 \tag{6.48}$$

exists only for $\mu > \mu_c$ (first quadrant), as one would normally expect. Note again that this corresponds to a correlation length exponent $\nu = 1$, as in the ordinary $(\varphi^2)^2$ field theory (Eq. (2.37)), though the form of the saddle point equations are different: in the supersymmetric theory this is the free field value. Moreover, the exponent is independent of u, though the term proportional to u is not negligible. If one takes into account the leading correction coming from regularization (expansion (2.13)), one finds

$$(1 + u/u_c)M = \mu - \mu_c + ua(3)M^2/\Lambda \, .$$

Unlike what happens in the usual φ^4 field theory, no value of u can cancel the leading correction to the relation (6.48), and therefore no IR fixed point can be identified.

The second solution

$$M = M_- = (\mu - \mu_c)/(1 - u/u_c) < 0$$

is very peculiar. There are two different situations depending on the position of u with respect to u_c:

(i) $u > u_c = 4\pi$ and then $\mu > \mu_c$: the solution is degenerate with another $O(N)$ symmetric solution M_+.

(ii) $u < u_c$ and then $\mu < \mu_c$: the solution is degenerate with a solution of broken $O(N)$ symmetry.

The phase structure is summarized in Fig. 6 in the first and second quadrant of the $\{\mu - \mu_c, u/u_c\}$ plane where the following different phases appear:

Region I : $\mu - \mu_c \geq 0 \quad , \quad u/u_c \leq 1$:
Here, there is only one $O(N)$ symmetric, supersymmetric ground state with $m_\psi = m_\varphi = M_+ = (\mu - \mu_c)/(u/u_c + 1)$ and $\varphi^2 = 0$.

Region II : $\mu - \mu_c \geq 0 \quad , \quad u/u_c \geq 1$:
There are two degenerate $O(N)$ symmetric ($\varphi = 0$) supersymmetric ground states with masses $m_\psi = m_\varphi = M_+ = (\mu - \mu_c)/(u/u_c + 1)$ and $m_\psi = m_\varphi = -M_- = (\mu - \mu_c)/(u/u_c - 1)$.

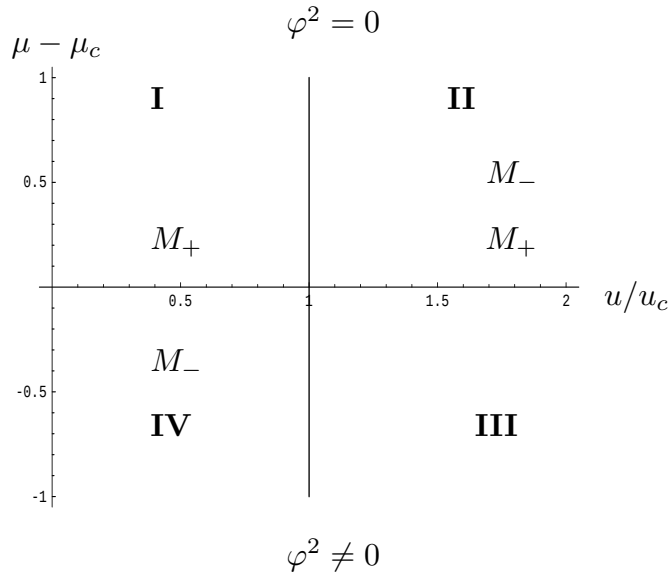

Fig. 6 Summary of the phases of the model in the $\{\mu - \mu_c, u\}$ plane. Here $m_\varphi = m_\psi = |M_\pm| = (\mu - \mu_c)/(u/u_c \pm 1)$. The lines $u = u_c$ and $\mu - \mu_c = 0$ are lines of first and second order phase transitions.

Region III : $\mu - \mu_c \leq 0$, $u/u_c \geq 1$:
There is one supersymmetric ground state, it is an ordered state with broken $O(N)$ symmetry ($\varphi^2 \neq 0$, $m_\psi = m_\varphi = 0$).

Region IV : $\mu - \mu_c \leq 0$, $u/u_c \leq 1$:
There are two degenerate ground states: an $O(N)$ symmetric , supersymmetric ground states with masses $m_\psi = m_\varphi = m_- = (\mu - \mu_c)/(u/u_c - 1)$ and $\varphi^2 = 0$. The second ground state is a supersymmetric, broken $O(N)$ symmetry state with $m_\psi = m_\varphi = 0$ and $\varphi^2 \neq 0$.

The action density. To exhibit the phase structure in terms of the variation of the action density $\mathcal{E}$, we plot the expression (6.33), but use only the fermion gap equation in Eq. (6.43a), in such a way that $\mathcal{E}$ remains a function of φ and λ, or equivalently φ and $m = \sqrt{M^2 + \lambda}$:

$$\frac{1}{N}\mathcal{E}(m,\varphi) = \tfrac{1}{2}M^2(m,\varphi)\frac{\varphi^2}{N} + \frac{1}{24\pi}\left[m - |M(m,\varphi)|\right]^2 \times \left(m_\varphi + 2\,|M(m,\varphi)|\right). \tag{6.49}$$

Two figures display the restriction of $\mathcal{E}$ to $\varphi = 0$:

$$\frac{1}{N}\mathcal{E}(m,\varphi = 0) = \frac{1}{24\pi}\left[m - |\mu - \mu_c - (u/u_c)m|\right]^2 \left(m + 2\,|\mu - \mu_c - (u/u_c)m|\right). \tag{6.50}$$

Several peculiar phase transitions can be easily traced now in Eq. (6.49). First, one notes the phase transitions that occur when $\mu - \mu_c$ changes sign. When

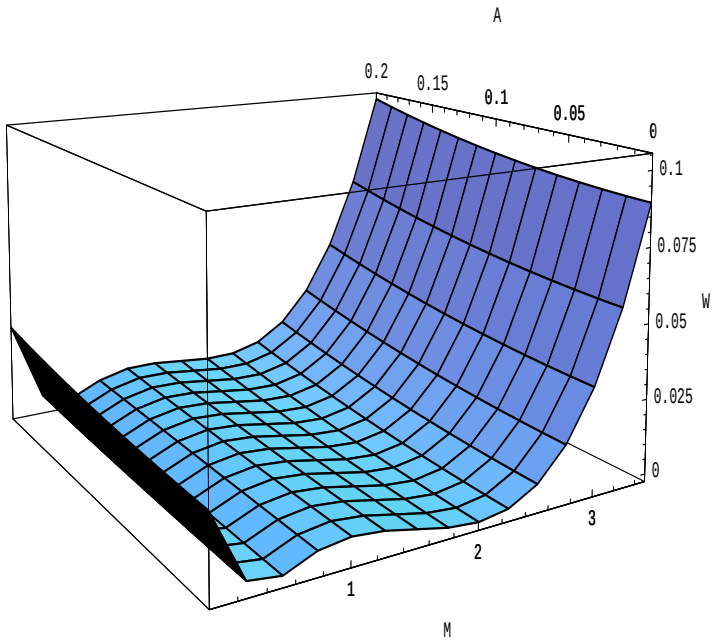

Fig. 7 **Region II** of Fig. 6: The energy density $W(m = \sqrt{M^2 + \lambda}, \varphi) \equiv \frac{1}{N}\mathcal{E}(m, \varphi)$ as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Two degenerate, $O(N)$ symmetric phases exist with massive bosons (and massive fermions). Here $\mu - \mu_c = 1$ (sets the mass scale) $u/u_c = 1.5$.

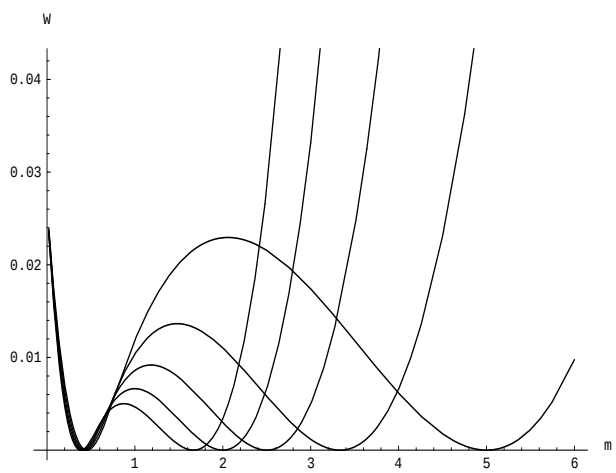

Fig. 8 The energy density $W(m) \equiv \frac{1}{N}\mathcal{E}(m, \varphi = 0)$ from Eq. (6.50) in region **II** of Fig. 6. Here $\mu - \mu_c = 1$ (sets the mass scale) and u/u_c= varies between 1.6 and 1.2. There are two degenerate $O(N)$ symmetric SUSY vacua with $\varphi = 0$ at $m_\psi = m_\varphi = M_+ = (\mu - \mu_c)/(u/u_c + 1)$ and at $m_\psi = m_\varphi = -M_- = (\mu - \mu_c)/(u/u_c - 1)$.

$0 < u < u_c$ and $\mu > \mu_c$, the system has a non-degenerate $O(N)$ symmetric ground state with bosons and fermions of mass $M = M_+$. As $\mu - \mu_c$ changes sign ($0 < u < u_c$ fixed), two degenerate ground states appear. Either $M = 0$ and $\varphi^2 = -N(\mu - \mu_c)/u$ or the system stays in an $O(N)$ symmetric ground state

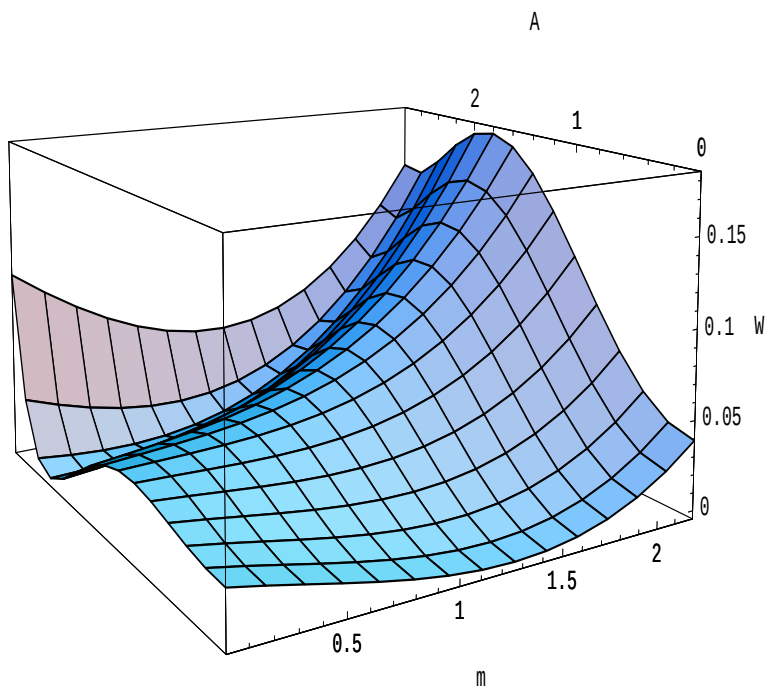

Fig. 9 **Region IV :** The energy density $W(m,\varphi) = \frac{1}{N}\mathcal{E}(m,\varphi)$ as given in Eq. (6.49) as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu - \mu_c = -1$, $u/u_c = 0.2$. As seen here there are two distinct degenerate phases. One is an ordered phase ($\varphi \neq 0$) with a massless boson and fermion, the other is a symmetric phase ($\varphi = 0$) with a massive ($m = |M_-|$) boson and fermion.

with a mass $|M_-|$ for the bosons and fermions. Similarly, when one goes from $\mu < \mu_c$ to $\mu > \mu_c$ at $u > u_c = 4\pi$ the $O(N)$ symmetry is restored but there are two degenerated ground states to choose from $M = M_\pm$.

In Fig. 7 and Fig. 8, $W(m = \sqrt{M^2+\lambda}, \varphi) \equiv \frac{1}{N}\mathcal{E}(m,\varphi)$ from Eq. (6.50) in region **II** ($\mu - \mu_c \geq 0$, $u/u_c \geq 1$) is plotted as a function of m and φ. An unusual transition takes place when one varies the coupling constant u. The transition from the degenerate vacua at $u/u_c = 1.4$ to a non-degenerate ground state at $u/u_c = 0.8$ is shown in Fig. 10 (from phase **II** to phase **I**).

For positive $\mu - \mu_c$, we find two degenerate ground states if $u > u_c$. As u is lowered (at fixed $\mu - \mu_c$), the ground state with mass $m_\varphi = m_\psi = -M_-$ disappears ($|M_-/M_+| \to \infty$) and only the $O(N)$ symmetric phase remains with $M = M_+$ (Figs. 8 and 10). Namely, suppose we consider at $\{u > u_c, \mu - \mu_c > 0\}$ a physical system in a state denoted by A and defined by $\{\varphi^2 = 0, M = M_-\}$, such a system will go into a state B defined by $\{\varphi^2 = 0, M = M_+\}$ when u **decreases** and passes the value $u = u_c = 4\pi$. Now, if we consider the reversed process; a physical system at $\{u < u_c, \mu - \mu_c > 0\}$ that is initially in the ground state B and u **increases** and passes $u = u_c$. There is no reason now for the system to go through the reversed transition from B to A since the $O(N)$ symmetric states with $M = M_-$ and $M = M_+$ are degenerate and the fact that supersymmetry is

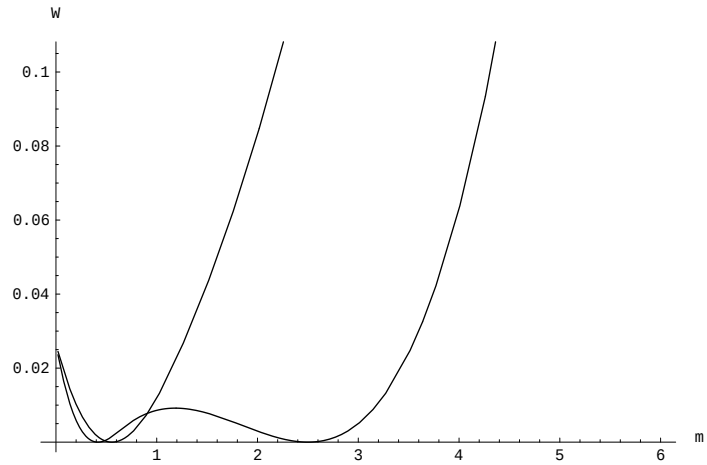

Fig. 10 The energy density $W(m) = \frac{1}{N}\mathcal{E}(m, \varphi = 0)$ as given in Eq. (6.50). Here $\mu - \mu_c = 1$ and $u/u_c = 1.2$ is changed to $u/u_c = 0.8$ (from region **II** to region **I** of Fig. 6). There are two degenerate $O(N)$ symmetric SUSY vacua at $u/u_c = 1.2$ with masses $m_\psi = m_\varphi = |M_\pm|$ where $|M_\pm| = \mu/(u/u_c \pm 1)$ while at $u/u_c = 0.8$ there is a non-degenerate vacuum at $m_\psi = m_\varphi = M_+$.

preserved will avoid that the energy of state A will go below zero. These peculiar phase transitions with the ($|M_-/M_+| \to \infty$) and with "infinite hysteresis" in the $A \to B$ transitions are due to the fact that supersymmetry is left unbroken in the leading order in $1/N$. If supersymmetry would have been broken by some small parameter, the lifted degeneracy would be, most probably, translated into a slow transitions between the otherwise degenerate ground states.

In section 8 we will study the transitions between the different phases of Fig. 6 as a function of the temperature .

Special situation. In general when $\mu = \mu_c$, the mass M vanishes. However, there is a special case when

$$u = u_c = 4\pi\,.$$

Then the value of M is left undetermined. An accumulation point of coexisting degenerate ground states exist in the phase structure shown in Fig. 6. The case $\mu = \mu_c$ represents a scale invariant theory where, however, the $O(N)$ fermionic and bosonic quanta can have a non-vanishing mass $m_\varphi = m_\psi = |M|$. Since u has not undergone any perturbative renormalization there is no explicit breaking of scale invariance at this point. Thus, the only scale invariance breaking comes from the solution of the gap equation for M, which leaves, however, its numerical value undetermined. We see a dimensional transmutation from the dimensionless coupling u that is fixed at a value of $u = u_c$ into an undetermined scale M. If $M \neq 0$ the spontaneous breaking of scale invariance will require the appearance of a Goldstone boson at the point $u = u_c$. The massless Goldstone boson is associated here with the spontaneous breaking of scale invariance. Moreover,

since the ground state is supersymmetric, we expect the appearance of a massless $O(N)$ singlet Goldstone boson (a dilaton) and its massless fermionic partner (a "dilatino"). In order to see all this, we now calculate the $\langle LL\rangle$ propagator that will enable us to see these poles in the appropriate four-point functions.

Finally, note that this analysis is only valid in the complete absence of cut-off corrections. Otherwise if $\mu = \mu_c$ and u is tuned as $u = u_c - M_0/\Lambda$, where M_0 is an arbitrary mass scale, one finds for the M_- solution:

$$M = -\frac{M_0}{a(3)u_c^2}\,.$$

6.5 The $\langle LL\rangle$ propagator and massless bound states of fermions and bosons

The $\langle LL\rangle$ propagator. We now calculate the $\langle LL\rangle$ propagator in the symmetric phase. Then, the propagators of the fields ϕ and L are decoupled. In the quartic potential example the R field can be eliminated by gaussian integration. The relevant part of the L-action then reads

$$-\frac{N}{2u}\int \mathrm{d}^3x\,\mathrm{d}^2\theta(L-\mu)^2 + \frac{1}{2}(N-1)\mathrm{Str}\,\ln\left(-\bar{\mathrm{D}}\mathrm{D}+2L\right).$$

The calculation of the $\langle LL\rangle$ propagator involves the super-propagator (6.19). For the inverse propagator one finds

$$\Delta_L^{-1}(p) = -\frac{N}{u}\delta^2(\theta'-\theta) - 2N\int\frac{\mathrm{d}^3k}{(2\pi)^3}\Delta(k,\theta,\theta')\Delta(p-k,\theta,\theta')$$

with here (see Eq. (6.6))

$$\Delta(k,\theta,\theta') = \frac{1}{k^2+M^2}\left[1+\tfrac{1}{2}M\delta^2(\theta'-\theta)\right]\mathrm{e}^{-i\bar{\theta}\not{k}\theta'}\ .$$

Then,

$$\Delta(k,\theta,\theta')\Delta(p-k,\theta,\theta') = \frac{\left[1+M\delta^2(\theta'-\theta)\right]\mathrm{e}^{-i\bar{\theta}\not{p}\theta'}}{(k^2+M^2)[(p+k)^2+M^2]}\,.$$

Notice the cancellation of the factor $\mathrm{e}^{-i\bar{\theta}\not{k}\theta'}$ which renders the integral more convergent that one could naively expect. The integral over k then yields the three-dimensional bubble diagram

$$\begin{aligned} B(p) &= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{(k^2+M^2)[(p+k)^2+M^2]} \\ &= \frac{1}{4\pi p}\mathrm{Arctan}(p/2|M|)\,. \end{aligned}$$

At leading order for p small, we need only $B(0) = 1/8\pi|M|$. Then,

$$\Delta_L^{-1} = -\frac{N}{4\pi|M|}\left[1 + (M + 4\pi|M|/u)\delta^2(\theta' - \theta)\right] \mathrm{e}^{-i\bar{\theta}\not{p}\theta'} .$$

Comparing with the expression (6.10), we conclude that for $M > 0$ small the LL propagator corresponds to a super-particle of mass $2M(1 + 4\pi/u)$. For $M < 0$ the mass is $2|M(1 - 4\pi/u)|$. For $|u - u_c|$ small, it is a bound state and at the special point $u = u_c$ the mass vanishes.

More generally, we find

$$\Delta_L(p) = \frac{2}{NB(p)}\frac{1}{p^2 + m^2(p)}\left[1 - \tfrac{1}{2}m(p)\delta^2(\theta' - \theta)\right] \mathrm{e}^{-i\bar{\theta}\not{p}\theta'} \tag{6.51}$$

with

$$m(p) = 2M + \frac{1}{uB(p)} .$$

We note that only a mass renormalization is required at leading order, a situation similar to the φ^4 scalar field theory. As a consequence, dimensions of fields are not modified.

Clearly, the propagation of the fields $M(x)$ and $\ell(x)$ (of Eq. (6.14)), as indicated in Eq. (6.51), when combined with the $L(\theta)\Phi^2$ interaction in Eq. (6.13), namely,

$$\int \mathrm{d}^2\theta\, L\Phi^2 = M(-\tfrac{1}{2}\bar{\psi}\psi + F\varphi) - \varphi\bar{\ell}\psi + \tfrac{1}{2}\lambda\varphi^2,$$

describes the bound states in the $\varphi\varphi$, $\psi\psi$ and $\psi\varphi$ scattering amplitudes. For example, in the supersymmetric ground state case and with $\mu - \mu_c = 0$, the $\psi\varphi$ scattering amplitude $T_{\psi\varphi,\psi\varphi}(p^2)$, in the limit $p^2 \to 0$ satisfies

$$T_{\psi\varphi,\psi\varphi}(p^2) \sim \frac{2u}{N}\left[1 + \frac{u}{4\pi}\frac{M}{|M|} + \frac{u}{2\pi}\frac{i\not{p}}{|M|}\right]^{-1} \to -\frac{4\pi i}{N}\frac{|M|}{\not{p}} \tag{6.52}$$

for $M < 0$ and $u \to u_c$

One notes here that the fermionic massless bound state pole appears when a non-zero solution (M) to the gap equation exists ($m_\varphi = m_\psi = |M|$) in the absence of any dimensional parameters ($\mu - \mu_c = 0$). This happens when the force between the massive ψ and φ quanta is determined by $u \to u_c$. The massless $O(N)$ singlet fermionic bound state excitation is associated with the spontaneous breaking of scale invariance. Similarly, the bosonic partner of this massless bound state excitation can be then seen, at the same value of the parameters, in $\varphi\varphi$ and $\psi\psi$ scattering amplitudes as Eq. (6.51) shows.

At $\mu = \mu_c$ in the generic situation $M = 0$, or for $|p| \to \infty$, we find

$$B(p) = \frac{1}{8p}$$

and, thus,

$$\Delta_L^{-1}(p) = -\frac{N}{u}\delta^2(\theta' - \theta) - \frac{N}{4p}\,\mathrm{e}^{-i\bar{\theta}\not{p}\theta'}\,.$$

As a consequence, the canonical dimension of the field L is 1, as in perturbation theory, and the interaction $L\Phi^2$ in Eq. (6.13) is renormalizable. The L^2 term thus is not negligible in the IR limit. Nevertheless, as we have noticed, the behaviour of physical quantities does not depend on u. Therefore, we expect that, at least for generic values of u, we can describe the same physics with the supersymmetric non-linear σ-model.

$O(N)\times O(N)$ *symmetric models.* The dynamics by which scale invariance can be broken in a theory which has no trace anomalies in perturbation theory has been studied also in $O(N)\times O(N)$ symmetric models [126,128]. Here one finds that spontaneous breaking of scale invariance is due to the breakdown of the internal $O(N)\times O(N)$ symmetry or to non-perturbative mass generation on a critical surface. The mass of the fermion and boson $O(N)$ quanta is arbitrary due to the appearance of flat directions in the action density. Also the ratios between the scales associated with breaking of the internal $O(N)\times O(N)$ symmetry and scale symmetries are arbitrary on the critical surface. Massless bound states of fermions and bosons appear due to the spontaneous breaking of scale invariance. Here, again, in the large N limit there is no explicit breaking of scale invariance and the perturbative β function vanishes. The variational ground state energy, which is calculable in the large N limit, has flat directions which allow spontaneous breaking of scale invariance by non-perturbative generation of mass scales. This results in lines of fixed points in the coupling constant plane which are associated with dynamical scale symmetry breaking. The interplay between internal symmetry, scale symmetry and supersymmetry is reflected in a rich phase structure. The novel issue in these models are phases in which the breaking of scale invariance is directly related to the breaking of the internal symmetry $O(N)\times O(N)\to O(N-1)\times O(N-1)$.

For the $O(N)\times O(N)$ symmetric potential $NU(\Phi_1^2/N, \Phi_2^2/N)$ where

$$U(R_1, R_2) = \mu(R_1 + R_2) + \tfrac{1}{2}u(R_1^2 + R_2^2) + vR_1R_2\,, \tag{6.53}$$

one again finds in $d = 3$, in the leading order in $1/N$, that only a mass renormalization is needed. Correspondingly, the critical value of the parameter μ now is $\mu_c = -2(u+v)\rho_c$. Here too, the interesting case is $\mu = \mu_c$. As in the $O(N)$ symmetric model, one finds that supersymmetry is left unbroken to leading order in $1/N$. The gap equations are now

$$\begin{aligned} m_1\left(\frac{u}{4\pi} + \mathrm{sgn}(m_{\psi_1})\right) + m_2\left(\frac{v}{4\pi}\right) &= u\varphi_1^2/N + v\varphi_2^2/N\,,\\ m_1\left(\frac{v}{4\pi}\right) + m_2\left(\frac{u}{4\pi} + \mathrm{sgn}(m_{\psi_2})\right) &= v{\varphi_1}^2/N + u{\varphi_2}^2/N\,, \end{aligned} \tag{6.54}$$

where non-zero solutions are obtained on lines $v = v(u)$, which is the condition for spontaneous breaking of scale invariance. As expected, this is also the condition for the appearance of massless dilaton and "dilatino" in the spectrum. One notes that scale invariance can be broken here as a consequence of the breakdown of the internal $O(N) \times O(N)$ symmetry. Indeed, the massless dilaton and "dilatino" appear either as bound states of massive "pions" and their supersymmetric partners or, when the internal symmetry is broken, they are mixed with the "sigma" boson and fermion particles. Thus, the non-zero scale that is responsible for the spontaneous breaking of scale invariance may be set here also by $\varphi^2 \neq 0$ rather than by $m_\psi = m_\varphi \neq 0$ only, as was the case in the $O(N)$ symmetric model. On the lines $v = v(u)$, one finds that $\mathcal{E}$ has flat directions in all dimensional parameters.

6.6 Dimensions $2 \leq d \leq 3$

In d space–time dimensions, $2 \leq d \leq 3$, one can keep in the space of γ matrices $\mathrm{tr}\,\mathbf{1} = 2$. From the point of view of power counting in perturbation theory the model is super-renormalizable since the dimension of u now is $3 - d$.

The calculations follow the same pattern discussed above. Eqs. (6.26) are replaced by (with the definition (2.12))

$$\begin{aligned} \rho - \varphi^2/N &= \Omega_d(m), \\ s - 2F\varphi/N &= 2M\left[\Omega_d(m) - \Omega_d(|M|)\right]. \end{aligned} \tag{6.55a}$$

Taking into account the other saddle point equations, and $\rho_c = \Omega_d(0)$, one obtains a generalization of Eq. (6.33):

$$2\mathcal{E}/N = M^2\varphi^2/N + (M^2 - m^2)\left[\Omega_d(m) - \Omega_d(0)\right] + \int_0^\lambda \mathrm{d}\lambda'\,\Omega_d\left(\sqrt{M^2 + \lambda'}\right) \tag{6.56}$$

and, thus, from Eqs. (2.13) and (2.14a):

$$\mathcal{E}/N = \tfrac{1}{2}M^2\varphi^2/N + K(d)\left[\tfrac{1}{2}(m^2 - M^2)m^{d-2} - \left(m^d - |M|^d\right)/d\right],$$

where

$$K(d) = -\frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}}.$$

Moreover, from Eq. (6.56),

$$\frac{\partial\mathcal{E}}{\partial m} = -\frac{N}{2m}(m^2 - M^2)\Omega_d'(m), \tag{6.57}$$

which has the sign of $m - |M|$ because $\Omega_d'(m)$ is negative. The function, therefore, has a minimum at $m = |M|$ for all values of d, and supersymmetry is maintained in the ground state for all $2 \leq d \leq 3$.

The critical exponents now are $\eta = 0$, $\beta = 1/2$, and for the mass in the unbroken phase one obtains

$$M = \mu - \mu_c - K(d)u|M|^{d-2} + O(M^2\Lambda^{d-4}).$$

We see that the l.h.s. is now negligible, the equation having a solution only for $(\mu - \mu_c)/u > 0$. The non-perturbative value $\nu = 1/(d-2)$ of the correlation exponent follows. One also verifies that the dimension of L remains 1, and thus the L^2 term now is negligible in the IR limit, leading immediately to the non-linear σ-model. However, again one finds no IR fixed point, no value of u cancels the leading correction to scaling, and therefore the argument leading to the non-linear σ-model is not as solid as for the usual $(\varphi^2)^2$ field theory.

Two dimensions. It may be interesting to consider the same model in two dimensions. The model now is super-renormalizable, but even at $\mu = 0$ it is not chiral invariant since a chiral transformation corresponds to a change $U \mapsto -U$.

The expression (6.57) is cut-off independent and has a limit for $d = 2$:

$$\mathcal{E}/N = \tfrac{1}{2}M^2\varphi^2/N + \frac{1}{8\pi}\left[m^2 - M^2 - 2M^2\ln(m/M)\right], \tag{6.58}$$

an expression which again has an absolute minimum at $m = |M|$. Then, a minimization with respect to M and φ yields $M\varphi = 0$. At the minimum $\mathcal{E}$ vanishes.

Taking into account $\lambda = 0$ and Eq. (6.55a) in the $d = 2$ limit, we obtain the gap equation

$$M = \mu + u\varphi^2/N + \frac{u}{2\pi}\left[\ln(\Lambda/|M|] + O(1)\right).$$

The solution $M^2 = \lambda = 0$, thus, is not acceptable. The $O(N)$ symmetry is never broken. When the mass M is small in the cut-off scale, the l.h.s. is negligible and

$$m = M \propto \Lambda\,\mathrm{e}^{2\pi\mu/u}.$$

This solution exists only when $-\mu/u$ is positive and large. Its behaviour suggests immediately a relation with the non-linear σ-model, which we now briefly examine.

6.7 A supersymmetric non-linear σ model at large N

We consider the supersymmetric non-linear σ model in d dimensions, $2 \le d \le 3$. The action [129–135]

$$\mathcal{S}(\Phi) = \frac{1}{2\kappa}\int \mathrm{d}^d x\,\mathrm{d}^2\theta\,\bar{\mathrm{D}}\Phi\cdot\mathrm{D}\Phi \tag{6.59}$$

involves an N-component scalar superfield Φ, which satisfies

$$\Phi\cdot\Phi = N.$$

This relation is implemented by introducing a superfield

$$L(x,\theta) = M(x) + \bar{\theta}\ell(x) + \tfrac{1}{2}\bar{\theta}\theta\lambda(x),$$

where $M(x)$, $\lambda(x)$ and $\ell(x)$ are the Lagrange multiplier fields, and adding to the action

$$\mathcal{S}_L = \frac{1}{\kappa}\int \mathrm{d}^d x\, \mathrm{d}^2\theta\, L(x,\theta)\left[\Phi^2(x,\theta) - N\right]. \tag{6.60}$$

The partition function is given by ($\mathcal{S}(\Phi, L) = \mathcal{S} + \mathcal{S}_L$):

$$\mathcal{Z} = \int [d\Phi][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}(\Phi,L)}\,.$$

In terms of component fields, the total action reads

$$\begin{aligned}\mathcal{S} = \frac{1}{\kappa}\int \mathrm{d}^d x\{&\tfrac{1}{2}\varphi(-\partial^2 + M^2)\varphi - \tfrac{1}{2}\bar{\psi}(\partial\!\!\!/ + M)\psi \\ &+ \tfrac{1}{2}\lambda(\varphi^2 - N) - \bar{\ell}(\bar{\psi}\cdot\varphi)\}.\end{aligned} \tag{6.61}$$

As in the case of the Φ^4 theory, we integrate out $N-1$ components leaving out $\phi = \Phi_1$:

$$\mathcal{Z} = \int [d\phi][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}_N(\phi,L)}$$

where

$$\mathcal{S}_N(\phi, L) = \frac{1}{\kappa}\int \mathrm{d}^d x\, \mathrm{d}^2\theta\left[\tfrac{1}{2}\bar{\mathrm{D}}\phi\cdot\mathrm{D}\phi + L\left(\phi^2 - N\right)\right] + \frac{N-1}{2}\mathrm{Str}\,\ln(-\bar{\mathrm{D}}\mathrm{D} + 2L). \tag{6.62}$$

By varying the effective action with respect to ϕ, one finds the saddle point equation

$$\bar{\mathrm{D}}\mathrm{D}\phi - 2L\phi = 0\,, \tag{6.63}$$

which implies for constant φ and $\psi = 0$:

$$F - M\varphi = 0 \qquad \text{and} \qquad MF + \lambda\varphi = 0\,. \tag{6.64}$$

When the large N action is varied with respect to L and using the expression in (6.19) for the ϕ propagator, one finds ($N \gg 1$)

$$\frac{1}{\kappa} - \frac{\phi^2}{N} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\left[\frac{1 + M\bar{\theta}\theta}{k^2 + M^2 + \lambda} - \frac{M\bar{\theta}\theta}{k^2 + M^2}\right]. \tag{6.65}$$

We now introduce the boson mass parameter $m = \sqrt{M^2 + \lambda}$. Eq. (6.65) in component form (for $\psi = 0$) then reads

$$1 - \frac{\varphi^2}{N} = \kappa\,\Omega_d(m), \tag{6.66a}$$

$$M\frac{\varphi^2}{N\kappa} = M\left[\Omega_d(|M|) - \Omega_d\,(m)\right]. \tag{6.66b}$$

Dimension $d = 3$. Introducing the critical (cut-off dependent) value

$$\kappa_c = 1/\Omega_3(0)\,,$$

we can write Eqs. (6.66) now as

$$\frac{1}{\kappa} - \frac{1}{\kappa_c} - \frac{\varphi^2}{N\kappa} = -\frac{m}{4\pi} \tag{6.67a}$$

$$M\frac{\varphi^2}{N\kappa} = \frac{1}{4\pi}M\left(m - |M|\right). \tag{6.67b}$$

The calculation of the ground state energy density of the non-linear σ model follows similar steps as in the $(\Phi^2)^2$ model:

$$\mathcal{E}/N = \frac{1}{2N\kappa}m^2\varphi^2 - \tfrac{1}{2}(m^2 - M^2)\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right) - \frac{1}{12\pi}\left(m^3 - |M|^3\right) \tag{6.68}$$

and, taking into account the saddle point equation (6.66a),

$$\mathcal{E}/N = \frac{1}{2N\kappa}M^2\varphi^2 + \frac{1}{24\pi}(m - |M|)^2(m + 2|M|), \tag{6.69}$$

an expression identical to (6.33), up to the normalization of φ. Again, if a supersymmetric solution exists it has the lowest energy. We thus look for supersymmetric solutions.

In the $O(N)$ symmetric phase ($\varphi = 0$) Eq. (6.67b) is satisfied while Eq. (6.67a) yields

$$m = |M| = 4\pi\left(\frac{1}{\kappa_c} - \frac{1}{\kappa}\right).$$

This phase exists for $\kappa \geq \kappa_c$.

In the broken phase $m = M = 0$, and Eq. (6.67b) is again satisfied. Eq. (6.67a) then yields

$$\varphi^2/N = 1 - \kappa/\kappa_c\,, \tag{6.70}$$

which is the solution for $\kappa \leq \kappa_c$.

Since we have found solutions for all values of κ, we conclude that the ground state is always supersymmetric. At κ_c, a transition occurs between a massless phase with broken $O(N)$ symmetry ($\varphi^2 \neq 0$) and a symmetric phase with massive fermions and bosons of equal mass. The supersymmetric non-linear σ model and the usual non-linear σ-model studied in section 3.1, thus, have analogous phase structures, which are less surprising than the structure of $(\Phi^2)^2$ field theory.

Renormalization group. From the point of view of power counting, the model is analogous to its non-supersymmetric version (section 3.1), and RG equations

take the same form with two RG functions $\beta(\kappa)$ and $\zeta(\kappa)$. Both have been calculated to four loop order.

At leading order for $N \to \infty$, one finds the same critical exponents as in the supersymmetric $(\Phi^2)^2$ field theory or the usual non-linear σ model: $\eta = 0$ and, in generic dimension d: $\nu = 1/(d-2)$. The β-function has at this order the same form as in the non-supersymmetric model:

$$\beta(\kappa) = (d-2)\kappa(1 - \kappa/\kappa_c) + O(1/N).$$

In the supersymmetric model the critical exponents ν and η have been calculated to order $1/N^2$ and $1/N^3$, respectively. To order $1/N$ one finds

$$\eta = \frac{X_1(d)}{N} + O(1/N^2)\,, \quad \nu = \frac{1}{d-2} + O(1/N^2),$$

where $X_1(d)$ is the constant (3.28), $X_1(3) = 8/\pi^2$.

Dimension $d = 2$. The phase structure of the supersymmetric non-linear σ model in two dimensions is rather simple, and again analogous to the structure of the usual non-linear σ-model. Eq. (6.67a) immediately implies that $M^2 + \lambda = 0$ is not a solution and thus the $O(N)$ symmetry remains unbroken ($\varphi^2 = 0$) for all values of the coupling constant κ. Then, with a suitable normalization of the cut-off Λ,

$$m = \Lambda\, \mathrm{e}^{-2\pi/\kappa}\,. \tag{6.71}$$

Correspondingly, the action density becomes

$$\mathcal{E}/N = -\frac{\lambda}{2\kappa} - \frac{1}{8\pi}\left[(M^2+\lambda)\ln(M^2+\lambda) - \lambda\ln\Lambda^2 - M^2\ln M^2 - \lambda\right] \tag{6.72}$$

and, taking into account Eq. (6.71),

$$\mathcal{E}/N = \frac{1}{8\pi}\left[m^2 - M^2 + 2M^2\ln(M/m)\right].$$

We recognize the expression (6.58), and conclude in the same way that a supersymmetric solution has the lowest energy. Supersymmetry remains unbroken, and the common boson and fermion mass is

$$M = m = \Lambda\, \mathrm{e}^{-2\pi/\kappa}\,. \tag{6.73}$$

The model, thus, is UV asymptotically free and the condition $M \ll \Lambda$ implies that non-trivial physics is concentrated near $\kappa = 0$.

7 Finite temperature field theory in the large N limit

In this section, we first recall a few general properties of Statistical Quantum Field Theory (QFT) at equilibrium, that is QFT at finite temperature [136–143]. We argue that when the temperature varies, one generally observes a crossover from a zero temperature d dimensional theory to an effective $d-1$ dimensional field theory at high temperature, a phenomenon called dimensional reduction. This corresponds also to a classical limit with a transition from a statistical quantum field theory to a statistical classical field theory. Note that high temperature here refers to an ultra-relativistic limit where the temperature, in energy unit, is much larger than the physical masses of particles, so that bosons can produce a classical field. In perturbation theory it is often impossible to describe this crossover and therefore large N techniques may again be useful.

We illustrate these ideas with several standard examples, ϕ^4 field theory, the non-linear σ model, the Gross–Neveu model and gauge theories.

7.1 Finite temperature QFT: general remarks

We first recall some general properties of QFT at thermal equilibrium in $(1, d-1)$ dimensions. We introduce the mode expansion of fields in the euclidean time variable, discuss the conditions under which statistical properties of finite temperature QFT in $(1, d-1)$ dimension can be described by an effective local classical statistical field theory in $d-1$ dimensions, and indicate how to construct it explicitly.

Partition function. Equilibrium properties of QFT at finite temperature can be derived from the partition function $\mathcal{Z} = \operatorname{tr} \mathrm{e}^{-H/T}$, where H is the quantum hamiltonian and T the temperature (interesting physics related to the addition of a chemical potential will not be discussed here). We will consider, first, a euclidean action $\mathcal{S}(\phi)$ describing a scalar boson field ϕ. The partition function is given by the functional integral

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi)\right], \tag{7.1}$$

where $\mathcal{S}(\phi)$ is the integral of the euclidean lagrangian density $\mathcal{L}(\phi)$:

$$\mathcal{S}(\phi) = \int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}^{d-1}x\, \mathcal{L}(\phi),$$

and the field ϕ satisfies *periodic boundary conditions* in the (euclidean or imaginary) time direction:

$$\phi(\tau = 0, x) = \phi(\tau = 1/T, x).$$

A QFT may also involve fermions. Fermion fields $\psi(\tau, x)$, by contrast, satisfy anti-periodic boundary conditions:

$$\psi(\tau = 0, x) = -\psi(\tau = 1/T, x).$$

Mode expansion. As a consequence of the finite temperature boundary conditions, fields have a Fourier series expansion in the euclidean time direction with quantized frequencies ω_n (also called Matsubara frequencies). For boson fields

$$\phi(t, x) = \sum_{n \in \mathbb{Z}} \mathrm{e}^{i\omega_n t}\, \phi_n(x), \quad \omega_n = 2n\pi T\,. \tag{7.2}$$

In the case of fermions, anti-periodic boundary conditions lead to the expansion

$$\psi(t, x) = \sum_{\omega_n = (2n+1)\pi T} \mathrm{e}^{i\omega_n t}\, \psi_n(x). \tag{7.3}$$

Remark. The mode expansion (7.2) is well suited to simple situations where fields belong to a linear space. In the case of non-linear σ models or non-abelian gauge theories, the decomposition in modes leads to problems, because it conflicts with the geometric structure.

Classical statistical field theory and thermal mass. The quantum partition function (7.1) can also be considered as the partition function of a classical statistical field theory in d dimensions. In this interpretation finite temperature for the quantum partition function (7.1) corresponds to a finite size $\beta = 1/T$ in one direction for the classical partition function. The zero temperature limit of the quantum theory corresponds to an infinite volume limit of the classical theory.

An important parameter then is the *thermal mass* m_T, inverse of the correlation length ξ_T, which characterizes the decay of correlations in space directions. A cross-over is expected between a d-dimensional behaviour when the correlation length ξ_T is small compared with β, that is the mass scale m_T is large compared with the temperature T, to the $(d-1)$-dimensional behaviour when m_T is small compared with T. Moreover, in this limit macroscopic properties and correlations for momenta much smaller than the temperature T or distances much larger than β, can be described by an effective $(d-1)$-dimensional local field theory. As we shall see, this corresponds to a classical limit, in the sense that classical fields replace quantum fields. Within this framework, the temperature then plays the role of a large momentum cut-off.

The ratio m_T/T can be expected to be small in several situations, at high temperature and near a finite temperature phase transition. Moreover, it is also small at low temperature in a third peculiar situation, when a symmetry is broken at zero temperature and no phase transition is possible at finite temperature.

Finite temperature renormalization group. From the classical statistical interpretation one learns that general results obtained in the study of finite size effects also apply here. Correlation functions satisfy the renormalization group (RG) equations of the corresponding d-dimensional classical theory [148,149]. Indeed, RG equations are related to short distance singularities and are, therefore, insensitive to finite sizes. A finite size affects only solutions of the RG equations, because a new dimensionless RG invariant parameter becomes available, here for instance the ratio m_T/T.

At high temperature, the ratio m/T, where m is the physical mass scale, goes to zero. This does not automatically imply that m_T/T is small because the parameter m_T is identical with m only in the tree approximation. By solving the RG equations in terms of the effective coupling at the temperature scale T, one infers that if the effective coupling goes to zero at high temperature, then the ratio m_T/T really becomes small. Two examples will be met: the first corresponds to theories where the free field theory is an IR fixed point, like ϕ_4^4 scalar field theory or QED_4, the second corresponds to UV asymptotically free field theories. Conversely, when a non-trivial IR fixed point is present the ratio m_T/T goes to a constant. At high temperature, one then has to rearrange the initial perturbation theory by adding and subtracting a mass term to suppress fictitious perturbative large IR contributions.

Zero mode and large distance behaviour. We consider first the example of a free scalar field theory with the action

$$\mathcal{S}(\phi) = \tfrac{1}{2}\int_0^{1/T} \mathrm{d}t \int \mathrm{d}^{d-1}x \left[(\partial_t\phi)^2 + (\nabla_x\phi)^2 + m^2\phi^2\right].$$

After introducing the mode expansion (7.2) into the action and integrating over time, one obtains a $(d-1)$-dimensional euclidean field theory with an infinite number of fields, the modes $\phi_n(x)$. The form of the action,

$$\mathcal{S}(\phi) = \frac{1}{2T}\int \mathrm{d}^{d-1}x \sum_n \left[|\nabla_x\phi_n(x)|^2 + (m^2 + 4\pi^2 n^2 T^2)|\phi_n(x)|^2\right],$$

shows that ϕ_n has a mass $\sqrt{m^2 + 4\pi^2 n^2 T^2}$. Therefore, at high temperature all modes become very massive except the zero-mode, whose mass m governs the decay of $\phi(t,x)$-field correlation functions in space directions (here $m_T = m$). The large distance, low momentum, physics can entirely described by an effective $(d-1)$-dimensional field theory involving only the zero mode.

Note, however, that all fermion modes become very massive at high temperature due to the anti-periodic boundary conditions (7.3)and generally decouple.

In an interacting theory, if m_T remains much smaller than the temperature, one expects still to be able to describe low momentum physics in terms of an effective $(d-1)$-dimensional euclidean local field theory. The thermal mass m_T

is the mass of the zero-mode in the effective theory, and all other scalar and fermion modes have masses at least of order T. The effective theory can thus be constructed by integrating out perturbatively all non-zero modes and performing a local expansion of the resulting effective action. As mentioned above, in this framework the temperature T acts as a large momentum cut-off.

Dimensional reduction. We now briefly outline the construction of the $(d-1)$-dimensional effective theory in the example of a general scalar QFT, assuming that the mass m_T of the zero-mode remains indeed much smaller than T [150,159]. We first separate the zero-mode $\varphi(x)$, setting

$$\phi(t,x) = \varphi(x) + \chi(t,x), \tag{7.4}$$

where χ the sum of all other modes (Eq. (7.2)):

$$\chi(t,x) = \sum_{n\neq 0} \mathrm{e}^{i\omega_n t}\, \phi_n(x), \quad \omega_n = 2n\pi T\,. \tag{7.5}$$

The action $\mathcal{S}_T(\varphi)$ of the effective theory is then obtained by integrating over χ:

$$\mathrm{e}^{-\mathcal{S}_T(\varphi)} = \int [\mathrm{d}\chi] \exp[-\mathcal{S}(\varphi+\chi)]. \tag{7.6}$$

Leading order. From the point of view of the χ integration, the tree approximation corresponds to setting $\chi = 0$ and one simply finds

$$\mathcal{S}_T(\varphi) = \frac{1}{T}\int \mathrm{d}^{d-1}x\, \mathcal{L}(\varphi), \tag{7.7}$$

an action that is obviously local.

We note that T plays, in this leading approximation, the formal role of $\hbar$, and the small T expansion corresponds to a loop expansion. If the ratio T/Λ, which is always assumed to be small, is the relevant expansion parameter, which means that the perturbative expansion is dominated by large momentum (UV) contributions, then the effective $(d-1)$ dimensional theory can still be studied by perturbative methods. This is expected when the number $d-1$ of space dimensions is large and field theories are non-renormalizable. However, another dimensionless ratio can be found, m/T, which at high temperature is small. This may be the relevant expansion parameter for theories that are dominated by small momentum (IR) contributions, a problem that arises in low dimensions. Then, perturbation theory is no longer possible or useful. Actually, the relevant parameter in the full effective theory is m_T/T. Therefore, the contributions to the mass of the zero-mode due to quantum and thermal fluctuations need to be investigated.

Loop corrections. The integration over non-zero modes generates non-local interactions. To study long wave length phenomena, one can, however, perform a *local expansion* of the effective action, expansion that breaks down at momenta of order T. In general, higher order corrections coming from the integration over non-zero modes generate terms that renormalize the terms already present at leading order, and additional interactions are suppressed by powers of $1/T$. Exceptions are provided by gauge theories where new low dimensional interactions are generated as a consequence of the explicit breaking of the $O(1, d-1)$ symmetry.

Renormalization in the effective theory. If the initial $(1, d-1)$ dimensional theory has been renormalized at $T = 0$, the complete theory is finite in the formal infinite cut-off limit because finite size effects do not affect divergences. However, as a consequence of the zero-mode subtraction, cut-off dependent terms may appear in the reduced $(d-1)$-dimensional action. These terms provide the necessary counter-terms that render the perturbative expansion of the effective field theory finite.

7.2 Scalar quantum field theory at finite temperature for N large

General finite size effects and, therefore, as we have already discussed, finite temperature physics involve crossover phenomena between different dimensions when the temperature varies from zero to infinity. This severely limits the applicability of perturbation theory (even with RG improvement), in particular because IR divergences are more severe in lower dimensions. Large N techniques, however, which are rather insensitive to changes in the number of dimensions are, therefore, particularly well suited to study a crossover situation [149].

We first consider again the self-interacting scalar field with $O(N)$ symmetric action, studied at zero temperature in section 2.1. The field ϕ is an N-component vector and the hamiltonian

$$\mathcal{H}(\Pi, \phi) = \frac{1}{2} \int \mathrm{d}^{d-1}x \, \Pi^2(x) + \Sigma(\phi) \tag{7.8}$$

with

$$\Sigma(\phi) = \int \mathrm{d}^{d-1}x \left\{ \tfrac{1}{2} \left[\nabla\phi(x)\right]^2 + NU\big(\phi^2(x)/N\big) \right\}. \tag{7.9}$$

A cut-off Λ, as usual, is implied that renders the QFT UV finite. The quantum partition function (7.1) at finite temperature T,

$$\mathcal{Z} = \int \left[\mathrm{d}\phi(x)\right] \exp\left[-\mathcal{S}(\phi)\right], \tag{7.10}$$

can then be expressed in terms of the action

$$\mathcal{S}(\phi) = \int_0^{1/T} \mathrm{d}t \left[\int \mathrm{d}^{d-1}x \tfrac{1}{2} (\mathrm{d}_t\phi)^2 + \Sigma(\phi) \right], \tag{7.11}$$

where the field ϕ satisfies periodic boundary conditions in the euclidean time direction.

Following Eq. (2.7), the action density at finite temperature and large N is

$$\frac{1}{N}\mathcal{F}(\rho,\sigma,m_T) = U(\rho) + \tfrac{1}{2}m_T^2(\sigma^2 - N\rho) + \frac{1}{2\beta V_{d-1}}\,\mathrm{tr}\ln\left[-\nabla^2 + m_T^2\right], \quad (7.12)$$

where V_{d-1} is the $d-1$ volume. The tr ln contribution is now modified by the boundary conditions:

$$\begin{aligned}\frac{1}{V_{d-1}}\,\mathrm{tr}\ln(-\nabla^2 + m_T^2) &= \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \sum_{n\in\mathbb{Z}} \ln(\omega_n^2 + k^2 + m_T^2) \\ &= 2\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \ln\left[2\sinh\big(\beta\omega(k)/2\big)\right] \qquad (7.13)\end{aligned}$$

with $\omega(k) = \sqrt{k^2 + m_T^2}$. The result in Eq. (7.13) is given up to a mass independent infinite constant. We have used $\omega_n = 2n\pi/\beta$ and the identity (A2.9).

Differentiating $\mathcal{F}$ in Eq. (7.12) with respect to σ, ρ and m_T, one obtains the saddle point equations. The algebraic transformations which produce the large N effective action are clearly insensitive to boundary conditions and, therefore, only the specific form of the saddle point equation in Eq. (2.10c) is modified. Eqs. (2.10) become

$$m_T^2\sigma = 0\,, \quad m_T^2 = 2U'(\rho)$$

and

$$\rho - \sigma^2/N = G_2(m_T, T) \qquad (7.14)$$

with

$$\begin{aligned}G_2(m_T,T) &= \frac{T}{(2\pi)^{d-1}} \sum_{n\in\mathbb{Z}} \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi nT)^2 + k^2 + m_T^2}, \\ &= \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega(k)} \left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\beta\omega(k)} - 1}\right). \qquad (7.15)\end{aligned}$$

The quantum ($T = 0$) and thermal ($T > 0$ finite) fluctuations are clearly separated when the two terms in Eq. (7.15) are written as

$$\begin{aligned}G_2 &= [G_2]_{\mathrm{quantum}} + [G_2]_{\mathrm{thermal}} \\ &= \frac{1}{(2\pi)^d} \int^{\Lambda} \frac{\mathrm{d}^d k}{k^2 + m_T^2} + \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^{d-1}k}{\omega(k)} \frac{1}{\mathrm{e}^{\beta\omega(k)} - 1}. \qquad (7.16)\end{aligned}$$

In particular, as expected, only the zero temperature contribution is cut-off dependent. For $T \to \infty$, the second contribution dominates and at leading order is identical to the zero-mode contribution.

Note that, alternatively, we could have used the finite size formalism of section 3.4 and the corresponding Jacobi function (3.38).

Symmetry breaking. In the broken symmetry phase $\sigma \neq 0$ and thus $m_T = 0$. Then, G_2 can be calculated explicitly:

$$G_2(0,T) = \rho_c + N_{d-1}\Gamma(d-2)\zeta(d-2)T^{d-2}, \tag{7.17}$$

where N_d is the loop factor (2.14*b*) and $\zeta(s)$ Riemann's ζ-function. The function $\zeta(s)$ has a pole for $s = 1$ and therefore, as expected, a phase transition for $T > 0$ is possible only for $d > 3$ when the result is IR finite. The result has a simple interpretation: at $d = 3$, IR divergences come from the contribution of the zero-mode in Eq. (7.15) and are those of a two-dimensional theory, where no phase transition is possible. This is a direct example of the property of *dimensional reduction* $d \mapsto d-1$.

The parameter ρ is the minimum of U and thus takes its $T = 0$ value. Finally, the expectation value σ is given by

$$\sigma^2/N = \rho(T=0) - \rho_c - N_{d-1}\Gamma(d-2)\zeta(d-2)T^{d-2}.$$

The field expectation value σ decreases with the temperature, which implies that the symmetry is broken at finite temperature only if it is already broken at zero temperature. The critical temperature T_c can be expressed, for $N \to \infty$, in terms of the zero temperature expectation value:

$$T_c \propto [\sigma(T=0)]^{2/(d-2)}\,. \tag{7.18}$$

A RG analysis actually shows that, in general, T_c can be related to the crossover scale between Goldstone and critical behaviour (section 7.3).

The symmetric phase. One now finds

$$\rho - \rho_c = \Omega_d(m_T) - \Omega_d(0) + T^{d-2} f_d(m_T^2/T^2), \tag{7.19}$$

where Ω_d is defined by Eq. (2.12) and

$$\begin{aligned} f_d(z) &= N_{d-1}\int_0^\infty \frac{x^{d-2}\mathrm{d}x}{\sqrt{x^2+z}}\frac{1}{\exp[\sqrt{x^2+z}]-1} \\ &= N_{d-1}\int_{\sqrt{z}}^\infty (y^2-z)^{(d-3)/2}\frac{\mathrm{d}y}{\mathrm{e}^y-1}\,. \end{aligned} \tag{7.20}$$

In particular,

$$f_d(0) = N_{d-1}\Gamma(d-2)\zeta(d-2), \quad f_d'(0) = -\tfrac{1}{2}(d-3)N_{d-1}\Gamma(d-4)\zeta(d-4). \tag{7.21}$$

The other equation is $m_T^2 = U'(\rho)$. In the special limit of a critical potential at $T = 0$ (the massless theory), which satisfies $U'(\rho_c) = 0$, the equation can be expanded as

$$m_T^2 \sim (\rho - \rho_c)U''(\rho_c)$$

if $U''(\rho_c)$ does not vanish. Then,

$$m_T^2/U''(\rho_c) \sim \Omega_d(m_T) - \Omega_d(0) + T^{d-2} f_d(m_T^2/T^2). \tag{7.22}$$

For $d > 4$, this equation implies

$$m_T/T \propto (T/\Lambda)^{(d-4)/2} \ll 1 \tag{7.23}$$

and, thus, dimensional reduction is justified.

For $d = 4$, the conclusion is the same because

$$\left(\frac{m_T}{T}\right)^2 \sim \frac{2\pi^2}{3\ln(\Lambda/T)} \ll 1\,. \tag{7.24}$$

Instead, for $d < 4$, the l.h.s. is negligible and, therefore, m_T is proportional to T. These results have a simple interpretation from the RG point of view in the framework of the $(\phi^2)^2$ field theory.

7.3 The $(\phi^2)^2$ field theory at finite temperature

We now specialize to the $(\phi^2)^2$ field theory, that is (section 2.3)

$$U(\rho) = \tfrac{1}{2} r\rho + \frac{u}{4!}\rho^2\,.$$

We first summarize what can be learned from a simple RG analysis, and then solve the large N saddle point equations in this case.

We define the quantity $r_c(u)$, which has the form of a mass renormalization, as the value of r at which the physical mass m of the field ϕ vanishes at $T = 0$. At $T = 0$, $r = r_c$, a transition occurs between a symmetric phase ($r > r_c$) and a broken symmetry phase ($r < r_c$). We recall that a QFT is meaningful only if the physical mass m is much smaller than the cut-off Λ. This implies either (the famous *fine tuning* problem) $|r - r_c| \ll \Lambda^2$ or, for $N > 1$, $r < r_c$ which corresponds to a spontaneously broken symmetry with massless Goldstone modes. The latter situation will be examined in section 7.5 within the more suitable formalism of the non-linear σ-model.

It is also convenient to introduce a dimensionless coupling

$$\lambda = u\Lambda^{d-4}/N\,, \tag{7.25}$$

where later $N\lambda$ will be assumed to take generic (i.e. not very small) values.

RG at finite temperature. As we have already stressed, some useful information can be obtained from a RG analysis, which also explains the nature of some of the results obtained in the large N limit.

Correlation functions at finite temperature satisfy the RG equations of the zero temperature QFT or the d-dimensional classical field theory in infinite volume. The dimension $d = 4$ is special, because then the ϕ_4^4 theory is just renormalizable. One important quantity is the ratio m_T/T, where m_T governs the decay of correlations in space directions and is also, after dimensional reduction, the mass of the zero-mode in the effective theory.

Higher dimensions. For $d > 4$, the theory is non-renormalizable, which means that the gaussian fixed point $u = 0$ is stable. The coupling constant $u = N\lambda\Lambda^{4-d}$ is small in the physical domain, and perturbation theory is applicable. At zero temperature, the physical mass in the symmetric phase has the scaling behaviour of a free or gaussian theory, $m \propto (r - r_c)^{1/2}$. The leading corrections to the two-point function due to finite temperature effects, are of order u. Therefore, in the symmetric phase, for dimensional reasons,

$$m_T \propto (r - r_c + \text{const. } \lambda\Lambda^{4-d}T^{d-2})^{1/2}.$$

If at zero temperature the symmetry is broken ($r < r_c$), a phase transition thus occurs at a temperature T_c, which scales like

$$T_c \propto \Lambda \left[(r_c - r)/\Lambda^2\right]^{1/(d-2)} \gg (r_c - r)^{1/2}\,.$$

The critical temperature is large compared with the $T = 0$ crossover mass scale (2.67).

At high temperature or in the massless theory ($r = r_c$), the thermal mass behaves like

$$m_T/T \propto (T/\Lambda)^{(d-4)/2} \ll 1\,,$$

a behaviour consistent with the form (7.23). The property $m_T/T \ll 1$ implies the validity of dimensional reduction.

Dimension $d = 4$. The $(\phi^2)^2$ theory is just renormalizable and RG equations take the form

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \frac{n}{2}\eta(\lambda) - \eta_2(\lambda)(r - r_c)\frac{\partial}{\partial r}\right]\Gamma^{(n)}(p_i; r, \lambda, T, \Lambda) = 0\,. \quad (7.26)$$

The ratio $m_T/T = F(\Lambda/T, \lambda, r/T^2)$ is dimensionless and RG invariant, and thus satisfies

$$\left[\Lambda\frac{\partial}{\partial\Lambda} + \beta(\lambda)\frac{\partial}{\partial\lambda} - \eta_2(\lambda)(r - r_c)\frac{\partial}{\partial r}\right]F = 0\,.$$

The solution can be written as

$$m_T/T = F(\Lambda/T, \lambda, (r - r_c)/T^2) = F\big(\ell\Lambda/T, \lambda(\ell), (r(\ell) - r_c)/T^2\big), \quad (7.27)$$

where ℓ is a scale parameter, and $\lambda(\ell), r(\ell)$ the corresponding running parameters (or effective parameters at scale ℓ):

$$\ell\frac{\mathrm{d}\lambda(\ell)}{\mathrm{d}\ell} = \beta\big(\lambda(\ell)\big), \quad \ell\frac{\mathrm{d}r(\ell)}{\mathrm{d}\ell} = -\big(r(\ell) - r_c\big)\eta_2\big(\lambda(\ell)\big).$$

The form of the RG β-function,

$$\beta(\lambda) = \frac{(N+8)}{48\pi^2}\lambda^2 + O\left(\lambda^3\right), \tag{7.28}$$

implies that the theory is IR free, that is $\lambda(\ell) \to 0$ for $\ell \to 0$. The effective coupling constant at the physical scale is logarithmically small, implying logarithmic deviations from naive scaling. To describe physics at the scale T, one has to choose $\ell = T/\Lambda \ll 1$ and, thus,

$$\lambda(T/\Lambda) \sim \frac{48\pi^2}{(N+8)\ln(\Lambda/T)}. \tag{7.29}$$

Therefore, RG improved perturbation theory can be used to derive the effective action of the reduced theory. In particular, for $r = r_c$ one recovers the behaviour (7.24):

$$\left(\frac{m_T}{T}\right)^2 \sim \frac{N+2}{72}\lambda(T/\Lambda) \sim \frac{2\pi^2(N+2)}{3(N+8)}\frac{1}{\ln(\Lambda/T)}.$$

Dimension $d = 3$. The three-dimensional classical theory has an IR fixed point λ^*. Then finite size scaling (Eq. (7.27)) predicts, in the symmetric phase,

$$m_T/T = f\big((r - r_c)/T^{1/\nu}\big),$$

where ν is the correlation exponent of the three-dimensional system. Therefore, m_T in general remains of order T at high temperature.

The zero-mode plays a special role only if there exist values such that the function f is small (relative to 1). This would happen near a phase transition, but in an effective two-dimensional theory a phase transition is impossible for $N > 2$.

The large N limit. We now specialize the results of section 7.2, and verify consistency with the general RG analysis.

Broken symmetry phase. In the broken symmetry phase $m_T = 0$, G_2 is given by Eq. (7.17):

$$G_2(0,T) = \rho_c + N_{d-1}\Gamma(d-2)\zeta(d-2)T^{d-2}.$$

We have already noted that a phase transition for $T > 0$ is possible only for $d > 3$ when the result is IR finite.

Then, using the last equation $U'(\rho) = 0$ in the example of the $(\phi^2)^2$ theory, one obtains

$$\sigma^2/N = \frac{6}{Nu}(r_c - r) - N_{d-1}\Gamma(d-2)\zeta(d-2)T^{d-2}$$

and finds a critical temperature

$$T_c \propto \Lambda\big((r_c - r)/\Lambda^2\big)^{1/(d-2)}.$$

The value of T_c can be compared with $m_{\rm cr}$, the mass scale at which a crossover between critical and Goldstone behaviours occurs.

For $d > 4$,

$$T_c \propto \Lambda(m_{\rm cr}/\Lambda)^{2/(d-2)} \gg m_{\rm cr}$$

(from Eq. (2.67)) and, thus, the temperature T_c is large compared to the relevant mass scale.

For $d = 4$, the explicit value is

$$T_c = (72/Nu)^{1/2}(r_c - r)^{1/2} \propto m_{\rm cr}\sqrt{\ln(\Lambda/m_{\rm cr})},$$

where Eq. (2.66) has been used. In all cases the critical temperature is large compared to the crossover mass.

The symmetric phase. The saddle point equations become

$$\begin{aligned} \rho - \rho_c &= \Omega_d(m_T) - \Omega_d(0) + T^{d-2} f_d(m_T^2/T^2), \\ m_T^2 &= (Nu/6)(\rho - \rho_c) + r - r_c\,. \end{aligned}$$

In the special limit $r = r_c$, which corresponds to the critical point at zero temperature, Eq. (7.22) now reads

$$(6/Nu)m_T^2 = \Omega_d(m_T) - \Omega_d(0) + T^{d-2} f_d(m_T^2/T^2).$$

The behaviour of m_T/T has already been discussed in section 7.2. The special dimension $d = 4$ will be examined in more detail in section 7.4 in the context of variational calculations, and the general large N limit again in section 7.5 from the point of view of the non-linear σ model.

7.4 Variational calculations in the $(\phi^2)^2$ theory

To further compare variational calculations with large N results, we generalize the zero-temperature calculations of section 2.8 to a finite temperature system. The variational principle is still based on the inequality (2.83), but is applied to a finite temperature system $T = 1/\beta$. For a system of N non-interacting massive

bosons at temperature T in $d-1$ space dimensions, the Helmholtz free energy density is given by

$$\mathcal{F}_0 = -\frac{1}{V\beta}\ln\mathcal{Z}_0 = -\frac{1}{V\beta}\ln\operatorname{tr}\mathrm{e}^{-\beta H_0}$$
$$= N\frac{1}{(2\pi)^{d-1}}\int \mathrm{d}^{d-1}k\left(\tfrac{1}{2}\omega(k) + T\ln\left(1-\mathrm{e}^{-\beta\omega(k)}\right)\right) \quad (7.30)$$

The inequality (2.84) is replaced by

$$\mathcal{F} \leq \mathcal{F}_{\text{var.}} = \mathcal{F}_0 + N\left\langle U\big(\phi^2(x)/N\big) - \tfrac{1}{2}m_T^2\big(\phi(x)-\phi_0\big)^2/N\right\rangle_0 , \quad (7.31)$$

where $\langle\bullet\rangle_0$ means average with respect to $\mathrm{e}^{-\mathcal{S}_0}$ in a finite temperature, infinite space volume system, and $\mathcal{S}_0$ again is the trial free action:

$$\mathcal{S}_0 = \int \mathrm{d}^d x\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m_T^2}{2}\big(\phi(x)-\phi_0\big)^2\right),$$

m_T^2, $\phi_0^2 = \sigma^2$ (in the notation of section 7.2) being the two variational parameters. We now evaluate the gaussian average in the large N limit:

$$\big\langle U\big(\phi^2(x)/N\big)\big\rangle_0 \sim U\big(\big\langle\phi^2(x)/N\big\rangle_0\big),$$

and introduce the parameter ρ, which is given by

$$\rho = \big\langle\phi^2(x)/N\big\rangle_0 = \phi_0^2/N + \frac{T}{(2\pi)^{d-1}}\sum_{n\in\mathbb{Z}}\int\frac{\mathrm{d}^{d-1}k}{(2\pi nT)^2+k^2+m_T^2},$$
$$= \sigma^2/N + G_2(m_T,T). \quad (7.32)$$

Then, the variational free energy density becomes

$$\mathcal{F}_{\text{var}} = \mathcal{F}_0 + NU(\rho) - \tfrac{1}{2}m_T^2(N\rho-\sigma^2). \quad (7.33)$$

The relation

$$\frac{\partial\mathcal{F}_0}{\partial m_T^2} = \tfrac{1}{2}(N\rho-\sigma^2),$$

still holds, in such a way that

$$\frac{\partial\mathcal{F}_{\text{var.}}}{\partial m_T^2} = N\frac{\partial\rho}{\partial m_T^2}\left[U'(\rho) - \tfrac{1}{2}m_T^2\right]. \quad (7.34)$$

The condition of stationarity of $\mathcal{F}_{\text{var.}}$ with respect to variations of m_T^2, together with the equation defining ρ, are again equivalent to the large N saddle point equations.

The $(\phi^2)^2$ field theory in dimension $d = 4$. We now discuss the example of the $(\phi^2)^2$ field theory in four dimensions:

$$U(\rho) = \tfrac{1}{2} r\rho + \frac{u}{4!}\rho^2.$$

From Eqs. (7.19,2.19), one finds

$$\rho(m_T^2) = \rho_c + \sigma^2/N - \frac{m_T^2}{8\pi^2}\ln(\Lambda/m_T) + \frac{T^2}{2} f_4(m_T^2/T^2), \tag{7.35}$$

where $f_d(z)$ is given in Eq. (7.20). The variational free energy of the system can be obtained from Eq. (7.33) or by integrating Eq. (7.34):

$$\begin{aligned}\frac{1}{N}\mathcal{F}_{\rm var}(\sigma, m_T^2) &= U(\rho) - \tfrac{1}{2}\int_0^{m_T^2} \mathrm{d}s\; s\frac{\partial\rho(s)}{\partial s} \\ &= \frac{1}{32\pi^2} m_T^4 \ln(\Lambda/m_T) + \frac{1}{4}T^4 \int_{m_T^2/T^2}^{\infty} \mathrm{d}z\, z f_4'(z) \\ &+ \frac{u}{24}\left\{\sigma^2/N + 6\frac{r - r_c}{u} - \frac{1}{8\pi^2} m_T^2 \ln(\Lambda/m_T) + \frac{1}{2}T^2 f_4(m_T^2/T^2)\right\}^2 \\ &- \frac{3}{2}\frac{r^2}{u}\,. \end{aligned} \tag{7.36}$$

The renormalized theory with finite non-zero renormalized coupling $u_{\rm r}$ can be reached only with a negative u in a metastable state mentioned in section 2.8. In the limit $\Lambda \to \infty$, $u \to 0^-$ and the renormalized free energy can be calculated in terms of the renormalized parameters $u_{\rm r}, m_T$ and using the gap equation. One finds

$$\begin{aligned}\frac{1}{N}\mathcal{F}_{\rm var}(\sigma, m_T^2) + \frac{3}{2}\frac{r^2}{u} = \frac{m_T^2}{2}(\sigma^2/N \pm 6M^2/u_{\rm r}) - \frac{r^2}{4}\left\{\frac{1}{u_{\rm r}} + \frac{1}{16\pi^2}\ln\left(\frac{\mathrm{e}^{\frac{3}{2}}\, M^2}{m_T^2}\right)\right\} \\ - \frac{1}{4}T^4 \int_{m_T^2/T^2}^{\infty} \mathrm{d}z\, f_4(z). \end{aligned} \tag{7.37}$$

Here M is a normalization scale and the choice $\pm$ sign depends on the sign of $M^2/u_{\rm r}$. The basic instability that originates from the negatively coupled theory is maintained also in the renormalized expression in Eq. (7.37). One finds that as the temperature increases, the $O(N)$ symmetric metastable state that exists at low temperature becomes even less stable, and above a certain critical temperature, disappears altogether. The theory now has only the $O(N)$ broken symmetry phase that we have found; it is unstable, as mentioned at the bottom of section 2.8, and thus renders the theory inconsistent.

7.5 The non-linear σ model at finite temperature

We now discuss another, related, example: the non-linear σ model because the presence of Goldstone modes introduces some new features in the analysis. In the perturbative framework, due the non-linear character of the group representation, one is confronted with difficulties which also appear in non-abelian gauge theories. Moreover, the non-linear σ model and non-abelian theories share another property: both are UV asymptotically free in the dimensions in which they are renormalizable.

Finally, we recall that it has been proven in section 3.1, within the framework of the $1/N$ expansion, that the non-linear σ model is equivalent to the $((\phi^2))^2$ field theory (at least for generic ϕ^4 coupling), both quantum field theories corresponding to two different perturbative expansions of the same physical model.

The non-linear σ model. The non-linear σ model has been studied at zero temperature in section 3.1. It is an $O(N)$ symmetric QFT, with an N-component scalar field $\mathbf{S}(t,x)$ which belongs to a sphere, that is that satisfies the constraint $\mathbf{S}^2(t,x)=1$.

The partition function of the non-linear σ model can be written as

$$\mathcal{Z} = \int [\mathrm{d}\mathbf{S}(t,x)\mathrm{d}\lambda(t,x)] \exp\left[-\mathcal{S}(\mathbf{S},\lambda)\right] \tag{7.38}$$

with

$$\begin{aligned}\mathcal{S}(\mathbf{S},\lambda) = \frac{1}{2g}\int_0^{1/T} \mathrm{d}t\, \mathrm{d}^{d-1}x \Big[&\big(\partial_t \mathbf{S}(t,x)\big)^2 \\ &+ \big(\nabla \mathbf{S}(t,x)\big)^2 + \lambda(t,x)\big(\mathbf{S}^2(t,x)-1\big)\Big],\end{aligned} \tag{7.39}$$

where the λ integration runs along the imaginary axis and enforces the constraint $\mathbf{S}^2(x)=1$. The parameter g is the coupling constant of the quantum model as well as the temperature of the corresponding d-dimensional classical theory.

As we have already explained, a finite temperature T leads, in the corresponding classical theory, to one finite size $\beta = 1/T$ with periodic boundary conditions.

Finite temperature saddle point equations. The non-linear σ model has been discussed in the large N limit at zero temperature in section 3.1 with a slightly different notation ($T \to g$). At finite temperature, the saddle point equation (3.8a) remains unchanged. The saddle point equation (3.8b) is modified because the frequencies in the time direction are quantized. It can be expressed in terms of the function

$$G_2(m_T,T) = \frac{T}{(2\pi)^{d-1}} \sum_{n\in\mathbb{Z}} \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi n T)^2 + k^2 + m_T^2}$$

defined earlier (Eq. (7.15)). In the symmetric phase $\langle \mathbf{S}(T) \rangle = 0$, one finds

$$1 = (N-1)\, g\, G_2(m_T, T). \tag{7.40}$$

Here, m_T is the thermal mass and $\xi_T = m_T^{-1}$ has the meaning of a correlation length in space directions.

In terms of the functions (2.12,7.20), Eq. (7.40) (for N large) can be rewritten as

$$1/Ng = \Omega_d(m_T) + T^{d-2} f_d(m_T^2/T^2). \tag{7.41}$$

One recovers that a phase transition is possible only if $f_d(0)$ is finite, which from Eq. (7.21) implies $d > 3$, a result that can be seen as a consequence of dimensional reduction.

Dimension $d = 2$. We first examine the case $d = 2$. This corresponds to a situation where even at zero temperature the $O(N)$ symmetry remains always unbroken. In the zero temperature QFT, or the infinite volume classical statistical system, the continuum limit corresponds to $g \ll 1$ and the physical mass m then is given by Eq. (3.16):

$$1/Ng = \Omega_2(m) \;\Rightarrow\; m = \xi_0^{-1} \propto \Lambda\, \mathrm{e}^{-2\pi/Ng}\,.$$

By subtracting this equation from Eq. (7.41) (the finite temperature gap equation), one finds

$$\ln(m_T/m) = \ln(\xi_0/\xi_T) = 2\pi f_2(m_T^2/T^2). \tag{7.42}$$

High temperature corresponds to $T/m \gg 1$ and thus one also expects $m_T \gg m$. The integral (7.20) then is dominated by the contribution of the zero-mode and, therefore,

$$\frac{T}{m_T} = \frac{1}{\pi} \ln(m_T/m) \sim \frac{1}{\pi} \ln(T/m). \tag{7.43}$$

The logarithmic decrease of the ratio m_T/T at high temperature corresponds to the UV asymptotic freedom of the classical non-linear σ model in two dimensions.

Dimensions $d > 2$. In higher dimensions the system has a phase transition for $T = 0$ at a value g_c of the coupling constant. The gap equation can then be rewritten as

$$\frac{1}{Ng} - \frac{1}{Ng_c} = \Omega_d(m_T) - \Omega_d(0) + T^{d-2} f_d(m_T^2/T^2). \tag{7.44}$$

For $g > g_c$, the equation can also be expressed in terms of the physical mass m (Eq. (3.12)) as

$$\left[\Omega_d(m) - \Omega_d(m_T)\right]/T^{d-2} = f_d(m_T^2/T^2). \tag{7.45}$$

The behaviour of the system then depends on the ratio T/m. To obtain more explicit results, one has to distinguish between various possible dimensions.

Dimension $d = 3$ [96,155]. The absence of phase transition in two dimensions prevents a phase transition at finite temperature. The gap equation has a scaling form, as predicted by finite size RG arguments. A short calculation yields

$$f_3(s) = -\frac{1}{2\pi} \ln\left(1 - \mathrm{e}^{-\sqrt{s}}\right)$$

and

$$\Omega_3(m_T) - \Omega_3(0) = -\frac{m_T}{4\pi}\,.$$

For $g > g_c$, after some simple algebra, the gap equation can be written as

$$2\sinh(m_T/2T) = \mathrm{e}^{m/2T}\,.$$

One verifies that for m/T large (low temperature) $m_T \to m$, and at high temperature $T \gg m$, m_T becomes proportional to T:

$$m_T/T \sim 2\ln\bigl((1+\sqrt{5})/2\bigr).$$

For $g < g_c$, that is when the symmetry is broken at zero temperature, one has to return to the general form

$$2\sinh(m_T/2T) = \exp\left[-\frac{2\pi}{NT}\left(\frac{1}{g} - \frac{1}{g_c}\right)\right]. \tag{7.46}$$

One can also introduce the mass scale

$$m_{\mathrm{cr}}(g) = \frac{1}{g} - \frac{1}{g_c}$$

(see Eq. (2.65)), which at zero temperature characterizes the crossover between critical and Goldstone mode behaviours. Then,

$$2\sinh(m_T/2T) = \mathrm{e}^{-2\pi m_{\mathrm{cr}}/NT}\,.$$

For $g < g_c$, the zero-mode dominates if the ratio m_T/T is small and thus if m_{cr}/T is large. This condition is realized for all temperatures if $|g - g_c|$ is not small because then $m_{\mathrm{cr}} = O(\Lambda) \gg T$: this is the situation of chiral perturbation theory, and corresponds to the deep IR (perturbative) region where only Goldstone particles propagate. It is also realized in the critical domain $|g - g_c|$ small, if $T \ll m_{\mathrm{cr}}$, that is at low (but non-zero) temperature. Then,

$$m_T \sim T\,\mathrm{e}^{-2\pi m_{\mathrm{cr}}/NT} = T\exp\left[-\frac{2\pi}{NT}\left(\frac{1}{g} - \frac{1}{g_c}\right)\right]. \tag{7.47}$$

Note that, when the coupling constant g or the temperature T go to zero, the mass m_T has the exponential behaviour characteristic of the dimension 2. This property of dimensional reduction at low temperature is somewhat surprising. It is in fact a precursor of the symmetry breaking at zero temperature.

Higher dimensions: the critical temperature. For $d > 3$, the quantity $f_d(0)$ is finite and, therefore, a phase transition at finite temperature is possible, in agreement with dimensional reduction and the property that a phase transition is possible in dimensions larger than two (in the case of continuous symmetries). From Eq. (7.44) one infers

$$T_c^{d-2} = \frac{1}{N f_d(0)} \left(\frac{1}{g} - \frac{1}{g_c} \right). \tag{7.48}$$

Since $f_d(0)$ is positive, this result confirms that a transition is possible only for $g < g_c$, that is if at zero temperature the symmetry is broken.

However, this result is meaningful only if $T \ll \Lambda$ and thus only for $|g - g_c|$ small. Then, T_c is proportional to the crossover mass scale $m_{\rm cr}(g)$ (Eq. (2.65)) between critical and Goldstone behaviours.

Dimension $d = 4$. Since $f_4(0) = 1/12$ one finds

$$T_c^2 = \frac{12}{N} \left(\frac{1}{g} - \frac{1}{g_c} \right) = \frac{12}{N} m_{\rm cr}^2. \tag{7.49}$$

Another limit of interest is the high temperature or massless limit. For $m_T \neq 0$, one finds an additional cut-off dependence:

$$\Omega_4(m_T) - \Omega_4(0) \sim -\frac{m_T^2}{8\pi^2} \ln(\Lambda/m_T).$$

At $g = g_c$, one finds that m_T/T decreases logarithmically with the cut-off. At leading order, using $f_4(0) = 1/12$, one obtains

$$(m_T/T)^2 = \frac{2\pi^2}{3\ln(\Lambda/T)},$$

in agreement with Eq. (7.24).

Dimension $d = 5$. From $f_5(0) = \zeta(3)/4\pi^2$, one infers the critical temperature T_c:

$$T_c^3 \sim \frac{4\pi^2}{N\zeta(3)} \left(\frac{1}{g} - \frac{1}{g_c} \right) = \frac{4\pi^2}{N\zeta(3)} m_{\rm cr}^3 \,.$$

In the massless limit $g = g_c$,

$$(m_T/T)^2 \sim \frac{\zeta(3)}{4\pi^2} \frac{T}{D_5(0)} \tag{7.50}$$

with $D_5(0) \propto \Lambda$, a result consistent with the behaviour (7.23).

7.6 The Gross–Neveu model at finite temperature

To gain some intuition about the role of fermions at finite temperature, we now examine a simple model of self-interacting fermions, the Gross–Neveu (GN [157,158]. The GN model is described in terms of a $U(\tilde{N})$ symmetric action for a set of $\tilde{N}$ massless Dirac fermions $\{\psi^i, \bar{\psi}^i\}$ (for details see section 4.3):

$$\mathcal{S}(\bar{\psi}, \psi) = -\int \mathrm{d}t\, \mathrm{d}^{d-1}x \left[\bar{\psi}(t,x) \cdot \partial\!\!\!/ \psi(t,x) + \tfrac{1}{2N} G \left(\bar{\psi}(t,x) \cdot \psi(t,x)\right)^2\right], \quad (7.51)$$

where $N = \tilde{N}\,\mathrm{tr}\,\mathbf{1}$ is the total number of fermion components.

The GN model has in all dimensions a discrete symmetry that prevents the addition of a mass term. In even dimensions it corresponds to a discrete chiral symmetry, and in odd dimensions to space reflection. Below, to simplify, we will speak about chiral symmetry, irrespective of dimensions.

The GN model is renormalizable in $d = 2$ dimensions, where it is asymptotically free and the chiral symmetry is always broken at zero temperature.

It has been proven in section 4.6 that within the $1/N$ expansion the GN model is equivalent to the GNY (Y for Yukawa) model, at least for generic couplings: the GNY model has the same symmetry, but contains an elementary scalar particle coupled to fermions through a Yukawa-like interaction, and is renormalizable in four dimensions. This equivalence provides a simple interpretation to some of the results that follow.

At finite temperature, due to the anti-periodic boundary conditions, fermions have no zero modes. Therefore, limited insight about the physics of the model at high temperature can be gained from perturbation theory; all fermions are simply integrated out. Instead, we study here the GN model within the framework of the $1/N$ expansion.

Effects due to the addition of a chemical potential will not be considered here.

The integration over fermions, leads to an effective non-local action for a periodic scalar field σ, which has already been discussed in the zero temperature limit in section 4.6:

$$\mathcal{S}_N(\sigma) = \frac{N}{2G} \int_0^{1/T} \mathrm{d}t \int \mathrm{d}^{d-1}x\, \sigma^2(t,x) - \tilde{N}\,\mathrm{tr}\ln\left(\partial\!\!\!/ + \sigma(\cdot)\right), \quad (7.52)$$

where T is the temperature, and the σ field satisfies periodic boundary conditions in the euclidean time variable.

In the situations in which the thermal mass of the σ field is small compared with the temperature, one can perform a mode expansion of the σ-field, integrate over the non-zero modes and obtain an effective local $(d-1)$-dimensional action for the zero-mode. It is important to realize that for $T > 0$, since the reduced action is local and symmetric in $\sigma \mapsto -\sigma$, it describes the physics of the Ising transition with *short range interactions*, unlike what happens at zero temperature

where the fermions are massless. The question that then arises is the possibility of a breaking of this remaining reflexion symmetry.

As we have seen, a non-trivial perturbation theory is obtained by expanding for large N. If a continuous order phase transition occurs at finite temperature, IR divergences generated by the σ zero-mode will appear in the $1/N$ perturbation theory at the transition temperature T_c for $d-1<4$. Below T_c, in the same way as at zero temperature, the decay of σ correlation functions in space directions is characterized by the saddle point value M_T of the field $\sigma(x)$. Above T_c, the correlation length is also finite in contrast with the zero temperature situation.

One then finds two regimes, which have to be dealt with differently. Near the transition temperature, $1/N$ perturbation theory is not useful for $d<5$. Instead, one has to perform a mode expansion of the σ-field and a local expansion of the dimensionally reduced action for the σ zero-mode. The effective field theory relevant for long distance properties is of σ^4 type (as in the case of the Ising model) with coefficients depending on coupling constant and temperature, which has to be studied by the usual RG methods. Note that this implies the absence of phase transition for $d=2$ at finite temperature. In the other regime where the σ correlation length is of order $1/T$, all modes are similar and $1/N$ perturbation theory is directly applicable.

7.7 The gap equation

Calling M_T the saddle point value of the field $\sigma(x)$, we obtain the action density at finite temperature and large N:

$$\frac{1}{N}\mathcal{F}(\rho, M_T) = \frac{M_T^2}{2G} - \frac{T}{V_{d-1}\,\mathrm{tr}\,\mathbf{1}}\,\mathrm{tr}\ln\bigl(\not\partial + M_T\bigr), \tag{7.53}$$

where V_{d-1} is the $d-1$ dimension volume and

$$\frac{1}{V_{d-1}}\,\mathrm{tr}\ln(\not\partial + M_T) = \frac{1}{2}\,\mathrm{tr}\,\mathbf{1}\int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\sum_{n\in\mathbb{Z}}\ln(\omega_n^2 + k^2 + M_T^2)$$

with $\omega_n = (2n+1)\pi T$. The sum over frequencies follows from the identity ($A2.9$), and one obtains

$$\frac{1}{V_{d-1}\,\mathrm{tr}\,\mathbf{1}}\,\mathrm{tr}\ln(\not\partial + M_T) = \int\frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\ln\bigl[2\cosh\bigl(\omega(k)/2T\bigr)\bigr] \tag{7.54}$$

with $\omega(k) = \sqrt{k^2 + M_T^2}$. Alternatively, one could use Schwinger's representation of the propagator and another function of elliptic type

$$\vartheta_1(s) = \sum_n \mathrm{e}^{-(n+1/2)^2\pi s}\,. \tag{7.55}$$

The gap equation at finite temperature, obtained by differentiating $\mathcal{F}$ with respect to M_T, again splits into two equations $M_T = 0$ and

$$1/G = \mathcal{G}_2(M_T, T) \tag{7.56}$$

with

$$\mathcal{G}_2(M_T, T) = \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \frac{1}{\omega(k)} \left(\frac{1}{2} - \frac{1}{\mathrm{e}^{\omega(k)/T} + 1} \right). \tag{7.57}$$

This is the fermion analogue of Eqs. (7.40,7.15) in which one recognizes again the sum of quantum and thermal contributions. Note, however, that the function $\mathcal{G}_2(M_T, T)$ has a regular expansion in M_T^2 at $M_T = 0$.

In terms of the function

$$\begin{aligned} g_d(s) &= N_{d-1} \int_0^{\infty} \frac{x^{d-2}\mathrm{d}x}{\sqrt{x^2+s}} \frac{1}{\exp[\sqrt{x^2+s}] + 1} \\ &= N_{d-1} \int_{\sqrt{s}}^{\infty} (y^2 - s)^{(d-3)/2} \frac{\mathrm{d}y}{\mathrm{e}^y + 1}, \end{aligned} \tag{7.58}$$

where N_d is given in (2.14*b*), the gap equation can be rewritten as

$$1/G = \Omega_d(M_T) - T^{d-2} g_d(M_T^2/T^2). \tag{7.59}$$

If $d > 2$, one can introduce the critical value G_c where $M \equiv M_{T=0}$ vanishes at zero temperature:

$$\frac{1}{G} - \frac{1}{G_c} = \Omega_d(M_T) - \Omega_d(0) - T^{d-2} g_d(M_T^2/T^2). \tag{7.60}$$

Finally, for $G > G_c$, the equation can be expressed in terms of the fermion physical mass $M \equiv m_\psi$ solution of Eq. (4.12),

$$1/G = \Omega_d(M), \tag{7.61}$$

and then reads

$$\Omega_d(M_T) - \Omega_d(M) = T^{d-2} g_d(M_T^2/T^2). \tag{7.62}$$

The σ two-point function. Since the correlation length ξ_σ of the σ zero-mode plays a crucial role, we also calculate the σ two-point function $\Delta_\sigma(p) \equiv \Delta_\sigma(p_0 = 0, p)$ (see Eq. (4.55)). Note that in what follows we use the notation m_σ for the corresponding temperature-dependent thermal mass: $m_\sigma(T) = 1/\xi_\sigma(T)$, which is also the mass of the σ field in the dimensionally reduced theory.

For $M_T = 0$, one finds

$$\frac{1}{N\Delta_\sigma(p)} = \frac{1}{G} - \mathcal{G}_2(0, T) + \frac{T}{2(2\pi)^{d-1}} p^2 \sum_n \int^{\Lambda} \frac{\mathrm{d}^{d-1}k}{(k^2 + \omega_n^2)\left[(p+k)^2 + \omega_n^2\right]}. \tag{7.63}$$

For $d > 2$, the propagator can be expressed in terms of the constant $g_d(0)$:

$$\frac{1}{N\Delta_\sigma(p)} = \frac{1}{G} - \frac{1}{G_c} + T^{d-2} g_d(0) + \frac{T}{2(2\pi)^{d-1}} p^2 \sum_n \int^\Lambda \frac{\mathrm{d}^{d-1}k}{(k^2+\omega_n^2)\left[(p+k)^2+\omega_n^2\right]}. \tag{7.64}$$

For $M_T \neq 0$, using the gap equation, one can write the propagator as

$$\frac{1}{N\Delta_\sigma(p)} = \frac{T\left(p^2+4M_T^2\right)}{2(2\pi)^{d-1}} \sum_n \int^\Lambda \frac{\mathrm{d}^{d-1}k}{(k^2+\omega_n^2+M_T^2)\left[(p+k)^2+\omega_n^2+M_T^2\right]}. \tag{7.65}$$

Therefore, when the symmetry is broken the mass $m_\sigma(T) = 2M_T$, generalizing the zero temperature result.

More detailed properties require distinguishing between dimensions.

Local expansion. When the σ thermal mass or expectation value is small compared with T, one can perform a mode expansion of the field σ, retaining only the zero mode and then a local expansion of the action (7.52), and study it to all orders in the $1/N$ expansion. In the reduced theory, T plays the role of a large momentum cut-off.

The first terms of the effective $(d-1)$-dimensional action are

$$\mathcal{S}_{d-1}(\sigma) = N \int \mathrm{d}^{d-1}x \left[\tfrac{1}{2} Z_\sigma (\partial_\mu \sigma)^2 + \tfrac{1}{2} r\sigma^2 + \frac{1}{4!} u\sigma^4\right], \tag{7.66}$$

where terms of order σ^6 and $\partial^2\sigma^4$ and higher have been neglected. The three parameters are given by

$$Z_\sigma = \tfrac{1}{2}\mathcal{G}_4(0,T)/T\,, \quad r = \left[1/G - \mathcal{G}_2(0,T)\right]/T\,, \quad u = 6\mathcal{G}_4(0,T)/T\,,$$

where $\mathcal{G}_4(m_T,T)$ can be calculated from

$$\mathcal{G}_4(m_T,T) = -\frac{\partial}{\partial m_T^2}\mathcal{G}_2(m_T,T).$$

For $d < 4$, $\mathcal{G}_4(0,T)$ is finite and thus proportional to T^{d-4}. For $d > 2$, after the shift of the coupling constant, one finds

$$rT = \frac{1}{G} - \frac{1}{G_c} - T^{d-2} g_d(0) = g_d(0)\left(T_c^{d-2} - T^{d-2}\right). \tag{7.67}$$

As already explained, the properties of this model are those of the critical ϕ^4 theory and, for $d < 5$, have to be studied by the usual non-perturbative techniques.

7.8 Phase structure for N large

Phase transition and critical temperature. One notes that $\Omega_d(M) - \Omega_d(M_T)$ is always negative and, thus, since Ω_d is a decreasing function, $|M| < |M_T|$. Therefore, when the chiral symmetry is unbroken at zero temperature, Eq. (7.62) has no solution, $M_T = 0$ is the minimum and the $\sigma \mapsto -\sigma$ symmetry is not broken at $N = \infty$.

We now assume a situation of symmetry breaking at zero temperature, which implies $G > G_c$. For $d > 2$,

$$g_d(0) = N_{d-1}\left(1 - 2^{3-d}\right)\Gamma(d-2)\zeta(d-2),$$

is finite and, thus, the gap equation (7.62) has a solution up to a temperature T_c where M_T vanishes:

$$T_c = \left[\frac{1}{g_d(0)}\left(\frac{1}{G_c} - \frac{1}{G}\right)\right]^{1/(d-2)} = \left[\frac{\Omega_d(0) - \Omega_d(M)}{g_d(0)}\right]^{1/(d-2)}. \tag{7.68}$$

Moreover, the $p = 0$ limit of the σ propagator (7.64) confirms that the symmetric phase is stable only if $T > T_c$. Therefore, T_c separates two Ising-like phases, a low temperature phase with symmetry breaking and a symmetric high temperature phase. Since $\mathcal{G}_2(M_T, T)$ is a regular function of M_T at $M_T = 0$, one finds that, near T_c,

$$M_T^2 \propto \Lambda^{4-d}(T_c^{d-2} - T^{d-2}) \propto \Lambda(T_c - T),$$

that is a quasi-gaussian or mean-field behaviour in all dimensions.

More specific results require considering various d dimensions separately, first $d > 2$, the case $d = 2$ requiring a special analysis.

High dimensions. For $d > 4$, the critical temperature scales like

$$T_c \propto M(\Lambda/M)^{(d-4)/(d-2)} \;\Rightarrow M \ll T_c \ll \Lambda\,,$$

and, thus, T_c is physical and large compared with the particle masses.

In the symmetric high temperature phase $T > T_c$, the σ thermal mass (σ inverse correlation length) behaves like

$$m_\sigma^2 \propto T^2(T/\Lambda)^{d-4}\left[1 - (T_c/T)^{d-2}\right],$$

and thus is small with respect to T, justifying dimensional reduction. For $T < T_c$ but $T \gg M$, one finds $|M_T^2 - M^2| \ll T^2$ and the property remains true.

Dimension $d = 4$. In the high temperature symmetric phase, the thermal mass m_σ is given by

$$m_\sigma^2 \propto \frac{1}{\ln(\Lambda/T)}\left(\frac{1}{G} - \frac{1}{G_c} + T^2 g_4(0)\right).$$

The thermal mass m_σ is physical (i.e. $m_\sigma \ll \Lambda$) only for $|1/G - 1/G_c| \ll \Lambda^2$, that is in the critical domain of the zero temperature theory. For $G > G_c$, one can introduce the critical temperature (Eq. (7.68)):

$$m_\sigma^2 \propto \frac{1}{\ln(\Lambda/T)}(T^2 - T_c^2).$$

Eventually, m_σ vanishes as $(T-T_c)^{1/2}$ a typical mean-field or gaussian behaviour, and a phase transition occurs. Eq. (7.68) yields T_c which, expressed in terms of the physical fermion mass M, is given by

$$(T_c/M)^2 \sim \frac{3}{\pi^2}\ln(\Lambda/M).$$

The critical temperature is thus large compared with the physical mass M.

In the broken symmetry phase, for T/M finite, the mass parameter M_T remains close to M and one finds

$$\left(\frac{M_T}{M}\right)^2 = 1 - 8\pi^2 g_4(M^2/T^2)\left(\frac{T}{M}\right)^2 \frac{1}{\ln(\Lambda/M)}\,.$$

Dimension $d = 3$ [96,156]. In the symmetric phase

$$m_\sigma^2 \propto \frac{T}{G} - \frac{T}{G_c} + T^2 g_3(0).$$

The mass parameter m_σ is physical only if $T(1/G - 1/G_c) \ll \Lambda^2$. At the transition coupling constant G_c, one finds as expected $m_\sigma \propto T$.

In the broken symmetry phase since at leading order $\Omega_d(M) - \Omega_d(0) = -M/4\pi$ and

$$g_3(s) = \frac{1}{2\pi}\ln\left(1 + \mathrm{e}^{-\sqrt{s}}\right),$$

the gap equation can be written as

$$2\cosh(M_T/2T) = \mathrm{e}^{M/2T},$$

an equation that has a scaling form expected for $d < 4$ from the correspondence between GN and GNY models, and the existence of an IR fixed point in the latter. The critical temperature is proportional to the fermion mass:

$$T_c/M = \frac{1}{2\ln 2}\,.$$

Dimension $d = 2$ [157,158]. The situation $d = 2$ is doubly special, since at zero temperature chiral symmetry is always broken and at finite temperature the

Ising symmetry is never broken. The GN model is renormalizable and UV free. For N large

$$\beta(G) = -\frac{1}{2\pi}G^2.$$

All masses are proportional to the RG invariant mass scale

$$\Lambda(G) \propto \Lambda \exp\left[-\int^G \frac{\mathrm{d}G'}{\beta(G')}\right].$$

In particular, the fermion physical mass has the form

$$M \propto \Lambda\, \mathrm{e}^{-2\pi/G}.$$

At finite temperature all masses, in the sense of inverse of the correlation length in the space direction, have a scaling property. For example, the σ mass has the form

$$m_\sigma/T = f(M/T).$$

For $T > M$, one can also express the scaling properties by introducing a temperature dependent coupling constant G_T defined by

$$\int_G^{G_T} \frac{\mathrm{d}G'}{\beta(G')} = -\ln(\Lambda/T).$$

At high temperature, G_T decreases like

$$G_T \sim \frac{2\pi}{\ln(T/M)}.$$

One expects, therefore, trivial high temperature physics with weakly interacting fermions.

At high temperature the mass parameter m_σ is proportional to T and, therefore, the zero-mode is not different from other modes. Eventually, it decreases when T approaches $T_c = M/\pi$. At leading order one finds

$$m_\sigma^2 \propto T^2 \ln(\pi T/M). \tag{7.69}$$

This result does not imply a phase transition since, for $m_\sigma/T \ll 1$, dimensional reduction is justified: the statistical system becomes essentially one-dimensional with short-range interactions and thus can have no phase transition. Due to fluctuations the correlation length $1/m_\sigma$ never diverges.

For $T < T_c$, the gap equation becomes

$$\ln(M/M_T) = 2\pi g_2(M_T^2/T^2).$$

The function $g_2(s)$ is positive, which again implies $M_T < M$, goes to ∞ for $s \to 0$ and goes to 0 for $s \to \infty$. At low temperature, M_T/M converges to 1 exponentially in M/T. At high temperature, M_T/T goes to zero and

$$g_2(s) \sim -\frac{1}{4\pi} \ln s \,.$$

The equation implies $M \propto T$ and thus has no solution for $T \to \infty$, but instead solutions at finite temperature, in agreement with Eq. (7.69), which shows that m_σ vanishes for some value $T_c \propto M$. The existence of non-trivial solutions to the gap equation here implies only that the σ potential has degenerate minima, but as a consequence of fluctuations the expectation value of σ nevertheless vanishes.

More precisely, the expansion (7.66) can be applied to the $d = 2$ example. One obtains a simple model in 1D quantum mechanics: the quartic anharmonic oscillator. Straightforward considerations show that the correlation length, inverse of the σ mass parameter, becomes small only when the coefficient of σ^2 is large and negative. This happens only at low temperature where the two lowest eigenvalues of the corresponding quantum hamiltonian are almost degenerate. Then, instantons relate the two classical minima of the potential and restore the symmetry. Calculating the difference between the two lowest eigenvalues of the hamiltonian, one obtains a behaviour of the form

$$m_\sigma/T \propto (\ln M/T)^{5/4}\, \mathrm{e}^{-\mathrm{const.}(\ln M/T)^{3/2}} \,.$$

Again, the property that m_σ/T is small at low temperature is a precursor of the zero temperature phase transition.

7.9 Abelian gauge theories

The presence of gauge fields introduces some new features in finite temperature field theories, because the $O(d)$ space symmetry is explicitly broken. This affects the mode decomposition of the gauge field and quantization. Therefore, we discuss here only QED-like theories, that is abelian gauge fields coupled to fermions with N flavours in $(1, d-1)$ dimensions, because the mode decomposition is gauge invariant and the quantization simpler as we now recall, before describing some explicit large N calculations [160].

Mode expansion. We first describe the effect of a mode expansion on an abelian gauge field A_μ. We set

$$A_\mu(t,x) = B_\mu(x) + Q_\mu(t,x),$$

where $B_\mu(x)$ is the zero-mode and, thus, $Q_\mu(t,x)$ satisfies

$$\int_0^\beta \mathrm{d}t\, Q_\mu(t,x) = 0 \,.$$

Gauge transformations then become

$$\delta A_\mu(t,x) = \partial_\mu \left[\varphi_1(t,x) + \varphi_2(t,x)\right] \tag{7.70}$$

with

$$\delta Q_\mu(t,x) = \partial_\mu \varphi_1(t,x), \quad \delta B_\mu(x) = \partial_\mu \varphi_2(t,x).$$

The constraints on B_μ and Q_μ then imply

$$\varphi_2(t,x) = F(x) + \Omega t\,, \quad \int_0^\beta \mathrm{d}t\, \varphi_1(t,x) = \text{const.}\,, \quad \varphi_1(0,x) = \varphi_1(\beta,x), \tag{7.71}$$

with Ω constant.

The interpretation of this result is simple, Q_μ transforms as a gauge field but the gauge function has no zero-mode, B_i transforms as a $(d-1)$-dimensional gauge field, B_0 behaves like a $(d-1)$-dimensional massless scalar, since a translation of B_0 by a constant Ω leaves the action unchanged.

Quantization. We may quantize by adding to the action a term quadratic in $\partial_\mu A_\mu$. Then,

$$\partial_\mu A_\mu = \partial_t Q_0(t,x) + \partial_i B_i(x) + \partial_i Q_i(t,x),$$

and, therefore,

$$\int_0^\beta \mathrm{d}t (\partial_\mu A_\mu)^2 = \beta\big(\partial_i B_i(x)\big)^2 + \int_0^\beta \mathrm{d}t (\partial_\mu Q_\mu)^2.$$

An integration over the non-zero modes, then leads to a $(d-1)$-dimensional gauge theory, quantized in the same covariant gauge, coupled to a neutral massless scalar field.

Massless QED. If one now considers a massless QED-like theory

$$\mathcal{S}(\bar\psi,\psi,A_\mu) = \int \mathrm{d}^d x \left[\tfrac{1}{4e^2} F_{\mu\nu}^2(x) - \bar\psi(x)\cdot(\not\partial + i\not A)\,\psi(x)\right], \tag{7.72}$$

quantized in the same covariant gauge, one can integrate out the fermions because, due to anti-periodic boundary conditions, they have no zero mode. At one-loop order after fermion integration, the leading local corrections take the same form as in the free case, only coefficients are modified.

Note, however, that one can change the fermion situation by introducing a chemical potential term.

Gauge transformations

$$\psi(t,x) = \mathrm{e}^{i\varphi(t,x)}\,\psi'(t,x),$$

must now preserve the anti-periodicity of the charged fields, and thus be periodic. This implies

$$\varphi(\beta, x) = \varphi(0, x) \pmod{2\pi}.$$

In terms of the decomposition $\varphi = \varphi_1 + \varphi_2$, this condition implies that the parameter Ω in (7.71) is quantized:

$$\Omega = 2\pi n T\,, \quad n \in \mathbb{Z}\,.$$

This affects the transformation of the scalar field B_0:

$$\delta B_0(x) = 2\pi n T$$

and, therefore, the thermodynamic potential is a periodic function of B_0. One consequence of this quantization, then, is that the field B_0, which is massless in the tree approximation, acquires a mass from quantum corrections, as we verify below by explicit calculations.

Temporal gauge. It is also instructive to quantize in the temporal gauge. To change gauges, we make a gauge transformation with a periodic function

$$\varphi(t, x) = \int_0^t \mathrm{d}t\, A_0(t', x) - \frac{t}{\beta} \int_0^\beta \mathrm{d}t\, A_0(t', x).$$

Then, A_0 is reduced to its zero-mode component, and the other modes only appear in the gauge fixing function $\partial_\mu A_\mu$. Integration over these modes reduces it to $\partial_i A_i$, where now only the zero-modes of the other fields A_i contribute. For what concerns zero-modes, the situation is the same as before, while the non-zero modes are now quantized in temporal gauge. This means that one finds $d-1$ families of vector fields with masses $2\pi n T$, $n \neq 0$, quantized in the unitary, and thus non-renormalizable gauge.

Let us confirm that the same result is obtained by quantizing in the temporal gauge $A_0(t, x) = 0$ directly:

$$\mathcal{S}(\bar{\psi}, \psi, A_\mu) = \int \mathrm{d}^{d-1}x\, \mathrm{d}t \left[\tfrac{1}{4e^2} \left(2\dot{A}_i^2 + F_{ij}^2(t, x) \right) - \bar{\psi}(t, x) \cdot (\not{\partial} + i\not{A})\, \psi(t, x) \right]. \tag{7.73}$$

One has to remember that Gauss's law has to be imposed. Therefore, a projector over gauge invariant states has to be introduced in the functional integral. This can be accomplished by imposing periodic, anti-periodic respectively, boundary conditions in the time direction up to a gauge transformation:

$$\begin{aligned} A_i(\beta, x) &= A_i(0, x) - \beta \partial_i \varphi(x), \\ \psi(\beta, x) &= -\,\mathrm{e}^{i\beta\varphi(x)}\, \psi(0, x), \\ \bar{\psi}(\beta, x) &= -\,\mathrm{e}^{i\beta\varphi(x)}\, \bar{\psi}(0, x). \end{aligned}$$

We thus set

$$\begin{aligned} A_i(t,x) &= A_i'(t,x) - t\partial_i\varphi(x), \\ \psi(t,x) &= \mathrm{e}^{it\varphi(x)}\,\psi'(t,x), \\ \bar\psi(t,x) &= \mathrm{e}^{-it\varphi(x)}\,\bar\psi'(t,x), \end{aligned}$$

where the fields A_i', $\psi', \bar\psi'$ now are periodic and anti-periodic, respectively. Two modifications appear in the action:

$$\begin{aligned} \int \mathrm{d}x\mathrm{d}t\,(\partial_t A_i)^2 \;&\mapsto\; \int \mathrm{d}x\mathrm{d}t\,(\partial_t A_i)^2 + \beta\int \mathrm{d}x\,(\partial_i\varphi(x))^2 \\ \int \mathrm{d}x\mathrm{d}t\,\bar\psi(t,x)\gamma_0\partial_t\psi(t,x) &\mapsto \int \mathrm{d}x\mathrm{d}t\,\bar\psi(t,x)\gamma_0(\partial_t + i\varphi(x))\psi(t,x). \end{aligned}$$

Therefore, $\varphi(x)$ is simply the zero-mode of the A_0 component. Not enforcing Gauss's law suppresses this additional mode.

The theory has a $(d-1)$-dimensional gauge invariance with the zero-mode of $A_i(t,x)$ as a gauge field.

Fermion integration and large N calculations. We now consider the action (7.72), and integrate over fermions, to render the N-dependence explicit:

$$\mathcal{S}_N(A_\mu) = \int \mathrm{d}^d x\, \tfrac{1}{4e^2}F_{\mu\nu}^2(x) - \tilde N\,\mathrm{tr}\ln\left(\not\partial + i\not A\right), \tag{7.74}$$

where $\tilde N$ is the number of charged fermions, and below $N = \tilde N\,\mathrm{tr}\,\mathbf{1}$.

The action density as a function of a constant field $\varphi \equiv A_0$ then is given by

$$\frac{1}{N}\mathcal{F}(\varphi) = -\frac{T}{2}\sum_n \frac{1}{(2\pi)^{d-1}}\int \mathrm{d}^{d-1}k\,\ln\Big[k^2 + \big(\varphi + (2n+1)\pi T\big)^2\Big].$$

The sum over n can be performed with the help of the identity (A2.9) and one obtains

$$\frac{1}{N}\mathcal{F}(\varphi) = -\frac{T}{2}\int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\,\ln\big(\cosh(\beta k) + \cos(\beta\varphi)\big). \tag{7.75}$$

One verifies that the difference $\mathcal{F}(\varphi) - \mathcal{F}(0)$ is UV finite and has a scaling form $T^d f(\varphi/T)$. This is not surprising since in the zero temperature limit no gauge field mass and quartic φ potential are generated.

The derivative

$$\frac{1}{N}\mathcal{F}'(\varphi) = \frac{1}{2}\frac{1}{(2\pi)^{d-1}}\sinh(\beta\varphi)\int \frac{\mathrm{d}^{d-1}k}{\cosh(\beta k) + \cos(\beta\varphi)},$$

is negative for $-\pi < \beta\varphi < 0$ and positive for $0 < \beta\varphi < \pi$. The action density has a unique minimum at $\varphi = 0$ in the interval $-\pi < \beta\varphi < \pi$.

A special case is $d = 2$ for which one finds [163]

$$\mathcal{F}'(\varphi) = \tfrac{1}{2}N\varphi/\pi \quad \text{for } |\varphi| < \pi \quad \text{and thus } \mathcal{E}(\varphi) = \tfrac{1}{4}N\varphi^2/\pi\,.$$

Neglecting all φ derivatives, one obtains a contribution to the action $\mathcal{S}_T$:

$$-\tilde{N}\,\mathrm{tr}\ln(\partial\!\!\!/ + iB\!\!\!/) \sim -\frac{1}{2}N\int \mathrm{d}^{d-1}x \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}\,\ln\bigl(\cosh(\beta k) + \cos(\beta\varphi(x))\bigr).$$

The coefficient $K_2(d)$ of $\frac{1}{2}\int \mathrm{d}x\,\varphi^2(x)$ follows:

$$\begin{aligned} K_2(d) &= NN_{d-1}\Gamma(d-1)\left(1 - d^{3-d}\right)\zeta(d-2)T^{d-3} \\ &= N\frac{8}{(4\pi)^{d/2}}\Gamma(d/2)\left(2^{d-3} - 1\right)\zeta(d-2)T^{d-3}. \end{aligned}$$

Discussion. At leading order, the mass term is thus proportional to $eT^{(d-2)/2}$. If e is generic, that is of order 1 at the microscopic scale $1/\Lambda$, then $e \propto \Lambda^{(4-d)/2}$ and the scalar mass m_T is proportional to $(\Lambda/T)^{(4-d)/2}T$. It is thus large with respect to the vector masses for $d < 4$ and small for $d > 4$.

If one takes into account loop corrections, one finds for $d > 4$ a finite coupling constant renormalization $e \mapsto e_{\mathrm{r}}$, and the conclusion is not changed. The zero-mode becomes massive but with a mass small compared with T, justifying mode and local expansions.

For $d = 4$, QED is IR free,

$$\beta_{e^2} = \frac{\tilde{N}}{6\pi^2}e^4 + O(e^6),$$

e_{r} has to be replaced by the effective coupling constant $e(T/\Lambda)$, which is logarithmically small:

$$e^2(T/\Lambda) \sim \frac{6\pi^2}{\tilde{N}\ln(\Lambda/T)},$$

and the scalar mass thus is still small, although only logarithmically:

$$m_\varphi^2 \propto \frac{T^2}{\ln(\Lambda/T)}.$$

The separation between zero and non-zero modes remains justified. High temperature QED shares some properties with high temperature ϕ^4 field theory, and a perturbative expansion for the same reason remains meaningful.

Note that if one is interested in IR physics only, one can in a second step integrate over the massive scalar field φ.

Finally for $d < 4$, one finds an IR fixed point and, therefore, one expects that in massless QED m_T becomes proportional to T and comparable to all other modes, in particular, to all gauge field non-zero modes that become massive vector fields.

8 $O(N)$ supersymmetric models at finite temperature

In this section we would like to find out which of the peculiar properties of the phase structure seen at $T = 0$ in section 6 are maintained at finite temperature and how the phase transition occurs as the temperature, rather than the coupling constant, is varied [121].

Supersymmetry is softly broken at finite temperature as bosons and fermions behave differently when interacting with a heat bath. Breakdown of supersymmetry at finite temperature has attracted much attention since the early interest in supersymmetry and involved much controversy on its consequences, appearance and phase structure [164,165]. Other related issues such as the restoration of broken internal symmetries at finite temperature supersymmetric theories were also debated [166]. More recently, there is a continued interest in the thermodynamics of supersymmetric Yang-Mills theory [167] and temperature effects on the minimal supersymmetric model [168].

8.1 The free energy at finite temperature

Following the notation of section 6 to which we refer for details, we now derive the free energy at finite temperature. The partition function is given by

$$\mathcal{Z} = \int [\mathrm{d}\Phi][\mathrm{d}\rho][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}(\Phi,L,R)} \tag{8.1}$$

with

$$\mathcal{S}(\Phi, L, R) = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \left\{ \tfrac{1}{2}\bar{\mathrm{D}}\Phi \cdot \mathrm{D}\Phi + NU(R) + L(\theta)\left[\Phi^2(\theta) - NR(\theta)\right]\right\},$$

where Φ, L, R are N-component scalar superfields, parametrized in the form Eq. (6.4):

$$\Phi(\theta, x) = \varphi + \bar{\theta}\psi + \tfrac{1}{2}\bar{\theta}\theta F$$

and (Eqs. (6.14,6.15))

$$L(\theta, x) = M + \bar{\theta}\ell + \tfrac{1}{2}\bar{\theta}\theta\lambda\,, \quad R(\theta, x) = \rho + \bar{\theta}\sigma + \tfrac{1}{2}\bar{\theta}\theta s\,. \tag{8.2}$$

The Grassmann coordinate θ is a two-component Majorana spinor; φ, M and ρ, are N component real scalar fields ψ, ℓ and σ are N component, two-component Majorana spinors F, λ and s are N component auxiliary fields. D is the covariant derivative, $\mathrm{D} = \partial/\partial\bar{\theta} - \not{\partial}\,\theta$, the integration measure is $\mathrm{d}^2\theta = \frac{i}{2}d\theta_2 d\theta_1$ and $\bar{\theta}_\alpha\theta_\beta = \frac{1}{2}\delta_{\alpha\beta}\bar{\theta}\theta$.

Integrating out $N-1$ superfield components of Φ and keeping $\Phi_1 \equiv \phi$ (the scalar component of the superfield ϕ is identified as $\varphi_1 \equiv \varphi$) one finds

$$\mathcal{Z} = \int [\mathrm{d}\phi][\mathrm{d}R][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}_N(\phi,R,L)}, \tag{8.3}$$

where the large N action is

$$\mathcal{S}_N = \int \mathrm{d}^3x\, \mathrm{d}^2\theta \left[\tfrac{1}{2}\bar{\mathrm{D}}\phi\mathrm{D}\phi + NU(R) + L\left(\phi^2 - NR\right)\right] + \tfrac{1}{2}(N-1)\mathrm{Str}\,\ln\left[-\bar{D}D + 2L\right]. \tag{8.4}$$

The two saddle point equations (6.18) are not changed. Only the third equation (6.21) is affected by the boundary conditions due to finite temperature:

$$2L\phi - \bar{D}D\phi = 0\,, \tag{8.5a}$$

$$L - U'(R) = 0\,, \tag{8.5b}$$

$$R - \phi^2/N = \frac{1}{N}\,\mathrm{tr}\,\Delta(k,\theta,\theta), \tag{8.5c}$$

$\Delta(k,\theta,\theta)$ is given by (6.20):

$$\Delta(k,\theta,\theta) = \frac{1}{k^2 + M_T^2 + \lambda} + \bar{\theta}\theta M_T\left(\frac{1}{k^2 + M_T^2 + \lambda} - \frac{1}{k^2 + M_T^2}\right). \tag{8.6}$$

where M_T is the expectation value of $M(x)$ at finite temperature T. When written in components, Eq. (8.5a) implies

$$F - M_T\varphi = 0\ , \tag{8.7a}$$

$$\lambda\varphi + M_T F = 0\,, \tag{8.7b}$$

from which the Goldstone condition $\varphi(\lambda + M_T^2) = 0$ follows.

Eq. (8.5b) implies:

$$M_T = U'(\rho)\ , \quad \lambda = sU''(\rho). \tag{8.8}$$

When calculating the trace in Eq. (8.5c), one has to take into account that bosons at finite temperature satisfy periodic, and fermions anti-periodic boundary conditions. Then, combining Eq. (7.15) with the $\theta = 0$ part of the finite temperature Eq. (8.5c), we obtain

$$\rho - \varphi^2/N = \int \frac{\mathrm{d}^2k}{(2\pi)^2}\frac{1}{\omega_\varphi(k)}\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\omega_\varphi(k)/T} - 1}\right)$$

with

$$\omega_\varphi(k) = \sqrt{M_T^2 + \lambda + k^2}.$$

It is convenient here to introduce the boson thermal mass

$$m_T = \sqrt{M_T^2 + \lambda}\,. \tag{8.9}$$

The thermal mass m_T characterizes the decay of correlation functions in space directions. Note that, on the other hand, M_T does not characterize the decay of fermion correlations, because fermions have no zero mode, the relevant parameter being $\sqrt{M_T^2+\pi^2T^2}$. Then,

$$\rho-\varphi^2/N=\rho_c-\frac{T}{2\pi}\ln\bigl(2\sinh(m_T/2T)\bigr) \tag{8.10}$$

where ρ_c has been defined in Eq. (6.27).

In the same way, the $\bar\theta\theta$ part of Eq. (8.5c) combined with Eq. (7.57) yields

$$\begin{aligned} s-2F\varphi/N=2M_T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\Biggl\{&\frac{1}{2\omega_\varphi(k)}\left(\frac12+\frac{1}{\mathrm{e}^{\omega_\varphi(k)/T}-1}\right)\\ &-\frac{1}{2\omega_\psi(k)}\left(\frac12-\frac{1}{\mathrm{e}^{\omega_\psi(k)/T}+1}\right)\Biggr\}\end{aligned}$$

with $\omega_\psi=\sqrt{M_T^2+k^2}$. Integrating we obtain

$$s-2F\varphi/N=\frac{TM_T}{\pi}\left[\ln\bigl(2\cosh(M_T/2)\bigr)-\ln\bigl(2\sinh(m_T/2T)\bigr)\right]. \tag{8.11}$$

The action density at finite temperature $\mathcal{F}$, to leading order in $1/N$, is given by ($\beta=1/T$)

$$\mathcal{F}=\mathcal{S}_N/V_2\beta\ , \tag{8.12}$$

where $\mathcal{S}_N$ is given in Eq. (8.4) at constant fields and V_2 is the two dimensional volume.

The supertrace term in Eq. (8.4) can be calculated at finite temperature by using the expressions Eqs. (7.13) and (7.54):

$$\begin{aligned}\frac{1}{V_2}\mathrm{Str}\ \ln\left[-\bar DD+2L\right]&=\frac{1}{V_2}\,\mathrm{tr}\ln(-\partial^2+M_T^2+\lambda)-\frac{1}{V_2}\,\mathrm{tr}\ln(\partial\!\!\!/+M_T)\\ &=2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\ln[2\sinh(\beta\omega_\varphi/2)]-2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\ln[2\cosh(\beta\omega_\psi/2)]\\ &=\frac{1}{T}\rho_c(m_T^2-M_T^2)-\frac{1}{6\pi T}\left(m_T^3-|M_T|^3\right)\\ &\quad+2\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\ln[1-\mathrm{e}^{-\beta\omega_\varphi}]-\ln[1+\mathrm{e}^{-\beta\omega_\psi}]\right\}.\end{aligned} \tag{8.13}$$

The explicitly subtracted part reduces for $M_T=M$ to the $T=0$ result. The other contributions to $\mathcal{E}$ are the same (up to the change $M\mapsto M_T$) as in Eq. (6.31) and therefore one finds

$$\begin{aligned}\frac{1}{N}\mathcal{F}=-\frac{F^2}{2N}+M_T\frac{F\varphi}{N}+\lambda\frac{\varphi^2}{2N}+\tfrac12 s(U'(\rho)-M_T)-\frac{1}{12\pi}\left(m_T^3-|M_T|^3\right)\\ +\tfrac12\lambda(\rho_c-\rho)+T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\ln[1-\mathrm{e}^{-\beta\omega_\varphi}]-\ln[1+\mathrm{e}^{-\beta\omega_\psi}]\right\}.\end{aligned} \tag{8.14}$$

In the limit $T = 0$, the free energy in Eq. (8.14) reduces to the action density of Eq. (6.31), $\mathcal{F}(T = 0) \equiv \mathcal{E}$.

Eq. (8.14) can be rewritten by using Eq. (8.10), which is also obtained by setting to zero the $\partial/\partial\lambda$ derivative of Eq. (8.14), as well as $M_T - U'(\rho) = 0$ from Eq. (8.8) (or equivalently, setting to zero the $\partial/\partial s$ derivative of Eq. (8.14)). One finds

$$\frac{1}{N}\mathcal{F} = \tfrac{1}{2}M_T^2\frac{\varphi^2}{N} + \frac{1}{24\pi}(m_T - |M_T|)^2(m_T + 2|M_T|) + \frac{T\lambda}{4\pi}\ln(1 - \mathrm{e}^{-m_T/T}) + T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\{\ln(1 - \mathrm{e}^{-\beta\omega_\varphi}) - \ln(1 + \mathrm{e}^{-\beta\omega_\psi})\}. \quad (8.15)$$

Eq. (8.15) is the finite energy version of Eq. (6.37).

Inserting Eqs. (8.10) and (8.11) (with $F = M_T\varphi$) into Eqs. (8.8) with $U(R) = \mu R + \frac{1}{2}uR^2$, one finds ($\mu_c = -u\rho_c$)

$$M_T = \mu - \mu_c + u\frac{\varphi^2}{N} - \frac{u}{2\pi}T\ln\left(2\sinh\left(\tfrac{1}{2}m_T/T\right)\right), \quad (8.16)$$

$$\frac{m_T^2 - M_T^2}{uM_T} = \frac{2\varphi^2}{N} - \frac{T}{\pi}\left[\ln\left(2\sinh\left(\tfrac{1}{2}m_T/T\right)\right) - \ln\left(2\cosh\left(\tfrac{1}{2}|M_T|/T\right)\right)\right], \quad (8.17)$$

which are the gap equations for $T \neq 0$.

Solutions to saddle point equations: One first notes that the r.h.s. of the gap Eq. (8.16) diverges when $m_T \to 0$. This phenomenon has been already discussed in the example of the scalar field theory. A finite temperature system in three dimensions has the property of a statistical system in two dimensions. Spontaneous breaking of a continuous symmetry is impossible in two dimensions due to the IR behaviour of a system with potential massless Goldstone particles. Therefore, the $O(N)$ symmetry is never broken, $\varphi = 0$, and thus

$$\frac{M_T}{T} = \frac{\mu - \mu_c}{T} - \frac{u}{2\pi}\ln\left(2\sinh\left(\tfrac{1}{2}m_T/T\right)\right), \quad (8.19)a$$

$$\frac{M_T^2 - m_T^2}{uTM_T} = \frac{1}{\pi}\left[\ln\left(2\sinh\left(\tfrac{1}{2}m_T/T\right)\right) - \ln\left(2\cosh\left(\tfrac{1}{2}M_T/T\right)\right)\right]. \quad (8.20)b$$

Note that while the expression of $\mathcal{F}$ is complicated, its derivative with respect to m_T^2 remains simple

$$\frac{1}{N}\frac{\partial\mathcal{F}}{\partial m_T^2} = \frac{(m_T^2 - M_T^2)}{16\pi m_T\tanh(m_T/2T)}. \quad (8.21)$$

Therefore, the minimum still occurs at $m_T = |M_T|$, but one verifies that $m_T = |M_T|$ is not a solution to the saddle point equations. We have only found a lower bound on the free energy density $\mathcal{F}$ as a function of M_T:

$$\mathcal{F} = NT\int\frac{\mathrm{d}^2k}{(2\pi)^2}\ln\tanh\left(\tfrac{1}{2}\beta\sqrt{M_T^2 + k^2}\right). \quad (8.22)$$

Its derivative with respect to M_T is

$$\frac{\partial \mathcal{F}}{\partial M_T} = -N \frac{T M_T}{2\pi} \ln \tanh(|M_T|/2T).$$

Therefore the derivative has the sign of M_T. The lower bound has a limit for $M_T \to 0$, which is thus an absolute lower bound:

$$\mathcal{F} = -\frac{7}{8\pi} \zeta(3) N T^3 \,. \tag{8.23}$$

Similarly, the derivative of $\mathcal{F}$ with respect to M_T at m_T fixed is

$$\begin{aligned} \frac{\partial \mathcal{F}}{\partial M_T} &= \frac{M_T T}{2\pi} \left[\ln\bigl(2\cosh(M_T/2T)\bigr) - \ln\bigl(2\sinh(m_T/2T)\bigr) \right] \\ &= \frac{m_T^2 - M_T^2}{2u} \,. \end{aligned}$$

We now have to find the solutions to the saddle point equations and compare their free energies.

Note a first solution in the regime $T \to 0$, $\mu < \mu_c$ with $|m_T| \ll T$ and $|M_T \ll T$ Then, from (8.16), we find the boson thermal mass

$$m_T \sim T \, \mathrm{e}^{-2\pi(\mu_c - \mu)/uT} \,.$$

Eq. (8.17) yields the other mass parameter

$$M_T \sim \frac{m_T^2}{\mu_c - \mu} \,.$$

Since both m_T and M_T are very small, the free energy is very close to the lower bound (8.23). The solution found here is a precursor of the zero temperature phase transition, and corresponds to $\varphi \neq 0$ in region III and IV of Fig. 6. Other solutions exist in this regime but they converge, up to exponential corrections, to the finite masses of the $T = 0$ spectrum, and thus their free energy is much larger from the lower bound (8.22).

It is expected that even for T larger and $\mu \geq \mu_c$ the continuation of the small mass solution remains the solution with lowest energy.

Variational calculations will produce here similar results to those obtained from the path integral at $N \to \infty$. We know from the general discussion at $T = 0$ in section 6.3 that one of the gap equations $M_T = U'(\rho)$ is immediately implied by demanding that the free energy density in the variational calculations remains finite for large cut-off Λ. Therefore, the variational free energy in Eq. (6.35) (with ρ and $\tilde{\rho}$ of Eqs. (6.34a) and (6.34b) replaced by their thermal expressions) takes the form identical to Eq. (8.15).

m_T and M_T being related, in what follows, we take as the independent variable $m \equiv m_T$ and, using Eq. (8.16) for $M_T = M_T(m, \varphi, \mu, u, T)$, we discuss

$$W(m,T) = \frac{1}{N}\mathcal{F}(m_T = m, M_T(m, \varphi, \mu, u, T), T). \tag{8.24}$$

Since, as described above, at $T \neq 0$ there is no breaking of $O(N)$ symmetry, we will discuss mainly $\varphi = 0$ in Eq. (8.24).

For $T = 0$, Eq. (8.24) results in $\frac{1}{N}\mathcal{F}(m, \varphi^2, T = 0) = \frac{1}{N}\mathcal{E}(m, \varphi^2)$ which has been analyzed in section 6. In particular, see the phase structure in Fig.6 and the interesting degeneracy found in Regions II and IV and exhibited in Figs.8–10.

Rather than changing the value of $\mu - \mu_c$ and the coupling u as done in section 6, we are interested here to see the temperature dependence while the parameters $\mu - \mu_c$ and u are held fixed.

At finite temperature, the effective field theory describes the interactions of the fermions and bosons with the heat bath. This interaction acts like a source of soft breaking of supersymmetry. The short distance behavior is cured at finite temperature in a similar way it happens at $T = 0$. We will see now that the general properties of the transitions as a function of the temperature T have a similar character as the transitions seen at $T = 0$ when the coupling constant was varied.

As seen in Fig. 6, at $T = 0$ there are two regions in the $\{\mu - \mu_c, u\}$ space where the vacua are degenerate. These are (region **II :** $\mu - \mu_c \geq 0$ and $u \geq u_c$ and region **IV :** $\mu - \mu_c \leq 0$ and $u \leq u_c$). Fig. 7 shows the ground state energy ($T = 0$) in region **II** and Fig. 9 shows the ground state energy in region **IV**.

We will discuss first region **(II)** $\mu - \mu_c \geq 0$ and $u \geq u_c$:
Here, as T increases the ground state with the smaller mass ($m_\varphi \equiv m = m_+$) has a lower free energy than the heavier one ($m_\varphi \equiv m = m_- > m_+$) due to a higher entropy contribution, (Fig. 11).

Peculiar transitions can occur in this system. If the system was initially at $T = 0$ in the ground state with a boson mass $m_\varphi = m_-$, it will eventually go, as T is increased, into the state with mass $m_\varphi = m_+$, namely, into the lower mass ground state. This is due to the entropy negative contribution to the free energy forcing the system to favors the lowest mass state. On the other hand, a system that started at a high temperature in the state of low mass $m_\varphi = m_+$ will stay in this state as the system is cooled and will never roll back into the $m = m_-$ high mass phase. Favoring the lowest mass phase as the temperature increases is a general effect that will occur in any physical system that is initially (at $T = 0$) mass degenerated. Here, supersymmetry imply that the m_+ and m_- vacua are at the same energy $E = 0$ at $T = 0$.
Region **(IV)** $\mu \leq 0$ and $u \leq u_c$: Here one finds at $T = 0$ two distinct degenerate phases. One is an ordered phase ($\varphi \neq 0$) with a zero boson and fermion mass,

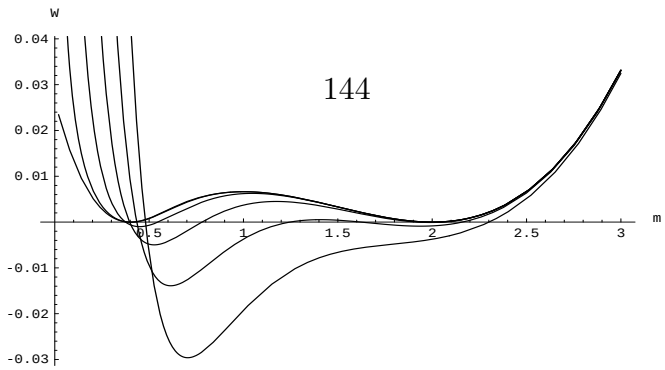

Fig. 11 The free energy $W(m,\varphi,T) = \frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ as given in Eq. (8.15) at $\varphi = 0$ as a function of the boson mass (m) at different temperatures. Here $\mu - \mu_c = 1$ (this sets the mass scale), $u/u_c = 1.5$ and T varies between $T = 0 - 0.5$. At $T = 0$ two degenerate $O(N)$ symmetric phases exist with a light $m = m_+$ and heavier $m = m_- > m_+$ massive boson (and fermion). As the temperature increases the light mass phase is stronger affected as its entropy increases faster and becomes the only ground state.

the other is a symmetric phase ($\varphi = 0$) with a massive ($m = m_-$) boson and fermion. (as seen in Fig. 9).

As mentioned above at $T \neq 0$ the ordered phase with the broken $O(N)$ symmetry, ($m = 0, \varphi^2 \neq 0$) disappears into $\varphi^2 = 0$ symmetric phase and a very small mass ground state $m_\varphi = m \geq 0$ is created (as seen in Fig. 12 and in Fig. 13).

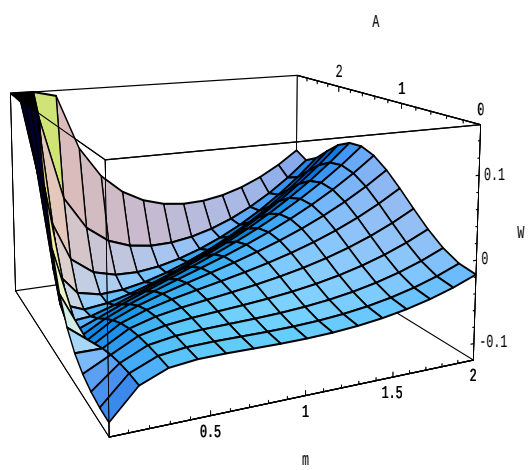

Fig. 12 Same as Fig. 9 in section 6 but the temperature has been increased from $T = 0$ (in Fig. 9) to $T = 0.7$ (here). $W(m,\varphi,T) = \frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ as given in Eq. (8.15) as a function of the boson mass (m) and A, where $A^2 = \varphi^2/u_c$. Here $\mu - \mu_c = -1$ and $u/u_c = 0.2$. A non-degenerate $O(N)$ symmetric ground state ($\varphi = 0$) appears with a very small boson mass (the non-zero mass is not seen here due to the limited resolution of the plot).

Eventually, also the other $O(N)$ symmetric vacua ($m = m_-, \varphi^2 = 0$) goes into the small mass $O(N)$ symmetric ground state.
As in **II**, a system that was initially, at $T = 0$ in the $O(N)$ symmetric phase ($m_\varphi = m_-, \varphi^2 = 0$) will decay into the smaller mass state when the temperature.

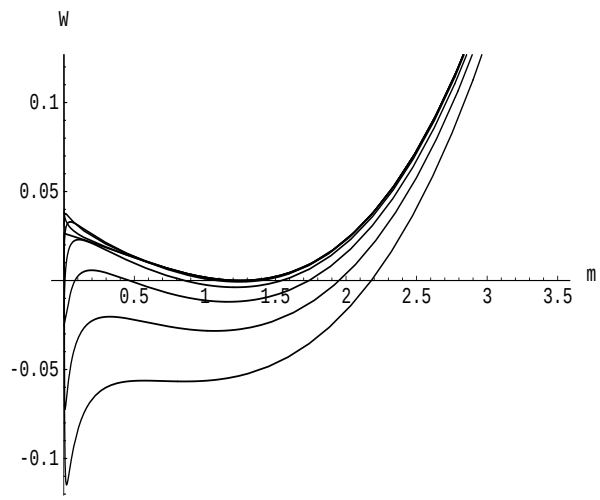

Fig. 13 This figure displays the effect of increasing the temperature from $T = 0$ in Fig. 9 to T that varies in the range $0 \leq T \leq 0.7$ (Fig. 12 has $T = 0.7$). The free energy $W(m, \varphi, T) = \frac{1}{N}\mathcal{F}(m \equiv m_\varphi, \varphi, T)$ as given in Eq. (8.15) at $\varphi = 0$ is plotted as a function of the boson mass (m) at different temperatures. Here $\mu - \mu_c = -1$, $u/u_c = 0.2$. At $T = 0$ there are two degenerate phases: An $O(N)$ symmetric phase, shown here, with a massive ($m = m_-$) boson and fermion and an ordered phase ($\varphi \neq 0$) with massless particles (both phases are shown in Fig. 9). At finite temperatures the $O(N)$ symmetry is restored (see Fig. 12) and a small mass ground state appears, the heavy mass state decays into the small mass ground state as seen here.

But upon cooling the system in the small mass phase will roll into the ordered phase ($m_\varphi = m = 0, \varphi^2 \neq 0$) at $T = 0$. The system will never roll back into the symmetric phase ($m_\varphi = m_-, \varphi = 0$).

8.2 The supersymmetric $O(N)$ non-linear σ model at finite temperature

The supersymmetric non-linear σ model we consider here has already been discussed at zero temperature in section 6.7 to which we refer for details.

The partition function of the $O(N)$ supersymmetric non-linear σ model in d dimensions is given by

$$\mathcal{Z} = \int [\mathrm{d}\Phi][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}(\Phi, L)} \tag{8.25}$$

with

$$\mathcal{S}(\Phi) = \frac{1}{2\kappa} \int \mathrm{d}^d x\, \mathrm{d}^2\theta\, \bar{D}\Phi \cdot D\Phi + L(\Phi^2 - N)\,, \tag{8.26}$$

where κ is a constant. The scalar superfield $\Phi(x, \theta)$ is an N-component vector:

$$\Phi(x, \theta) = \varphi + \bar{\theta}\psi + \tfrac{1}{2}\bar{\theta}\theta F\,, \tag{8.27}$$

and the scalar superfield $L(x,\theta)$ is given by

$$L(x,\theta) = M(x) + \bar{\theta}\ell(x) + \tfrac{1}{2}\bar{\theta}\theta\lambda(x). \tag{8.28}$$

After integrating out $N-1$ superfield components of Φ, leaving out $\phi = \Phi_1$, one obtains

$$\mathcal{Z} = \int [d\phi][\mathrm{d}L]\, \mathrm{e}^{-\mathcal{S}_N(\phi,L)}$$

where

$$\mathcal{S}_N(\phi,L) = \frac{1}{\kappa}\int \mathrm{d}^d x\, \mathrm{d}^2\theta \left[\tfrac{1}{2}\bar{D}\phi\cdot D\phi + L\left(\phi^2 - N\right)\right] + \frac{N-1}{2}\mathrm{Str}\,\ln(-\bar{D}D + 2L). \tag{8.29}$$

Varying the action $\mathcal{S}_N(\phi,L)$ by varying the superfields ϕ and L, one obtains Eq. (6.63) and the generalization of Eq. (6.65):

$$2L\phi - \bar{D}D\phi = 0\,, \tag{8.30a}$$

$$\frac{N}{\kappa} - \frac{\phi^2}{N} = \mathrm{tr}\,\Delta(k,\theta,\theta). \tag{8.30b}$$

The first equation is equivalent to $F = M_T\varphi$ and $\lambda\varphi + FM_T = 0$ while the second equation can be compared with Eq. (8.5c), and thus leads to
(for $N \gg 1$ and $\psi = 0$)

$$\frac{1}{\kappa} - \frac{\varphi^2}{N} = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{\omega_\varphi(k)}\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\omega_\varphi(k)/T} - 1}\right), \tag{8.31a}$$

$$-\frac{2F\varphi}{N} = 2M_T\int \frac{\mathrm{d}^d k}{(2\pi)^d}\left\{\frac{1}{2\omega_\varphi(k)}\left(\frac{1}{2} + \frac{1}{\mathrm{e}^{\omega_\varphi(k)/T} - 1}\right) - \frac{1}{2\omega_\psi(k)}\left(\frac{1}{2} - \frac{1}{\mathrm{e}^{\omega_\psi(k)/T} + 1}\right)\right\} \tag{8.31b}$$

with $\omega_\varphi(k) = \sqrt{m_T^2 + k^2}$ and $\omega_\psi(k) = \sqrt{M_T^2 + k^2}$.

Dimension $d = 3$. As it has been discussed already, the $d = 3$ finite temperature theory is analogous from the point of phase transitions to a two-dimensional theory. Therefore the $O(N)$ symmetry remains unbroken and $\varphi = 0$. We thus write the two gap equations only in this limit

$$\frac{1}{\kappa_c} - \frac{1}{\kappa} = \frac{T}{2\pi}\ln\bigl(2\sinh(m_T/2T)\bigr) \tag{8.32a}$$

$$0 = \frac{TM_T}{2\pi}\left[\ln\bigl(2\sinh(m_T/2T)\bigr) - \ln\bigl(2\cosh(M_T/2T)\bigr)\right]. \tag{8.32b}$$

The calculation of the energy density to leading order for $N \to \infty$ of the non-linear σ model at finite temperature follows the similar steps of the $(\Phi^2)^2$ field theory. The free energy at finite temperature is given by

$$\mathcal{F} = T\mathcal{S}_N/V_2\,, \tag{8.33}$$

where $\mathcal{S}_N$ is given by Eq. (8.29) at constant fields and V_2 is the two dimensional volume. The expression has been calculated in section 6.7 but here the supertrace term in Eq. (8.29) has to be replaced by the finite temperature form as it appears in Eq. (8.14). The free energy then is given by

$$\begin{aligned}\frac{1}{N}\mathcal{F} &= \tfrac{1}{2}(M_T^2 - m_T^2)\left(\frac{1}{\kappa} - \frac{1}{\kappa_c}\right) - \frac{1}{12\pi}\left(m_T^3 - |M_T|^3\right)\\ &\quad + T\int \frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\ln[1-\mathrm{e}^{-\beta\omega_\varphi}] - \ln[1+\mathrm{e}^{-\beta\omega_\psi}]\right\}.\end{aligned} \tag{8.34}$$

The free energy in Eq. (8.34) can be compared with Eq. (6.68) which gives the ground state energy $\mathcal{F}$ at $T = 0$.

In section 6.7 we have verified that after using the zero temperature limit of Eq. (8.32a) the energy densities of the $(\Phi^2)^2$ theory and the non-linear σ model become identical, up to a possible rescaling of the field. It is now clear that the same mechanism works here. Using Eq. (8.32a) one indeed finds the expression (8.15) (for $\varphi = 0$):

$$\begin{aligned}\frac{1}{N}\mathcal{F} &= \frac{1}{24\pi}(m_T - |M_T|)^2(m_T + 2|M_T|) + \frac{T}{4\pi}(m_T^2 - M_T^2)\ln(1-\mathrm{e}^{-m_T/T})\\ &\quad + T\int\frac{\mathrm{d}^2k}{(2\pi)^2}\{\ln(1-\mathrm{e}^{-\beta\omega_\varphi}) - \ln(1+\mathrm{e}^{-\beta\omega_\psi})\}.\end{aligned} \tag{8.35}$$

Solutions. The derivative of $\mathcal{F}$ with respect to $|M_T|$ at m_T fixed is

$$\frac{\partial\mathcal{F}}{\partial|M_T|} = \frac{|M_T|T}{2\pi}\left[\ln\bigl(2\cosh(M_T/2T)\bigr) - \ln\bigl(2\sinh(m_T/2T)\bigr)\right].$$

It is convenient to introduce the notation

$$X(\kappa, T) = \exp\left[\frac{2\pi}{T}\left(\frac{1}{\kappa_c} - \frac{1}{\kappa}\right)\right].$$

Then the behaviour of $\mathcal{F}$ depends on the position of X with respect to 2:

$$X = 2 \;\Leftrightarrow\; T = \frac{2\pi}{\ln 2}\left(\frac{1}{\kappa_c} - \frac{1}{\kappa}\right),$$

an equation that has a solution only for $\kappa > \kappa_c$.

For $X < 2$, the derivative vanishes at $M_T = 0$ and is positive for all $|M_T| > 0$. Instead for $X > 2$, the derivative vanishes both at $M_T = 0$ and

$$|M_T| = 2T \ln \left[\tfrac{1}{2}(X + \sqrt{X^2 - 4}) \right], \tag{8.36}$$

which are the two solutions of Eq. (8.32a). The minimum of $\mathcal{F}$ is located at the second value (8.36).

We find therefore an interesting non-analytic behaviour:

$$\begin{cases} M_T = 0 & \text{for } X < 2, \\ M_T = 2T \ln \left[\tfrac{1}{2}(X + \sqrt{X^2 - 4}) \right] & \text{for } X > 2 \ . \end{cases}$$

Eq. (8.32a) then yields directly the boson thermal mass

$$m_T = 2T \ln \left[\tfrac{1}{2}(X + \sqrt{X^2 + 4}) \right].$$

For $\kappa < \kappa_c$ and $T \to 0$, we find the asymptotic behaviour

$$m_T \sim TX(\kappa, T),$$

which is exponentially small, and $M_T = 0$.

For $\kappa > \kappa_c$ and $T \to 0$, both m_T and M_T converge toward the finite $T = 0$ limit with exponentially small corrections.

In the opposite high temperature limit $T \gg |1/\kappa - 1/\kappa_c|$, we find that m_T is asymptotically proportional to T:

$$m_T \sim 2T \ln\big((1 + \sqrt{5})/2\big),$$

while $M_T = 0$.

It is not clear whether such a result can survive $1/N$ corrections.

Dimension $d = 2$. In generic dimensions $2 \leq d \leq 3$ a phase transition is not even possible at finite temperature and $\varphi = 0$ (in two dimensions it is even impossible at zero temperature). The gap equations take the form

$$\frac{1}{\kappa} = \Omega_d(m_T) + T^{d-2} f_d(m_T^2/T^2), \tag{8.37}$$

$$0 = M_T \left[\Omega_d(|M_T|) - T^{d-2} g_d(M_T^2/T^2) - \Omega_d(m) - T^{d-2} f_d(m_T^2/T^2) \right], \tag{8.38}$$

where f_d and g_d are defined in Eqs. (7.20) and (7.58) respectively.

For $d = 2$ it is then convenient to introduce the physical mass m solution of

$$\frac{1}{\kappa} = \Omega_2(m).$$

The equations can then be rewritten as

$$\Omega_2(m) = \Omega_2(m_T) + f_2(m_T^2/T^2) \tag{8.39}$$

and either $M_T = 0$ or

$$\Omega_2(|M_T|) - g_2(M_T^2/T^2) - \Omega_2(m_T) - f_2(m_T^2/T^2) = 0\,. \tag{8.40}$$

The first equation, which determines m_T, is identical to the Eq. (7.42) already obtained for the usual non-linear σ model. At low temperature $m_T = m$ up to exponentially small corrections. At high temperature

$$\frac{T}{m_T} \sim \frac{1}{\pi}\ln(m_T/m) \sim \frac{1}{\pi}\ln(T/m),$$

a consequence of the domination of the zero mode and the UV asymptotic freedom of the non-linear σ-model.

Combining both gap equations we find

$$\Omega_2(|M_T|) - g_2(M_T^2/T^2) = \Omega_2(m),$$

which is identical to Eq. (7.62).

The analysis of section 7.8 for $d = 2$ has shown that for T large the equation has no solution and thus $M_T = 0$, but it has a solution for T small. The situation therefore is similar to what has been encountered in three dimensions. Again an analysis of LL propagator and $1/N$ corrections is required to understand whether this result survives beyond the large N limit.

9 Weakly interacting Bose gas and large N techniques

As it is well known a free Bose gas undergoes at a low but finite temperature a transition called Bose–Einstein condensation. The condensed low temperature phase is characterized by a macroscopic occupation of the one-particle ground state.

The effect of a weak repulsive two-body interaction on the transition temperature of a dilute Bose gas at fixed density has been controversial for a long time [170-176]. In ref. [177] it has first been argued theoretically that the transition temperature T_c increases linearly with the strength of the interaction, parametrized in terms of the scattering length a. However, the coefficient cannot be obtained from perturbation theory. A simple self-consistent approximation was thus used to derive an explicit estimate.

A better understanding of the physics of the weakly interacting Bose gas came from the recognition that the universal properties of the system under study, like the helium superfluid transition, can be described by a particular example of the general N vector model, for N =2 [178]. Renormalization group (RG) arguments then allowed to prove the existence, besides the universal IR behaviour common with the superfluid transition, of a universal large momentum (UV) behaviour peculiar to systems with a very weak two-body interaction. The long wavelength properties of the weakly interacting Bose gas can be described by the three-dimensional, super-renormalizable, euclidean $(\phi^2)^2$ field theory. A more direct and general RG derivation of the linear behaviour of the shift of the critical temperature and the universality of the coefficient followed.

However, the calculation of the coefficient remained a non-perturbative problem. A possible method to calculate the temperature shift is to generalize the problem to arbitrary N. This generalization makes new tools available; in particular, the coefficient of $\Delta T_c/T_c$ can be calculated by carrying out an expansion in $1/N$. The leading contribution to $\Delta T_c/T_c$ already requires a $1/N$ calculation [178], as we show explicitly below. The result happens to be independent of N, for non-trivial reasons. The calculation involves subtle technical points, which are most easily dealt with by dimensional regularization. More recently, a $1/N^2$ calculation has been performed that yields the $1/N$ correction to $\Delta T_c/T_c$ [179]. The relative correction for $N = 2$ is about 26%, a correction which is smaller than what is typically found in the calculation of critical exponents.

9.1 Quantum field theory and Bose–Einstein condensation

We consider a system of identical non-relativistic bosons of mass m, at temperature $T = 1/\beta$ close to the critical temperature T_c. When the two-body potential V_2 is short-range and one is interested only in long wavelength phenomena, one can approximate the potential by a δ-function pseudo-potential,

$$V_2(x-y) = G\,\delta^{(d)}(x-y),$$

regularized at short distance because such a potential is singular for $d > 1$.

The partition function

$$\mathcal{Z}(\beta, \mu) = \operatorname{tr} \mathrm{e}^{-\beta(H-\mu N)},$$

where H is the hamiltonian in Fock space, N the particle number operator (which commutes with H), and μ the chemical potential, can then be expressed as a functional integral

$$\mathcal{Z} = \int [\mathrm{d}\varphi \mathrm{d}\bar{\varphi}] \exp[-\mathcal{S}(\varphi, \bar{\varphi})], \tag{9.1}$$

where $\mathcal{S}(\varphi, \bar{\varphi})$ is a non-relativistic *local* action:

$$\begin{aligned}\mathcal{S}(\varphi, \bar{\varphi}) = \int_0^\beta \mathrm{d}t \int \mathrm{d}^d x \Bigg[\bar{\varphi}(x,t) \left(-\frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \nabla^2 - \mu \right) \varphi(x,t) \\ + \frac{G}{2} \big(\bar{\varphi}(x,t) \varphi(x,t) \big)^2 \Bigg],\end{aligned} \tag{9.2}$$

and the fields satisfy periodic boundary conditions in the euclidean time direction,

$$\varphi(x, \beta = 1/T) = \varphi(x, 0) \quad \bar{\varphi}(x, \beta = 1/T) = \bar{\varphi}(x, 0).$$

The strength G of the interaction must be positive, corresponding to a repulsive interaction, for the boson system to be stable.

It is customary to parametrize the strength G of the pair potential in terms of the scattering length a. In three dimensions $a = mG/4\pi\hbar^2$. Furthermore, we assume that $a \ll \lambda$, where

$$\lambda = \hbar \sqrt{2\pi / m k_B T}$$

is the thermal wavelength. (In the following we set $k_B = \hbar = 1$, and write simply $\lambda^2 = 2\pi/mT$.)

Note that, unlike what would happen for a fluid, in a very dilute system even though the pair-potential is weak, the N-body potentials, $N > 2$, are even much smaller and can thus be totally neglected.

To compute the effects of the interactions on the transition temperature, we write the equation of state, the relation between particle number density, temperature and chemical potential. The particle number density can be expressed as a sum of the single particle Green's function over Matsubara frequencies $\omega_\nu = 2\pi\nu T$:

$$n = \frac{T}{\Omega} \frac{\partial \ln \mathcal{Z}}{\partial \mu} = \langle \bar{\varphi}(x,t) \varphi(x,t) \rangle = T \int \frac{\mathrm{d}^d k}{(2\pi)^d} \sum_{\nu \in \mathbb{Z}} \tilde{G}^{(2)}(\omega_\nu, k) \tag{9.3}$$

(Ω is the space volume). Above the transition, the single particle Green's function can be parametrized as

$$\left[\tilde{G}^{(2)}(\omega, k) \right]^{-1} = -i\omega - \mu + \frac{k^2}{2m} + \Sigma(\omega, k). \tag{9.4}$$

The Bose–Einstein condensation temperature is determined by the point where $[\tilde{G}^{(2)}(0,0)]^{-1} = 0$, that is where $\Sigma(0,0) = \mu$. At this point,

$$\left[\tilde{G}^{(2)}(\omega,k)\right]^{-1} = -i\omega + \frac{k^2}{2m} - [\Sigma(\omega,k) - \Sigma(0,0)]\,. \tag{9.5}$$

At (T_c, μ_c) the Fourier transform of the two-point correlation function at zero frequency diverges at zero momentum, and so does the correlation length.

In the absence of interactions,

$$\sum_{\nu\in\mathbb{Z}} \tilde{G}^{(2)}(k,\omega_\nu) = \frac{\beta}{\mathrm{e}^{\beta(k^2/2m-\mu)} - 1},$$

$\mu = 0$ at the transition, and

$$n = \zeta(d/2)/\lambda_c^d\,, \tag{9.6}$$

where

$$\lambda_c^2 = 2\pi/mT_c^0\,,$$

and T_c^0 is the condensation temperature of the ideal gas.

In the presence of weak interactions, the temperature of the Bose–Einstein condensation becomes the critical temperature of the interacting model, and is shifted by the interactions. From the theory of critical phenomena we know that the variation of the critical temperature in systems with dimension d below four depends primarily on contributions from the small momenta or large distance (which we refer to as the infrared, or IR) region. This property, which we later verify explicitly for $d = 3$, simplifies the problem, since to leading order the IR properties are only sensitive to the $\omega_\nu = 0$ component.

It follows from the relations (9.5,9.6) that at leading order in the dilute limit, where only the $\omega_\nu = 0$ Matsubara frequency contributes, the shift in the transition temperature at fixed density, $\Delta T_c = T_c - T_c^0$, can be related to the change Δn in the density at fixed T_c by [177]

$$\frac{\Delta T_c}{T_c} = -\frac{2}{d}\frac{\Delta n}{n}, \tag{9.7}$$

where

$$\Delta n = \frac{4\pi}{\lambda^2} N_d \int_0^\infty \mathrm{d}k\, k^{d-1} \left(\frac{1}{k^2 + \mathcal{M}(k)} - \frac{1}{k^2}\right),$$

N_d is the usual loop factor (2.14*b*) ($N_3 = 1/2\pi^2$), and

$$\mathcal{M}(k) \equiv 2m\,[\Sigma(k,0) - \Sigma(0,0)]\,.$$

Restricting the fields φ and $\bar{\varphi}$ to their zero Matsubara frequency components, corresponds to take a classical field limit. Equivalently, the time dependence of the fields $\varphi(x,t), \bar{\varphi}(x,t)$ in (9.2) can be neglected. It is then convenient to rescale the field φ in order to introduce the field theory normalizations used elsewhere in this review, and to parametrize it in terms of two real fields ϕ_1, ϕ_2:

$$\varphi(t,x) \sim \sqrt{mT}\big(\phi_1(x) + i\phi_2(x)\big).$$

The partition function then reads

$$\mathcal{Z} = \int [\mathrm{d}\phi(x)] \exp\left[-\mathcal{S}(\phi)\right], \tag{9.8}$$

where now $\mathcal{S}(\phi) = \mathcal{H}/T$ is given by

$$\mathcal{S}\phi) = \int \left\{ \frac{1}{2}\left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r\phi^2(x) + \frac{u}{4!}\left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x\,, \tag{9.9}$$

with $r = -2m\mu$, and for $d = 3$:

$$u = 96\pi^2 \frac{a}{\lambda^2} = 12 m^2 T G. \tag{9.10}$$

where G is the strength of the pair potential in expression (9.2). In Eq. (9.9) we have kept the dimension d of the spatial integration arbitrary in order to use dimensional regularization later. The two-point correlation function $\tilde{G}^{(2)}(p)$ is related to the two-point vertex function $\tilde{\Gamma}^{(2)}(p)$ of the classical statistical field theory by

$$\tilde{G}^{(2)}(\omega = 0, p) = \frac{2mT}{\tilde{\Gamma}^{(2)}(p)}.$$

The model described by the euclidean action (9.9) reduces to the ordinary $O(2)$ symmetric $(\phi^2)^2$ field theory, which indeed describes the universal properties of the superfluid Helium transition. As it stands this field theory suffers from more UV divergences than the original theory, the higher frequency modes providing a large momentum cutoff $\sim \sqrt{mT} \sim 1/\lambda$. This cutoff may be restored when needed, for instance, by replacing the propagator by the regularized propagator:

$$\frac{2mT}{k^2} \to \frac{1}{\mathrm{e}^{k^2/2mT} - 1}\,.$$

In fact, as we show later, since the shift of the critical temperature is dominated by long distance properties it is independent of the precise cutoff procedure, that is universal.

A second effect of the non-zero frequency modes is to renormalize the effective coefficients of the euclidean action. This problem can be explored by returning to

the functional integral representation (9.2) of the complete quantum theory and integrating over the non-zero modes perturbatively. The corrections generated are of higher order in a and can thus be neglected.

Because the interactions are weak, one may imagine calculating the change in the transition temperature by perturbation theory. However, the perturbative expansion for a critical theory does not exist for any fixed dimension $d < 4$; IR divergences prevent a straightforward calculation. If one introduces an infrared cutoff k_c to regulate the momentum integrals, one finds that perturbation theory breaks down when $k_c \sim a/\lambda^2$, all terms being then of the same order of magnitude. To discuss this problem in more detail, we now generalize the model to N component fields with an $O(N)$ symmetric hamiltonian.

9.2 The N-vector model. Renormalization group

We consider the $O(N)$ symmetric generalization (2.26) of the model corresponding to the euclidean action (9.9):

$$\mathcal{S}(\phi) = \int \left\{ \frac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + \frac{1}{2} r \phi^2(x) + \frac{1}{4!} \frac{u}{N} \left[\phi^2(x)\right]^2 \right\} \mathrm{d}^d x \,, \tag{9.11}$$

where the field $\phi(x)$ now has N components (note the change in the normalization of the interaction strength $u \mapsto u/N$). The advantage of this generalization is that it provides us with a tool, the large N expansion, which allows calculating directly at the critical point.

The goal is to obtain the leading order non-trivial contribution at criticality (in the massless theory) to

$$n = 2mT \sum_{i=1}^{N} \left\langle \phi_i^2 \right\rangle \equiv 2mT\, N\, \rho$$

with

$$\rho = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{\tilde{\Gamma}^{(2)}(k)} \,. \tag{9.12}$$

Here, $\delta_{ij}/\tilde{\Gamma}^{(2)}(k)$ is the connected two-point correlation function.

We first recover, by a simple renormalization group analysis, the result of [177] that the change in ρ due to the interaction is linear in the coupling constant. We introduce a large momentum cutoff $\Lambda \propto \sqrt{mT} \sim 1/\lambda$, and the dimensionless coupling constant

$$g = \Lambda^{d-4} \frac{u}{N} \propto \left(\frac{a}{\lambda}\right)^{d-2} \,. \tag{9.13}$$

At T_c the inverse two-point function in momentum space satisfies the renormalization group equation [2]

$$\left(\Lambda \frac{\partial}{\partial \Lambda} + \beta(g) \frac{\partial}{\partial g} - \eta(g) \right) \tilde{\Gamma}^{(2)}(p, \Lambda, g) = 0 \,.$$

This equation combined with dimensional analysis implies that the two-point function has the general form

$$\tilde{\Gamma}^{(2)}(p,\Lambda,g) = p^2 Z(g) F\big(p/\Lambda(g)\big). \tag{9.14}$$

On dimensional grounds $\Lambda(g)$ is proportional to Λ, with

$$\beta(g)\frac{\partial \ln Z(g)}{\partial g} = \eta(g), \tag{9.15}$$

$$\beta(g)\frac{\partial \ln \Lambda(g)}{\partial g} = -1\,. \tag{9.16}$$

Since

$$\beta(g) = -(4-d)g + (N+8)g^2/48\pi^2 + \mathcal{O}(g^3),$$

$\beta(g)$ is of order g for small g in $d<4$; similarly

$$\eta(g) = (N+2)g^2/(72(8\pi^2)^2) + \mathcal{O}(g^3).$$

Therefore,

$$Z(g) = \exp\int_0^g \frac{\eta(g')}{\beta(g')}\mathrm{d}g' = 1 + \mathcal{O}(g^2).$$

Thus to leading order $Z(g)$ =1. The function $\Lambda(g)$ is then obtained by integrating Eq. (9.16):

$$\Lambda(g) = g^{1/(4-d)}\Lambda \exp\left[-\int_0^g \mathrm{d}g'\left(\frac{1}{\beta(g')} + \frac{1}{(4-d)g'}\right)\right].$$

In the generic situation (like in a fluid) $g = O(1)$, the scale $\Lambda(g)$ is indistinguishable from the cut-off scale Λ. Universal behaviour exists only for momenta $|p| \ll \Lambda$ where

$$\tilde{\Gamma}^{(2)}(p) \propto p^{2-\eta} \quad \text{for } p \ll \Lambda(g) = O(\Lambda). \tag{9.17}$$

However, for $g \ll 1$ one finds

$$\Lambda(g) \sim g^{1/(4-d)}\Lambda \ll \Lambda\,.$$

There exists therefore an intermediate scale $\Lambda^{-1}(g)$ between the IR and the microscopic scales. The scale $\Lambda(g)$ corresponds to a crossover separating a universal long-distance regime governed by the non-trivial zero g^* of the β-function, from a universal short distance regime governed by the gaussian fixed point $g=0$, where

$$\tilde{\Gamma}^{(2)}(p) \propto p^2 \qquad \Lambda(g) \ll p \ll \Lambda\,.$$

In a generic situation g is of order unity and the universal large momentum region is absent.

Since g is equal to a/λ (see Eq. (9.13)) which is $\ll 1$, this condition is satisfied in the present situation. Moreover, unlike what is seen in ordinary critical phenomena, no $(\phi^2)^3$ term is here present, because this would correspond to three-body interactions, which for such dilute systems are even smaller.

In the language of the running or effective coupling constant, the situation can be described as follows: because the initial coupling constant at the microscopic scale is very close to the (unstable) gaussian fixed point value, it first moves very slowly away from the fixed point. For a dilatation λ such that $g(\lambda)$ all irrelevant couplings are already negligible, justifying the existence of a universal small distance, large momentum regime, whose physics here is the physics of the Bose–Einstein condensation. Then, very rapidly, $g(\lambda)$ moves toward the IR fixed point g^*, where the universal IR behaviour of the superfluid transition is seen.

We now show that with this condition, $\Delta T_c \propto \Lambda(g)$. First, from the $g = 0$ limit we infer that $F(\infty) = 1$ (see Eq. (9.14).) The function $F(p)$ behaves for large p as

$$F(p) = 1 + O(p^{2d-8}),$$

up to $\ln p$ factors, as can be verified directly from perturbation theory ($g = 0$ is the UV fixed point). Therefore, the first correction to the density (9.12) is convergent at large momentum and independent of the cutoff procedure, that is universal,

$$\delta\rho = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2} \left(\frac{1}{F(p/\Lambda(g))} - 1 \right).$$

Similarly, the IR scaling result (Eq. (9.17)) implies that this integral is IR convergent. Setting $p = \Lambda(g)k$, we then find the general form

$$\delta\rho = [\Lambda(g)]^{d-2} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{1}{k^2} \left(\frac{1}{F(k)} - 1 \right), \tag{9.18}$$

the g dependence is entirely contained in $\Lambda(g)$. For g small we conclude

$$\frac{\delta\rho}{\rho} \propto [\Lambda(g)/\Lambda]^{d-2} \propto g^{(d-2)/(4-d)}.$$

In the physical dimension 3 we recover

$$\frac{\delta\rho}{\rho} \propto g \propto a n^{1/3},$$

in agreement with [177]. It is important to note that both the perturbative large momentum region and the non-perturbative IR region contribute to the integral in Eq. (9.18). Therefore, we cannot evaluate it from a perturbative calculation

of the function $F(p)$. However, as we now show, we can calculate $\delta\rho$ in the form of an $1/N$ expansion.

Finally, the result for generic dimension d shows that the linear behaviour found for $d = 3$, that could lead to believe that the result is in some way perturbative, is just accidental.

9.3 The large N expansion at order $1/N$

We now use the techniques of large N expansion explained in section 2, where the large N limit is taken at Nu fixed. To leading order the critical two-point function has simply its free field form. However, a non-trivial correction is generated at order $1/N$; one finds the inverse two-point function (2.77) [13,2],

$$\tilde{\Gamma}^{(2)}(p) = p^2 + \frac{2}{N}\int \frac{\mathrm{d}^d q}{(2\pi)^d}\frac{1}{(6/u) + B_\Lambda(q,0)}\left(\frac{1}{(p+q)^2} - \frac{1}{q^2}\right) + O(1/N^2), \quad (9.19)$$

where $B_\Lambda(p,0)$ is the one-loop function defined by Eq. (2.59):

$$B_\Lambda(q,0) = \int^\Lambda \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{1}{k^2(k+q)^2} \underset{q\to 0}{=} b(d)q^{d-4} - \frac{6}{Ng^*}\Lambda^{d-4} + \mathcal{O}(1/\Lambda^2),$$

and the large N value of g^* has been used. We recall (Eq. (2.60))

$$b(d) = -\frac{\pi}{\sin(\pi d/2)}\frac{\Gamma^2(d/2)}{\Gamma(d-1)}N_d\,.$$

For $d = 3$ one finds $b(3) = 1/8$.

Note that in the large N limit, the chemical potential μ is proportional to $1/N$.

In the large N limit, the β-function takes the simple form $\beta(g) = (d-4)g(1 - g/g^*)$. Therefore, the leading cutoff-dependent correction to $B_\Lambda(q,0)$ combines with $6/u$ to yield $(6/u)(1 - g/g^*)$, as expected from renormalization group arguments. This cut-off dependent correction, however, can be neglected because $g \ll g^*$.

We evaluate

$$\delta\rho = -\frac{2}{N}\int \frac{\mathrm{d}^d p}{(2\pi)^{2d}}\frac{1}{p^4}\frac{\mathrm{d}^d q}{(6/u) + b(d)q^{d-4}}\left(\frac{1}{(p+q)^2} - \frac{1}{q^2}\right) \quad (9.20)$$

by keeping the dimension d generic and using dimensional regularization. The integral over p is

$$\begin{aligned}\int \frac{\mathrm{d}^d p}{(2\pi)^d}\frac{1}{p^4}\frac{1}{(p+q)^2} &= \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(3-d/2)\Gamma(d/2-1)\Gamma(d/2-2)}{\Gamma(d-3)}q^{d-6}\,. \\ &= \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(d/2-1)}{\Gamma(d-3)}\frac{\pi}{\sin\pi d/2}\,q^{d-6}\,.\end{aligned}$$

Note that the singularity at $d = 3$, which would apparently entail the vanishing of the integral, is cancelled in the subsequent q integral, which reduces to

$$\int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{q^{d-6}}{(6/u) + b(d) q^{d-4}} = \frac{N_d}{4-d} \quad \frac{\pi}{\sin\left(\pi(d-2)/(4-d)\right)} \times b^{(2d-6)/(4-d)} \left(\frac{6}{u}\right)^{(2-d)/(4-d)} .$$

In the $d = 3$ limit the two integrations in Eq. (9.20) yield $(1/32\pi^2)(u/6)$. As expected, $\delta\rho \propto u$:

$$\delta\rho = -Ku/N\,, \quad K = \frac{1}{96\pi^2}\,,$$

or in terms of the original parameters $(u/N) = 96\pi^2 a/\lambda^2$,

$$\delta\rho = -\frac{a}{\lambda^2}. \tag{9.21}$$

Using this result in Eq. (9.7), we finally obtain the change in the transition temperature [178]:

$$\begin{aligned} \frac{\Delta T_c}{T_c} &= \frac{8\pi}{3\zeta(3/2)} \frac{a}{\lambda} \\ &= \frac{8\pi}{3\zeta(3/2)^{4/3}} a n^{1/3} = 2.33\, a n^{1/3}. \end{aligned} \tag{9.22}$$

Note that although the final result does not depend on N and, therefore, replacing N by two is easy, the result is only valid for N large.

Conclusion. The properties of the weakly interacting Bose gas remain dominated by the UV fixed point of the renormalization group equations up to very large length scales; this is why we can still refer to the Bose–Einstein condensation when discussing the phase transition of the dilute interacting Bose gas. Renormalization group arguments also confirm directly that the shift of the transition temperature at fixed density is proportional to the dimensionless combination $an^{1/3}$ for weak interactions. This result is non-perturbative, and the proportionality coefficient cannot be obtained from perturbation theory. However, a non-perturbative method, the large N expansion, allows a systematic calculation of this coefficient as a power series in $1/N$, where eventually one has to set $N = 2$.The explicit calculation of the leading order contribution has been given in this section. The first correction is formally of order $1/N$ multiplied by a function of the product aN which is fixed in the large N limit. Because the final result in three dimensions is linear in a, the $1/N$ factor somewhat surprisingly cancels, and the result is independent of N. As mentioned earlier the $1/N$ correction has now been calculated [179], and yields a rather low 26% correction for $N = 2$. Moreover, the value found agrees within a factor 2 with the most recent numerical estimates [180–185].

10 Multicritical points and double scaling limit

We use now the general formalism presented in section 2.1 to discuss various additional issues concerning the general N-vector model with one scalar field, in the large N limit [186]. One obvious application concerns multi-critical points. Of particular interest are the subtleties involved in the stability of the phase structure at critical dimensions. The example of the tricritical $(\phi^2)^3$ theory will illustrate explicitly, however, the limitations of the large N method.

Another issue involves the so-called *double scaling limit*. Statistical mechanical properties of random surfaces as well as randomly branched polymers can be analyzed within the framework of large N expansion. In the same manner in which matrix models in their double scaling limit provide representations of dynamically triangulated random surfaces summed on different topologies [187], $O(N)$ symmetric vector models represent discretized branched polymers in this limit [188–193], where $N \to \infty$ and the coupling constant $g \to g_c$ in a correlated manner. The surfaces in the case of matrix models, and the randomly branched polymers in the case of vector models are classified by the different topologies of their Feynman graphs and thus by powers of $1/N$. Though matrix theories attract most attention, a detailed understanding of these theories exists only for dimensions $d \leq 1$ [194]. On the other hand, in many cases, the $O(N)$ vector models can be successfully studied also in dimensions $d > 1$, and thus, provide us with intuition for the search for a possible description of quantum field theory in terms of extended objects in four dimensions, which is a long lasting problem in elementary particle theory.

The phase structure of $O(N)$ vector quantum field theories at $N \to \infty$ is generally well understood, there are, however, certain cases where it is still unclear which of the features survives at finite N, and to what extent. One such problem is the multicritical behaviour of $O(N)$ models *at critical dimensions*. Here, one finds that in the $N \to \infty$ limit, there exists a non-trivial UV fixed point, scale invariance is spontaneously broken, and the one parameter family of ground states contains a massive vector and a massless bound state, a Goldstone boson–dilaton. However, since it is unclear whether this structure is likely to survive for finite N, one would like to know whether it is possible to construct a local field theory of a massless dilaton via the double scaling limit, where all orders in $1/N$ contribute. The double scaling limit is viewed as the limit at which the attraction between the $O(N)$ vector quanta reaches a value at $g \to g_c$, at which a massless bound state is formed in the $N \to \infty$ limit, while the mass of the vector particle stays finite. In this limit, powers of $1/N$ are compensated by IR singularities and thus all orders in $1/N$ contribute.

The special case of field theory in two dimension is discussed in section 10.1. In higher dimensions a new phenomenon arises: the possibility of a spontaneous breaking of the $O(N)$ symmetry of the model, associated to the Goldstone phenomenon. Before discussing a possible double scaling limit, the critical and

multicritical points of the $O(N)$ vector model are re-examined in section 10.2. In particular, a certain sign ambiguity that appears in the expansion of the gap equation is noted, and related to the existence of the IR fixed point in dimensions $2 < d < 4$ discussed in section 2.4. In section 10.3, the interesting physical example of the tricritical point in three dimension is discussed. Some insight in the problem is obtained there from variational calculations.

In section 10.4, we discuss the subtleties and conditions for the existence of an $O(N)$ singlet massless bound state along with a small mass $O(N)$ vector particle excitation. It is pointed out that the correct massless effective field theory is obtained after the massive $O(N)$ scalar is integrated out. Section 10.5 is devoted to the double scaling limit with a particular emphasis on this limit in theories at their critical dimensions.

10.1 The 2D $O(N)$ symmetric field theory in the double scaling limit

We first re-examine and summarize the results for the $O(N)$ symmetric field theory with a potential $NU(\phi^2/N)$, where ϕ is N-component field, in the large N limit in two dimensions. Indeed, there are no phase transitions in two dimensions and we therefore discuss this case separately. The action is

$$\mathcal{S}(\phi) = N \int \mathrm{d}^2x \left\{ \tfrac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + U\left(\phi^2/N\right) \right\}, \tag{10.1}$$

where an implicit cut-off Λ is always assumed below. Whenever the explicit dependence in the cut-off will be relevant, we shall assume a Pauli–Villars's type regularization replacing the propagator by a regularized form (2.11).

As in section 2.1, one introduces two fields $\rho(x)$ and $\lambda(x)$ and uses the identity (2.81). The large N action obtained by integrating over the field ϕ is then

$$\mathcal{S}_N = N \int \mathrm{d}^2x \left[U(\rho) - \tfrac{1}{2}\lambda\rho\right] + \tfrac{1}{2} N \operatorname{tr} \ln(-\nabla^2 + \lambda). \tag{10.2}$$

The integral is evaluated for N large by the steepest descent method. The saddle point value λ is the ϕ-field mass squared, and thus we set in general $\lambda = m^2$. Since in two dimensions there is no phase transition, $\langle\phi\rangle = 0$, the three saddle point equations (2.10) reduce to two:

$$U'(\rho_s) = \tfrac{1}{2} m^2 \,, \tag{10.3a}$$

$$\rho_s = \Omega_2(m) \,, \tag{10.3b}$$

where the function $\Omega_d(m)$ is defined by Eq. (2.12).

For $m \ll \Lambda$, one finds

$$\Omega_2(m) = \frac{1}{2\pi} \ln(\Lambda/m) + \frac{1}{4\pi} \ln(8\pi K) + O(m^2/\Lambda^2),$$

where K is a regularization dependent constant.

As was discussed in the case of quantum mechanics in ref. [186], a critical point is characterized by the vanishing at zero momentum of the determinant of second derivatives of the action at the saddle point. The mass-matrix has then a zero eigenvalue which, in field theory, corresponds to the appearance of a new massless excitation other than ϕ (this implies also that the forces between ϕ quanta are attractive, a serious problem in the case of bosons).

In order to obtain the effective action for this scalar massless mode, we must integrate over one of the fields. In the field theory case the resulting effective action can no longer be written in a local form. In order to discuss the order of the critical point we only need, however, the action for space independent fields, and thus, for example, we can eliminate λ using the λ saddle point equation. The action density $\mathcal{E}(\rho)$ (Eq. (2.9)) can then be written as

$$\frac{1}{N}\mathcal{E}(\rho) = U(\rho) - \frac{1}{2}\int^{\lambda(\rho)} \mathrm{d}\lambda'\, \lambda' \frac{\partial}{\partial\lambda'}\Omega_d(\sqrt{\lambda'}), \tag{10.4}$$

where at leading order for Λ large

$$\lambda(\rho) = 8\pi K\Lambda^2\, \mathrm{e}^{-4\pi\rho}\,.$$

The expression (10.4) is valid for any d and will be used also in section 10.5. Here, it reduces to

$$\frac{1}{N}\mathcal{E}(\rho) = U(\rho) + K\Lambda^2\, \mathrm{e}^{-4\pi\rho} = U(\rho) + \tfrac{1}{8\pi}m^2\, \mathrm{e}^{-4\pi(\rho-\rho_s)}\,.$$

A multicritical point is defined by the condition

$$\mathcal{E}(\rho) - \mathcal{E}(\rho_s) = O\left((\rho-\rho_s)^n\right). \tag{10.5}$$

This implies the conditions

$$U^{(k)}(\rho_s) = \tfrac{1}{2}(-4\pi)^{k-1}m^2 \qquad \text{for} \quad 1 \le k \le n-1\,.$$

Note that the coefficients $U^{(k)}(\rho_s)$ are the coupling constants renormalized at leading order for N large. If $U(\rho)$ is a polynomial of degree $n-1$ (the minimal polynomial model), the multicritical condition in Eq. (10.5) determines the critical values of renormalized coupling constants as well as ρ_s.

When the fields are space-dependent it is, instead, simpler to eliminate ρ because the corresponding field equation

$$U'\big(\rho(x)\big) = \tfrac{1}{2}\lambda(x) \tag{10.6}$$

is local. This equation can be solved by expanding $\rho(x)-\rho_s$ in a power series in $\lambda(x)-m^2$:

$$\rho(x)-\rho_s=\frac{1}{2U''(\rho_s)}\big(\lambda(x)-m^2\big)+O\left((\lambda-m^2)^2\right). \tag{10.7}$$

The resulting action for the field $\lambda(x)$ remains non-local but because, as we shall see, adding powers of λ as well as adding derivatives make terms less relevant, only the few first terms of a local expansion of the effective action are important.

If in the local expansion of the determinant we keep only the two first terms, we obtain an action containing at leading order a kinetic term proportional to $(\partial_\mu\lambda)^2$ and the interaction $(\lambda(x)-m^2)^n$:

$$\mathcal{S}_N(\lambda)\sim N\int \mathrm{d}^2x\left[\frac{1}{96\pi m^4}(\partial_\mu\lambda)^2+\tfrac{1}{n!}S_n\big(\lambda(x)-m^2\big)^n\right],$$

where the neglected terms are of order $(\lambda-m^2)^{n+1}$, $\lambda\partial^4\lambda$, and $\lambda^2\partial^2\lambda$ and

$$S_n=\frac{1}{N}\mathcal{E}^{(n)}(\rho_s)[2U''(\rho_s)]^{-n}=\frac{1}{N}\mathcal{E}^{(n)}(\rho_s)(-4\pi m^2)^{-n}.$$

Moreover, we note that, together with the cut-off Λ, m now also acts as a cut-off in the local expansion.

To eliminate the N dependence in the action we have, as in the example of quantum mechanics [186], to rescale both the field $\lambda-m^2$ and space:

$$\lambda(x)-m^2=\sqrt{48\pi}m^2N^{-1/2}\varphi(x)\,,\quad x\mapsto N^{(n-2)/4}x\,. \tag{10.8}$$

We obtain

$$\mathcal{S}_N(\varphi)\sim\int \mathrm{d}^2x\left[\tfrac{1}{2}(\partial_\mu\varphi)^2+\tfrac{1}{n!}g_n\varphi^n\right].$$

In the minimal model, where the polynomial $U(\rho)$ has exactly degree $n-1$, we find $g_n=6(48\pi)^{(n-2)/2}m^2$.

As anticipated, we observe that derivatives and powers of φ are affected by negative powers of N, justifying a local expansion. However, we also note that the cut-offs (Λ or the mass m) are now also multiplied by $N^{(n-2)/4}$. Therefore, the large N limit becomes also a large cut-off limit.

Double scaling limit. The existence of a double scaling limit relies on the existence of IR singularities due to the massless or small mass bound state which can compensate the $1/N$ factors appearing in the large N perturbation theory. We refer the reader to the examples in ref. [186] of a simple integral ($d=0$) and a quantum mechanical ($d=1$) example of the double scaling limit.

We now add to the action relevant perturbations

$$\delta_kU=v_k(\rho(x)-\rho_s)^k,\quad 1\le k\le n-2\,,$$

proportional to $\int \mathrm{d}^2x(\lambda - m^2)^k$:

$$\delta_k \mathcal{S}_N(\lambda) = N S_k \int \mathrm{d}^2x \big(\lambda(x) - m^2\big)^k,$$

where the coefficients S_k are functions of the coefficients v_k. After the rescaling of Eq. (10.8), one finds

$$\delta_k \mathcal{S}_N(\varphi) = \tfrac{1}{k!} g_k N^{(n-k)/2} \int \mathrm{d}^2x\, \varphi^k(x) \qquad 1 \le k \le n-2\,.$$

However, unlike quantum mechanics [186], it is not sufficient to scale the coefficients g_k with the power $N^{(k-n)/2}$ in order to obtain a finite scaling limit. Indeed, perturbation theory is affected by UV divergences, and we have just noticed that the cut-off diverges with N. In two dimensions the nature of divergences is very simple: it is entirely due to the self-contractions of the interaction terms and only one divergent integral appears

$$\big\langle \varphi^2(x) \big\rangle = \frac{1}{4\pi^2} \int \frac{\mathrm{d}^2 q}{q^2 + \mu^2}\,,$$

where μ is the small mass of the bound state, required as an IR cut-off to define perturbatively the double scaling limit. We can then extract the N dependence:

$$\big\langle \varphi^2(x) \big\rangle = \frac{1}{8\pi}(n-2)\ln N + O(1).$$

Therefore, the coefficients S_k have also to cancel these UV divergences, and thus have a logarithmic dependence in N superposed to the natural power obtained from power counting arguments. In general, for any potential, (Eq. (A3.4))

$$V(\varphi) =: V(\varphi) : + \left[\sum_{k=1} \frac{1}{2^k k!} \big\langle \varphi^2 \big\rangle^k \left(\frac{\partial}{\partial \varphi} \right)^{2k} \right] : V(\varphi) :\,, \tag{10.9}$$

where $: V(\varphi) :$ is the potential from which self-contractions have been subtracted (it has been normal-ordered). For example, for $n = 3$

$$\varphi^3(x) =: \varphi^3(x) : + 3 \big\langle \varphi^2 \big\rangle \varphi(x),$$

and thus the double scaling limit is obtained with

$$N g_1 + \frac{1}{16\pi} \ln N g_3 \qquad \text{held fixed as } N \to \infty\,.$$

For the example $n = 4$ ($\varphi^4(x) =: \varphi^4(x) : + 6 \big\langle \varphi^2 \big\rangle \varphi(x)^2 - 3 \big\langle \varphi^2 \big\rangle^2$) one finds that the double scaling limit is obtained when

$$g_1 N^{3/2} \quad \text{and} \quad N g_2 + \frac{g_4}{8\pi} \ln N \quad \text{are held fixed as} \quad N \to \infty.$$

10.2 The $O(N)$ symmetric model in higher dimensions: phase transitions

In higher dimensions, a phase transition associated with the spontaneous breaking of the $O(N)$ symmetry is possible. In a first part we thus study the $O(N)$ symmetric $NU(\phi^2/N)$ field theory, in the large N limit in order to explore the possible phase transitions and identify the corresponding multicritical points.

The action is

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} \left[\partial_\mu \phi(x)\right]^2 + NU\left(\phi^2/N\right) \right\}, \tag{10.10}$$

where an implicit cut-off Λ is again assumed.

Following the strategy of section 2.1, to which we refer for detail, we again introduce two auxiliary fields $\lambda(x)$, $\rho(x) = \phi^2(x)/N$, integrate over $N-1$ components of ϕ, and obtain the large N action

$$\mathcal{S}_N = N \int \mathrm{d}^d x \left[\tfrac{1}{2} \left(\partial_\mu \sigma\right)^2 + U(\rho) + \tfrac{1}{2}\lambda \left(\sigma^2/N - \rho\right) \right] + \tfrac{1}{2}(N-1)\,\mathrm{tr}\ln(-\nabla^2 + \lambda), \tag{10.11}$$

with $\sigma(x) \equiv \phi_1(x)$.

The saddle point equations: the $O(N)$ critical point. The large N saddle point equations are given by Eqs. (2.10):

$$m^2 \sigma = 0\,, \tag{10.12a}$$

$$U'(\rho) = \tfrac{1}{2} m^2\,, \tag{10.12b}$$

$$\sigma^2/N = \rho - \Omega_d(m)\,. \tag{10.12c}$$

In the ordered phase $\sigma \neq 0$ and thus m vanishes. Eq. (10.12c) has a solution only for $\rho > \rho_c$,

$$\rho_c = \frac{1}{(2\pi)^d} \int^\Lambda \frac{\mathrm{d}^d k}{k^2} \quad \Rightarrow \quad \sigma = \sqrt{\rho - \rho_c}\,.$$

Eq. (10.12b) which reduces to $U'(\rho) = 0$ then yields the critical temperature. Setting

$$U(\rho) = V(\rho) + \tfrac{1}{2} r \rho,$$

one finds

$$r_c = -2V'(\rho_c).$$

In order to find the magnetization critical exponent β, we need the relation between the r and ρ near the critical point.

In the disordered phase, $\sigma = 0$, Eq. (10.12c) relates ρ to the ϕ-field mass m. For $m \ll \Lambda$, ρ approaches ρ_c, and the relation becomes (Eq. (2.13))

$$\rho - \rho_c = -K(d) m^{d-2} + a(d) m^2 \Lambda^{d-4} + O\left(m^4 \Lambda^{d-6}, m^d \Lambda^{-2}\right). \tag{10.13}$$

The constant $K(d)$ is universal (Eq. (2.14a)). The constant $a(d)$, which also appears in Eq. (2.13), on the other hand, depends on the cut-off procedure (Eq. (2.15)).

For $2 < d < 4$, (the situation we assume below except when stated otherwise) the universal non-analytic part in m^{d-2} dominates.

For $d = 4$, Eq. (10.13) becomes

$$\rho - \rho_c = \tfrac{1}{8\pi^2} m^2 \left(\ln m/\Lambda + \text{ const.}\right),$$

and for $d > 4$ the analytic contribution dominates and

$$\rho - \rho_c \sim a(d) m^2 \Lambda^{d-4}.$$

Critical point. In a generic situation $V''(\rho_c) = U''(\rho_c)$ does not vanish, a situation we have examined in section 2.2. We find in the low temperature phase

$$t = r - r_c \sim -2U''(\rho_c)(\rho - \rho_c) \;\Rightarrow\; \beta = \tfrac{1}{2}\,. \tag{10.14}$$

This is the case of an ordinary critical point. Stability implies $U''(\rho_c) > 0$ so that $t < 0$.

At high temperature, in the disordered phase, the ϕ-field mass m is given by $2V'(\rho) + r = m^2$ and thus, using (10.13), at leading order

$$t \sim 2U''(\rho_c) K(d) m^{d-2}.$$

Of course, the simplest realization of this situation is to take $U(\rho)$ quadratic, and we recover the $(\phi^2)^2$ field theory.

Multicritical points. A new situation arises if we can adjust the parameters of the potential in such a way that $U''(\rho_c) = 0$. This can be achieved only if the potential U is at least cubic. We then expect a tricritical behaviour [200]. Higher critical points can be obtained when more derivatives vanish. We shall examine the general case though, from the point of view of real phase transitions, higher order critical points are not especially interesting. Indeed, for continuous symmetries phase transitions occur only for $d > 2$ and quasi-gaussian behaviour is then obtained for all dimensions $d \geq 3$. The analysis will, however, be useful in the study of double scaling limit.

Assuming that the first non-vanishing derivative is $U^{(n)}(\rho_c)$, we expand further Eq. (10.12b). In the ordered low temperature phase, we now find

$$t = -\frac{2}{(n-1)!} U^{(n)}(\rho_c)(\rho - \rho_c)^{n-1}, \;\Rightarrow\; \sigma \propto (-t)^{\beta}, \quad \beta = \frac{1}{2(n-1)}\,, \tag{10.15}$$

which is the magnetic exponent obtained in the mean field approximation for such a multicritical point. We have in addition the condition $U^{(n)}(\rho_c) > 0$.

In the high temperature phase, instead,

$$m^2 = t + (-1)^{n-1} \frac{2}{(n-1)!} U^{(n)}(\rho_c) K^{n-1}(d) m^{(n-1)(d-2)}. \tag{10.16}$$

For $d > 2n/(n-1)$, for m small the equation reduces to $m^2 = t$, which yields a simple gaussian behaviour, as expected above the upper-critical dimension.

For $d < 2n/(n-1)$, we find a peculiar phenomenon, the term in the r.h.s. is always dominant, but depending on the parity of n the equation has solutions for $t > 0$ or $t < 0$. For n even, t is positive and we find

$$m \propto t^{\nu}, \qquad \nu = \frac{1}{(n-1)(d-2)}, \tag{10.17}$$

which is a non gaussian behaviour below the critical dimension. However, for n odd (this includes the tricritical point), t must be negative, in such a way that we have now two competing solutions at low temperature. We have to find out which one is stable. We verify below that only the ordered phase is stable, so that the correlation length of the ϕ-field in the high temperature phase remains always finite. Although these dimensions do not correspond to physical situations because $d < 3$, the result is peculiar and inconsistent with the ε-expansion.

For $d = 2n/(n-1)$, we find a gaussian behaviour without logarithmic corrections, provided the condition

$$U^{(n)}(\rho_c) < \Omega_c\,, \quad \Omega_c = \tfrac{1}{2}(n-1)!\,[K(2n/(n-1))]^{1-n}\,, \qquad K(3) = 1/(4\pi), \tag{10.18}$$

is met. In particular, we will see that the special point

$$U^{(n)}(\rho_c) = \Omega_c \tag{10.19}$$

has several peculiarities.

We examine, in section 10.3, in more detail, the most interesting example: the tricritical point.

Discussion. In the tree approximation, the dominant configuration is given by the minimum of the function $U(\rho) \propto \rho^n$. For n odd, the function is not bounded from below, but $\rho = 0$ is the minimum because by definition $\rho \geq 0$. Here, however, we are in the situation where $U(\rho) \sim (\rho - \rho_c)^n$ with ρ_c is positive. Thus, this extremum of the potential is likely to be unstable for n odd. To check the global stability requires further work. This difficulty shed immediately some doubts about the possibility of studying such multicritical points by the large N method.

Another point to notice concerns renormalization group: the $n = 2$ example is peculiar in the sense that the large N limit exhibits a non-trivial IR fixed point. For higher values of n, no coupling renormalization arises in the large N limit

and only the gaussian fixed point is found. We are in a situation quite similar to usual perturbation theory, the β function can only be calculated perturbatively in $1/N$ and below the upper-critical dimension the IR fixed point is outside the $1/N$ perturbative regime.

Local stability and the mass matrix. The matrix of the general second partial derivatives of the action (10.11) is

$$N \begin{pmatrix} p^2+m^2 & 0 & \sigma \\ 0 & U''(\rho) & -\frac{1}{2} \\ \sigma & -\frac{1}{2} & -\frac{1}{2}B_\Lambda(p,m) \end{pmatrix}, \tag{10.20}$$

where $B_\Lambda(p,m)$ is defined in (2.56).

We are in position to study the local stability of the critical points. Since the integration contour for $\lambda = m^2$ should be parallel to the imaginary axis, a necessary condition for stability is that the determinant remains negative.

The disordered phase. Then $\sigma = 0$ and thus we have only to study the 2×2 matrix $\mathbf{M}$ of the ρ, m^2 subspace. Its determinant must remain negative, which implies

$$\det \mathbf{M} < 0 \;\Leftrightarrow\; 2U''(\rho)B_\Lambda(p,m) + 1 > 0\,. \tag{10.21}$$

For Pauli–Villars's type regularization, the function $B_\Lambda(p,m)$ is decreasing so that this condition is implied by the condition at zero momentum

$$\det \mathbf{M} < 0 \;\Leftarrow\; 2U''(\rho)B_\Lambda(0,m) + 1 > 0\,.$$

For m small, we use Eq. (2.58) and at leading order the condition becomes

$$K(d)(d-2)m^{d-4}U''(\rho) + 1 > 0\,.$$

This condition is always satisfied by a normal critical point since $U''(\rho_c) > 0$. For a multicritical point, and taking into account Eq. (10.13), one finds

$$(-1)^n \frac{d-2}{(n-2)!} K^{n-1}(d) m^{n(d-2)-d} U^{(n)}(\rho_c) + 1 > 0\,. \tag{10.22}$$

We obtain a result consistent with our previous analysis. For n even it is always satisfied; for n odd, it is always satisfied above the critical dimension and never below. At the upper-critical dimension we find a condition on the value of $U^{(n)}(\rho_c)$, which we recognize to be identical to condition (10.18) because then $2/(n-1) = d-2$.

The ordered phase. Now $m^2 = 0$ and the determinant Δ of the complete matrix is

$$-\Delta > 0 \;\Leftrightarrow\; 2U''(\rho)B_\Lambda(p,0)p^2 + p^2 + 4U''(\rho)\sigma^2 > 0\,. \tag{10.23}$$

We recognize a sum of positive quantities, and the condition is always satisfied. Therefore, in the case where there is a competition with a disordered saddle point, only the ordered one can be stable.

10.3 The tricritical point: variational analysis

The most interesting physical example is the tricritical point in three dimensions. Some insight in the problem can be obtained from the variational calculations of section 2.8. Eq. (2.87) becomes

$$\rho - \rho_c = \frac{\sigma^2}{N} - \frac{m}{4\pi}\,.$$

The variational action density for large N then reads

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{\sim} -\frac{m^3}{12\pi} + U(\rho) - \tfrac{1}{2}m^2(\rho-\rho_c)\,,$$

The polynomial of minimal degree near a tricritical point can be parametrized as

$$U(\rho) = \frac{1}{3!}U'''(\rho_c)(\rho-\rho_c)^3 + \frac{1}{2}t(\rho-\rho_c).$$

In the disordered phase, after elimination of $\rho-\rho_c$, the variational action density becomes

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{\sim} \frac{m^3}{24\pi}\left(1 - U'''(\rho_c)/\Omega_c\right) - \frac{1}{8\pi}mt\,.$$

For $U'''(\rho_c)/\Omega_c > 1$, the action is unbounded from below and, in the absence of other interactions, m is of the order of the cut-off, independently of the value of t, in such that way that the transition has disappeared. For $U'''(\rho_c)/\Omega_c < 1$, instead, and $t > 0$, one finds a minimum such that

$$m^2 = \left(1 - U'''(\rho_c)/\Omega_c\right)^{-1/2} t^{1/2},$$

in agreement with Eq. (10.16). For $t < 0$ the minimum corresponds to $m = 0$.

It is also easy to understand from the same arguments what happens for general d. The variational action density reads (Eq. (10.13))

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{\sim} -\frac{1}{3!}U'''(\rho_c)K^3(d)m^{3(d-2)} + \frac{d-2}{2d}K(d)m^d - \frac{1}{2}K(d)tm^{d-2}.$$

In the formal situation $d < 3$ and for $t = 0$, the negative term proportional to $m^{3(d-2)}$ dominates for m small. The minimum is obtained for

$$m^{2(3-d)} = K^2(d)U'''(\rho_c),$$

that is for a mass of the order of the cut-off. Of course, when m is of the order of the cut-off, several expressions based on a small m expansion are no longer valid. For $t > 0$, Eq. (10.16) now has a different interpretation: it has a solution because m again is of the order of the cut-off and m^2 is larger than $m^{2(d-2)}$.

Therefore, no particle propagates in the high temperature phase. For $t < 0$, by contrast, the term in tm^{d-2} dominates for m small and $\mathcal{E}_{\text{var.}}$ first increases. The solution of Eq. (10.16) yields a small value for m, but it corresponds to a maximum. Thus, for all values of $t \ll \Lambda^2$, the contribution proportional to t is negligible at the minimum of the action density and m remains of the order of the cut-off. This situation is clearly pathological and invalidates the large N expansion.

By contrast for $d > 3$, the term proportional to $U'''(\rho_c)$ is negligible, the minimum is given by Eq. (10.16), and the quasi-gaussian (or mean-field) theory applies.

In the ordered phase, for $d = 3$,

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{\sim} \frac{1}{3!}U'''(\rho_c)\frac{\sigma^6}{N^3} + \frac{t}{2}\frac{\sigma^2}{N}\,.$$

For $t > 0$, the minimum corresponds to $\sigma = 0$. For $t < 0$, the minimum is given by

$$\sigma^2/N = [-t/U'''(\rho_c)]^{1/2}.$$

Therefore, in contrast of what is seen in $d < 3$, the variational analysis confirms for $U'''(\rho_c)/\Omega_c < 1$ a non-pathological situation.

Dimension three: the end-point $U'''(\rho_c) = \Omega_c$. In the disordered phase the action density reduces to one term. It is interesting to add the first correction for m small

$$\mathcal{E}_{\text{var.}}/N \underset{N\to\infty}{\sim} -\frac{1}{8\pi}mt + \frac{a(3)}{4}\frac{m^4}{\Lambda}\,.$$

The results now depend on the sign of $a(3)$. A comment concerning the non-universal constant $a(d)$, given in (2.15) , is here in order because, while its absolute value is irrelevant, its sign plays a role in the discussion of multicritical points. The relevance of this sign to the RG properties of the large N limit of the simple $(\phi^2)^2$ field theories has already mentioned (section 2.4). For the simplest Pauli–Villars's type regularization, $D(k^2)$ is an increasing function and thus $a(d)$ is finite and positive in dimensions $2 < d < 4$, but this clearly is not a universal feature.

If $a(3)$ is positive, for $t > 0$ the action density has a minimum at a non-trivial value of m:

$$m \sim \left(\frac{t\Lambda}{8\pi a(3)}\right)^{1/3}\,.$$

For t small, m is small, justifying the analysis, but has a different scaling behaviour, $\nu = 1/3$, a property one would expect from an IR unstable fixed point. For $t < 0$, the minimum occurs at $m = 0$, as one expects for a phase transition.

If $a(3)$ is negative, the action density, in this small m approximation, is not bounded from below and the situation is pathological.

The variational analysis has clarified some peculiarities of the large N limit. However, it does not allow to investigate whether the results survive at N finite. This is here specially relevant because, for example, the point $U'''(\rho_c) = \Omega_c$ is at the boundary of a pathological region in parameter space, and its peculiar properties are expected to be especially sensitive to $1/N$ corrections. To calculate them, one has to return to the steespest descent method.

10.4 The scalar bound state

In this section, we study the limit of stability in the disordered phase ($\sigma = 0$). This is a problem which only arises when n is odd, the first case being provided by the tricritical point.

The mass-matrix has then a zero eigenvalue, which corresponds to the appearance of a new massless excitation other than ϕ. Let us denote by $\mathbf{M}$ the ρ, m^2 2×2 submatrix. Then,

$$\det \mathbf{M} = 0 \;\Leftrightarrow\; 2U''(\rho)B_\Lambda(0,m) + 1 = 0\,.$$

In the two-space the corresponding eigenvector has components $(\frac{1}{2}, U''(\rho))$.

The small mass region. In the small m limit, the equation can be rewritten in terms of the constant $K(d)$ defined in (2.13) as

$$(d-2)K(d)m^{d-4}U''(\rho) + 1 = 0\,. \tag{10.24}$$

Eq. (10.24) tells us that $U''(\rho)$ must be small. We are thus close to a multicritical point. Using the result of the stability analysis, we obtain

$$(-1)^{n-1}\frac{d-2}{(n-2)!}K^{n-1}(d)m^{n(d-2)-d}U^{(n)}(\rho_c) = 1\,. \tag{10.25}$$

We immediately notice that this equation has solutions only for $n(d-2) = d$, that is at the critical dimension. The compatibility then fixes the value of $U^{(n)}(\rho_c)$. We find again the point (10.19), $U^{(n)}(\rho_c) = \Omega_c$. If we take into account the leading correction to the small m behaviour we find, instead,

$$U^{(n)}(\rho_c)\Omega_c^{-1} - 1 \sim (2n-3)\frac{a(d)}{K(d)}\left(\frac{m}{\Lambda}\right)^{4-d}. \tag{10.26}$$

This means that when $a(d) > 0$, there exists a small region $U^{(n)}(\rho_c) > \Omega_c$ where the vector field is massive with a small mass m and the bound-state massless. The value Ω_c is a fixed point value.

The scalar field at small mass. We now extend the analysis to a situation where the scalar field has a small but non-vanishing mass M and m is still small. The

goal is in particular to explore the neighbourhood of the special point (10.19). Then, the vanishing of the determinant of $\mathbf{M}$ implies

$$1 + 2U''(\rho)B_\Lambda(iM, m) = 0\,. \tag{10.27}$$

Because M and m are small, this equation still implies that ρ is close to a point ρ_c where $U''(\rho)$ vanishes. Since reality imposes $M < 2m$, it is easy to verify that this equation has also solutions for only the critical dimension. Then,

$$U^{(n)}(\rho_c)f(m/M) = \Omega_c\,, \tag{10.28}$$

where we have set

$$f(z) = \int_0^1 \mathrm{d}x\,\left[1 + (x^2-1)/(4z^2)\right]^{d/2-2}\,, \qquad \tfrac{1}{2} < z\,. \tag{10.29}$$

In dimension three, it reduces to

$$f(z) = z\ln\left(\frac{2z+1}{2z-1}\right).$$

$f(z)$ is a decreasing function which diverges for $z = \frac{1}{2}$ because $d \leq 3$. Thus, we find solutions in the whole region $0 < U^{(n)}(\rho_c) < \Omega_c$, that is, when the multicritical point is locally stable.

Evaluating the propagator near the pole, we find the matrix

$$\mathbf{\Delta} = \frac{2}{G^2}\left[N\,\frac{\mathrm{d}B_\Lambda(p,m)}{\mathrm{d}p^2}\bigg|_{p^2=-M^2}\right]^{-1}\frac{1}{p^2+M^2}\begin{pmatrix}1 & G\\ G & G^2\end{pmatrix}, \tag{10.30}$$

where we have set

$$G = \frac{2(-K)^{n-2}\mathcal{E}^{(n)}}{N(n-2)!}m^{4-d}\,.$$

For m/M fixed, the residue goes to zero with m as m^{d-2} because the derivative of B is of the order of m^{d-6}. Thus the bound state decouples on the multicritical line.

10.5 Stability and double scaling limit

In order to discuss more thoroughly the stability issue and the double scaling limit, we now construct the effective action for the scalar bound state. We consider first only the massless case. We need the action only in the IR limit, and in this limit we can integrate out the vector field and the second massive eigenmode.

Integration over the massive modes. As we have already explained in section 10.1, we can integrate over one of the fields, the second being fixed, and we need the result only at leading order. Therefore, we replace in the functional integral

$$\mathrm{e}^{\mathcal{Z}} = \int [\mathrm{d}\rho \mathrm{d}\lambda] \exp\left[-\tfrac{1}{2}N \operatorname{tr}\ln(-\nabla^2+\lambda) + N\int \mathrm{d}^d x \left(-U(\rho)+\tfrac{1}{2}\rho\lambda\right)\right] \quad (10.31)$$

one of the fields by the solution of the field equation. It is useful to discuss the effective potential of the massless mode first. This requires calculating the action only for constant fields. It is then simpler to eliminate λ. We assume in this section that m is small (the vector propagates). For $\lambda \ll \Lambda$, the λ-equation reads ($d<4$)

$$\rho - \rho_c = -K(d)\lambda^{(d-2)/2}. \quad (10.32)$$

It follows that the resulting action density $\mathcal{E}(\rho)$, obtained from Eq. (10.4), is

$$\frac{1}{N}\mathcal{E}(\rho) = U(\rho) + \frac{d-2}{2d(K(d))^{2/(d-2)}}(\rho_c-\rho)^{d/(d-2)}. \quad (10.33)$$

In the sense of the double scaling limit, the criticality conditions are

$$\mathcal{E}(\rho) = O\big((\rho-\rho_s)^n\big).$$

It follows

$$U^{(k)}(\rho_s) = -\frac{K^{1-k}(d)}{2}\frac{\Gamma\big(k-d/(d-2)\big)}{\Gamma\big(-2/(d-2)\big)}m^{d-k(d-2)}\,, \quad 1\le k\le n-1\,.$$

For the potential U of minimal degree, we find

$$\frac{1}{N}\mathcal{E}(\rho) \sim \frac{K^{1-n}(d)}{2n!}\frac{\Gamma\big(n-d/(d-2)\big)}{\Gamma\big(-2/(d-2)\big)}m^{d-n(d-2)}(\rho-\rho_s)^n.$$

The double scaling limit. We recall here that quite generally one verifies that a non-trivial double scaling limit may exist only if the resulting field theory of the massless mode is super-renormalizable, that is below its upper-critical dimension $d=2n/(n-2)$, because perturbation theory has to be IR divergent. Equivalently, to eliminate N from the critical theory, one has to rescale

$$\rho-\rho_s \propto N^{-2\theta}\varphi\,, \quad x\mapsto xN^{(n-2)\theta} \quad \text{with} \quad 1/\theta = 2n-d(n-2),$$

where θ has to be positive.

We now specialize to dimension three, since $d<3$ has already been examined, and the expressions above are valid only for $d<4$. The normal critical point ($n=3$), which leads to a φ^3 field theory, and can be obtained for a quadratic

potential $U(\rho)$ (the $(\phi^2)^2$ field theory) has been discussed in section 2.3. We thus concentrate on the next critical point $n = 4$ where the minimal potential has degree 3.

The $d = 3$ tricritical point. The action density then becomes

$$\frac{1}{N}\mathcal{E}(\rho) = U(\rho) + \tfrac{8\pi^2}{3}(\rho_c - \rho)^3. \tag{10.34}$$

If the potential $U(\rho)$ has degree larger than 3, we obtain after a local expansion and a rescaling of fields,

$$\rho - \rho_s = -\frac{1}{32\pi^2\rho_c}(\lambda - m^2) \propto \varphi/N\,, \quad x \mapsto Nx\,, \tag{10.35}$$

a simple super-renormalizable $\varphi^4(x)$ field theory. If we insist, on the other hand, that the initial theory should be renormalizable, then we remain with only one candidate, the renormalizable $(\phi^2)^3$ field theory, also relevant for the tricritical phase transition with $O(N)$ symmetry breaking. Inspection of $\mathcal{E}(\rho)$ immediately shows a remarkable feature: because the term added to $U(\rho)$ is itself a polynomial of degree 3, the critical conditions lead to an action density $\mathcal{E}(\varphi)$ that vanishes identically. This result reflects the property that the two saddle point equations ($\partial\mathcal{S}/\partial\rho = 0$, $\partial\mathcal{S}/\partial\lambda = 0$ in Eqs. (10.12)) are proportional and thus have a continuous one-parameter family of solutions. This results in a flat effective potential for $\varphi(x)$. The effective action for φ depends only on the derivatives of φ, like in the $O(2)$ non-linear σ model.

We conclude that no non-trivial double scaling limit can be obtained in this way. In 3 dimensions with a $(\phi^2)^3$ interaction, we can generate at most a normal critical point $n = 3$, but then a simple $(\phi^2)^2$ field theory suffices.

Remark. The problem of the sign of $a(d)$ discussed in section 10.2 has an interesting appearance in $d = 3$ in the small m^2 region. If one keeps the extra term proportional to $a(d)$ in Eq. (10.33), one finds

$$\frac{1}{N}\mathcal{E}(\rho) = U(\rho) + \frac{8\pi^2}{3}(\rho_c - \rho)^3 + \frac{a(3)}{\Lambda}4\pi^2(\rho_c - \rho)^4.$$

Using now Eq. (10.32) and, as mentioned in section 10.4, the fact that in the small m^2 region the potential is proportional to $(\rho - \rho_c)^3$, we can solve for m^2. Since $m^2 > 0$, the appearance of a phase with small mass depends on the sign of $a(d)$. Clearly, this shows a non-commutativity of the limits of $m^2/\Lambda^2 \to 0$ and $N \to \infty$. The small m^2 phase can be reached by a special tuning and cannot be reached with an improper sign of $a(d)$. Calculated in this way, m^2 can be made proportional to the deviation of the coefficient of ρ^3 in $U(\rho)$ from its critical value $16\pi^2$.

A few concluding remarks. In this section, we studied of several subtleties in the phase structure of $O(N)$ vector models around multicritical points of odd and even orders. One of the main topics is the understanding of the multicritical behaviour of these models at their critical dimensions and the effective field theory of the $O(N)$-singlet bound state obtained in the $N \to \infty$, $g \to g_c$ correlated limit. It was pointed out that the integration over massive $O(N)$ singlet modes is essential in order to extract the correct effective field theory of the small mass scalar excitation. After performing this integration, it has been established here that the double scaling limit of $(\phi^2)^K$ vector model in its critical dimension $d = 2K/(K-1)$ results in a theory of a free massless $O(N)$ singlet bound state. This fact is a consequence of the existence of flat directions at the scale invariant multicritical point in the effective action. In contrast to the case $d < 2K/(K-1)$ where IR singularities compensate powers of $1/N$ in the double scaling limit, at $d = 2K/(K-1)$ there is no such compensation and only a non-interacting effective field theory of the massless bound state is left.

Another interesting issue in this study is the ambiguity of the sign of $a(d)$. The coefficient of $m^2\Lambda^{d-4}$ denoted by $a(d)$ in the expansion of the gap equation in Eqs. (10.12c) and (10.13) seems to have a surprisingly important role in the approach to the continuum limit ($\Lambda^2 \gg m^2$). The existence of an IR fixed point at $g \sim O(N^{-1})$, as seen in the β function for the unrenormalized coupling constant (section 2.4), depends on the sign of $a(d)$. Moreover, the existence of a phase with a small mass m for the $O(N)$ vector quanta and a massless $O(N)$ scalar depends also on the sign of $a(d)$. It may very well be that the importance of the sign of $a(d)$ is a mere reflection of the limited coupling constant space used to described the model.

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Appendices

A1 Spontaneous breaking of scale invariance, non-trivial fixed points

The subject of scale invariance breaking has a long history in particle physics. Under very general conditions, a d-dimensional field theory that is scale invariant, is also invariant under the transformations of the conformal group $SO(d+1,1)$ (for $d > 2$). The scale and conformal current, S_μ and $K_{\mu\nu}$ are then constructed [205] from the "improved" energy momentum tensor $\tilde{T}_{\mu\nu}$. The divergences of these currents are proportional to the trace of the energy momentum tensor $\tilde{T}^\mu_\mu$. They are:

$$S^\mu \;=\; x_\lambda \tilde{T}^{\mu\lambda} \qquad \text{and} \qquad K^{\mu\nu} \;=\; x^2 \tilde{T}^{\mu\nu} \;-\; 2x^\mu x^\lambda \tilde{T}^\nu_\lambda. \qquad (A1.1)$$

There are two reasonably well understood mechanisms for breaking these symmetries in a theory which is scale and conformal invariant at tree level (see, for example, Ref. [206]):

(a) Spontaneous breaking of scale invariance of the Nambu–Jona-Lasinio type encountered in the BCS theory of superconductivity.

(b) Explicit breaking of scale invariance, which is expressed at the quantum level by the anomaly in the trace of the energy momentum tensor, as the result of radiative corrections.

In conventional quantum field theories the two mechanisms occur simultaneously and the spontaneous breaking of scale symmetry is not normally accompanied by the appearance of a massless Nambu–Goldstone boson. Thus, for example, we do not find a massless dilaton in QCD. An interesting possibility exists, however, in case of theories with an hierarchy of scales in which a light dilaton will appear as a pseudo-Goldstone boson associated with the spontaneous breaking of scale symmetry.

Finite theories like $N = 2$ and $N = 4$ supersymmetric Yang–Mills theories are especially interesting. In these theories there is no trace anomaly and thus mechanism **b** mentioned above does not apply; but the alternative dynamical generation of mass scales in these cases is unclear. Mechanisms **a** and **b** and related physical issues can be also analyzed in solvable large N vector theories, $O(N)$ symmetric scalar, and supersymmetric theories. In particular, we will mention in this appendix the case of spontaneous breaking of scale invariance, unaccompanied by explicit breaking (namely, **a** without **b**). This will be

demonstrated in the leading large N analysis of the euclidean action of an $O(N)$ symmetric quantum field theory in three dimensions.

We will recall first a few general points on scale invariance. The scale current is conserved at tree level for a scale invariant potential

$$V(\phi) = g\phi^n$$

if the trace of the improved energy momentum tensor vanishes, namely

$$\partial_\mu S^\mu = \tilde{T}^\nu{}_\nu = 0 \qquad (A1.2)$$

where, in general,

$$\tilde{T}^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \mathcal{S}(\phi). \qquad (A1.3)$$

Here, $\mathcal{S}(\phi)$ is the action on a Riemannian manifold:

$$\mathcal{S}(\phi) = \int \sqrt{-g} \mathrm{d}^d x \tfrac{1}{2} \{ g^{\alpha\beta}(x) \nabla_\alpha \phi(x) \nabla_\beta \phi(x) - m^2 \phi^2(x) - \xi R(x) \phi^2(x) \}, \quad (A1.4)$$

$g = \det\{g_{\alpha\beta}\}$, ∇_α is the covariant derivative (which reduces to ∂_α for the scalar ϕ) and $R(x)$ is the curvature scalar (trace of the Ricci curvature tensor). Varying with respect to $h_{\mu\nu}$ near flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, one then obtains from Eq. $(A1.3)$:

$$\tilde{T}^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \eta^{\mu\nu} \left(\tfrac{1}{2} (\partial\phi)^2 - V(\phi) \right) + \xi (\eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) \phi^2. \qquad (A1.5)$$

Using the equation of motion, the trace is given by

$$\begin{aligned} \partial_\mu S^\mu &= \tilde{T}^\nu{}_\nu \\ &= (\partial\phi)^2 \left[1 - d/2 + 2\xi(d-1) \right] + \phi \partial^2 \phi \left[-d/n + 2\xi(d-1) \right]. \end{aligned} \quad (A1.6)$$

So, in d dimensions

$$\xi = \frac{1}{4} \left(\frac{d-2}{d-1} \right) \quad \text{and} \quad n = \frac{2d}{d-2}. \qquad (A1.7)$$

(ξ is independent of the potential, e.g. $\xi = \frac{1}{6}$ in $d = 4$ [205]).

Following the variational method described in section 2.8, one defines the on-shell single particle state of momentum $\vec{p}$ and mass m to be used as a variational parameter as $|\Psi_{\vec{p}}\rangle = a^\dagger(\vec{p})|0\rangle$ where we normalize by

$$[a(\vec{p}_1), a^\dagger(\vec{p}_2)] = (2\pi)^{d-1} 2\omega \delta^{d-1}(\vec{p}_1 - \vec{p}_2).$$

The energy is $\omega = (\vec{p}^{\,2} + m^2)^{1/2}$ and the field operators are

$$\phi(t,\vec{x}) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}2\omega}\{\mathrm{e}^{ikx}\, a^\dagger(\vec{k}) + \mathrm{e}^{-ikx}\, a(\vec{k})\}. \qquad (A1.8)$$

One finds from Eq.$(A1.5)$

$$\langle\vec{p}_2|\,\tilde{T}^{\mu\nu}\,|\vec{p}_1\rangle = p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - \eta^{\mu\nu} p_1\cdot p_2 - 2\xi(q^2\eta^{\mu\nu} - q^\mu q^\nu) + \eta^{\mu\nu}\,\langle\vec{p}_2|\,V(\phi)\,|\vec{p}_1\rangle \qquad (A1.9)$$

where $q^\mu = p_2^\mu - p_1^\mu$.

As an illustration, we will discuss the $d = 3$, $O(N)$ symmetric model with the Euclidean action

$$\mathcal{S}(\phi) = \int \mathrm{d}^3x\left[\tfrac{1}{2}(\partial_\mu\phi)^2 \;+ V(\phi)\right] \qquad (A1.10)$$

and the potential

$$V(\phi) = \frac{{\mu_0}^2}{2}\phi^2 + \frac{\lambda_0}{4N}(\phi^2)^2 + \frac{\eta_0}{6N^2}(\phi^2)^3. \qquad (A1.11)$$

Following section 2.8, the variational ground state energy density can be calculated from

$$\begin{aligned}\mathcal{E}_{\mathrm{var.}}/N \underset{N\to\infty}{\to} & \frac{1}{2(2\pi)^d}\int \mathrm{d}^dk\,\ln[(k^2+m^2)/k^2] - \tfrac{1}{2}m^2\Omega_3(m) + U(\langle\phi^2(x)/N\rangle_0) \\ &= \int_0^m s ds\Omega_3(s) - \tfrac{1}{2}m^2\Omega_3(m) + U\big(\Omega_3(m) + \frac{\sigma^2}{N}\big). \qquad (A1.12)\end{aligned}$$

The only divergence (see Eqs. (2.12,2.85)) occurs in

$$\langle\phi^2/N\rangle_0 = \Omega_3(m) + \frac{\sigma^2}{N} = \int^\Lambda \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{k^2+m^2} + \frac{\sigma^2}{N} = \Omega_3(0) - \frac{|m|}{4\pi} + \frac{\sigma^2}{N}, \quad (A1.13)$$

and the only renormalization needed in order to obtain a finite ground state energy density $\mathcal{E}(m^2,\sigma^2)$ is

$$\mu^2 = {\mu_0}^2 + \lambda_0\Omega_3(0) + \eta_0(\Omega_3(0))^2, \quad \lambda = \lambda_0 + 2\eta_0\Omega_3(0)\,, \quad \eta = \eta_0\,. \qquad (A1.14)$$

The ground state energy density is then given, at leading order for $N\to\infty$, by

$$\begin{aligned}\frac{1}{N}\mathcal{E}(m^2,\sigma^2) = & \frac{m^3}{24\pi} + \frac{\mu^2}{2}\left(\frac{\sigma^2}{N} - \frac{|m|}{4\pi}\right) + \frac{\lambda}{4}\left(\frac{\sigma^2}{N} - \frac{|m|}{4\pi}\right)^2 \\ & + \frac{\eta}{6}\left(\frac{\sigma^2}{N} - \frac{|m|}{4\pi}\right)^3. \qquad (A1.15)\end{aligned}$$

In what follows, we discuss the region of parameter space where the renormalized dimensional parameters are set to zero ($\mu^2 = 0$, $\lambda = 0$). We show that the non-trivial critical end-point ($\mu^2 = 0$, $\lambda = 0$, $\eta = \eta_c = (4\pi)^2$) can govern the continuum limit of the theory. At this point the mass m of the quanta created by $\phi(x)$, as found from the gap equation, is non zero though all dimensional parameters are set to zero. The appearance of a propagating mass m is associated with spontaneous breakdown of scale invariance. This is shown by the appearance of a massless bound state as $\eta \to \eta_c$ and the vanishing of the trace of the energy momentum tensor.

Explicit calculation. The term $\eta^{\mu\nu}\langle\vec{p}_2|V(\phi)|\vec{p}_1\rangle$ in Eq. ($A1.9$) contributes to $\langle\vec{p}_2|\tilde{T}^{\mu\nu}|\vec{p}_1\rangle$ at the tree level a term

$$\begin{aligned}
&\mu_0^2 + \frac{\lambda_0}{N}\left\langle\phi^2\right\rangle_0 + \frac{\eta_0}{N^2}\left\langle\phi^2\right\rangle_0^2 \\
&= \mu^2 + \lambda\left(\frac{\sigma^2}{N} - \frac{|m|}{4\pi}\right) + \eta\left(\frac{\sigma^2}{N} - \frac{|m|}{4\pi}\right)^2 . \qquad (A1.16)
\end{aligned}$$

Using the gap equation, the contribution to $\langle\vec{p}_2|\tilde{T}^{\mu\nu}|\vec{p}_1\rangle$ in Eq. ($A1.16$) is given, simply, by $\eta^{\mu\nu}m^2$. In the leading order in an $1/N$ calculation of $\langle\vec{p}_2|\tilde{T}^{\mu\nu}|\vec{p}_1\rangle$, one encounters an effective four-point vertex:

$$\frac{\bar{\lambda}}{N} \equiv \frac{\lambda_0}{N} + 2\frac{\eta}{N^2}\left\langle\phi^2\right\rangle_0 = \frac{\lambda}{N} - \eta\frac{|m|}{2\pi N} . \qquad (A1.17)$$

Summing all bubble graphs, one finds (using the Euclidean action in Eq. ($A1.10$))

$$\begin{aligned}
\left\langle\vec{p}_2|\tilde{T}^{\mu\nu}|\vec{p}_1\right\rangle &= p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - \eta^{\mu\nu}p_1\cdot p_2 - \eta^{\mu\nu}m^2 \\
&\quad - \left[\tfrac{1}{4}(q^2\eta^{\mu\nu} - q^\mu q^\nu) + \bar{\lambda}I^{\mu\nu}(q)\right]\left(\frac{1}{1+\bar{\lambda}B(q)}\right), \qquad (A1.18)
\end{aligned}$$

where

$$\begin{aligned}
B(q) &= \frac{1}{(2\pi)^3}\int\frac{\mathrm{d}^3k}{(k^2+m^2)\left[(q+k)^2+m^2\right]} \\
&= \frac{1}{8\pi}\int_0^1 \mathrm{d}\alpha\frac{1}{\sqrt{\alpha(1-\alpha)q^2+m^2}} = \frac{1}{4\pi q}\arctan\left(\frac{q}{2|m|}\right), \qquad (A1.19)
\end{aligned}$$

and

$$\begin{aligned}
I^{\mu\nu}(q) &= \int\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{(q+k)^\mu k^\nu + k^\mu(q+k)^\nu - \eta^{\mu\nu}(q+k)\cdot k - \eta^{\mu\nu}m^2}{(k^2+m^2)\left[(q+k)^2+m^2\right]} \\
&= \frac{q^\mu q^\nu - q^2\eta^{\mu\nu}}{4\pi}\int_0^1 \mathrm{d}\alpha\frac{\alpha(\alpha-1)}{\sqrt{m^2+\alpha(\alpha-1)q^2}} . \qquad (A1.20)
\end{aligned}$$

As mentioned above, we are interested in the theory when all dimensional, renormalized parameters are set to zero and the dimensionless η at its critical value:

$$\mu = \lambda = 0 \quad \text{and} \quad \eta = 16\pi^2.$$

The first line in Eq. $(A1.18)$ contributes $\frac{1}{2}q^2 - 2m^2$ to the trace of the energy momentum tensor. As a result of spontaneous breaking of scale invariance, we will find that the term in the second line in Eq. $(A1.18)$ contributes $-\frac{1}{2}q^2 + 2m^2$ and ensures a traceless energy momentum tensor and a conserved scale current. Indeed, one notices that the factor $(1+\bar{\lambda}B(q))^{-1}$ is exactly the term that appears in a four-point function of the ϕ field. Thus, the appearance of a massless bound state in this amplitude is directly associated with the vanishing of the trace of $T^{\mu\nu}$ as expected.

In Eq. $(A1.18)$, the induced effective coupling ($\bar{\lambda}$ in Eq. $(A1.17)$) is

$$\bar{\lambda} = -8\pi|m|$$

and thus $1 + \bar{\lambda}B(q) = 0$ at $q^2 = 0$. A massless $\phi - \phi$ bound state is created at this effective binding strength. Eq. $(A1.18)$ can be written as

$$\left\langle \vec{p}_2 | \tilde{T}^{\mu\nu} | \vec{p}_1 \right\rangle = p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - \eta^{\mu\nu} p_1 \cdot p_2 - \eta^{\mu\nu} m^2 + \frac{1}{4}(q^\mu q^\nu - q^2 \eta^{\mu\nu})$$
$$\times \left[1 - 8m \int_0^1 \mathrm{d}\alpha \frac{\alpha(1-\alpha)}{\sqrt{m^2 + \alpha(1-\alpha)q^2}} \right] \left[1 - m \int_0^1 \mathrm{d}\alpha \frac{1}{\sqrt{m^2 + \alpha(1-\alpha)q^2}} \right]^{-1}. \tag{A1.21}$$

Indeed, due to the following simple identity

$$\frac{1}{4} \left[1 - 8m \int_0^1 \mathrm{d}\alpha \frac{\alpha(1-\alpha)}{\sqrt{m^2 + \alpha(1-\alpha)q^2}} \right]$$
$$= \left(\frac{1}{4} - \frac{m^2}{q^2} \right) \left(1 - m \int_0^1 \mathrm{d}\alpha \frac{1}{\sqrt{m^2 + \alpha(1-\alpha)q^2}} \right), \tag{A1.22}$$

as clearly seen now in Eq. $(A1.21)$, the term in the trace of $\langle \vec{p}_2 | \tilde{T}^{\mu\nu} | \vec{p}_1 \rangle$ at tree level is exactly canceled by the induced quantum correction term due to the bound state massless Goldstone boson–dilaton pole. that appears at $\eta = \eta_c$ when $\mu^2 = \lambda = 0$.

More can be seen on Eq. $(A1.15)$: Clearly, at $\mu^2 = \lambda = 0$ one finds $\mathcal{E}(m^2, \sigma^2) \to -\infty$ as $m \to \infty$ unless $0 < \eta < \eta_c = (4\pi)^2$. One also notices that it is the

large negative renormalization of $\vec{\phi}^2$ that makes the energy unbounded from below. In the regularized theory positivity of $\vec{\phi}^2$ is maintained, but the instability at $\Lambda \to \infty$ is now reflected in the solution of the gap equation. In general, $m = \Lambda f(\eta_0)$ where $f(\eta_0)$ is a function of the only (unrenormalized) dimensionless coupling constant in the theory. Thus, $m \to \infty$ as $\Lambda \to \infty$ unless the theory has an UV fixed point exhibited here by a zero of $f(\eta_0)$ at some $\eta = \eta_c$. In this case, as seen in Eq. $(A1.15)$, the ground state energy density $\mathcal{E}(m^2, \sigma^2)$ is flat, namely, is m independent after $\Lambda \to \infty$ has been removed. A cutoff independent, finite physical mass can appear in the ground state spectrum. Namely,

$$m = \Lambda f(\eta_0) \to \text{ finite value as } \Lambda \to \infty \quad \text{since} \quad \eta_0(\Lambda) \to {\eta_c}^- \text{ as } \Lambda \to \infty\,. \tag{A1.23}$$

One finds

$$m = \Lambda f(\eta_0) = \Lambda \frac{\pi}{2}\left(1 - \sqrt{\frac{\eta_c}{\eta_0}}\right). \tag{A1.24}$$

This gives (from $\partial m/\partial\Lambda = 0$) a $\beta(\eta_0)$ function with a UV fixed point at $\eta_0 = \eta_c$. The bare coupling constant is then solved and one finds

$$\eta_0(\Lambda) = \eta_c + \frac{\tilde{\mu}}{\Lambda} \quad \text{and thus} \quad m = \frac{\pi}{4\eta_c}\tilde{\mu}\,, \tag{A1.25}$$

where $\tilde{\mu}$ is a new normalization scale. The above is just the manifestation of dimensional transmutation at the non-trivial UV fixed point.

Summary : Two facts should be noted now: **(a)** In the massive phase described above, scale invariance has been broken only spontaneously. As seen from $\eta_0 = \eta$, there is no explicit breaking of scale invariance (at $\mu^2 = \lambda = 0$), and the perturbative $\beta(\eta)$ function vanishes in the large N limit. Indeed, one finds that the trace of the energy momentum tensor stays zero. A massless dilaton — the Goldstone boson associated with this breaking — appears in the ground state spectrum as a reflection of the Goldstone realization of scale symmetry. $\mathcal{E}(m^2, \sigma^2)$ is m independent. **(b)** The normal ordering of ϕ^6 induces a new $\bar{\lambda}\phi^4$ interaction, where $\bar{\lambda} = \lambda - \eta_0[\frac{m}{2\pi}]$. This new interaction guarantees the appearance of the dilaton pole in the physical amplitudes as $\eta_0 \to \eta_c$.

Finally, higher orders in $1/N$ introduce an explicit breaking of scale invariance and probably destabilize the finite N scalar theory in Eq. $(A1.10)$.

A2 One-loop diagrams

We present some technical details concerning one-loop calculations relevant to the gap equation at leading order.

A2.1 The regularized one-loop diagrams

We expand here for $m \ll \Lambda$ the regularized one-loop diagram (2.12)

$$\Omega_d(m) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \tilde{\Delta}_\Lambda(k) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{m^2 + k^2 D(k^2/\Lambda^2)} = \Lambda^{d-2} \omega_d(m/\Lambda). \tag{A2.1}$$

The function $D(z)$, $D(0) = 1$, is strictly positive for $z > 0$, analytic in the neighbourhood of the real positive semi-axis, and increasing faster than $z^{(d-2)/2}$ for $z \to +\infty$.

To extract the coefficients in the expansion we use the Mellin transform and dimensional continuation (see section 2.5). We define

$$M(s) = \int_0^\infty \mathrm{d}z\, z^{-1-s} \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{z + k^2 D(k^2)}\,.$$

If the k integral has a contribution of the form z^α in a small z expansion, the Mellin transform has a pole at $s = \alpha$ and the coefficient can be identified.

We change variables $z \mapsto z k^2 D(k^2)$ and obtain

$$\begin{aligned} M(s) &= \int_0^\infty \mathrm{d}z\, \frac{z^{-1-s}}{1+z} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \big(k^2 D(k^2)\big)^{-1-s} \\ &= -\frac{\pi}{\sin(\pi s)} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \big(k^2 D(k^2)\big)^{-1-s}. \end{aligned}$$

$$M(s) = \int_0^\infty \mathrm{d}z\, \frac{z^{-1-s}}{1+z} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \big(k^2 D(k^2)\big)^{-1-s} = -\frac{\pi}{\sin(\pi s)} I_d(s)$$

with

$$I_d(s) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \big(k^2 D(k^2)\big)^{-1-s} = N_d \int_0^\infty \mathrm{d}k\, k^{d-3-2s} \big(D(k^2)\big)^{-1-s}.$$

We find two series, the first corresponding to the zeros of the sine function with s non-negative integer (the poles with $s < 0$ corresponding to the singularities in the large z behaviour)

$$M(s) \underset{s \to n}{\sim} (-1)^n I_d(n) \frac{1}{n-s}.$$

Note that the analytic continuation of $I_n(d)$ for low dimensions is obtained by subtracting to D^{-1-n} the first terms of its Taylor series at $k=0$.

The second corresponds to the divergence at $k=0$ of the k integral $I(s)$. The residues are obtained by integrating near $k=0$ and expanding $D(k^2)$ in powers of k^2

$$\begin{aligned} I_d(s) &\sim N_d \int_0^1 \mathrm{d}k\, k^{d-3-2s} \big(D(k^2)\big)^{-1-s} \\ &\sim \frac{N_d}{d-2-2s} - (1+d/2)D'(0)\frac{N_d}{d-2s} + \cdots . \end{aligned}$$

In general the poles are at $s=\frac{1}{2}(d-2)+n$ with n non-negative integer. Therefore the small z expansion of $\omega_d(\sqrt{z})$ has the general form

$$\omega_d(\sqrt{z}) = \sum_{n=0} \alpha_n(d) z^{(d-2)/2+n} + \beta_n(d) z^n .$$

Parametrization. We parametrize the expansion for $z \to 0$ and $d>2$ as

$$\omega_d(z) = \omega_d(0) - K(d) z^{d-2} + a(d) z^2 + O\left(z^d, z^4\right). \qquad (A2.2)$$

The first contribution is proportional to ρ_c (equation (2.20))

$$\omega_d(0) = \frac{1}{(2\pi)^d} \int \frac{\mathrm{d}^d k}{k^2 D(k^2)}. \qquad (A2.3)$$

The constant $K(d)$ is independent of the cut-off procedure because it involves only the leading behaviour of the integrand for $k \to 0$ and thus $D(0)=1$:

$$N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \qquad (A2.4a)$$

$$K(d) = -\frac{\pi}{2\sin(\pi d/2)} N_d = -\frac{1}{(4\pi)^{d/2}} \Gamma(1-d/2)\,, \qquad (A2.4b)$$

where we have introduced the usual loop factor N_d.

The constant $a(d)$, which characterizes the leading correction in equation (2.13), depends explicitly on the regularization, i.e. the way large momenta are cut,

$$a(d) = \begin{cases} -\dfrac{1}{(2\pi)^d} \displaystyle\int \frac{\mathrm{d}^d k}{k^4} \frac{1}{D^2(k^2)} & \text{for } d>4 \\ \dfrac{1}{(2\pi)^d} \displaystyle\int \frac{\mathrm{d}^d k}{k^4} \left(1 - \frac{1}{D^2(k^2)}\right) & \text{for } d<4 \end{cases} . \qquad (A2.5)$$

For $d=4$ the integrals diverge at $k=\infty$. For $\varepsilon = 4-d \to 0$

$$a(d) \underset{\varepsilon=4-d\to 0}{\sim} 1/(8\pi^2\varepsilon). \qquad (A2.6)$$

A logarithmic contribution then appears in the expansion $(A2.2)$:

$$\omega_d(z) - \omega_d(0) \sim \frac{1}{8\pi^2} z^2 \ln z\,. \qquad (A2.7)$$

A2.2 Finite temperature

Let us add a few remarks concerning the calculation of Feynman diagrams. General methods explained in the framework of finite size scaling can also be used here, involving Jacobi's elliptic functions. However, more specific techniques are also available in finite temperature quantum field theory. The idea is the following: in the mixed $(d-1)$-momentum, time representation the propagator is the two-point function $\Delta(t,p)$ of the harmonic oscillator with frequency $\omega(p) = \sqrt{p^2+m^2}$ and time interval $\beta = 1/T$:

$$\frac{1}{p^2+\omega^2+m^2} \mapsto \Delta(t,p) = \frac{1}{2\omega(p)} \frac{\cosh\left[(\beta/2-|t|)\omega(p)\right]}{\sinh\big(\beta\omega(p)/2\big)}.$$

Summing over all frequencies is equivalent to set $t=0$. For the simple one-loop diagram one finds

$$\frac{1}{2\pi}\int \frac{\mathrm{d}\omega}{\omega^2+p^2+m^2} \mapsto \frac{1}{2\omega(p)} \frac{\cosh\big(\beta\omega(p)/2)\big)}{\sinh\big(\beta\omega(p)/2\big)},$$

This expression can be written in a way that separates quantum and thermal contributions:

$$\frac{1}{2\omega(p)\tanh(\beta\omega(p)/2)} = \frac{1}{2\omega(p)} + \frac{1}{\omega(p)(\mathrm{e}^{\beta\omega(p)}-1)},$$

where the first term is the zero-temperature result, and the second term, which involves the relativistic Bose statistical factor, decreases exponentially at large momentum.

Finally in the example of fermions or gauge theories we can use a more general identity that can be proven by replacing the sum by a contour integral,

$$\sum_n \frac{x}{(n+\nu)^2+x^2} = \pi \frac{\sinh(2\pi x)}{\cosh(2\pi x)-\cos(2\pi\nu)}, \qquad (A2.8)$$

and thus after integration,

$$\ln\big(\cosh(2\pi x)-\cos(2\pi\nu)\big)-\ln 2 = \lim_{N\to\infty}\sum_{n=-N}^{N}\left[\ln\big((n+\nu)^2+x^2\big)-2\ln(n+1/2)\right]. \qquad (A2.9)$$

A2.3 Γ, ψ, ζ functions: a few useful identities

Two useful identities on the Γ function are

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)\,, \quad \Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi\,. \qquad (A2.10)$$

They translate into a relations for the $\psi(z)$ function, $\psi(z) = \Gamma'(z)/\Gamma(z)$

$$2\psi(2z) = 2\ln 2 + \psi(z) + \psi(z+1/2), \quad \psi(z) - \psi(1-z) + \pi/\tan(\pi z) = 0 \quad (A2.11)$$

where $\zeta(s)$ is the standard Riemann ζ-function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}\,. \qquad (A2.12)$$

It is also useful to remember the reflection formula

$$\zeta(s)\Gamma(s/2) = \pi^{s-1/2}\Gamma\big((1-s)/2\big)\zeta(1-s), \qquad (A2.13)$$

which can be written in different forms using Γ-function relations. Moreover

$$\sum_{n=1} \frac{(-1)^n}{n^s} = (2^{1-s}-1)\zeta(s), \qquad (A2.14)$$

Finally

$$\zeta(1+\varepsilon) = \frac{1}{\varepsilon} - \psi(1) + O(\varepsilon), \qquad (A2.15)$$

$$\zeta(\varepsilon) = -\tfrac{1}{2}(2\pi)^{\varepsilon} + O(\varepsilon^2). \qquad (A2.16)$$

A3 Gaussian measure and normal product

We assume that we are given some gaussian measure $\exp[-\frac{1}{2}\sum_{ij} x_i A_{ij} x_j]$ for a finite or infinite number of random variables x_i. In the following applications the index i will describe a parameter that can be either discrete (e.g. group theoretic index) or continuous (e.g. space-time). We denote the expectation values

$$\langle x_{i_i} \ldots x_{i_\ell} \rangle = \mathcal{N}(\mathbf{A}) \int \mathrm{d}x\, x_{i_i} \ldots x_{i_\ell} \exp[-\tfrac{1}{2}\sum_{ij} x_i A_{ij} x_j],$$

where the normalization $\mathcal{N}(\mathbf{A})$ is such that $\langle 1 \rangle = 1$. We denote by $\mathbf{\Delta}$ the inverse of the symmetrix (or operator) $\mathbf{A}$ and, therefore,

$$\Delta_{ij} = \langle x_i x_j \rangle\,.$$

We now define the normal ordering of a formal power series in the variables x_i as a linear map, with the following property: If $V(x)$ is a homogeneous polynomial of degree n, the normal-ordered polynomial $:V(x):$ is a polynomial of degree n such that

$$V(x) - :V(x): = O(|x|^{n-2}),$$

and

$$\langle :V(x): x_{i_1} x_{i_2} \dots x_{i_\ell} \rangle = 0 \qquad \forall \ell < n\,.$$

The normal ordering amounts to subtract from $V(x)$ all terms corresponding to self-contractions in the sense of Wick's theorem.

We now establish an explicit expression for the normal order of any formal power series in the variables x_i. First, we consider an exponential $\exp\left[\sum_i h_i x_i\right]$ and we generate expectation values by a generating function $\exp\left[\sum_i g_i x_i\right]$. We, therefore, calculate

$$\begin{aligned} \mathcal{Z}(g,h) &= \mathcal{N}(\mathbf{A}) \int \mathrm{d}x \, \exp\left[-\frac{1}{2} \sum_{ij} x_i A_{ij} x_j + \sum_i (g_i + h_i) x_i \right] \\ &= \exp\left[\frac{1}{2} \sum_{ij} (g_i + h_i) \Delta_{ij} (g_j + h_j) \right]. \end{aligned} \tag{A3.1}$$

The suppression of the self-contractions is now easy; we divide by the value for $g = 0$. Indeed, then the remaining dependence in h, in the exponential, is linear in g. A non vanishing result can only be obtained by differentiating at least as many times with respect to g as to h before taking the $g = h = 0$ limit. Therefore,

$$:\mathrm{e}^{\mathbf{h}\cdot\mathbf{x}}: = \mathrm{e}^{-\mathbf{h}\cdot\boldsymbol{\Delta}\mathbf{h}/2}\, \mathrm{e}^{\mathbf{h}\cdot\mathbf{x}}\,.$$

We now write any function as a Laplace transform

$$V(x) = \int \mathrm{d}h \, \tilde{V}(h) \, \mathrm{e}^{\mathbf{h}\cdot\mathbf{x}}$$

and use the linearity of the normal order operation to get

$$:V(x): = \exp\left[-\frac{1}{2} \sum_{ij} \Delta_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right] V(x). \tag{A3.2}$$

Note that analogous expressions can be derived by the same method for complex or grassmannian gaussian measures.

We now discuss a few applications.

Local polynomials in field theory. In the exemple of a gaussian functional measure over a field φ, $\mathbf{\Delta}$ is the propagator. If the field theory is invariant under space translations the propagator is of the form $\Delta(x-y)$. If V is a local function of a field of the form $V[\varphi(x)]$, the normal order takes the form

$$: V(\varphi) := \exp\left[-\frac{1}{2}\left\langle\varphi^2(0)\right\rangle\left(\frac{\partial}{\partial\varphi}\right)^2\right]V(\varphi), \qquad (A3.3)$$

justifying equation (10.9).

By acting with $\partial/\partial\varphi$ on both sides of the equation, we infer

$$\frac{\partial}{\partial\varphi} : V(\varphi) :=: \frac{\partial}{\partial\varphi}V(\varphi) : \qquad .$$

Namely, Eq. $(A3.3)$ implies that the normal ordering commutes with differentiation. Therefore, the relation $(A3.3)$ can be inverted as

$$V(\varphi) =: \exp\left[\frac{1}{2}\left\langle\varphi^2(0)\right\rangle\left(\frac{\partial}{\partial\varphi}\right)^2\right]V(\varphi) : \qquad . \qquad (A3.4)$$

$O(N)$ *invariant theories.* With an $O(N)$ invariant gaussian measure and N component field φ the two-point function takes the form

$$\Delta_{ij}(x) = \delta_{ij}\Delta(x) = \frac{\delta_{ij}}{N}\left\langle\varphi(x)\cdot\varphi(0)\right\rangle.$$

The normal ordered polynomial is then given by

$$: V(\varphi) := \exp\left[-\frac{1}{2N}\left\langle\varphi^2(0)\right\rangle\sum_i\left(\frac{\partial}{\partial\varphi_i}\right)^2\right]V(\varphi). \qquad (A3.5)$$

In case of an $O(N)$ invariant local function of the form $U[\varphi^2(x)/N]$,

$$\sum_i\left(\frac{\partial}{\partial\varphi_i}\right)^2 U(\varphi^2/N) = 2U'(\varphi^2/N) + \frac{4}{N^2}\varphi^2 U''(\varphi^2/N),$$

and, thus, setting $\varphi^2/N = \rho$:

$$: U(\rho) := \exp\left\{-\frac{1}{2N}\left\langle\varphi^2(0)\right\rangle\left[2\frac{\partial}{\partial\rho} + \frac{4\rho}{N}\left(\frac{\partial}{\partial\rho}\right)^2\right]\right\}U(\rho).$$

The exponential of a first order derivative is a translation operator. Moreover, $\langle\varphi^2(0)\rangle/N = \langle\rho\rangle$. Therefore, the normal-ordered polynomial can be rewritten

$$:U(\rho):= \exp\left[-\frac{2}{N}\langle\rho\rangle\,\rho\left(\frac{\partial}{\partial\rho}\right)^2\right]U\big(\rho-\langle\rho\rangle\big). \qquad (A3.6)$$

In this way the normal ordered expression can be expanded in powers of $1/N$. At leading order for $N\to\infty$:

$$:U(\rho): \underset{N\to\infty}{\sim} U\big(\rho-\langle\rho\rangle\big). \qquad (A3.7)$$

This relation, when written in the form $:U(\rho+\langle\rho\rangle):\sim U\big(\rho)$ immediately implies the result (2.86).