



HAL
open science

What is the probability of connecting two points ?

Christian Tanguy

► **To cite this version:**

Christian Tanguy. What is the probability of connecting two points ?. Journal of Physics A: Mathematical and Theoretical, 2007, 40, pp.14099-14116. 10.1088/1751-8113/40/47/005 . hal-00122268v1

HAL Id: hal-00122268

<https://hal.science/hal-00122268v1>

Submitted on 28 Dec 2006 (v1), last revised 3 Jul 2008 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

What is the probability of connecting two points ?

C. TANGUY

France Telecom Division R&D CORE, 38-40 rue du Général Leclerc, 92794 Issy-les-Moulineaux Cedex, France *

PACS 89.20.-a – Interdisciplinary applications of physics
PACS 05.50.+q – Lattice theory and statistics (Ising, Potts, etc.)
PACS 02.10.0x – Combinatorics; graph theory

Abstract. - The two-terminal reliability, known as the pair connectedness or connectivity function in percolation theory, may actually be expressed as a product of transfer matrices in which the probability of operation of each link and site is exactly taken into account. When link and site probabilities are p and ρ , it obeys an asymptotic power-law behavior, for which the scaling factor is the transfer matrix's eigenvalue of largest modulus. The location of the complex zeros of the two-terminal reliability polynomial exhibits structural transitions as $0 \leq \rho \leq 1$.

Introduction. – Since the original work of Moore and Shannon [1], network reliability has been a field devoted to the calculation of the connection probability between different sites of a network constituted by edges (links, bonds) and nodes (vertices, sites), each of them having a probability of operating correctly (the reliability). This field, although mainly developed in an applied background [2], is strongly related to graph theory [3, 4], combinatorics and algebraic structures [5, 6], percolation theory [7, 8], as well as numerous lattice models in statistical physics [9–12]. For instance, the all-terminal reliability Rel_A , i.e., the probability that all nodes are connected, is derived from the Tutte polynomial, an invariant of the associated graph, when all edges have the same reliability ($0 \leq p \leq 1$). This polynomial appears in the partition function for various Potts models, and has been calculated for several families of graphs [9–11]; the location of its complex zeros has also been studied [10, 11, 13]. The two-terminal reliability $\text{Rel}_2(s \rightarrow t)$, the probability that source s and a destination t are connected, is known as the connectivity function or pair connectedness in percolation theory. It has been used in modeling epidemics or fire propagation [7, 8]. This approach is complementary to the effort recently devoted on complex networks, in which the network resilience, i.e., its robustness against link or node failures (sometimes following deliberate attacks) has been studied for “scale-free” random graphs [14].

Exact reliability calculations are known to be very difficult [15], except for series-parallel reducible graphs for which only successive simplifications $\{p_{\text{series}} = p_1 p_2, p_{\text{parallel}} = p_1 // p_2 = p_1 + p_2 - p_1 p_2\}$ are needed. Even for planar graphs with identical edge reliabilities p and perfect nodes (i.e., $p_{\text{node}} \equiv 1$), their algorithmic complexity has been classified as #P-hard [5, 16]. Yet, the development of Internet traffic makes it important for telecom operators to assess the overall reliability of network connections,

when links *and* nodes may fail.

In this Letter, we show that for a network represented by an undirected graph G , the two-terminal reliability may be expressed as a product of transfer matrices, where *individual* edge and node reliabilities are exactly taken into account. Such a factorization, already observed for graph coloring polynomials [4, 11], 2D-percolation in square strips [17] or all-terminal reliability polynomials [9, 10], originates with the underlying algebraic structure of the graph. We apply our method to the two examples (K_n is the complete graph with n nodes) of fig. 1. The K_4 ladder, for which we explicitly give the 5×5 transfer matrix, describes a generic architecture for long-haul connections, while the K_3 cylinder slightly generalizes the “sponge model” of width three by Seymour and Welsh [18]. When edge and node reliabilities are respectively equal to p and ρ , a unique transfer matrix is involved; its largest eigenvalue determines the asymptotic power-law behavior of reliability as a function of the ladder length. The location of the complex zeros of $\text{Rel}_2(p)$ exhibits striking structure transitions as $0 \leq \rho \leq 1$.

Graph decomposition. – The gist of our method is to simplify the graph by removing links of the n^{th} (last) elementary cell of the network, namely the edges and nodes indexed by n , a procedure called pivotal decomposition or deletion-contraction [5]. If the end terminal t (which can be regarded as perfect) is connected to node u through edge e , with respective reliabilities p_u and p_e , then

$$\begin{aligned} \text{Rel}_2(G) = & (1 - p_e) \text{Rel}_2(G \setminus e) + p_e p_u \text{Rel}_2(G \cdot e) \\ & + p_e (1 - p_u) \text{Rel}_2(G \setminus u), \end{aligned} \quad (1)$$

where $G \setminus e$ and $G \setminus u$ are the graphs where e or u have been deleted, and $G \cdot e$ the graph where t and u have been merged through the “contraction” of e ; eq. (1) merely sums probabilities of disjoint events. This procedure, along with standard series-parallel reductions, is

*E-mail: christian.tanguy@orange-ftgroup.com

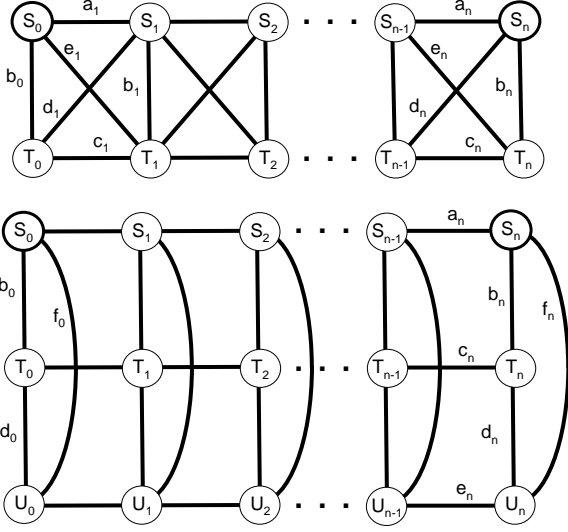


Fig. 1: Generic network architectures: (top) K_4 ladder (bottom) K_3 cylinder. Links and nodes are identified by their reliabilities: a_n, b_n , etc. for links, S_n, T_n , and U_n for nodes. The source is S_0 , the possible destinations are S_n, T_n , or U_n . A missing link or node's reliability is simply set to zero.

repeated for the three (instead of the usual two) secondary graphs in order to take advantage of a structural recursivity of the graph. After a finite number of such reductions, we get replicas of the original graph, albeit with one less elementary cell and with the $(n-1)^{\text{th}}$ cell's edge and node reliabilities possibly renormalized by those of the n^{th} cell, or set to either zero or one. In order to ensure the existence of a recursion relation, the graph structure must be *closed* under successive applications of eq. (1); it may initially require the use of extra edges with symbolic reliabilities, so that all nodes of an elementary cell are connected pair-wise, even if such links do not exist in the graph under consideration. At this point, a recursion hypothesis is needed, giving for instance $\text{Rel}_2(S_0 \rightarrow S_n)$ as a sum over specific polynomials in the reliabilities indexed by n ; they are often obvious from the $n=2$ value. Going from $n-1$ to n provides the transfer matrix linking the prefactors of the polynomials, because Rel_2 is an affine function of each component reliability; the (often trivial) $n=1$ case serves as the initial condition of the recurrence.

Application to the K_4 ladder. – Let us first illustrate this method by calculating $\mathcal{R}_n = \text{Rel}_2(S_0 \rightarrow S_n)$ for the K_4 ladder (top of fig. 1). Following the guidelines of the preceding section, we first consider b_n for deletion, apply eq. (1) twice, and find two contributions. The first one is a sum of \mathcal{R}_{n-1} -like terms with prefactors, in which the “old” a_{n-1}, \dots, T_{n-1} are renormalized by one or more of the “new” a_n, \dots, T_n . The second one is a sum of $\text{Rel}_2(S_0 \rightarrow T_{n-1})$ -like terms. There is no need for coupled recursion relations for the two destinations S_n and T_n , since they are essentially identical through the permutations $a_n \leftrightarrow e_n, c_n \leftrightarrow d_n$, and $S_n \leftrightarrow T_n$. \mathcal{R}_n may be

expressed as the sum of five polynomials in a_n, \dots, T_n (see below). The five prefactors at step n are obtained from those at step $n-1$ by a recursion relation which translates as a 5×5 transfer matrix (such calculations are routinely performed by mathematical software). The value of \mathcal{R}_1 leads to

$$\mathcal{R}_n = (1 \ 0 \ 0 \ 0 \ 0) M_n M_{n-1} \cdots M_1 M_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (2)$$

where M_n 's coefficients M_{ij} are ($\bar{x} \equiv 1-x$)

$$\begin{aligned} M_{11} &= (a_n + b_n e_n T_n - a_n b_n e_n T_n) S_n, \\ M_{12} &= (d_n + b_n c_n T_n - d_n b_n c_n T_n) S_n, \\ M_{13} &= a_n d_n S_n + b_n (\chi_n + c_n e_n) S_n T_n, \\ M_{14} &= -M_{44} = a_n e_n M_{42}, \\ M_{15} &= -M_{45} = c_n d_n M_{41}, \\ M_{21} &= (e_n + b_n a_n S_n - e_n b_n a_n S_n) T_n, \\ M_{22} &= (c_n + b_n d_n S_n - c_n b_n d_n S_n) T_n, \\ M_{23} &= c_n e_n T_n + b_n (\chi_n + a_n d_n) S_n T_n, \\ M_{24} &= -M_{54} = a_n e_n M_{52}, \\ M_{25} &= -M_{55} = c_n d_n M_{51}, \\ M_{31} &= -(a_n b_n + a_n e_n + b_n e_n - 2 a_n b_n e_n) S_n T_n, \\ M_{32} &= -(b_n c_n + b_n d_n + c_n d_n - 2 b_n c_n d_n) S_n T_n, \\ M_{33} &= ((1 - 2 b_n) \chi_n - b_n (c_n e_n + a_n d_n)) S_n T_n, \\ M_{34} &= -M_{14} - M_{24}, \\ M_{35} &= -M_{15} - M_{25}, \\ M_{41} &= \bar{a}_n \bar{b}_n e_n S_n T_n, \\ M_{42} &= \bar{b}_n c_n \bar{d}_n S_n T_n, \\ M_{43} &= \bar{b}_n (\chi_n + c_n e_n) S_n T_n, \\ M_{51} &= a_n \bar{b}_n \bar{e}_n S_n T_n, \\ M_{52} &= \bar{b}_n \bar{c}_n d_n S_n T_n, \\ M_{53} &= \bar{b}_n (\chi_n + a_n d_n) S_n T_n, \end{aligned} \quad (3)$$

with $\chi_n = \bar{a}_n \bar{c}_n d_n e_n + a_n c_n \bar{d}_n \bar{e}_n - a_n c_n d_n e_n$. In the $n=0$ case, $a_0 = 1$ and $c_0 = d_0 = e_0 = 0$. The five abovementioned polynomials are actually given by the first row of M_n . $\text{Rel}_2(S_0 \rightarrow T_n)$ is given by eq. (2) if the left vector is $(0 \ 1 \ 0 \ 0 \ 0)$. We have here another useful instance of a product of random matrices [19].

The case of identical reliabilities $a_n = \dots = e_n = p$ (unless $n=0$, see the restriction above) and $S_n = T_n = \rho$ is worth investigating, since only the n^{th} power of one matrix needs be taken. Because of the recursion relation between successive values of \mathcal{R}_n , the generating function $\mathcal{G}(z) = \sum_{n=0}^{\infty} \mathcal{R}_n z^n$ is a rational fraction of z . Its denominator $\mathcal{D}(z)$ is derived from the characteristic polynomial of the transfer matrix, taken at $1/z$. $\mathcal{G}(z)$ is deduced from the known first terms of the $\mathcal{G}(z) \mathcal{D}(z)$'s expansion:

$$\mathcal{G}(z) = \frac{1}{2} \rho (1 - p \rho) + \frac{\mathcal{N}(z)}{\mathcal{D}(z)}, \quad (4)$$

$$\begin{aligned} \mathcal{N}(z) &= \frac{1}{2} \rho (1 + p \rho) \\ &\quad - \frac{1}{2} p^2 \rho^3 (2 - 10p + 13p^2 - 4p^3 - p^3 \rho) z \\ &\quad + (1-p)^2 p^5 (2 - 4p + p^2) (1-\rho) \rho^5 z^2, \end{aligned} \quad (5)$$

$$\begin{aligned} \mathcal{D}(z) &= 1 - p \rho (2 + 4p \rho - 14p^2 \rho + 13p^3 \rho - 4p^4 \rho) z \\ &\quad + 2 (1-p) p^3 \rho^3 (2 - 7p + 4p^2 + 7p^2 \rho \\ &\quad \quad - 10p^3 \rho + 5p^4 \rho - p^5 \rho) z^2 \\ &\quad - 4 (2-p) (1-p)^3 p^6 (1-\rho) \rho^5 z^3. \end{aligned} \quad (6)$$

Equations (4–6) are simpler for perfect nodes, because the denominator is of degree 2 in z ; a partial fraction decomposition provides

$$\mathcal{R}_n = \frac{1-p}{2} \delta_{n,0} + a_+ \lambda_+^n + a_- \lambda_-^n, \quad (7)$$

$$\lambda_{\pm} = \frac{p}{2} \left[2 + 4p - 14p^2 + 13p^3 - 4p^4 \pm \sqrt{\mathcal{A}} \right], \quad (8)$$

$$a_{\pm} = \frac{1+p}{4} \pm \frac{2+2p+10p^2-27p^3+19p^4-4p^5}{4\sqrt{\mathcal{A}}}, \quad (9)$$

$$\begin{aligned} \mathcal{A} &= 4 + 32p^2 - 204p^3 + 452p^4 - 516p^5 \\ &\quad + 329p^6 - 112p^7 + 16p^8. \end{aligned} \quad (10)$$

As n grows, $\mathcal{R}_n \approx a_+ \lambda_+^n$: the two-terminal reliability exhibits a power-law behavior, the scaling factor being λ_+ , the eigenvalue of largest modulus. Alternatively, $\mathcal{R}_n \sim \exp(-n/\xi)$, where $\xi = -1/\ln(\lambda_+)$ is the correlation length of percolation theory [7].

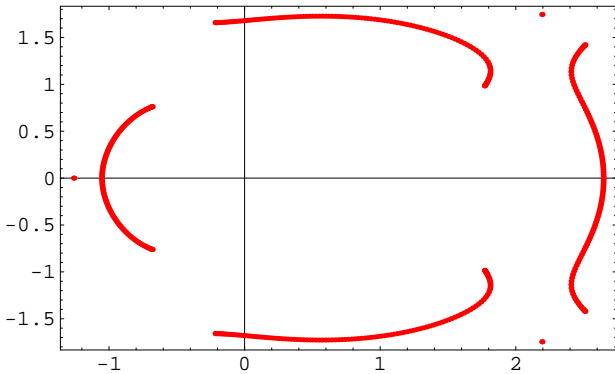


Fig. 2: Location of the complex zeros of the two-terminal reliability polynomial \mathcal{R}_n for $n = 120$ and $\rho = 0.05$.

The location of the zeros of $\text{Rel}_2(p)$ in the complex plane is also worth investigating. The situation differs from that for chromatic [4, 11] and all-terminal polynomials [9], because $\text{Rel}_2(p)$ is not a graph invariant. The new twist lies in the extra parameter at our disposal, the node reliability ρ , which has a deep impact on the curves to which the zeros of $\text{Rel}_2(p)$ converge as $n \rightarrow \infty$. The critical values of ρ at which shape transitions occur may be deduced [20] from the three roots of $\mathcal{D}(1/z)$ and will be detailed elsewhere [21]. A sample of the richness of behavior is displayed in fig. 2 for the K_4 ladder and $\rho = 0.05$. We see (i)

well-separated “curves” (ii) a triplet of doublets, whose separation vanishes exponentially with n , converging to three roots of $2 + 2\rho + 4(3\rho + 1)\rho p - (40\rho + 11)\rho p^2 + (45\rho + 4)\rho p^3 - 20\rho^2 p^4 + 3\rho^2 p^5 = 0$. Actually, this triplet is present only for $\rho < \rho_{c1} \approx 0.17522138$ (ρ_{c1} is a root of a polynomial of degree 10); if $\rho_{c1} < \rho < \rho_{c2} \approx 0.4066578$, only the two rightmost isolated points are present. The leftmost one, located on the real negative axis, is asymptotically given by $-(2\rho)^{-1/3} + 25/24 + O(\rho^{1/3})$; for the other two, ρ must be replaced by $\rho e^{\pm 2i\pi}$. By contrast, the algebraic curves’ asymptotic limit is a circle of radius $(2\rho)^{-1/4}$ centered at $(27/32, 0)$.

K_3 cylinder. – In the second architecture of fig. 1, S_0 is still the source while S_n , T_n , and U_n are the three possible destinations (the last two are equivalent through a permutation of variables). The crucial point is to take $f_n \neq 0$, because in the successive applications of eq. (1), the merging of nodes entails a secondary graph in which S_{n-1} and U_{n-1} are connected. As mentioned above, the dummy — with respect to the Manhattan-like strip — link f_n between S_n and U_n must therefore be present right at the start; this allows us to unveil the *coupled* recursion relations between the source and *all* the destinations. Each source-destination reliability is a sum of eight polynomials in reliabilities indexed by n . This could lead to a 24×24 transfer matrix \tilde{M}_n ; rearrangements of terms reduce its size to 13×13 , even when $f_n = 0$. The final expressions are much lengthier than eq. (3) and will be provided elsewhere [21]. For perfect nodes and $f_n \equiv 0$, $\tilde{\mathcal{G}}(S_0 \rightarrow U_n)$ is given by $\tilde{\mathcal{N}}/(\tilde{\mathcal{D}}_1 \tilde{\mathcal{D}}_2)$:

$$\begin{aligned} \tilde{\mathcal{N}} &= p^2 - (1-p) p^4 (3 + 3p - 4p^2) z \\ &\quad + (1-p)^3 p^6 (2 + 11p - 3p^2 - 2p^3) z^2 \\ &\quad + (1-p)^3 p^8 (2 - 4p + 3p^2 + 11p^3 - 13p^4 + 3p^5) z^3 \\ &\quad - (1-p)^4 p^{10} (3 + 6p - 12p^2 + 10p^3 - 10p^4 + 4p^5) z^4 \\ &\quad + (1-p)^6 p^{12} (1 + 8p - p^2 - 5p^3 - p^4 + p^5) z^5 \\ &\quad - (1-p)^8 p^{15} (2 + 5p - 4p^2) z^6 + (1-p)^{10} p^{18} z^7, \end{aligned} \quad (11)$$

$$\begin{aligned} \tilde{\mathcal{D}}_1 &= 1 - (1-p^2) p (1 + p - p^2) z \\ &\quad + (1-p)^2 p^3 (1 + p + p^2 - 2p^3) z^2 - (1-p)^4 p^6 z^3, \end{aligned} \quad (12)$$

$$\begin{aligned} \tilde{\mathcal{D}}_2 &= 1 - p (2 + 2p + p^2 - 9p^3 + 5p^4) z \\ &\quad + (1-p) p^2 (1 + 5p + 5p^2 - 6p^3 - 15p^4 \\ &\quad \quad + 13p^5 + p^6 - 2p^7) z^2 \\ &\quad - (1-p)^2 p^4 (2 + 6p + 6p^2 - 26p^3 + 17p^4 \\ &\quad \quad - 18p^5 + 27p^6 - 16p^7 + 3p^8) z^3 \\ &\quad + (1-p)^4 p^6 (1 + 6p + 4p^2 - p^3 - 17p^4 \\ &\quad \quad + 9p^5 + 3p^6 - 2p^7) z^4 \\ &\quad - (1-p)^6 p^9 (2 + 4p + p^2 - 7p^3 + 3p^4) z^5 \\ &\quad + (1-p)^8 p^{12} z^6. \end{aligned} \quad (13)$$

The eigenvalue of greatest modulus λ_{\max} involved in the asymptotic power-law behavior obeys $\widehat{\mathcal{D}}_2(1/\lambda_{\max}) = 0$ (if $f_n \equiv p$, the denominator of $\widehat{\mathcal{G}}$ is a product of polynomials of degrees 3 and 4 in z). The degree of the denominator leads us to expect that the “width” of the network should drastically affect the size of the transfer matrices.

Transfer matrices for the all-terminal reliability Rel_A . – Nodes may be viewed here as perfect, and simpler calculations may be done because eq. (1) has one less term. For the K_4 ladder, the transfer matrix is 2×2 :

$$\text{Rel}_A(n) = (10) \widehat{M}_n \cdots \widehat{M}_{n-1} \widehat{M}_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad (14)$$

The matrix elements $(\widehat{M}_n)_{ij}$ of \widehat{M}_n are

$$(\widehat{M}_n)_{11} = [(a_n + e_n)(c_n + d_n) - 2a_n c_n d_n e_n] \overline{b}_n + [(a_n/e_n) + (c_n/d_n)] b_n, \quad (15)$$

$$(\widehat{M}_n)_{12} = a_n c_n d_n e_n \left[\frac{1}{a_n} + \frac{1}{c_n} + \frac{1}{d_n} + \frac{1}{e_n} - 3 \right] \overline{b}_n + [(a_n/e_n)(c_n/d_n)] b_n, \quad (16)$$

$$(\widehat{M}_n)_{21} = [(a_n/c_n) + (d_n/e_n) - 2(a_n e_n/c_n d_n)] \overline{b}_n - (\widehat{M}_n)_{11}, \quad (17)$$

$$(\widehat{M}_n)_{22} = (c_n + d_n - 2c_n d_n)(a_n + e_n - 2a_n e_n) \overline{b}_n - (\widehat{M}_n)_{12}; \quad (18)$$

in \widehat{M}_0 , $a_0 = 1$ and $c_0 = d_0 = e_0 = 0$. This is a special case of a multivariate Tutte polynomial [22]. If $a_n = \dots = e_n \equiv p$, we recover Chang and Shrock’s result (appendix 4.2 of [9]): $\widehat{\mathcal{G}}_A(z) = \widehat{\mathcal{N}}_A(z)/\widehat{\mathcal{D}}_A(z)$ with $\widehat{\mathcal{N}}_A(z) = p + p^3(1-p)(4-3p)z$ and $\widehat{\mathcal{D}}_A(z) = 1 - p^2(12 - 26p + 21p^2 - 6p^3)z + 2p^5(1-p)^3(2-p)z^2$. The asymptotic power-law scaling factor is $\zeta_+ = \frac{1}{2}p^2(12 - 26p + 21p^2 - 6p^3 + \sqrt{\mathcal{B}})$ with $\mathcal{B} = 144 - 640p + 1236p^2 - 1308p^3 + 793p^4 - 260p^5 + 36p^6$.

Conclusion and perspectives. – The two-terminal reliability of undirected networks may be expressed by a product of transfer matrices, in which each edge and node reliability is exactly taken into account. This result is easily extended to the all-terminal reliability with nonuniform links, as well as to directed graphs. We can now go beyond series-parallel simplifications and look for new (wider) families of exactly solvable, meshed architectures that may be useful for general reliability studies (as building blocks for more complex networks), for the enumeration of self-avoiding walks on lattices, and for percolation with imperfect bonds *and* sites. Since the true generating function is itself a rational fraction, Padé approximants should provide efficient upper or lower bounds for these studies. Moreover, individual reliabilities can be viewed as average values of random variables. Having access to *each* edge or node allows the introduction of disorder or correlations in calculations. Finally, in a more applied perspective, we should mention that the failure frequency ν

of a given connection is another important performance index of networks. If equipment i of reliability p_i has a failure rate λ_i , $\nu = \sum_i \lambda_i p_i \partial \text{Rel}_2 / \partial p_i$. The matrix factorization makes the calculation straightforward, since each p_i appears in one transfer matrix only.

Valuable suggestions by N. Bontemps, R. Combescot, F. Glas and G. Patriarche are gratefully acknowledged.

REFERENCES

- [1] E. F. MOORE and C. E. SHANNON, *Journal Franklin Institute*, **262** (1956) 191; *ibid.*, **262** (1956) 281.
- [2] C. SINGH and R. BILLINTON, *System reliability modelling and evaluation* (Hutchinson, London) 1977.
- [3] F. Y. WU, *J. Phys. A: Math. Gen.*, **15** (1982) L395.
- [4] N. L. BIGGS and G. H. J. MEREDITH, *J. Combin. Theory B*, **20** (1976) 5; N. BIGGS, *Algebraic graph theory* (Cambridge University Press, Cambridge) 1993; N. L. BIGGS, *J. Combin. Theory B*, **82** (2001) 19.
- [5] C. J. COLBOURN, *The combinatorics of network reliability* (Oxford University Press, Oxford) 1987.
- [6] D. R. SHIER, *Network reliability and algebraic structures* (Clarendon Press, Oxford) 1991.
- [7] G. GRIMMETT, *Percolation* (Springer, Berlin) 1999.
- [8] B. D. HUGHES, *Random walks and random environments: Random Environments* (Clarendon Press, Oxford) 1996.
- [9] S.-C. CHANG and R. SHROCK, *Int. J. Mod. Phys.*, **B15** (2001) 443.
- [10] S.-C. CHANG and R. SHROCK, *J. Statist. Phys.*, **112** (2003) 1019; *Physica A*, **364** (2006) 231.
- [11] J. SALAS and A. D. SOKAL, *J. Statist. Phys.*, **104** (2001) 609; J. JACOBSEN, J. SALAS, and A. SOKAL, *J. Statist. Phys.*, **112** (2003) 921.
- [12] D. J. A. WELSH and C. MERINO, *J. Math. Phys.*, **41** (2000) 1127.
- [13] G. F. ROYLE and A. D. SOKAL, *J. Combin. Theory B*, **91** (2004) 345.
- [14] R. ALBERT and A.-L. BARABÁSI, *Rev. Mod. Phys.*, **74** (2002) 47.
- [15] J. OXLEY and D. WELSH, *Combin. Probab. Comput.*, **11** (2002) 403.
- [16] D. WELSH, *Complexity: Knots, colourings and counting* (Cambridge University Press, Cambridge) 1993.
- [17] B. DERRIDA and J. VANNIMENUS, *J. Phys. (Lettres)*, **41** (1980) L473.
- [18] P. D. SEYMOUR and D. J. A. WELSH, *Ann. Discr. Math.*, **3** (1978) 227.
- [19] A. CRISANTI, G. PALADIN, and A. VULPIANI, *Products of random matrices in statistical physics* (Springer, Berlin) 1999.
- [20] S. BERAHA, J. KAHANE, and N. J. WEISS, *Studies in Foundations and Combinatorics, Advances in Mathematics Supplementary Studies*, edited by G.-C. ROTA, Vol. **1** (Academic Press, New York) 1978, p. 213.
- [21] C. TANGUY, in preparation.
- [22] A. D. SOKAL, *Surveys in Combinatorics 2005*, edited by B. S. WEBB (Cambridge University Press, Cambridge) 2005, p. 173.