

CONVEX CHAINS IN \mathbb{Z}^2

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Abstract: A detailed combinatorial analysis of lattice convex polygonal lines of \mathbb{N}^2 joining 0 to (n, n) is presented. We derive consequences on the line having the largest number of vertices as well as the cardinal and limit shape of lines having few vertices. The proof refines a statistical physical method used by Sinai to obtain the typical behavior of these lines, allied to some Fourier analysis. Limit shapes of convex lines joining 0 to (n, n) and having a given total length are also characterized.

1. INTRODUCTION

In 1926, Jarník found an equivalent of the maximal number of integral points that a portion of length n of a strictly convex function can interpolate. He obtained an explicit constant times $n^{2/3}$. This work was at the origin of many works of Diophantine analysis, and we refer the reader to the papers of W. Schmidt [11] and Bombieri and Pila [6] for more recent results, discussions and open questions on this subject.

One may slightly change Jarník's framework, and consider the set of integral points which are interpolated by the graph on $[0, n]$ of an increasing and strictly convex function satisfying $f(0) = 0$ and $f(n) = n$. It turns out that this question is related to another family of works we shall discuss now.

In 1979, Arnol'd [1] considered the question of the number of equivalence classes of convex lattice polygons having a given integer as area (we say that two polygons having their vertices on \mathbb{Z}^2 are equivalent if one is the image of the other by an automorphism of \mathbb{Z}^2). Later, Vershik changed the constraint in this question and raised the question of the number, and typical shape, of convex lattice polygons included in a large box $[-N, N]^2$. In 1994, three different solutions to this problem were found by Bárány [3], Vershik [13] and Sinai [12]. To be more precise, let Π_n be the set of convex polygonal lines Γ joining 0 to (n, n) , we shall sometimes refer at *convex chains*, such that the vertices of Γ are points of \mathbb{Z}^2 and the angle between each side of Γ and the horizontal axis is in the interval $[0, \pi/2]$. They prove, firstly, that the cardinal of Π_n is of the order $\exp(n^{2/3}(3(\zeta(3)/\zeta(2))^{1/3} + o(1)))$, when n goes to infinity. Secondly, they prove that the number of vertices constituting a typical line is equivalent, when n goes to infinity, to $\frac{n^{2/3}}{(\zeta^2(3)\zeta(2))^{1/3}}$. Finally, they prove that the limit shape of a typical convex polygonal line is the arc of a parabola, which maximizes the affine perimeter. Later, Vershik and Zeitouni [15] proved, for a class of analogous problems, a large deviation principle involving the affine perimeter of the line. Finally, Bárány and Prodromou, while considering the maximal number of vertices for a lattice polygon included in a convex set, proved recently in [6] the analog of Jarník's result, and they showed that the largest number of vertices for an element of Π_n is equivalent to $3(\frac{n}{\pi})^{2/3}$.

The nature of these results shows that this problem is related to both affine differential geometry and geometry of numbers. Indeed, the parabola found as limit shape coincides with the convex curve inside the square having the largest affine perimeter. Furthermore, the appearance of the values of the Riemann zeta function shows the arithmetic aspects

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of the problem. One could show indeed that if the lattice \mathbb{Z}^2 was replaced by a Poisson Point Process having intensity one (which can be thought as the most isotropic “lattice” one can imagine), the constants $(\zeta^2(3)\zeta(2))^{-1/3} = 0,749$ and $3(\zeta(3)/\zeta(2))^{1/3} = 2.702$ would be merely raised respectively to 1 and 3 in probability (cf. the concluding remarks at the end of the paper).

In section 3, we precise the combinatorial aspect of the result of [3], [12], [13]. We obtain, for any positive number \mathbf{c} , a logarithmic equivalent of the number of lines having $\lfloor \mathbf{c}n^{2/3} \rfloor$ vertices. This question is reminiscent of other ones considered, for instance, by Arratia and Tavaré [2] or Vershik and Yakubovich [14] who were studying combinatorial objects (permutations, polynomials over finite field, Young tableaux...) having a specified number of summands (according to the setting, we call summands, cycles, irreducible divisors etc...). In both cases, like in Sinai’s approach, the method, borrowed from classical ideas of statistical physics, relies on the introduction of a grand canonical ensemble which endows the considered combinatorial object with a parametrized probability measure. Then, the strategy consists in calibrating the parameters of the probability in order to fit with the constraints one has to deal with. Concretely, in our question, it turns out that one can add one parameter in Sinai’s probability distribution that makes it possible to take into account, not only the location of the extreme point of the chain but also the number of vertices it contains. The asymptotic of the number of lines having $\lfloor \mathbf{c}n^{2/3} \rfloor$ vertices can finally be expressed in terms of the polylogarithm functions Li_2 and Li_3 .

In section 4, we deduce from this analysis the result of [6] about the largest number of vertices for an element of Π_n . We show how the proof of this result, using our analysis, suggests the more direct argument “à la Jarník” presented in [6].

In section 5, we derive results about the cardinal and limit shape of lines having few vertices, answering a question of Vershik: it turns out that there is a limit shape for lines having for instance $\lfloor \sqrt{n} \rfloor$ vertices which is, a little bit surprisingly, still the arc of parabola.

In section 6, we go back to Jarník’s problem. In addition to Jarník’s result we recover, we give the asymptotic of the number of chains, typical number of vertices, and limit shape, which is an arc of a circle, in this different framework.

Furthermore, one may mix both type of conditions and the statistical physical method still applies. In section 7, we obtain, for the convex lines joining 0 to (n, n) and having a given total length, a continuous family of convex limit shapes which interpolates the diagonal of the square and the two sides of the square, going through the above arc of parabola and arc of circle.

2. A ONE-TO-ONE CORRESPONDENCE

We start this paper by reminding the correspondence between finite convex polygonal lines issuing from 0 whose vertices define increasing sequences in both coordinates and finite subsets of the set of pairs of coprime positive integers each of them affected with a positive integer.

More precisely, let Π denote the set of finite planar convex polygonal lines Γ issuing from 0 such that the vertices of Γ are points of \mathbb{Z}^2 and the angle between each side of Γ and the horizontal axis is in the interval $[0, \pi/2]$. Now consider the set X of all pairs (x_1, x_2) of coprime positive integers including the pairs $(0, 1)$ and $(1, 0)$.

We present the following correspondence which is borrowed from Sinai [12]:

Lemma 1. [12] *The space Π is in one-to-one correspondence with the space $C_0(X)$ of finite nonnegative integer-valued functions $\nu(x)$ on X (where $\nu(x)$ is said to be finite if $\nu(x) \neq 0$ only for finitely many x).*

Proof: An element of Π is defined by the set of its sides since the sides are ordered according to their slope. Now, each side is defined by a vector (y_1, y_2) one can write $y \cdot (\frac{y_1}{y}, \frac{y_2}{y})$ where $y := \text{g.c.d.}(y_1, y_2)$. Now $(\frac{y_1}{y}, \frac{y_2}{y}) \in X$ and we define $\nu(\frac{y_1}{y}, \frac{y_2}{y}) = \text{g.c.d.}(y_1, y_2)$.

Conversely, given $\nu \in C_0(X)$, we order the (finite) set of points x such that $\nu(x) > 0$, according to the increasing values of $\frac{x_2}{x_1}$, and obtain a finite sequence of points $x^{(j)}$. Now, the polygon with consecutive sides $\nu(x^{(j)})x^{(j)}$ is convex. \square

3. A DETAILED COMBINATORIAL ANALYSIS

Let us introduce now the following parametrized probability distribution on $C_0(X)$, which, by the correspondence of Lemma 1, corresponds to a probability distribution on Π :

$$\begin{aligned} \forall \nu \in C_0(X), \quad Q_{z_1, z_2, \lambda}(\nu) &= \prod_{(x_1, x_2) \in X} \frac{(z_1^{x_1} z_2^{x_2})^{\nu(x)} \lambda^{1_{\nu(x) \neq 0}}}{1 + \frac{\lambda z_1^{x_1} z_2^{x_2}}{1 - z_1^{x_1} z_2^{x_2}}} \\ &= \prod_{(x_1, x_2) \in X} (z_1^{x_1} z_2^{x_2})^{\nu(x)} \lambda^{1_{\nu(x) \neq 0}} \frac{1 - z_1^{x_1} z_2^{x_2}}{1 + (\lambda - 1) z_1^{x_1} z_2^{x_2}} \end{aligned}$$

where $z_1, z_2 \in [0, 1[$, and $\lambda > 0$.

Taking $\lambda = 1$, the probability $Q_{z_1, z_2, 1}$ is nothing but the two-parameter probability distribution introduced by Sinai. Under $Q_{z_1, z_2, \lambda}$, the variables $(\nu(x))_{x \in X}$ are still independent, like in Sinai's framework, but follow a geometric law only for $\lambda = 1$.

Loosely speaking, compared to Sinai's probability distribution, $Q_{z_1, z_2, \lambda}$ corresponds to the introduction of a *penalization* of the probability by a factor λ each time a vertex appears.

Now, we define $\Pi(n_1, n_2, N)$ the set of polygons with endpoint (n_1, n_2) having N vertices, and we denote by $\mathcal{N}(n_1, n_2, N)$ its cardinal.

Since $Q_{z_1, z_2, \lambda}(\nu)$ depends only on $\sum_{x \in X} x_1 \nu(x)$, $\sum_{x \in X} x_2 \nu(x)$, and $\sum_{x \in X} 1_{\nu(x) \neq 0}$, we deduce that the conditional distribution induced by $Q_{z_1, z_2, \lambda}$ on $\Pi(n_1, n_2, N)$ is uniform, and we can write

$$\mathcal{N}(n_1, n_2, N) = z_1^{-n_1} z_2^{-n_2} \lambda^{-N} \prod_{(x_1, x_2) \in X} \left(\frac{1 - z_1^{x_1} z_2^{x_2}}{1 + (\lambda - 1) z_1^{x_1} z_2^{x_2}} \right)^{-1} Q_{z_1, z_2, \lambda}(\Pi(n_1, n_2, N)) \quad (\star)$$

In order to get a logarithmic equivalent of $\mathcal{N}(n_1, n_2, N)$, our strategy is to choose the three parameters so that

$$E_{z_1, z_2, \lambda} \left(\sum_{x \in X} x_1 \nu(x) \right) = \sum_{(x_1, x_2) \in X} \frac{x_1 \lambda z_1^{x_1} z_2^{x_2}}{(1 - z_1^{x_1} z_2^{x_2})(1 + (\lambda - 1) z_1^{x_1} z_2^{x_2})} = n_1 \quad (1)$$

$$E_{z_1, z_2, \lambda} \left(\sum_{x \in X} x_2 \nu(x) \right) = \sum_{(x_1, x_2) \in X} \frac{x_2 \lambda z_1^{x_1} z_2^{x_2}}{(1 - z_1^{x_1} z_2^{x_2})(1 + (\lambda - 1) z_1^{x_1} z_2^{x_2})} = n_2 \quad (2)$$

$$E_{z_1, z_2, \lambda} \left(\sum_{x \in X} 1_{\nu(x) \neq 0} \right) = \sum_{(x_1, x_2) \in X} \frac{\lambda z_1^{x_1} z_2^{x_2}}{1 + (\lambda - 1) z_1^{x_1} z_2^{x_2}} = N \quad (3)$$

This will indeed lead to an estimation of $Q_{z_1, z_2, \lambda}(\Pi(n_1, n_2, N))$ which will be much bigger than exponentially small (see lemma 2 below).

Now, our asymptotic setting is the following: $n_1 = n_2 = n \rightarrow +\infty$, and $N/n^{2/3} \rightarrow \mathbf{c}$. Let us note that we could have chosen a more general setting, taking n_2/n_1 converging to an

arbitrary constant, but this does not add any difficulty and just make the computations less readable, whereas the final result can be guessed from the case $n_1 = n_2$.

Theorem 1. For all $\lambda > 0$,

$$\ln(\mathcal{N}(n, n, [\mathbf{c}(\lambda)n^{2/3}])) = n^{2/3}(\mathbf{e}(\lambda) + o(1))$$

where $\mathbf{c}(\lambda)$ and $\mathbf{e}(\lambda)$ are positive functions on $]0, +\infty[$, defined by :

$$\mathbf{c}(\lambda) = \frac{\lambda \text{Li}_2(1-\lambda)}{1-\lambda} \times \frac{1}{\zeta(2)^{1/3}(\zeta(3) - \text{Li}_3(1-\lambda))^{2/3}}$$

$$\mathbf{e}(\lambda) = 3 \left(\frac{\zeta(3) - \text{Li}_3(1-\lambda)}{\zeta(2)} \right)^{1/3} - \frac{\lambda \ln(\lambda) \text{Li}_2(1-\lambda)}{1-\lambda} \times \frac{1}{\zeta(2)^{1/3}(\zeta(3) - \text{Li}_3(1-\lambda))^{2/3}}$$

Remark: The function $\mathbf{e}(\lambda)$ is maximal for $\lambda = 1$, and, in this case,

$$\mathbf{c}(1) = \frac{1}{(\zeta(2)\zeta(3)^2)^{1/3}}$$

$$\mathbf{e}(1) = 3 \left(\frac{\zeta(3)}{\zeta(2)} \right)^{1/3}$$

This implies the results of [3], [13] and [12] about the typical behavior of a convex chain, in a straighter manner than by the original arguments.

We refer the reader to Figure 1 which represents the graphic of $(\mathbf{c}(\lambda), \mathbf{e}(\lambda))$ for $\lambda \in]0, +\infty[$.

Proof: Let us consider parameters z_i , so that $z_1 = z_2 = z$ with z of the form $z = 1 - \frac{\delta}{n^{1/3}}$ where the new parameter δ is itself depending on n , but is bounded from above and below by two strictly positive constants. Equality (1) then writes

$$n^{1/3} \sum_{(t_1, t_2) \in \frac{1}{n^{1/3}}X} \frac{\lambda t_1 (1 - \frac{\delta}{n^{1/3}})^{n^{1/3}(t_1+t_2)}}{(1 - (1 - \frac{\delta}{n^{1/3}})^{n^{1/3}(t_1+t_2)})(1 + (\lambda - 1)(1 - \frac{\delta}{n^{1/3}})^{n^{1/3}(t_1+t_2)})} = n$$

Now, using Theorem 459 of the book of Hardy and Wright [8], saying that the set X has density $\frac{1}{\zeta(2)}$ in \mathbb{N}^2 , we get that the previous sum is equivalent to

$$\left(\frac{1}{\zeta(2)} \int_0^{+\infty} \int_0^{+\infty} \frac{\lambda t_1 e^{-\delta(t_1+t_2)}}{(1 - e^{-\delta(t_1+t_2)})(1 + (\lambda - 1)e^{-\delta(t_1+t_2)})} dt_1 dt_2 \right) \times n$$

Using a linear change of variable, $s_i := \delta t_i$, and the identity

$$\frac{\lambda X}{(1-X)(1+(\lambda-1)X)} = \frac{1}{1-X} - \frac{1}{1-(1-\lambda)X}$$

the previous expression writes

$$\left(\frac{1}{\zeta(2)\delta^3} \int_0^{+\infty} \int_0^{+\infty} \frac{s_1}{1 - e^{-(s_1+s_2)}} - \frac{s_1}{1 - (1-\lambda)e^{-(s_1+s_2)}} ds_1 ds_2 \right) \times n$$

Using the series expansion $\frac{1}{1-ae^{-(s_1+s_2)}} = \sum_{k \geq 0} a^k e^{-k(s_1+s_2)}$, we get finally the following expressions:

$$\frac{n}{\zeta(2)\delta^3} \left(\sum_{k \geq 1} \frac{1}{k^3} - \sum_{k \geq 1} \frac{(1-\lambda)^k}{k^3} \right) = \frac{n}{\zeta(2)\delta^3} (\zeta(3) - \text{Li}_3(1-\lambda))$$

(The series expansion on the left hand side does make sense only if the value of λ is restricted to $[0, 1]$, but the expression in terms of the polylogarithm function Li_3 is well defined for all $\lambda > 0$, and coincide with the computed integral)

We therefore get a first condition on the parameters δ and λ :

$$\delta^3 \underset{\infty}{\sim} \frac{1}{\zeta(2)} (\zeta(3) - \text{Li}_3(1 - \lambda)) \quad (5)$$

Of course, in our symmetric setting, the equality (2) yields the same condition. Let us now write equality (3):

$$\sum_{(t_1, t_2) \in \frac{1}{n^{1/3}} X} \frac{\lambda(1 - \frac{\delta}{n^{1/3}})^{n^{1/3}(t_1+t_2)}}{1 + (\lambda - 1)(1 - \frac{\delta}{n^{1/3}})^{n^{1/3}(t_1+t_2)}} = [\mathbf{c}n^{2/3}]$$

In the same way as previously, we deduce

$$\left(\frac{1}{\zeta(2)} \int_0^{+\infty} \int_0^{+\infty} \frac{\lambda e^{-\delta(t_1+t_2)}}{1 - (\lambda - 1)e^{-\delta(t_1+t_2)}} dt_1 dt_2 \right) \times n^{2/3} \underset{\infty}{\sim} \mathbf{c}n^{2/3}$$

The left hand side of the equality writes:

$$\frac{\lambda}{\delta^2 \zeta(2)} \sum_{k \geq 1} \frac{(1 - \lambda)^{k-1}}{k^2} \times n^{2/3} = \frac{\lambda}{\delta^2 \zeta(2)} \frac{\text{Li}_2(1 - \lambda)}{1 - \lambda} n^{2/3}$$

Hence, the second condition on the parameters δ and λ writes:

$$\frac{\lambda}{\delta^2 \zeta(2)} \frac{\text{Li}_2(1 - \lambda)}{1 - \lambda} \underset{\infty}{\sim} \mathbf{c}$$

Now, back to $\mathcal{N}(n_1, n_2, N)$, we use (\star) and replacing z_1 and z_2 by $(1 - \frac{\delta}{n^{1/3}})$, we get:

$$\begin{aligned} \mathcal{N}(n, n, [\mathbf{c}n^{2/3}]) &= (1 - \frac{\delta}{n^{1/3}})^{-2n} \lambda^{-[\mathbf{c}n^{2/3}]} \times \\ &\exp \left(\sum_{(t_1, t_2) \in \frac{1}{n^{1/3}} X} \ln \left(1 - (1 - \lambda) \left(1 - \frac{\delta}{n^{1/3}} \right)^{n^{1/3}(t_1+t_2)} \right) - \ln \left(1 - \left(1 - \frac{\delta}{n^{1/3}} \right)^{n^{1/3}(t_1+t_2)} \right) \right) \times \\ &Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^{2/3}])) \end{aligned}$$

This leads to

$$\begin{aligned} \mathcal{N}(n, n, [\mathbf{c}n^{2/3}]) &= Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^{2/3}])) \times \\ &\exp \left(n^{2/3} \left(2\delta - \mathbf{c} \ln \lambda + \frac{1}{\zeta(2)} \int_{\mathbb{R}_+^2} \ln(1 - (1 - \lambda)e^{-\delta(t_1+t_2)}) - \ln(1 - e^{-\delta(t_1+t_2)}) dt_1 dt_2 + o(1) \right) \right) \end{aligned}$$

Using again the linear change of variable, $s_i := \delta t_i$ and afterwards the series expansion $\ln(1 - ae^{-\delta(t_1+t_2)}) = - \sum_{k \geq 1} a^k \frac{e^{-k\delta(t_1+t_2)}}{k}$, we get

$$\begin{aligned} \mathcal{N}(n, n, [\mathbf{c}n^{2/3}]) &= Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^{2/3}])) \times \\ &\exp \left(n^{2/3} \left(2\delta - \mathbf{c} \ln \lambda + \frac{1}{\delta^2 \zeta(2)} (\zeta(3) - \text{Li}_3(1 - \lambda)) + o(1) \right) \right) \end{aligned}$$

Using (5) this can be simplified into:

$$\mathcal{N}(n, n, [\mathbf{c}n^{2/3}]) = Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^{2/3}])) \times \exp \left(n^{2/3} (3\delta - \mathbf{c} \ln \lambda + o(1)) \right)$$

We now have to remark that, according to a type of local limit theorem proved few lines below, the quantity $Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^{2/3}]))$ decreases polynomially. We derive the

expression of $\mathbf{c}(\lambda)$ from (5) and Barany, and the corresponding value of the exponent $\mathbf{e}(\lambda) = 3\delta - \mathbf{c}(\lambda) \ln(\lambda)$. \square

Let us now come back to the estimation of $Q_{z_1, z_2, \lambda}(\Pi(n, n, [cn^{2/3}]))$ which is governed by a kind of local limit theorem for the variables $\sum_{x \in X} x_1 \nu(x)$, $\sum_{x \in X} x_2 \nu(x)$, and $\sum_{x \in X} 1_{\nu(x) \neq 0}$.

Lemma 2. *In the framework of the proof of Theorem 1, $Q_{z_1, z_2, \lambda}(\Pi(n, n, [cn^{2/3}])) = O(\frac{1}{n^{5/3}})$*

Proof: like in the case of the usual local limit theorem for the sum of iid random variables (see [7]), the proof is based on Fourier analysis. However, a dominated convergence argument, like in the local limit theorem, doesn't work and we have to be finer.

We use a real test function h on \mathbb{R}^3 , whose Fourier transform has a compact support, and we are led to the proof of

$$E[h(\sum_X x_1 \nu(x) - n, \sum_X x_2 \nu(x) - n, \sum_X 1_{\{\nu(x) \neq 0\}} - [cn^{2/3}])] = O(\frac{1}{n^{5/3}})$$

In the sequel, we will use, for short, the notations X_1, X_2, X_3 for the random variables $\sum_X x_1 \nu(x)$, $\sum_X x_2 \nu(x)$, $\sum_X 1_{\{\nu(x) \neq 0\}}$, and the overline notation \overline{Z} to denote the centered variable $Z - E[Z]$ associated to a random variable Z . We will also use the notation C to denote a positive constant whose value may change along the lines.

We can now transform the previous expectation into $\int_{\mathbb{R}^3} \hat{h}(t_1, t_2, t_3) d\hat{\mu}_{\overline{X_1}, \overline{X_2}, \overline{X_3}}(t_1, t_2, t_3)$ where $\hat{\mu}_{\overline{X_1}, \overline{X_2}, \overline{X_3}}$ denotes the Fourier transform of the distribution measure $\mu_{\overline{X_1}, \overline{X_2}, \overline{X_3}}$ of the triplet of random variables $(\overline{X_1}, \overline{X_2}, \overline{X_3})$, which is a 2π -periodic function in its three variables, since we deal with integer valued random variables.

Therefore, we have to deal with $\int_{[-\pi, \pi]^3} l(t_1, t_2, t_3) d\hat{\mu}_{\overline{X_1}, \overline{X_2}, \overline{X_3}}(t_1, t_2, t_3)$ where $l(t_1, t_2, t_3) = \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \hat{h}(t_1 + k_1, t_2 + k_2, t_3 + k_3)$ (let us remark that this sum is actually finite).

Now, since all the variables $\nu(x)$ are independent,

$$d\hat{\mu}_{\overline{X_1}, \overline{X_2}, \overline{X_3}}(t_1, t_2, t_3) = (\prod_X \overline{\phi}_x(t_1, t_2, t_3)) dt_1 dt_2 dt_3$$

where $\overline{\phi}_x(t_1, t_2, t_3) = E[e^{i(t_1 x_1 \overline{\nu}(x) + t_2 x_2 \overline{\nu}(x) + t_3 1_{\nu(x) \neq 0})}]$.

Now, denoting by $\rho_x = z_1^{x_1} z_2^{x_2}$, we get

$$Var(\nu(x)) = \frac{\lambda \rho_x ((1 + \rho_x)(1 + (\lambda - 1)\rho_x) - \lambda \rho_x)}{(1 - \rho_x)^2 (1 + (\lambda - 1)\rho_x)^2}$$

$$Cov(1_{\nu(x) \neq 0}, \nu(x)) = \frac{\lambda \rho_x}{(1 + (\lambda - 1)\rho_x)^2}$$

and

$$Var(1_{\nu(x) \neq 0}) = \frac{\lambda \rho_x (1 - \rho_x)}{(1 + (\lambda - 1)\rho_x)^2}$$

From these formulas, we deduce the expression of the covariance matrix Γ_x of the random vector $(x_1 \nu(x), x_2 \nu(x), 1_{\nu(x) \neq 0})$. Now, after summing the matrices Γ_x , and approximating the sums on the set X by integrals, we obtain that the covariance matrix of the vector $\overline{X_1}, \overline{X_2}, \overline{X_3}$, is equivalent when n goes to infinity to

$$\Gamma_n = \begin{pmatrix} n^{4/3} A & n^{4/3} B & nD \\ n^{4/3} B & n^{4/3} A & nD \\ nD & nD & n^{2/3} E \end{pmatrix}$$

(one can state indeed a result of the form: for two given integers k, l , and for a continuous function f , $\sum_X x_1^k x_2^l f(\rho_x) \sim \left(\frac{6}{\pi^2} \int_{\mathbb{R}_+^2} s_1^k s_2^l f(e^{-\delta(s_1+s_2)}) ds_1 ds_2 \right) n^{\frac{2+k+l}{3}}$ as soon as the integral on the right hand side is finite.)

Our task is now to prove that $\int_{[-\pi, \pi]^3} l(\vec{t}) d\hat{\mu}_{\overline{X}_1, \overline{X}_2, \overline{X}_3}(\vec{t}) \sim \int_{[-\pi, \pi]^3} l(\vec{t}) \exp(-\frac{1}{2}\langle \vec{t}, \Gamma_n \vec{t} \rangle) d\vec{t}$ which is easily seen, after a linear change of variable, to be equivalent to some constant times $\frac{l(0)}{n^{5/3}}$.

Let $I_n := [-n^{-2/3} \log n, n^{-2/3} \log n]^2 \times [n^{-1/3} \log n, n^{-1/3} \log n]$. We partition the domain $[-\pi, \pi]^3$ into $I_n \cup ([-\pi, \pi]^3 \setminus I_n)$.

On I_n ,

$$\begin{aligned} |\prod_X \overline{\phi}_x(\vec{t}) - \exp(-\frac{1}{2}\langle \vec{t}, \Gamma_n \vec{t} \rangle)| &\leq \left| \exp(-\frac{1}{2}\langle \vec{t}, \Gamma_n \vec{t} \rangle) - \exp(-\frac{1}{2} \sum_X \langle \vec{t}, \Gamma_x \vec{t} \rangle) \right| \\ &\quad + \exp(-\frac{1}{2}\langle \vec{t}, \sum_X \Gamma_x \vec{t} \rangle) \left| \exp(\frac{1}{2} \sum_X \langle \vec{t}, \Gamma_x \vec{t} \rangle + \ln \overline{\phi}_x(\vec{t})) - 1 \right| \end{aligned}$$

Since $\Gamma_n \sim \sum_X \Gamma_x$, the integration of the first term of the rhs contributes only like $o(\frac{1}{n^{5/3}})$.

Concerning the second term of the rhs,

$$\left| \exp(\frac{1}{2} \sum_X \langle \vec{t}, \Gamma_x \vec{t} \rangle + \ln \overline{\phi}_x(\vec{t})) - 1 \right| \leq \left| \frac{1}{2} \sum_X \langle \vec{t}, \Gamma_x \vec{t} \rangle + \ln \overline{\phi}_x(\vec{t}) \right| e^{|\frac{1}{2} \sum_X \langle \vec{t}, \Gamma_x \vec{t} \rangle + \ln \overline{\phi}_x(\vec{t})|}$$

Now, using the fact that, on I_n , $|\overline{\phi}_x|$ is bounded away from 0, we get, on I_n , that

$$|\ln \overline{\phi}_x(\vec{t}) - (1 - \overline{\phi}_x(\vec{t}))| \leq |1 - \overline{\phi}_x(\vec{t})|^2$$

and

$$\begin{aligned} \left| \frac{1}{2} \sum_X \langle \vec{t}, \Gamma_x \vec{t} \rangle + \ln \overline{\phi}_x(\vec{t}) \right| &\leq \sum_X |1 - \overline{\phi}_x(\vec{t})|^2 + \frac{1}{6} \sum_X E[|\langle \vec{t}, (x_1 \overline{\nu}(x), x_2 \overline{\nu}(x), \overline{1_{\nu(x) \neq 0}}) \rangle|^3] \\ &\leq C \left(\sum_X (t_1^4 x_1^4 + t_2^4 x_2^4 + t_3^4) f(\rho_x) + \sum_X (|t_1|^3 x_1^3 + |t_2|^3 x_2^3 + |t_3|^3) g(\rho_x) \right) \end{aligned}$$

for suitable functions f and g .

This last expression is therefore dominated by a constant times $t_1^4 n^2 + t_2^4 n^2 + t_3^4 n^{2/3} + |t_1|^3 n^{5/3} + |t_2|^3 n^{5/3} + |t_3|^3 n^{2/3}$, which is, on I_n , a $o(t_1^2 n^{4/3} + t_2^2 n^{4/3} + t_3^2 n^{2/3}) = o(\langle \vec{t}, \Gamma_n \vec{t} \rangle)$.

Therefore, the integral on I_n of $|\prod_X \overline{\phi}_x(\vec{t}) - \exp(-\frac{1}{2}\langle \vec{t}, \Gamma_n \vec{t} \rangle)|$ contributes like $o(\frac{1}{n^{5/3}})$.

Finally, it remains to prove that the integral of $|\prod_X \overline{\phi}_x(\vec{t})|$ over $[-\pi, \pi]^3 \setminus I_n$ is $o(\frac{1}{n^{5/3}})$.

Actually, we will prove that, outside I_n , for all $x \in X$, $|\prod_X \overline{\phi}_x(\vec{t})| \leq \exp(-C(\ln n)^2)$.

If we denote $t' := (t_1, t_2)$, so that $\vec{t} = (t', t_3)$, a straightforward computation gives

$$|\overline{\phi}_x(\vec{t})| = \frac{1 + \frac{4\rho_x}{(1+(\lambda-1)\rho_x)^2} \sin^2 \frac{t'.x}{2} - \frac{4\lambda\rho_x}{(1+(\lambda-1)\rho_x)^2} \sin^2 \frac{t'.x+t_3}{2} + \frac{4\lambda\rho_x^2}{(1+(\lambda-1)\rho_x)^2} \sin^2 \frac{t_3}{2}}{1 + \frac{4\rho_x}{(1-\rho_x)^2} \sin^2 \frac{t'.x}{2}}$$

Hence,

$$\left| \prod_X \overline{\phi}_x(\vec{t}) \right| \leq \exp \left(-C \sum_X u(\rho_x) \sin^2 \frac{t'.x}{2} + v(\rho_x) \sin^2 \frac{t'.x+t_3}{2} - \rho_x v(\rho_x) \sin^2 \frac{t_3}{2} \right)$$

with

$$u(\rho_x) := \frac{4\lambda\rho_x^2(2 + (\lambda - 2)\rho_x)}{(1 - \rho_x)^2(1 + (\lambda - 1)\rho_x)^2}$$

$$v(\rho_x) := \frac{4\lambda\rho_x}{(1 + (\lambda - 1)\rho_x)^2}$$

From the simple inequality $u(\rho_x) \geq 2\rho_x v(\rho_x)$, we deduce

$$\left| \prod_X \bar{\phi}_x(\vec{t}) \right| \leq \exp \left(-C \sum_X \rho_x v(\rho_x) \left(2 \sin^2 \frac{t'.x}{2} + \frac{1}{\rho_x} \sin^2 \frac{t'.x + t_3}{2} - \sin^2 \frac{t_3}{2} \right) \right)$$

Now, for $M > 2$, the quadratic form $2x^2 + M(x + y)^2 - y^2$ is positive definite, and one can deduce that there exists $C > 0$, such that,

$$\forall x, y \in \mathbb{R}, \quad 2 \sin^2 x + M \sin^2(x + y) - \sin^2 y > C \sin^2 x.$$

Let us consider, firstly, the case $|t'| \in [\frac{\ln n}{n^{2/3}}, \frac{1}{n^{1/3}}]$. We first observe that, since the functions $|\bar{\phi}_x|$ are smaller than 1, it is enough to get an upper bound of $\prod_{X \cap [A_\delta, n^{2/3}, +\infty]^2} |\bar{\phi}_x(\vec{t})|$, where A_δ

is a positive real number which satisfies $e^{-2\delta A_\delta} < 1/3$. This implies that, on the considered domain, $\frac{1}{\rho_x} > 3$, and that, for $|t'| \in [\frac{\ln n}{n^{2/3}}, \frac{1}{n^{1/3}}]$,

$$\prod_{X \cap [A_\delta, n^{2/3}, +\infty]^2} |\bar{\phi}_x(\vec{t})| \leq \exp \left(-C n^{2/3} \int_{[A_\delta, +\infty]^2} u(e^{-\delta(s_1+s_2)}) \sin^2(n^{1/3}(t'.\vec{s})) ds_1 ds_2 \right)$$

$$\leq \exp(-C(\ln n)^2)$$

In the case $|t'| > \frac{1}{n^{1/3}}$, and $|t_3| < \frac{\ln n}{n^{1/3}}$, the number of points of X such that $t'.x$ belongs to the same period of the sinus is less than $n^{1/3}$, so that when we estimate the sum on X by an integral on $[0, +\infty]^2$, the terms $\sin^2 \frac{t'.x+t_3}{2}$ and $\sin^2 \frac{t'.x}{2}$ can be replaced by some positive constant times their mean value i.e. $\frac{1}{2}$.

Hence,

$$\prod_{X \cap [A_\delta, n^{2/3}, +\infty]^2} |\bar{\phi}_x(\vec{t})| \leq \exp \left(-C n^{2/3} \int_{[A_\delta, +\infty]^2} u(e^{-\delta(s_1+s_2)}) ds_1 ds_2 \right)$$

$$\leq \exp(-C n^{2/3})$$

The last case, when $|t_3| \geq \frac{\ln n}{n^{1/3}}$ and $|t'| \leq \frac{\ln n}{n^{2/3}}$, is treated in the same way as the first case, after observing that, for $M > 2$,

$$\forall x, y \in \mathbb{R}, \quad 2 \sin^2 x + M \sin^2(x + y) - \sin^2 y > C \sin^2 y.$$

□

4. THE CARDINAL OF THE LONGEST CONVEX CHAIN

In this section we are seeking an equivalent of the maximal number of vertices for an element of Π_n , or equivalently, the maximal number of integral points that an increasing convex function satisfying $f(0) = 0$ and $f(n) = n$ can interpolate. This question and its counterpart, concerning the maximal convex lattice polygons inscribed in a convex set, was recently solved in [6].

We present a first proof, using Theorem 1, and afterwards the proof of [6], which is more direct. Moreover, we show how the first proof can suggest the second one. Let us mention here that the aim of [6] was to get an equivalent of the maximal cardinal of the vertices of

a lattice polygon included into a given convex set. The following result was the stepping stone of their proof.

Theorem 2. [6] *Let $L \in \Pi_n$. We denote by $\text{card}(L)$ the cardinal of the set of vertices of L . Then*

$$\max_{L \in \Pi_n} \text{card}(L) \underset{n \rightarrow +\infty}{\sim} 3 \left(\frac{n}{\pi} \right)^{2/3} \simeq 1,40.n^{2/3}$$

Proof 1 (using Theorem 1): we first notice that $\mathbf{e}(\lambda)$ vanishes when λ goes to infinity. In the same time,

$$\mathbf{c}(\lambda) \underset{\lambda \rightarrow +\infty}{\sim} \frac{-\text{Li}_2(1-\lambda)}{\zeta(2)^{1/3}(-\text{Li}_3(1-\lambda))^{2/3}}$$

Now, we use the equivalent

$$\text{Li}_n(x) \underset{x \rightarrow -\infty}{\sim} \frac{-(\ln(-x))^n}{\Gamma(n+1)}$$

$$\text{to obtain } \lim_{\lambda \rightarrow +\infty} \mathbf{c}(\lambda) = \frac{\Gamma(4)^{2/3}}{\left(\frac{\pi^2}{6}\right)^{1/3} \Gamma(3)} = \frac{3}{\pi^{2/3}}$$

Since, for large values of λ , $\mathbf{c}(\lambda)$ becomes arbitrary close to $\frac{3}{\pi^{2/3}}$, and the corresponding values $\mathbf{e}(\lambda)$ remain strictly positive, we get obviously that

$$\liminf_{n \rightarrow +\infty} \frac{\max_{L \in \Pi_n} \text{card}(L)}{n^{2/3}} \geq \frac{3}{\pi^{2/3}}$$

Now, let $\varepsilon > 0$, and suppose

$$\limsup_{n \rightarrow +\infty} \frac{\max_{L \in \Pi_n} \text{card}(L)}{n^{2/3}} \geq \frac{3(1+2\varepsilon)}{\pi^{2/3}}$$

Then, for arbitrary large n , there is a chain $L \in \Pi_n$ having cardinal $[(1+\varepsilon) \times 3 \left(\frac{n}{\pi}\right)^{2/3}]$.

But, for $\delta > 0$, we observe that by just picking $[(1-\delta)3 \left(\frac{n}{\pi}\right)^{2/3}]$ vertices among the vertices of L , we get a set of convex chains having cardinal $[(1-\delta)3 \left(\frac{n}{\pi}\right)^{2/3}]$ whose cardinal is equal

to $\left(\frac{[(1+\varepsilon)3 \left(\frac{n}{\pi}\right)^{2/3}]}{[(1-\delta)3 \left(\frac{n}{\pi}\right)^{2/3}]} \right) = \exp \left(3 \frac{\varepsilon + \delta}{\pi^{2/3}} n^{2/3} (1 + o(1)) \right)$. Choosing δ small enough to get

$$\mathbf{e} \left(\frac{3(1-\delta)}{\pi^{2/3}} \right) < 3 \frac{\varepsilon - \delta}{\pi^{2/3}}$$

we obtain a contradiction □

Proof 2 (direct [6]): Let us denote by Ψ the function defined by $\Psi(n) = \max_{L \in \Pi_n} \text{card}(L)$.

This function is increasing, and we shall prove the existence of a sequence a_n of integers such that: $a_n \rightarrow +\infty$, $\frac{a_{n+1}}{a_n} \rightarrow 1$, and $\Psi(a_n) \underset{n \rightarrow +\infty}{\sim} 3 \left(\frac{a_n}{\pi}\right)^{2/3}$.

In order to define a_n , we introduce the set $X_n = X \cap \{(x, y) \in \mathbb{R}_+^2, \quad x + y \leq n\}$.

(The introduction of the set X_n is also suggested by Proof 1. Indeed, under $Q_{z(\lambda), z(\lambda), \lambda}$, the limiting law of the variables $\nu(x)$ when $\lambda \rightarrow \infty$, is δ_1 for $x \in X_n$ and δ_0 for $x \notin X_n$.)

We define now (a_n, a_n) as the sum of the elements of X_n .

Since the set X has an asymptotic density $\frac{6}{\pi^2}$, we obtain by projection along the line $y = x$:

$$\sqrt{2}.a_n \underset{n \rightarrow +\infty}{\sim} \frac{6}{\pi^2} \int_0^{n/\sqrt{2}} 2x^2 dx = \sqrt{2} \cdot \frac{n^3}{\pi^2}$$

Now, the polygonal line corresponding to the function $\nu_n := \mathbf{1}_{X_n} \in C_0(X)$ defines an element of Π_{a_n} which realizes the maximal number of vertices for an element of Π_{a_n} . Indeed, the function ν_n corresponds to the choice of the $\text{card}(X_n)$ first vectors of X , whose projections along the line $y = x$ are the smallest, affected with a minimal weight.

Therefore, $\Psi(a_n) = \text{card}(X_n)$.

Now,

$$\text{card}(X_n) \underset{n \rightarrow +\infty}{\sim} \frac{6}{\pi^2} \times \frac{n^2}{2} = \frac{3}{\pi^2} \times n^2$$

□

5. CHAINS WITH FEW VERTICES

We turn now to the case of chains having a number of vertices of order n^s with $s \in]0, 2/3[$.

Proposition 1. *For all $\lambda > 0$, and $s \in]0, 2/3[$,*

$$\ln(\mathcal{N}(n, n, [cn^s])) = ((2 - 3s)\mathbf{c} \ln(n) + 2\mathbf{c} - 3\mathbf{c} \ln \mathbf{c} + o(1))n^s$$

Proof: In this context, we will choose the parameters z_i so that $z_1 = z_2 = z$ with z of the form $z = 1 - \frac{\delta}{n^{1-s}}$ and $\lambda = \lambda_0 n^{3s-2}$. Equality (1) then writes

$$n^{2s-1} \sum_{(t_1, t_2) \in \frac{1}{n^{1-s}} X} \frac{\lambda_0 t_1 (1 - \frac{\delta}{n^{1-s}})^{n^{1-s}(t_1+t_2)}}{(1 - (1 - \frac{\delta}{n^{1-s}})^{n^{1-s}(t_1+t_2)})(1 + (\lambda - 1)(1 - \frac{\delta}{n^{1-s}})^{n^{1-s}(t_1+t_2)})} = n$$

Now, since λ tends to 0, the sum on the left hand side is equivalent to

$$\left(\frac{1}{\zeta(2)} \int_0^{+\infty} \int_0^{+\infty} \frac{\lambda_0 t_1 e^{-\delta(t_1+t_2)}}{(1 - e^{-\delta(t_1+t_2)})^2} dt_1 dt_2 \right) \times n$$

Using a linear change of variable, $s_i := \delta t_i$ and the series expansion $\frac{x}{(1-x)^2} = \sum_{k \geq 1} kx^k$, the

expression of the last integral reduces to $\frac{\lambda_0}{\delta^3}$.

As a result the first condition (1) yields $\lambda_0 \underset{\infty}{\sim} \delta^3$.

Let us turn to condition (3):

$$n^{3s-2} \sum_{(t_1, t_2) \in \frac{1}{n^{1-s}} X} \frac{\lambda_0 (1 - \frac{\delta}{n^{1-s}})^{n^{1-s}(t_1+t_2)}}{1 + (\lambda - 1)(1 - \frac{\delta}{n^{1-s}})^{n^{1-s}(t_1+t_2)}} = [cn^s]$$

In the same way as in the proof of Theorem 1, we deduce

$$\left(\frac{1}{\zeta(2)} \int_0^{+\infty} \int_0^{+\infty} \frac{\lambda_0 e^{-\delta(t_1+t_2)}}{1 - e^{-\delta(t_1+t_2)}} dt_1 dt_2 \right) \times n^s \underset{\infty}{\sim} cn^s$$

But the expression of the last integral reduces to $\frac{\lambda_0}{\delta^2}$. Hence, condition (3) yields $\lambda_0 \underset{\infty}{\sim} \delta^2 \mathbf{c}$.

We get finally $\delta \underset{\infty}{\sim} \mathbf{c}$ and $\lambda_0 \underset{\infty}{\sim} \mathbf{c}^3$.

$$\mathcal{N}(n, n, [\mathbf{c}n^s]) = \left(1 - \frac{\delta}{n^{1-s}}\right)^{-2n} \lambda^{-[\mathbf{c}n^s]} \times \exp \left(\sum_{(t_1, t_2) \in \frac{1}{n^{1-s}}X} \ln \left(1 - (1 - \lambda) \left(1 - \frac{\delta}{n^{1-s}}\right)^{n^{1-s}(t_1+t_2)} \right) - \ln \left(1 - \left(1 - \frac{\delta}{n^{1-s}}\right)^{n^{1-s}(t_1+t_2)} \right) \right) \times Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^s]))$$

This leads to

$$\mathcal{N}(n, n, [\mathbf{c}n^s]) = Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^s])) \times (\lambda_0 n^{3s-2})^{-[\mathbf{c}n^s]} \exp(n^s (2\delta + o(1)))$$

which gives the result after taking into account both conditions on δ and λ_0 , and proving $Q_{z_1, z_2, \lambda}(\Pi(n, n, [\mathbf{c}n^s])) = O\left(\frac{1}{n^{\frac{3s}{2}}}\right)$ which is a result analogous to Lemma 2. \square

Let us observe that we can generalize Proposition 1, in the following way:

Proposition 2. *Let u_n be a sequence of positive integers such that $u_n = o(n^{\frac{2}{3}})$ and $u_n \rightarrow \infty$,*

$$\mathcal{N}(n, n, u_n) = \left(\frac{n^2}{u_n^3}\right)^{u_n(1+o(1))}$$

Let us now investigate the limit shape of lines of $\Pi(n, n, [\mathbf{c}n^s])$.

Let us denote first by $P_{(n, n, [\mathbf{c}n^s])}$ the uniform measure on $\Pi(n, n, [\mathbf{c}n^s])$.

We introduce the arc of parabola

$$\mathcal{L} = \{(x, y) : \sqrt{y} = 1 - \sqrt{1-x}, 0 \leq x \leq 1\} = \{(x, y) : y = l(x) := (1 - \sqrt{1-x})^2, 0 \leq x \leq 1\}$$

and its " δ, n "-neighborhood $U_{\delta, n}$,

$$U_{\delta, n} := \{(x, y) : |y - n.l(x/n)| \leq n\delta\}$$

The following result answers a question asked by Vershik at the end of [13] (question 3, in the concluding remarks).

Theorem 3. *For all $\delta > 0$, $c > 0$, and $s \in]0, 2/3[$, $P_{(n, n, [\mathbf{c}n^s])}(\Gamma \subset U_{\delta, n})$ tends to 1 as $n \rightarrow \infty$*

Proof: Let us choose a subdivision of the interval $[0, \frac{\pi}{2}]$, $\theta_1^N < \theta_2^N < \dots < \theta_N^N$ such that $\max_{N \rightarrow \infty} (\theta_{j+1}^N - \theta_j^N) \rightarrow 0$. Consider $Q_{z_1, z_2, \lambda}$ where $z_1 = z_2 = z$ with z of the form $z = 1 - \frac{\delta}{n^{1-s}}$ and $\lambda = \lambda_0 n^{3s-2}$, the parameters z and λ being chosen according to \mathbf{c} and s like in the proof of Proposition 1. We introduce the following random variables under $Q_{z_1, z_2, \lambda}$

$$\zeta^{j, N} = \sum_{x \in X : \theta_j^N < \arctan(\frac{x_2}{x_1}) < \theta_{j+1}^N} \sqrt{x_1^2 + x_2^2} \nu(x).$$

The mathematical expectation of $\zeta^{j, N}$ writes

$$n^{2s-1} \sum_{\substack{(t_1, t_2) \in \frac{1}{n^{1-s}}X \\ \theta_j^N < \arctan(\frac{t_2}{t_1}) < \theta_{j+1}^N}} \frac{\lambda_0 \sqrt{t_1^2 + t_2^2} \left(1 - \frac{\delta}{n^{1-s}}\right)^{n^{1-s}(t_1+t_2)}}{\left(1 - \left(1 - \frac{\delta}{n^{1-s}}\right)^{n^{1-s}(t_1+t_2)}\right) \left(1 + (\lambda - 1) \left(1 - \frac{\delta}{n^{1-s}}\right)^{n^{1-s}(t_1+t_2)}\right)}$$

and is equivalent when $n \rightarrow \infty$, to

$$\left(\frac{1}{\zeta(2)} \int_{\theta_j^N < \theta < \theta_{j+1}^N} \frac{\lambda_0 \sqrt{t_1^2 + t_2^2} e^{-\delta(t_1+t_2)}}{(1 - e^{-\delta(t_1+t_2)})^2} dt_1 dt_2 \right) \times n$$

which is itself equivalent, when $\max_{N \rightarrow \infty} (\theta_{j+1}^N - \theta_j^N) \rightarrow 0$, to

$$C \times \frac{\theta_{j+1}^N - \theta_j^N}{(\cos \theta_j^N + \sin \theta_j^N)^3} \times n = C' \times \frac{\theta_{j+1}^N - \theta_j^N}{\cos(\theta_j^N - \frac{\pi}{4})^3} \times n.$$

Furthermore, by moment arguments we don't develop since they involve easy but rather long computations, one can check that for all positive η , $Q_{z_1, z_2, \lambda}(|\zeta^{j, N} - E[\zeta^{j, N}]| > \eta) = o(\frac{1}{n^k})$, for all k .

Therefore, if $d(O, w(\Gamma))$ denotes the distance to the origin of the endpoint of Γ , we deduce the existence of a limit shape for $\frac{1}{d(O, w(\Gamma))} \times \Gamma$ under $Q_{z_1, z_2, \lambda}$.

In the frame based at the intersection of this limit shape with the second diagonal of the unit square, with a first axis parallel to the first diagonal and a second axis parallel to the second diagonal, the limit shape has the following parametrized description:

$$x(\phi) = C_1 \cdot \int_0^\phi \frac{\cos u}{(\cos u)^3} du = \tan \phi, \quad y(\phi) = C_1 \cdot \int_0^\phi \frac{\sin u}{(\cos u)^3} du = C_1 \tan^2 \phi \quad (-\frac{\pi}{4} \leq \phi \leq \frac{\pi}{4}),$$

with $C_1 = x(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

$$\text{Hence, } y(\phi) = (x(\phi))^2 / \sqrt{2}.$$

This curve coincides precisely with \mathcal{L} , and, $Q_{z_1, z_2, \lambda}(\Gamma \not\subset U_{\delta, n}) = o(\frac{1}{n^k})$, for all k .

Now, by the same arguments as in Lemma 2, one obtains

$$Q_{z_1, z_2, \lambda}(\Gamma \in \Pi_{(n, n, [\lambda n^s])}) = O(\frac{1}{n^{\frac{5s}{2}}}).$$

Since the projection of $Q_{z_1, z_2, \lambda}$ on $\Pi_{(n, n, [\lambda n^s])}$ is precisely $P_{(n, n, [\lambda n^s])}$, we have got for all k

$$P_{(n, n, [\lambda n^s])}(\Gamma \not\subset U_{\delta, n}) = o(\frac{1}{n^k})$$

□

6. BACK TO JARNÍK'S PROBLEM

In [9], Jarník gives an equivalent of the maximum possible number of vertices of a convex polygonal line having its vertices in \mathbb{N}^2 , a *total length* smaller than n , and whose segments make an angle with the x -axis between 0 and $\frac{\pi}{4}$. What he finds is $3 \cdot \frac{n^{2/3}}{2 \cdot (2\pi)^{1/3}}$. If, in order to be closer to our setting, we ask the segments to make an angle with the x -axis between 0 and $\frac{\pi}{2}$, Jarník's equivalent is changed into $3 \cdot \frac{n^{2/3}}{2 \cdot \pi^{1/3}}$ (which is twice the above result for $n/2$).

In this section, we want to present a detailed combinatorial analysis of this set of lines, which leads to Jarník's result as well as to the asymptotic of the *typical* cardinal of such lines which is the analog of Bárány, Sinai and Vershik's result when the constraint concerns the total length.

Let us first describe Jarník's argument, which is a good application of the correspondence described in section 2. It says the following: the function ν realizing the maximum can be taken among the functions taking their values in $\{0, 1\}$. Indeed, by changing the non-zero values of a function ν into 1, one can obtain a chain with the same number of vertices, but with a shorter length. Now, if the number of vertices N is given, the convex chain having minimal length, will be defined by the function ν which associates 1 to the N points of X which are the closest to the origin. Since the set X has an asymptotic density $\frac{6}{\pi^2}$, when N is big, this set of points is asymptotically equivalent to the intersection of X with the disc of center O having radius R satisfying $\frac{6}{\pi^2} \cdot \frac{\pi R^2}{4} = N$ i.e. $R = (\frac{2\pi}{3} N)^{1/2}$. The total length

of the line is equivalent to $L = \int_0^R r \times \frac{6}{\pi^2} \cdot \frac{\pi}{2} r dr = \frac{R^3}{\pi} = \frac{(\frac{2\pi}{3}N)^{3/2}}{\pi}$. This yields precisely $N = 3 \cdot \frac{L^{2/3}}{2 \cdot \pi^{1/3}} \simeq 1,02L^{2/3}$.

Let us notice here, that the limit shape of the line described above is an arc of a circle and not an arc of a parabola like in the previous problem.

Let us turn now to a more precise analysis. We first introduce the following parametrized probability distribution on $C_0(X)$:

$$\begin{aligned} \forall \nu \in C_0(X), \quad \tilde{Q}_{z,\lambda}(\nu) &= \prod_{(x_1, x_2) \in X} \frac{(z\sqrt{x_1^2+x_2^2})^{\nu(x)} \lambda^{1_{\nu(x) \neq 0}}}{1 + \frac{\lambda z \sqrt{x_1^2+x_2^2}}{1 - z\sqrt{x_1^2+x_2^2}}} \\ &= \prod_{(x_1, x_2) \in X} (z\sqrt{x_1^2+x_2^2})^{\nu(x)} \lambda^{1_{\nu(x) \neq 0}} \frac{1 - z\sqrt{x_1^2+x_2^2}}{1 + (\lambda - 1)z\sqrt{x_1^2+x_2^2}} \end{aligned}$$

where $z \in [0, 1]$, and $\lambda > 0$.

Now, we define $\tilde{\Pi}(n, N)$ the set of the polygonal lines of Π having N vertices and a total length between n and $n + 1$, and we denote by $\tilde{\mathcal{N}}(n, N)$ its cardinal.

Since $\tilde{Q}_{z,\lambda}(\nu)$ depends only on $\sum_{x \in X} \sqrt{x_1^2 + x_2^2} \nu(x)$ and $\sum_{x \in X} 1_{\nu(x) \neq 0}$, we deduce that the conditional distribution induced by $\tilde{Q}_{z,\lambda}$ on $\tilde{\Pi}(n, N)$ is *almost* uniform. More precisely, its density with respect to the uniform measure is bounded between z and 1. So that, we can write

$$\mathcal{N}(n, N) = c(z) z^{-n} \lambda^{-N} \prod_{(x_1, x_2) \in X} \left(\frac{1 - z\sqrt{x_1^2+x_2^2}}{1 + (\lambda - 1)z\sqrt{x_1^2+x_2^2}} \right)^{-1} \tilde{Q}_{z,\lambda}(\tilde{\Pi}(n, N)) \quad (\star)$$

with $c(z) \in [1, \frac{1}{z}]$. In order to get a logarithmic equivalent of $\tilde{\mathcal{N}}(n, N)$, our strategy is to choose the parameters z and λ according to

$$\tilde{E}_{z,\lambda} \left(\sum_{x \in X} \sqrt{x_1^2 + x_2^2} \nu(x) \right) = \sum_{(x_1, x_2) \in X} \frac{\lambda \sqrt{x_1^2 + x_2^2} z \sqrt{x_1^2+x_2^2}}{(1 - z\sqrt{x_1^2+x_2^2})(1 + (\lambda - 1)z\sqrt{x_1^2+x_2^2})} = n \quad (7)$$

$$E_{z,\lambda} \left(\sum_{x \in X} 1_{\nu(x) \neq 0} \right) = \sum_{(x_1, x_2) \in X} \frac{\lambda z \sqrt{x_1^2+x_2^2}}{1 + (\lambda - 1)z\sqrt{x_1^2+x_2^2}} = N \quad (8)$$

Again, this will indeed lead to an estimation of $\tilde{Q}_{z,\lambda}(\tilde{\Pi}(n, N))$ which will be much bigger than exponentially small.

Now, our asymptotic setting is the following: $n \rightarrow +\infty$, and $N/n^{2/3} \rightarrow \mathbf{c}$. Similar computations as in section 3 yield:

Theorem 4. For all $\lambda > 0$,

$$\ln(\tilde{\mathcal{N}}(n, [\mathbf{c}_J(\lambda)n^{2/3}])) = n^{2/3}(\mathbf{e}_J(\lambda) + o(1))$$

where $\mathbf{c}_J(\lambda)$ and $\mathbf{e}_J(\lambda)$ are positive functions on $]0, +\infty[$, defined by :

$$\mathbf{c}_J(\lambda) = \frac{\pi^{1/3}}{2} \mathbf{c}(\lambda), \quad \mathbf{e}_J(\lambda) = \frac{\pi^{1/3}}{2} \mathbf{e}(\lambda)$$

where $\mathbf{c}(\lambda)$ and $\mathbf{e}(\lambda)$ are the functions introduced in Theorem 1.

Corollary 1. *Let $\delta > 0$. We denote by $\tilde{\Pi}_{n,n+\delta}$ the set of the polygonal lines of Π having a total length between n and $n + \delta$, and, by $T_{n,n+\delta}$, the typical number of vertices of an element of $\tilde{\Pi}_{n,n+\delta}$. We have got the three following asymptotics:*

$$\text{card}(\tilde{\Pi}_{n,n+\delta}) = \exp\left(\frac{3^{4/3}(\zeta(3))^{1/3}}{(4\pi)^{1/3}}n^{2/3}(1 + o(1))\right).$$

$$T_{n,n+\delta} \underset{n \rightarrow \infty}{\sim} \left(\frac{3}{4\pi\zeta(3)^2}\right)^{1/3} n^{2/3} \simeq 0,55n^{2/3}.$$

$$\max_{L \in \tilde{\Pi}_{n,n+\delta}} \text{card}(L) \underset{n \rightarrow \infty}{\sim} \left(\frac{3}{2\pi^{1/3}}\right) n^{2/3} \simeq 1,02n^{2/3}.$$

Remark: The third asymptotic is a slight refinement of Jarník's result, since the condition on the total length to belong to $[n, n + \delta]$ is more restrictive than in Jarník's case.

7. MIXING CONSTRAINTS AND FINDING NEW LIMIT SHAPES

In this section, we consider the set $\bar{\Pi}_{n,L}$ of convex polygonal lines of Π_n whose total length belongs to $[Ln, Ln + 1]$, for $L \in]\sqrt{2}, 2[$. We introduce $\bar{P}_{n,L}$, the uniform distribution on $\bar{\Pi}_{n,L}$, and we are investigating the limit shape of a line $\frac{1}{n}\Gamma$ under $\bar{P}_{n,L}$. The shapes corresponding to the two extreme values of L , $\sqrt{2}$ and 2 are respectively the diagonal and the two sides of the square. We want now to explicit a family of convex curves $(\mathcal{C}_L)_{L \in]\sqrt{2}, 2[}$ interpolating these extreme situations (see Figure 2).

- For $L \in]\frac{\pi}{2}, 2[$, there exists a unique $\alpha \in [-\frac{1}{\sqrt{2}}, +\infty[$ such that

$$L = \sqrt{2} \frac{\int_0^{\frac{\pi}{4}} \frac{1}{(\alpha + \cos u)^3} du}{\int_0^{\frac{\pi}{4}} \frac{\cos u}{(\alpha + \cos u)^3} du}.$$

We define the convex curve \mathcal{C}_L of length L , joining O to $(1, 1)$, and symmetric with respect to the second diagonal of the unit square by the following parametrized description:

$$x_L(\phi) = \sqrt{2} \frac{\int_0^\phi \frac{\cos u}{(\alpha + \cos(u - \frac{\pi}{4}))^3} du}{\int_0^{\frac{\pi}{2}} \frac{\cos u}{(\alpha + \cos u)^3} du}, \quad y_L(\phi) = \sqrt{2} \frac{\int_0^\phi \frac{\sin u}{(\alpha + \cos(u - \frac{\pi}{4}))^3} du}{\int_0^{\frac{\pi}{2}} \frac{\cos u}{(\alpha + \cos u)^3} du} \quad (0 \leq \phi \leq \frac{\pi}{2}).$$

When L tends to 2, α tends to $-\frac{1}{\sqrt{2}}$, and the curve \mathcal{C}_L converges to the two sides of the square.

When $L = 1 + \frac{\ln(1+\sqrt{2})}{\sqrt{2}} \simeq 1,623$, $\alpha = 0$ and the limit shape is the arc of parabola found in [3], [12], [13].

When L tends to $\frac{\pi}{2}^+$, α goes to infinity, and \mathcal{C}_L converges to the arc of the unit circle joining O to $(1, 1)$.

- For $L \in]\sqrt{2}, \frac{\pi}{2}[$, there exists a unique $\beta \in [1, +\infty[$ such that

$$L = \sqrt{2} \frac{\int_0^{\frac{\pi}{4}} \frac{1}{(\beta - \cos u)^3} du}{\int_0^{\frac{\pi}{4}} \frac{\cos u}{(\beta - \cos u)^3} du}.$$

We define the convex curve \mathcal{C}_L of length L , joining O to $(1, 1)$, and symmetric with respect to the second diagonal of the unit square by the following parametrized description:

$$x_L(\phi) = \sqrt{2} \frac{\int_0^\phi \frac{\cos u}{(\beta - \cos(u - \frac{\pi}{4}))^3} du}{\int_0^{\frac{\pi}{2}} \frac{\cos u}{(\beta - \cos u)^3} du}, \quad y_L(\phi) = \sqrt{2} \frac{\int_0^\phi \frac{\sin u}{(\beta - \cos(u - \frac{\pi}{4}))^3} du}{\int_0^{\frac{\pi}{2}} \frac{\cos u}{(\beta - \cos u)^3} du} \quad (0 \leq \phi \leq \frac{\pi}{2}).$$

When L tends to $\sqrt{2}$, β tends to 1, and the curve \mathcal{C}_L converges to the first diagonal of the square.

When L tends to $\frac{\pi}{2}$, β goes to infinity, and \mathcal{C}_L converges to the arc of the unit circle joining O to $(1, 1)$.

We can now define the " n, δ "-neighbourhood $V_{L,n,\delta}$ of \mathcal{C}_L by

$$V_{L,n,\delta} = \{(x, y) : d((\frac{x}{n}, \frac{y}{n}), \mathcal{C}_L) \leq \delta\}$$

and state the limit following limit shape result:

Theorem 5. For all $\delta > 0$, $L \in]\sqrt{2}, 2[$, $\bar{P}_{n,L}(\Gamma \subset V_{L,n,\delta})$ tends to 1 as $n \rightarrow \infty$

Proof: We introduce the following parametrized probability distribution $\bar{Q}_{y,z}$ on $C_0(X)$ which takes into account both the total length and the endpoint of Γ : for $\nu \in C_0(X)$, we define

$$\bar{Q}_{y,z}(\nu) = \prod_{(x_1, x_2) \in X} (y^{(x_1+x_2)} z^{\sqrt{x_1^2+x_2^2}} \nu(x) (1 - (y^{(x_1+x_2)} z^{\sqrt{x_1^2+x_2^2}})))$$

where y and z satisfy: for all (x_1, x_2) in X , $y^{(x_1+x_2)} z^{\sqrt{x_1^2+x_2^2}} \in [0, 1[$. (The introduction of an extra parameter λ in the definition of the probability, that would "count" the vertices is useless in a question of limit shape)

It turns out here that the computation of the parameters according to the conditions that the expectations of the endpoint and total length of the chain are respectively equal to (n, n) and Ln , are very heavy in this framework. This implies heavy formulas in the combinatorics of these lines.

However, we know that y and z will be of the form $1 + \frac{\gamma}{n^{1/3}}$ and $1 + \frac{\delta}{n^{1/3}}$.

$$\zeta^{j,N} = \sum_{x \in X : \theta_j^N < \arctan(\frac{x}{x_1}) < \theta_{j+1}^N} \sqrt{x_1^2 + x_2^2} \nu(x).$$

The mathematical expectation of $\zeta^{j,N}$ writes

$$n^{1/3} \sum_{\substack{(t_1, t_2) \in \frac{1}{n^{1/3}} X \\ \theta_j^N < \arctan(\frac{t_2}{t_1}) < \theta_{j+1}^N}} \frac{\sqrt{t_1^2 + t_2^2} (1 + \frac{\gamma}{n^{1/3}})^{n^{1/3}(t_1+t_2)} (1 + \frac{\delta}{n^{1/3}})^{n^{1/3} \sqrt{t_1^2+t_2^2}}}{(1 - (1 + \frac{\gamma}{n^{1/3}})^{n^{1/3}(t_1+t_2)} (1 + \frac{\delta}{n^{1/3}})^{n^{1/3} \sqrt{t_1^2+t_2^2}})}$$

and is equivalent when $n \rightarrow \infty$, to

$$\left(\frac{1}{\zeta(2)} \int_{\theta_j^N < \theta < \theta_{j+1}^N} \frac{\sqrt{t_1^2 + t_2^2} e^{\gamma(t_1+t_2) + \delta \sqrt{t_1^2+t_2^2}}}{1 - e^{\gamma(t_1+t_2) + \delta \sqrt{t_1^2+t_2^2}}} dt_1 dt_2 \right) \times n$$

which is itself equivalent, when $\max_{N \rightarrow \infty} (\theta_{j+1}^N - \theta_j^N) \rightarrow 0$, to

$$C \times \frac{\theta_{j+1}^N - \theta_j^N}{(\gamma(\cos \theta_j^N + \sin \theta_j^N) + \delta)^3} \times n = C' \times \frac{\theta_{j+1}^N - \theta_j^N}{(\gamma \cos(\theta_j^N - \frac{\pi}{4}) + \delta)^3} \times n.$$

The concentration of the normalized random chain around a limit shape \mathcal{C}_L , which is symmetric with respect to the second diagonal, follows like in the proof of Theorem 3. We derive the announced parametrization of \mathcal{C}_L from the above estimation of $E[\zeta^{j,N}]$.

When $L > \frac{\pi}{2}$, we have to take $\gamma > 0$, the variable α equals $\frac{\delta}{\gamma}$, and the condition on α comes from the condition we had to impose on y and z . We proceed similarly for $L < \frac{\pi}{2}$. \square

8. SOME REMARKS WHEN THE “LATTICE” IS RANDOM

Let us replace now the $(n+1)^2$ points of $[0, n]^2 \cap \mathbb{Z}^2$, by a set of n^2 points independently and uniformly distributed in $[0, n]^2$. It was observed in [5] that the probability for k points independently and uniformly distributed in the square $[0, 1]^2$ to form, with $(0, 0)$ and $(1, 1)$, a convex chain is exactly equal to $\frac{1}{k!(k+1)!}$. From this result, we can derive an analog of Theorems 1 and 3, for the expected number of chains. Indeed, one can easily deduce that for $s \in [0, 1]$ and $\mathbf{c} > 0$, the expectation of the number of chains, having a cardinal equal to $\lceil \mathbf{c}n^s \rceil$, is equal to

$$\frac{\binom{n^2}{\lceil \mathbf{c}n^s \rceil}}{(\lceil \mathbf{c}n^s \rceil)!(\lceil \mathbf{c}n^s \rceil + 1)!} = n^{\mathbf{c}(2-3s)n^s} \exp((3\mathbf{c} - 3\mathbf{c} \ln \mathbf{c})n^s + o(n^s)) \quad (9)$$

which is maximal for $\mathbf{c} = 1$ and $s = 2/3$, and, in this case, the expected number of convex chains of cardinal $\lceil n^{2/3} \rceil$ is equal to $\exp(3n^{2/3} + o(n^{2/3}))$. It is straightforward to see that the expected number of the total number of convex chains is also equal to $\exp(3n^{2/3} + o(n^{2/3}))$, and a moment method show that, in probability, the cardinal of a typical convex chain is equivalent to $n^{2/3}$. Let us remark finally that, in the case of chains having $\mathbf{c}n^s$ vertices, with $s < 2/3$, the main term $n^{\mathbf{c}(2-3s)n^s}$ is common to both Theorem 3 and formula (9).

Now, the question of the largest cardinal of a convex chain is delicate in the random case. We believe but we are not able to prove that there is an equivalent, in probability, of this quantity when n goes to infinity. Indeed, Kingman’s sub-additive ergodic theorem is of no direct help here, since the concatenation of two convex chains is not necessarily convex. Nevertheless, from (9), one can say, that, with an asymptotically overwhelming probability, this cardinal is less than $e.n^{2/3}$.

One can even slightly ameliorate this result in the following manner: suppose there exists $d \in]0, e[$, such that with an asymptotically positive probability, there will be a chain having length $\lceil dn^{2/3} \rceil$. When such a chain exists, since every sub-chain of a convex chain is also a convex chain, we get that, for all $c \in]0, d[$, there are, with an asymptotically positive probability, at least $\binom{\lceil dn^{2/3} \rceil}{\lceil cn^{2/3} \rceil}$ convex chains having cardinal $\lceil cn^{2/3} \rceil$. Therefore, the expected number of such chains is at least $\exp\left(\left((d-c) \ln \frac{d}{d-c} + c \ln \frac{d}{c}\right)n + o(n)\right)$. Now, using (9), the real d must satisfy that, for all $c \in]0, d[$,

$$\left((d-c) \ln \frac{d}{d-c} + c \ln \frac{d}{c}\right) \leq 3c(1 - \ln c)$$

Some numerical computations yield $d \leq 2,65$.

We deduce that, in the context of n random uniformly distributed points, the longest convex chain, is with an asymptotically overwhelming probability, smaller than $2,65.n^{2/3}$, which is, as we intuitively expected, bigger than the $1,40.n^{2/3}$ found in Theorem 2. Note that the lower bound obtained by just considering the extremal points of the convex hull of the points of the Poisson process which are situated above the parabola, which is, by a result

of Rényi and Sulanke [10] and their followers, equivalent to $\frac{2}{(3\pi)^{1/3}}\Gamma(5/3)n^{2/3} = 0,85.n^{2/3}$, does not give a better lower bound than the typical behavior.

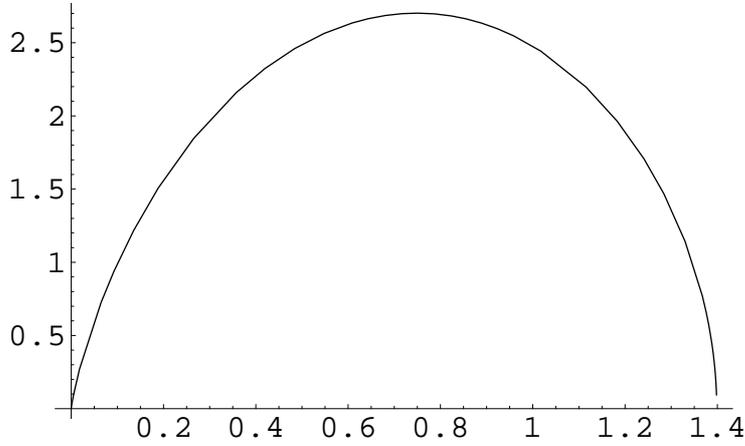


Fig. 1: The graph of $(c(\lambda), e(\lambda))$.

The point of maximal e -coordinate corresponds to typical chains [3][12][13].
 The point of maximal c -coordinate corresponds to chains of maximal cardinal [6].

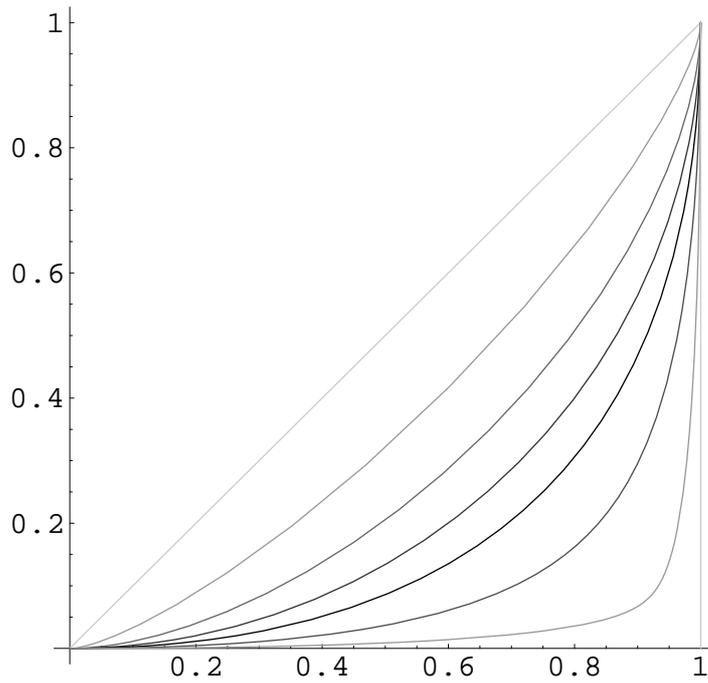


Fig. 2: Limit shapes of different lengths. Successively:
 $L = \sqrt{2}$ (diagonal), $L = 1,454$, $L = 1,516$, $L = \frac{\pi}{2}$ (circle), $L = 2^{2/3}$ (parabola),
 $L = 1,716$, $L = 1,861$, and $L = 2$ (sides of the square).

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