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# Topological derivative for wave-based identification of penetrable scatterers.

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*ABSTRACT. This communication, which builds on previous work on cavity identification, is concerned with the use of topological derivative as a tool for wave-based probing of elastic or acoustic media for buried objects. A formulation for computing the topological derivative field, based on an adjoint solution, is presented. Numerical results is included to illustrate the utility of topological derivative for outlining the inclusion location and size.*

*RÉSUMÉ. Cette communication, qui fait suite à des travaux antérieurs sur l'identification de cavités, développe le gradient topologique pour la détection d'inclusions pénétrables par mesures élastodynamiques ou acoustiques. Une formulation permettant son calcul du champ de gradient topologique au moyen d'un état adjoint est présentée. Des exemples numériques illustrent l'aptitude de cet outil à estimer l'emplacement et la taille d'une inclusion.*

*KEYWORDS: identification of inclusions, elastodynamics, inverse scattering, adjoint field method, topological derivative.*

*MOTS-CLÉS : identification d'inclusions, élastodynamique, diffraction inverse, méthode de l'état adjoint, dérivée topologique.*

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**1 Preliminaries.** This communication builds on previous work on cavity identification [BON 04]. Let  $\Omega$  denote a finite elastic body bounded by the external surface  $S$ , divided into complementary subsets  $S_N$  and  $S_D$  supporting prescribed tractions and displacements, respectively. An unknown inclusion (or a set thereof)  $B^{\text{true}}$ , bounded by the closed surface(s)  $\Gamma^{\text{true}}$  and characterized by elastic moduli  $\mathbf{C}^{\text{true}}$  and mass density  $\rho^{\text{true}}$ , is embedded in  $\Omega$  with perfect bonding conditions on  $\Gamma^{\text{true}}$ . The reference medium, i.e. that of the matrix  $\Omega \setminus B^{\text{true}}$  surrounding the inclusion, has elastic moduli  $\mathbf{C}$  and mass density  $\rho$ . On applying a steady-state traction  $\mathbf{p}$  on  $S_N$  with angular frequency  $\omega$  (chosen so as not to be an eigenfrequency of any of the boundary-value

problems appearing in the ensuing developments), an elastodynamic state  $\mathbf{u}^{\text{true}}$  arises which solves the problem  $\mathcal{P}(B^{\text{true}})$ , where  $\mathcal{P}(B)$  denotes the transmission problem for a generic inclusion  $B$ , defined by the equations

$$\mathcal{P}(B) : \begin{cases} \mathcal{L}(\omega)\mathbf{v} = \mathbf{0} \text{ (in } \Omega \setminus B), & \mathcal{L}^*(\omega)\mathbf{v} = \mathbf{0} \text{ (in } B) \\ \mathbf{t}[\mathbf{v}] = \mathbf{p} \text{ (on } S_N), & \mathbf{v} = \mathbf{0} \text{ (on } S_D), \quad \llbracket \mathbf{v} \rrbracket = \llbracket \mathbf{t}[\mathbf{v}] \rrbracket = \mathbf{0} \text{ (on } \Gamma) \end{cases} \quad [1]$$

Here  $\mathcal{L}(\omega)$  is the Navier linear partial differential operator for the reference medium, i.e.  $\mathcal{L}(\omega)\mathbf{w} = \nabla \cdot (\mathbf{C} : \nabla \mathbf{w}) + \rho\omega^2\mathbf{w}$ , and  $\mathcal{L}^*(\omega)$  is defined similarly for the inclusion medium;  $\mathbf{t}[\mathbf{v}] \equiv \boldsymbol{\sigma}[\mathbf{v}] \cdot \mathbf{n}$  denotes the traction vector associated with the displacement  $\mathbf{v}$  through Hooke's law with the relevant elastic moduli.

For the inverse scattering problem of interest, where the location, topology, geometry and material parameters of  $B^{\text{true}}$  are being sought, the trace of  $\mathbf{u}^{\text{true}}$  on  $S$ , denoted by  $\mathbf{u}^{\text{obs}}$ , is assumed to be available over the measurement region  $S^{\text{obs}} \subset S_N$ . Let  $\mathbf{u}^*$  denote the solution to the forward problem for a given excitation  $\mathbf{p}$  and a trial inclusion  $B^*$  bounded by  $\Gamma^*$ :  $\mathbf{u}^*$  is then governed by problem  $\mathcal{P}(B^*)$ , equation [1]. To solve such inverse problem, a misfit cost function is set up in order to minimize the difference between  $\mathbf{u}^{\text{obs}}$  and  $\mathbf{u}^*$ . Generic cost functions having the format

$$\mathcal{J}(B^*, \mathbf{C}^*, \rho^*) = \int_{S^{\text{obs}}} \varphi(\mathbf{u}^*(\boldsymbol{\xi}), \boldsymbol{\xi}) \, d\Gamma_{\boldsymbol{\xi}} \quad [2]$$

are considered. The commonly used output least-squares cost function corresponds to the particular case where  $2\varphi(\mathbf{w}, \boldsymbol{\xi}) = (\mathbf{w}(\boldsymbol{\xi}) - \mathbf{u}^{\text{obs}}(\boldsymbol{\xi})) \cdot (\mathbf{w}(\boldsymbol{\xi}) - \mathbf{u}^{\text{obs}}(\boldsymbol{\xi}))$ .

**2 Topological derivative.** To aid a subsequent gradient-based minimization of  $\mathcal{J}(\Omega)$  for identifying  $B^{\text{true}}, \mathbf{C}^{\text{true}}, \rho^{\text{true}}$  on the basis of  $\mathbf{u}^{\text{obs}}$ , the development of topological derivative for the cost functionals of form [2], which would facilitate a rational selection of the necessary initial ‘‘guess’’ for  $B^{\text{true}}, \mathbf{C}^{\text{true}}, \rho^{\text{true}}$ , is investigated. To this end, let  $B_{\varepsilon}(\mathbf{x}^{\circ}) = \mathbf{x}^{\circ} + \varepsilon\mathcal{B}$ , where  $\mathcal{B} \subset \mathbb{R}^3$  is a fixed bounded open set with boundary  $\mathcal{S}$  and volume  $|\mathcal{B}|$  containing the origin, define the region of space occupied by an inclusion of (small) size  $\varepsilon > 0$  containing a fixed sampling point  $\mathbf{x}^{\circ}$ . Following e.g. [GAR 01], one is in particular interested in the asymptotic behavior of  $\mathcal{J}(B_{\varepsilon}, \mathbf{C}^*, \rho^*)$  for infinitesimal  $\varepsilon > 0$ . With reference to this limiting behavior, the topological derivative  $\mathcal{T}(\mathbf{x}^{\circ})$  of the cost functional  $\mathcal{J}(B_{\varepsilon}, \mathbf{C}^*, \rho^*)$  at  $\mathbf{x}^{\circ}$  for an inclusion-free body is defined through an expansion of the form:

$$\mathcal{J}(B_{\varepsilon}, \mathbf{C}^*, \rho^*) = \mathcal{J}(\emptyset, \mathbf{C}^*, \rho^*) + f(\varepsilon) |\mathcal{B}| \mathcal{T}(\mathbf{x}^{\circ}) + o(f(\varepsilon)) \quad (\varepsilon \ll \text{Diam}(\Omega)) \quad [3]$$

where the function  $f$  defines the asymptotic behavior of  $\mathcal{J}(B_{\varepsilon}, \mathbf{C}^*, \rho^*)$  for  $\varepsilon \ll \text{Diam}(\Omega)$  and is such that  $f(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This definition is not restricted to spherical infinitesimal inclusions (for which  $\mathcal{B}$  is the unit ball,  $\mathcal{S}$  the unit sphere and  $|\mathcal{B}| = 4\pi/3$ ). In general, the value  $\mathcal{T}(\mathbf{x}^{\circ})$  is expected to depend on the shape of  $\mathcal{B}$ . One is here interested to find locations  $\mathbf{x}^{\circ}$  where  $\mathcal{T}(\mathbf{x}^{\circ})$  attains lowest negative values.

With reference to [3], the evaluation of  $\mathcal{J}(B_{\varepsilon}, \mathbf{C}^*, \rho^*)$  requires the knowledge of the elastodynamic solution  $\mathbf{u}^{\varepsilon}$  to the forward problem  $\mathcal{P}(B_{\varepsilon}(\mathbf{x}^{\circ}))$ , equation [1]. It is thus convenient to decompose the total displacement field  $\mathbf{u}^{\varepsilon}$  as  $\mathbf{u}^{\varepsilon} = \mathbf{u} + \tilde{\mathbf{u}}^{\varepsilon}$ , where

$\tilde{\mathbf{u}}^\varepsilon$  denotes the *scattered field* and  $\mathbf{u}$  is the *free field* defined as the response of the inclusion-free (reference) solid due to the given excitation  $\mathbf{p}$ , so that

$$\mathcal{L}(\omega)\mathbf{u} = \mathbf{0} \text{ (in } \Omega), \quad \mathbf{t}[\mathbf{u}] = \mathbf{p} \text{ (on } S_N), \quad \mathbf{u} = \mathbf{0} \text{ (on } S_D), \quad [4]$$

For infinitesimal  $\varepsilon$  the scattered field is expected to vanish, i.e.  $\lim_{\varepsilon \rightarrow 0} |\tilde{\mathbf{u}}^\varepsilon(\mathbf{x})| = 0$  ( $\mathbf{x} \in \Omega_c$ ), whereas the free-field, by its definition [4], does not depend on  $\varepsilon$ . One may expand  $\mathcal{J}(\Omega_\varepsilon)$  with respect to  $\tilde{\mathbf{u}}^\varepsilon$  as

$$\begin{aligned} \mathcal{J}(B_\varepsilon, \mathbf{C}^*, \rho^*) &= \int_{S^{\text{obs}}} \varphi(\mathbf{u}^\varepsilon(\boldsymbol{\xi}), \boldsymbol{\xi}) \, d\Gamma_\xi \\ &= \mathcal{J}(\emptyset, \mathbf{C}^*, \rho^*) + \int_{S^{\text{obs}}} \text{Re} \left( \frac{\partial \varphi}{\partial \mathbf{u}}(\mathbf{u}(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \tilde{\mathbf{u}}^\varepsilon(\boldsymbol{\xi}) \right) \, d\Gamma_\xi + o(\|\tilde{\mathbf{u}}^\varepsilon\|) \end{aligned} \quad [5]$$

where  $\mathcal{J}(\emptyset, \mathbf{C}^*, \rho^*)$  is the value of  $\mathcal{J}$  for the inclusion-free medium, and with the convention

$$\frac{\partial \varphi}{\partial \mathbf{w}} \equiv \frac{\partial \varphi}{\partial \mathbf{w}_R} - i \frac{\partial \varphi}{\partial \mathbf{w}_I} \quad (\mathbf{w}_R = \text{Re}(\mathbf{w}), \quad \mathbf{w}_I = \text{Im}(\mathbf{w})) \quad [6]$$

By means of [3] and [5], the topological derivative of  $\mathcal{J}(\Omega)$  can be recast as:

$$\mathcal{T}(\mathbf{x}^o) = \lim_{\varepsilon \rightarrow 0} \frac{1}{f(\varepsilon) |\mathcal{B}|} \int_{S^{\text{obs}}} \text{Re} \left( \frac{\partial \varphi}{\partial \mathbf{u}}(\mathbf{u}(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \tilde{\mathbf{u}}^\varepsilon(\boldsymbol{\xi}) \right) \, d\Gamma_\xi. \quad [7]$$

One now has to evaluate the leading contribution of the integral in [7] for  $\varepsilon \ll 1$ .

**3 Adjoint field approach.** Define the *adjoint field*  $\hat{\mathbf{u}}(\boldsymbol{\xi})$  as the solution to the elastodynamic boundary-value problem

$$\mathcal{L}(\omega)\hat{\mathbf{u}} = \mathbf{0} \text{ (in } \Omega), \quad \hat{\mathbf{t}} = \frac{\partial \varphi}{\partial \mathbf{u}} \text{ (on } S^{\text{obs}}), \quad \hat{\mathbf{t}} = \mathbf{0} \text{ (on } S_N \setminus S^{\text{obs}}), \quad \hat{\mathbf{u}} = \mathbf{0} \text{ (on } S_D). \quad [8]$$

where  $\mathbf{u}$  is the free-field defined by [4], the prescribed traction is defined in terms of the cost function density function, and the convention [6] is employed. Now, since  $\mathcal{L}(\omega)\mathbf{u} = \mathcal{L}(\omega)\hat{\mathbf{u}} = \mathbf{0}$  in  $\Omega$ , the elastodynamic reciprocity theorem yields the identity

$$\int_S [\hat{\mathbf{u}} \cdot \mathbf{t} - \mathbf{u} \cdot \hat{\mathbf{t}}] \, d\Gamma = 0 \quad [9]$$

In addition, the following reciprocity identity can be established for  $\hat{\mathbf{u}}$  and  $\mathbf{u}^\varepsilon$  over  $\Omega$ :

$$\int_S [\hat{\mathbf{u}} \cdot \mathbf{t}^\varepsilon - \mathbf{u}^\varepsilon \cdot \hat{\mathbf{t}}] \, d\Gamma = \int_{B_\varepsilon} [\nabla \hat{\mathbf{u}} : (\mathbf{C}^* - \mathbf{C}) : \nabla \mathbf{u}^\varepsilon - (\rho^* - \rho) \omega^2 \hat{\mathbf{u}} \cdot \mathbf{u}^\varepsilon] \, d\Omega \quad [10]$$

On subtracting [10] from [9] and using the boundary conditions [8b,c,d], one obtains

$$\int_{S^{\text{obs}}} \tilde{\mathbf{u}}^\varepsilon \cdot \frac{\partial \varphi}{\partial \mathbf{u}}(\mathbf{u}, \cdot) \, d\Gamma = \int_{B_\varepsilon} [\nabla \hat{\mathbf{u}} : (\mathbf{C}^* - \mathbf{C}) : \nabla \mathbf{u}^\varepsilon - (\rho^* - \rho) \omega^2 \hat{\mathbf{u}} \cdot \mathbf{u}^\varepsilon] \, d\Omega \quad [11]$$

Substituting this identity into [7] leads to the following expression of the topological derivative:

$$\mathcal{T}(\mathbf{x}^o) = \lim_{\varepsilon \rightarrow 0} \frac{1}{f(\varepsilon) |\mathcal{B}|} \int_{B_\varepsilon} \text{Re} [\nabla \hat{\mathbf{u}} : (\mathbf{C}^* - \mathbf{C}) : \nabla \mathbf{u}^\varepsilon - (\rho^* - \rho) \omega^2 \hat{\mathbf{u}} \cdot \mathbf{u}^\varepsilon] \, d\Omega \quad [12]$$

One is then left with evaluating the leading asymptotic behavior of the above domain integral for vanishing  $\varepsilon$ . This task in turn requires to investigate the asymptotic behav-

ior of  $\mathbf{u}^\varepsilon = \mathbf{u} + \tilde{\mathbf{u}}^\varepsilon$ . This is done on the basis of an integral equation formulation of problem  $\mathcal{P}(B_\varepsilon(\mathbf{x}^\circ))$ , the details being left out of this communication because of space constraints. As a result of this analysis, The scattered field  $\tilde{\mathbf{u}}^\varepsilon$  in  $B_\varepsilon(\mathbf{x}^\circ)$  is found to have the form

$$\tilde{\mathbf{u}}^\varepsilon(\mathbf{x}) = \varepsilon \mathbf{U}\left(\frac{\mathbf{x} - \mathbf{x}^\circ}{\varepsilon}\right) + o(\varepsilon)$$

where the auxiliary field  $\mathbf{U}(\mathbf{z})$  is such that  $\mathbf{U}(\mathbf{z}) + \nabla \mathbf{u}(\mathbf{x}^\circ) \cdot \mathbf{z}$  solves the *elastostatic* transmission problem for the normalized inclusion  $\mathcal{B}$  embedded in an infinite space under the constant stress  $\boldsymbol{\sigma}[\mathbf{u}](\mathbf{x}^\circ)$  at infinity.

If  $\mathcal{B}$  has spherical or ellipsoidal shape, it is well-known [MUR 82] that such field  $\mathbf{U}(\mathbf{z})$  has constant strains inside  $\mathcal{B}$ , which can be expressed in terms of  $\mathcal{S}$ , the Eshelby tensor of  $\mathcal{B}$ , as  $\varepsilon_z[\mathbf{U}] = \mathcal{S} : [(\mathcal{C}^* - \mathcal{C}) : \mathcal{S} + \mathcal{C}]^{-1} : (\mathcal{C}^* - \mathcal{C}) : \mathcal{C}^{-1}$ . On substituting this result into [12], one then finds that  $f(\varepsilon) = \varepsilon^3$  and

$$\mathcal{T}(\mathbf{x}^\circ) = \text{Re}[\nabla \hat{\mathbf{u}} : \mathcal{A} : \nabla \mathbf{u}^\varepsilon - (\rho^* - \rho)\omega^2 \hat{\mathbf{u}} \cdot \mathbf{u}^\varepsilon](\mathbf{x}^\circ) \quad [13]$$

where the *constant* fourth-order tensor  $\mathcal{A} = \mathcal{A}(\mathcal{B}, \mathcal{C}, \mathcal{C}^*, \rho, \rho^*)$  is given by

$$\mathcal{A}(\mathcal{B}, \mathcal{C}, \mathcal{C}^*, \rho, \rho^*) = [(\mathcal{C}^* - \mathcal{C}) : \mathcal{S} + \mathcal{C}]^{-1} : (\mathcal{C}^* - \mathcal{C}) : \mathcal{C}^{-1} \quad [14]$$

The topological derivative is established on the basis of a choice for  $(\mathcal{B}, \mathcal{C}^*, \rho^*)$  and depends on that choice. In particular, one has the option of tuning the value of some or all of  $(\mathcal{B}, \mathcal{C}^*, \rho^*)$  so as to obtain a *pointwise* optimal value  $\mathcal{T}^{\text{opt}}(\mathbf{x}^\circ)$ . For isotropic reference and inclusion media, detailed expressions for  $\mathcal{A}(\mathcal{B}; \mu, \nu, \rho, \mu^*, \nu^*, \rho^*)$ , not shown for brevity, are readily established in terms of the elastic moduli  $\mu, \nu, \mu^*, \nu^*$ .

**4 The linear acoustic case.** A formula similar to [13] can be established for the linear acoustic case. Consider a reference medium  $\Omega$  (wavenumber  $k$ , mass density  $\rho$ ) housing a penetrable inclusion  $B^{\text{true}}$  (wavenumber  $k^{\text{true}} = k/\gamma$ , mass density  $\rho^{\text{true}} = \rho/\beta$ ). The total field  $u^{\text{true}}$  (e.g. the acoustic pressure) is governed by problem  $\mathcal{P}_k(B^{\text{true}})$ , where  $\mathcal{P}_k(B)$  denotes the generic transmission problem defined by the set of equations

$$\mathcal{P}_k(B) : \begin{cases} (\Delta + k^2)v = 0 \text{ (in } \Omega \setminus B), & (\Delta + \gamma^2 k^2)v = 0 \text{ (in } B) \\ v_n = p \text{ (on } S_N), & v = 0 \text{ (on } S_D), \quad \llbracket v \rrbracket = \llbracket \rho^{-1} v, n \rrbracket = \mathbf{0} \text{ (on } \Gamma) \end{cases} \quad [15]$$

and can also be formulated in the form of a generalization of the Lippman-Schwinger integral equation [MAR 03]. Considering cost functions of the form

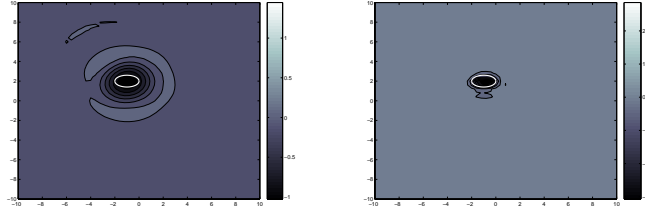
$$\mathcal{J}(B^*, \beta, \gamma) = \int_{S^{\text{obs}}} \varphi(u^*, \boldsymbol{\xi}) \, d\Gamma_\xi \quad [16]$$

where  $v = u^*$  solves the transmission problem  $\mathcal{P}_k(B^*)$ , equation [15], one then finds

$$\mathcal{J}(B_\varepsilon, \beta, \gamma) = \mathcal{J}(\emptyset, \beta, \gamma) + \varepsilon^3 |\mathcal{B}| \mathcal{T}(\mathbf{x}^\circ) + o(\varepsilon^3) \quad [17]$$

where the topological derivative  $\mathcal{T}(\mathbf{x}^\circ)$  is given this time by

$$\mathcal{T}(\mathbf{x}^\circ) = \text{Re}\{\nabla \hat{\mathbf{u}} : \mathcal{A}(\mathcal{B}, \beta, \gamma) \cdot \nabla u + (\beta\gamma^2 - 1)k^2 \hat{\mathbf{u}} u\}(\mathbf{x}^\circ) \quad [18]$$



**Figure 1.** Identification of an ellipsoidal penetrable obstacle (constitutive parameters  $\beta = 0.45, \gamma = 0.7$ , center  $(-1, 2, -3)$ , semiaxes  $(1, 0.6, 0.4)$  aligned with the Cartesian coordinate frame) embedded in an acoustic half-space: distribution of  $\mathcal{T}$  in the horizontal plane  $x_3 = -3$ , for  $k = 1$  (left) or  $k = 4$  (right). The horizontal contour of the “true” obstacle is shown (white ellipse).

In [18],  $u$  is the acoustic free-field while the adjoint field  $\hat{u}$  solves

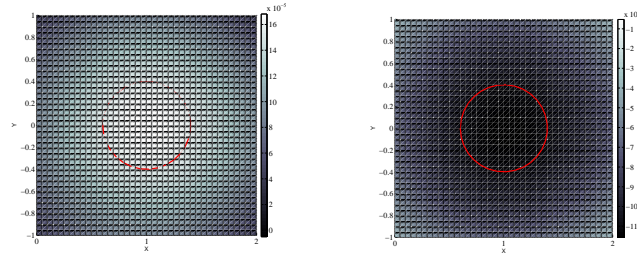
$$(\Delta + k^2)\hat{u} = 0 \text{ (in } \Omega), \quad \hat{u}_{,n} = \frac{\partial \varphi}{\partial u} \text{ (on } S^{\text{obs}}), \quad \hat{u}_{,n} = 0 \text{ (on } S_N \setminus S^{\text{obs}}), \quad \hat{u} = 0 \text{ (on } S_D)$$

and the second-order tensor  $\mathbf{A}(\mathcal{B}, \beta, \gamma)$  has been established for any inclusion shape  $\mathcal{B}$  and constitutive parameters  $(\beta, \gamma)$ . For the simplest case where  $\mathcal{B}$  is the unit sphere, one has

$$A_{ij}(\mathcal{B}, \beta, \gamma) = \frac{3(1 - \beta)}{2 + \beta} \delta_{ij} \quad [19]$$

The limiting situation  $\beta = 0$  in [18,19] yields the expression of the topological derivative for the case of a hard (i.e. rigid) obstacle of vanishing size  $\varepsilon$ .

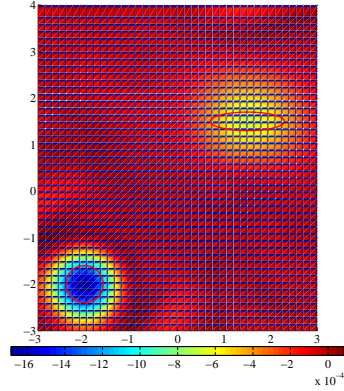
**5 Numerical example in acoustics.** A penetrable ellipsoidal inclusion  $B^{\text{true}}$  is embedded in the half-space  $x_3 \leq 0$ . The cost function [16] with  $2\varphi(w) = (w - u^{\text{obs}})(w -$



**Figure 2.** Identification of a spherical inclusion (centre  $(1, 0, -3)$ , radius  $0.4$ , parameters  $\mu^{\text{true}} = 5\mu, \nu^{\text{true}} = 1.5\nu, \rho^{\text{true}} = 1.3\rho$ ) embedded in an elastic half-space, with  $\omega = 1$ : distribution of  $\mathcal{T}(\mathbf{x}^o)$  in the horizontal plane  $x_3 = -3$ , computed with  $\mathcal{A}(\mathcal{B}; \mu, \nu, \rho, 0, 0)$ , left, or  $\mathcal{A}(\mathcal{B}; \mu, \nu, \rho, \mu^{\text{true}}, \nu^{\text{true}}, \rho^{\text{true}})$ , right. The horizontal contour of the “true” obstacle is shown (circle).

$u^{\text{obs}}$ ) is set up on the basis of the synthetic measurement  $u^{\text{obs}}$  generated on the basis of  $B^{\text{true}}$ . The source and measurement grids, both regular with size  $(21 \times 21)$ , are located in the square area  $-5 \leq x_1, x_2 \leq 5$  on the external surface  $S = \{x_3 = 0\}$ . The distribution of topological derivative  $\mathcal{T}(\mathbf{x}^\circ)$  in the plane  $x_3 = -3$ , displayed in figure 1, is consistent with the location of the “true” inclusion, despite the fact that the latter is of finite size whereas the asymptotic formula [17] only holds in the limit  $\varepsilon \rightarrow 0$ .

**6 Numerical example in elastodynamics.** The reference body  $\Omega$  is a half-space (free surface  $x_3 = 0$ ) with isotropic and homogeneous elastic moduli, and isotropic inclusions are considered. Time-harmonic point forces are applied on the free surface, and displacements recorded at measurement stations on the same surface. The case of a spherical inclusion is first considered. Figure 2 shows the distribution of  $\mathcal{T}(\mathbf{x}^\circ)$  with  $\mathcal{A}$  defined on the basis of either an infinitesimal spherical cavity (left) or an infinitesimal spherical inclusion with the correct parameters  $\mu^{\text{true}}, \nu^{\text{true}}, \rho^{\text{true}}$  (right). The latter distribution is consistent with the actual inclusion, whereas the former is not, which emphasizes the important role played by the reference parameters in the computation of  $\mathcal{T}(\mathbf{x}^\circ)$ . Finally, in figure 3, the distribution of  $\mathcal{T}^{\text{opt}}(\mathbf{x}^\circ)$  obtained by optimizing  $\mathcal{T}(\mathbf{x}^\circ)$  pointwise with respect to  $\mu^*$ , for the case of two dissimilar obstacles (a spherical elastic inclusion and an ellipsoidal cavity), exhibits two negative minima at the correct locations.



**Figure 3.** Identification of an ellipsoidal inclusion (center  $(1.5, 1.5, -3)$ , axes  $(1.6, 0.4, 1.2)$ , parameters  $\mu^{\text{true}} = 5\mu, \nu^{\text{true}} = 1.5\nu, \rho^{\text{true}} = 1.3\rho$ ) and a spherical cavity (center  $(-2, -2, -3)$ , radius  $0.4$ ) embedded in an elastic half-space, with  $\omega = 2$ : distribution of  $\mathcal{T}^{\text{opt}}(\mathbf{x}^\circ)$  in the horizontal plane  $x_3 = -3$ . The horizontal contour of the “true” obstacles are shown.

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#### 7 References

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