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# CALABI-YAU OBJECTS IN TRIANGULATED CATEGORIES

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ABSTRACT. We introduce the Calabi-Yau (CY) objects in a Hom-finite Krull-Schmidt triangulated  $k$ -category, and notice that the structure of the minimal, consequently all the CY objects, can be described. The relation between indecomposable CY objects and Auslander-Reiten triangles is provided. Finally we classify all the CY modules of self-injective Nakayama algebras, determining this way the self-injective Nakayama algebras admitting indecomposable CY modules. In particular, this result recovers the algebras whose stable categories are Calabi-Yau, which have been obtained in [BS].

## 1. Introduction

Calabi-Yau (CY) categories have been introduced by Kontsevich [Ko]. They provide a new insight and a wide framework for topics as in mathematical physics ([Co]), non-commutative geometry ([B], [Gin1], [Gin2]), and representation theory of Artin algebras ([BS], [ES], [IR], [Ke], [KR1], [KR2]).

Triangulated categories with Serre dualities ([BK], [RV]) and CY categories have important global naturality. On the other hand, even in non CY categories, inspired by [Ko], one can introduce CY objects. It turns out that they arise naturally in non CY categories and enjoy “local naturality” and interesting properties (Prop. 4.4, Theorems 3.2, 4.2, 5.5 and 6.1).

The first aim of this paper is to study the properties of such objects in a Hom-finite Krull-Schmidt triangulated  $k$ -category with Serre functor  $F$ . We give the relation between indecomposable CY objects and the Auslander-Reiten triangles (§3), and describe all the  $d$ -th CY objects via the minimal ones, which are exactly the direct sum of all the objects in finite  $\langle [-d] \circ F \rangle$ -orbits of  $\text{Ind}(\mathcal{A})$  (§4). We classify all the  $d$ -th CY modules of self-injective Nakayama algebras for any integer  $d$  (§5). Finally, we determine all the self-injective Nakayama algebras which admit indecomposable CY modules. In particular, this recovers the algebras whose stable categories are Calabi-Yau (§6), included in the work of Białkowski and Skowroński [BS]. Note that the CY modules are invariant under stable equivalences between self-injective algebras, with a very few exceptions (Prop.3.1). Consequently our results on self-injective Nakayama algebras extend to the one on the wreath-like algebras ([GR]), which contains the Brauer tree algebras ([J]).

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*Key words and phrases.* Serre functor, Calabi-Yau object, Auslander-Reiten triangle, stable module category, self-injective Nakayama algebra.

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This also raises an immediate question. Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category with a Serre functor. If all objects are  $d$ -th CY with the same  $d$ , whether  $\mathcal{A}$  is a Calabi-Yau category?

## 2. Backgrounds and Preliminaries

2.1. Let  $k$  be a field and  $\mathcal{A}$  a Hom-finite  $k$ -category. Recall from Bondal and Kapranov [BK] that a  $k$ -linear functor  $F: \mathcal{A} \rightarrow \mathcal{A}$  is a *right Serre functor* if there exist  $k$ -isomorphisms

$$\eta_{A,B}: \text{Hom}_{\mathcal{A}}(A, B) \longrightarrow D \text{Hom}_{\mathcal{A}}(B, FA), \quad \forall A, B \in \mathcal{A},$$

which are natural both in  $A$  and  $B$ , where  $D = \text{Hom}_k(-, k)$ . Such an  $F$  is unique up to a natural isomorphism, and fully-faithful; if it is an equivalence, then a quasi-inverse  $F^{-1}$  is a left Serre functor; in this case we call  $F$  a Serre functor. Note that  $\mathcal{A}$  has a Serre functor if and only if it has both right and left Serre functor. See Reiten and Van den Bergh [RV].

For triangulated categories we refer to [Har], [V], and [N]. Let  $\mathcal{A}$  be a Hom-finite triangulated  $k$ -category. Following Happel [Hap1], an *Auslander-Reiten triangle*  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$  of  $\mathcal{A}$  is a distinguished triangle satisfying:

- (AR1)  $X$  and  $Z$  are indecomposable;
- (AR2)  $h \neq 0$ ;
- (AR3) If  $t: Z' \rightarrow Z$  is not a retraction, then there exists  $t': Z' \rightarrow Y$  such that  $t = gt'$ .

Note that (AR3) is equivalent to

- (AR4) If  $Z'$  is indecomposable and  $t: Z' \rightarrow Z$  is a non-isomorphism, then  $ht = 0$ .

Under (AR1) and (AR2), (AR3) is equivalent to

- (AR3') If  $s: X \rightarrow X'$  is not a section, then there exists  $s': Y \rightarrow X'$  such that  $s = s'f$ .

Also, (AR3') is equivalent to

- (AR4') If  $X'$  is indecomposable and  $s: X \rightarrow X'$  is a non-isomorphism, then  $s \circ h[-1] = 0$ .

In an Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ , the object  $X$  is uniquely determined by  $Z$ . Write  $X = \tau_{\mathcal{A}}Z$ . In general  $\tau_{\mathcal{A}}$  is *not* a functor. By definition  $\mathcal{A}$  has right Auslander-Reiten triangles if there exists an Auslander-Reiten triangle  $X \rightarrow Y \rightarrow Z \rightarrow X[1]$  for any indecomposable  $Z$ ; and  $\mathcal{A}$  has Auslander-Reiten triangles if  $\mathcal{A}$  has right and left Auslander-Reiten triangles. We refer to [Hap1], [XZ] and [A] for the Auslander-Reiten quiver of a triangulated category.

A Hom-finite  $k$ -category is *Krull-Schmidt* if the endomorphism algebra of any indecomposable is local. In this case any object is uniquely decomposed into a direct sum of indecomposables, up to isomorphisms and up to the order of indecomposable direct summands (Ringel [R], p.52).

Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category. Theorem 1.2.4 in [RV] says that  $\mathcal{A}$  has a right Serre functor  $F$  if and if  $\mathcal{A}$  has right Auslander-Reiten triangles. In this case,  $F$  coincides with  $[1] \circ \tau_{\mathcal{A}}$  on objects, up to isomorphisms.

2.2. Let  $\mathcal{A}$  be a Hom-finite triangulated  $k$ -category with Serre functor  $F$ . Denote by  $[1]$  the shift functor of  $\mathcal{A}$ . Following Kontsevich [Ko],  $\mathcal{A}$  is a *Calabi-Yau category* if there is a natural isomorphism  $F \cong [d]$  of functors for some  $d \in \mathbb{Z}$ .

Denote by  $o([1])$  the order of  $[1]$ . If  $o([1]) = \infty$  then the integer  $d$  above is unique, and is called *the CY dimension* of  $\mathcal{A}$ ; if  $o([1])$  is finite then we call the minimal non-negative integer  $d$  such that  $F \cong [d]$  *the CY dimension* of  $\mathcal{A}$ . Denote by  $\text{CYdim}(\mathcal{A})$  the CY dimension of  $\mathcal{A}$ .

For example, if  $A$  is a symmetric algebra and  $\mathcal{P}$  is the category of projective modules, then the homotopy category  $K^b(\mathcal{P})$  is of CY dimension 0. Moreover, if  $\mathcal{A}$  is of CY dimension  $d$ , then  $\text{Ext}_{\mathcal{A}}^i(X, Y) \cong D \circ \text{Ext}_{\mathcal{A}}^{d-i}(Y, X)$ ,  $X, Y \in \mathcal{A}$ ,  $i \in \mathbb{Z}$ , where  $\text{Ext}_{\mathcal{A}}^i(X, Y) := \text{Hom}_{\mathcal{A}}(X, Y[i])$ . Thus, if  $A$  is a CY algebra ([B], [Gin2]), i.e. the bounded derived category  $D^b(A\text{-mod})$  is Calabi-Yau of CY dimension  $d$ , then  $\text{gl. dim}(A\text{-mod}) = d$  (see [B]).

2.3. Let  $\mathcal{A}$  and  $\mathcal{B}$  be triangulated categories. A *triangle functor* from  $\mathcal{A}$  to  $\mathcal{B}$  is a pair  $(F, \eta^F)$ , where  $F: \mathcal{A} \rightarrow \mathcal{B}$  is an additive functor and  $\eta^F: F \circ [1] \rightarrow [1] \circ F$  is a natural isomorphism, such that if  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$  is a distinguished triangle of  $\mathcal{A}$  then  $FX \xrightarrow{Ff} FY \xrightarrow{Fg} FZ \xrightarrow{\eta_X^F \circ Fh} (FX)[1]$  is a distinguished triangle of  $\mathcal{B}$ . Triangle functors  $(F, \eta^F)$  and  $(G, \eta^G)$  are *natural isomorphic* if there is a natural isomorphism  $\xi: F \rightarrow G$  such that the following diagram commutes for any  $A \in \mathcal{A}$

$$\begin{array}{ccc} F(A[1]) & \xrightarrow{\eta_A^F} & F(A)[1] \\ \xi_{A[1]} \downarrow & & \downarrow \xi_{A[1]} \\ G(A[1]) & \xrightarrow{\eta_A^G} & G(A)[1]. \end{array}$$

As Keller pointed out, the pair  $([n], (-1)^n \text{Id}_{[n+1]}) : \mathcal{A} \rightarrow \mathcal{A}$  is a triangle functor for  $n \in \mathbb{Z}$ . However,  $([n], \text{Id}_{[n+1]})$  may be *not*.

We need the following important result. A nice proof given by Van den Bergh is in the Appendix of [B].

**Lemma 2.1.** (*Bondal-Kapranov [BK]; Van den Bergh [B]*) *Let  $F$  be a Serre functor of a Hom-finite triangulated  $k$ -category  $\mathcal{A}$ . Then there exists a natural isomorphism  $\eta^F: F \circ [1] \rightarrow [1] \circ F$  such that  $(F, \eta^F): \mathcal{A} \rightarrow \mathcal{A}$  is a triangle functor.*

From [Ke, 8.1] and [B, A.5.1] one has the following

**Proposition 2.2.** (*Keller; Van den Bergh*) *Let  $\mathcal{A}$  be a Hom-finite triangulated  $k$ -category with Serre functor  $F$ . Then  $\mathcal{A}$  is a Calabi-Yau category if and only if there exists a natural isomorphism  $\eta^F: F \circ [1] \rightarrow [1] \circ F$ , such that  $(F, \eta^F)$  is a triangle functor and  $(F, \eta^F) \rightarrow ([d], (-1)^d \text{Id}_{[d+1]})$  is a natural isomorphism of triangle functors, for some integer  $d$ .*

**Proof.** For convenience we justify the “only if” part. By assumption we have a natural isomorphism  $\xi: F \cong [d]$ . Define  $\eta_A^F: F(A[1]) \rightarrow (FA)[1]$  for  $A \in \mathcal{A}$  by  $\eta_A^F := (-1)^d (\xi_A)^{-1}[1] \circ \xi_{A[1]}$ . Then  $\xi_{A[1]} \circ \eta_A^F = (-1)^d \xi_{A[1]}$ . The naturality of  $\eta^F: F \circ [1] \rightarrow [1] \circ F$  follows from the one of  $\xi$ . It remains to show that  $(F, \eta^F): \mathcal{A} \rightarrow \mathcal{A}$  is a triangle functor. Let  $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$  be a distinguished triangle. Since  $X[d] \xrightarrow{f[d]} Y[d] \xrightarrow{g[d]} Z[d] \xrightarrow{(-1)^d h[d]} X[d+1]$  is a distinguished triangle,

it suffices to prove that the following diagram is commutative

$$\begin{array}{ccccccc}
F(X) & \xrightarrow{F(f)} & F(Y) & \xrightarrow{F(g)} & F(Z) & \xrightarrow{\eta_X^F \circ F(h)} & (F(X))[1] \\
\xi_X \downarrow \wr & & \xi_Y \downarrow \wr & & \xi_Z \downarrow \wr & & \xi_{X[1]} \downarrow \wr \\
X[d] & \xrightarrow{f[d]} & Y[d] & \xrightarrow{g[d]} & Z[d] & \xrightarrow{(-1)^d h[d]} & X[d+1].
\end{array}$$

By the naturality of  $\xi$  the first and the second square are commutative. We also have

$$\xi_{X[1]} \circ \eta_X^F \circ F(h) = (-1)^d \xi_{X[1]} \circ F(h) = (-1)^d h[d] \circ \xi_Z. \quad \blacksquare$$

2.4. Let  $A$  be a self-injective  $k$ -algebra,  $A\text{-mod}$  the category of finite-dimensional left  $A$ -modules, and  $A\text{-mod}$  the stable category of  $A\text{-mod}$  modulo projective modules. Then the Nakayama functor  $\mathcal{N} := D(A) \otimes_A -$ , Heller's syzygy functor  $\Omega$ , and the Auslander-Reiten translate  $\tau \cong \Omega^2 \circ \mathcal{N} \cong \mathcal{N} \circ \Omega^2$  ([ARS], p.126), and are endo-equivalences of  $A\text{-mod}$  ([ARS], Chap. IV). Note that  $A\text{-mod}$  is a Hom-finite Krull-Schmidt triangulated  $k$ -category with  $[1] = \Omega^{-1}$  ([Hap1], p.16). By the bi-naturality of the Auslander-Reiten isomorphisms ([AR])

$$\underline{\text{Hom}}(X, Y) \cong D \circ \text{Ext}_A^1(Y, \tau X) \cong D \circ \underline{\text{Hom}}(Y, [1] \circ \tau X),$$

where  $\underline{\text{Hom}}(X, Y) := \text{Hom}_{A\text{-mod}}(X, Y)$ , one gets the Serre functor  $F := [1] \circ \tau \cong \Omega \circ \mathcal{N}$  of  $A\text{-mod}$ . It follows that  $A\text{-mod}$  is Calabi-Yau if and only if  $\mathcal{N} \cong \Omega^{-(d+1)}$  for some  $d$  ([Ke, 8.3]). In this case denote by  $\text{CYdim}(A)$  the CY dimension of  $A\text{-mod}$ . Note that  $\Omega$ ,  $F$ ,  $\mathcal{N}$ ,  $\tau$  are pairwise commutative as functors of  $A\text{-mod}$ . This follows from Lemma 2.1.

2.5. Let  $A$  be a finite-dimensional  $k$ -algebra. Recall that  $A$  is a Nakayama algebra if any indecomposable is uniserial, i.e. it has a unique composition series ([ARS], p.197). In this case  $A$  is representation-finite. If  $k$  is algebraically closed then any connected self-injective Nakayama algebra is Morita equivalent to  $\Lambda(n, t)$ ,  $n \geq 1$ ,  $t \geq 2$  ([GR], p.243), which is defined below.

Let  $\mathbb{Z}_n$  be the cyclic quiver with vertices indexed by the cyclic group  $\mathbb{Z}/n\mathbb{Z}$  of order  $n$ , and with arrows  $a_i : i \rightarrow i+1$ ,  $\forall i \in \mathbb{Z}/n\mathbb{Z}$ . Let  $k\mathbb{Z}_n$  be the path algebra of the quiver  $\mathbb{Z}_n$ ,  $J$  the ideal generated by all arrows, and  $\Lambda = \Lambda(n, t) := k\mathbb{Z}_n/J^t$  with  $t \geq 2$ . Denote by  $\gamma_i^l$  the path starting at vertex  $i$  and of length  $l$ , and  $e_i := \gamma_i^0$ . We write the conjunction of paths from right to left. Then  $\{\gamma_i^l \mid 0 \leq i \leq n-1, 0 \leq l \leq t-1\}$  is a basis of  $\Lambda$ ; while  $\{P(i) := \Lambda e_i \mid 0 \leq i \leq n-1\}$  is the set of pairwise non-isomorphic indecomposable projective modules, and  $\{I(i) := D(e_i \Lambda) \mid 0 \leq i \leq n-1\}$  is the set of pairwise non-isomorphic indecomposable injective modules, with  $P(i) \cong I(i+t-1)$ . Note that  $\Lambda$  is a Frobenius algebra, and  $\Lambda$  is symmetric if and only if  $n \mid (t-1)$ . Write  $S(i) := P(i)/\text{rad } P(i)$ , and  $S_i^l := \Lambda \gamma_{i+l-t}^{t-l}$ . Then  $S_i^l$  is the indecomposable with top  $S(i)$  and the Loewy length  $l$ , and  $\{S_i^l \mid 0 \leq i \leq n-1, 1 \leq l \leq t\}$  is the set of pairwise non-isomorphic indecomposable modules, with  $S_i^t = P(i)$  and  $\text{soc}(S_i^l) = S(i+l-1)$ . For the Auslander-Reiten quiver of  $\Lambda$  see [GR], Section 2, and [ARS], p.197. In particular, the stable Auslander-Reiten quiver of  $\Lambda$  is  $\mathbb{Z}A_{t-1}/\langle \tau^n \rangle$ .

### 3. Indecomposable Calabi-Yau objects

The purpose of this section is to introduce the Calabi-Yau objects and to give the relation between indecomposable Calabi-Yau objects and Auslander-Reiten triangles.

3.1. Let  $\mathcal{A}$  be a Hom-finite triangulated  $k$ -category. A non-zero object  $X$  is called a *Calabi-Yau object* if there exists a natural isomorphism

$$(3.1) \quad \mathrm{Hom}_{\mathcal{A}}(X, -) \cong D \circ \mathrm{Hom}_{\mathcal{A}}(-, X[d])$$

for some integer  $d$ .

By Yoneda Lemma, such a  $d$  is unique up to a multiple of the relative order  $o([1]_X)$  of  $[1]$  respect to  $X$ . Recall that  $o([1]_X)$  is the minimal positive integer such that  $X[o([1]_X)] \cong X$ , otherwise  $o([1]_X) = \infty$ . If  $o([1]_X) = \infty$  then  $d$  in (3.1) is unique and is called *the CY dimension* of  $X$ . If  $o([1]_X)$  is finite then the minimal non-negative integer  $d$  in (3.1) is called *the CY dimension* of  $X$ . We denote  $\mathrm{CYdim}(X)$  the CY dimension. Thus, if  $o([1]) < \infty$  then  $o([1]_X) \mid o([1])$  and  $0 \leq \mathrm{CYdim}(X) < o([1]_X)$ .

Let  $A$  be a finite-dimensional self-injective algebra. An  $A$ -module  $M$  without projective direct summands is called a *Calabi-Yau module* of CY dimension  $d$ , if it is a Calabi-Yau object of  $\underline{A\text{-mod}}$  with  $\mathrm{CYdim}(M) = d$ .

Note that  $\mathrm{CYdim}(X)$  is usually not easy to determine. In case (3.1) holds for some  $d$ , we say that  $X$  is a  *$d$ -th CY object*. Of course, if  $o([1]_X) < \infty$  then  $o([1]_X) \mid (d - \mathrm{CYdim}(X)) \geq 0$ .

If  $\mathcal{A}$  has right Serre functor  $F$ , then by Yoneda Lemma a non-zero object  $X$  is a  $d$ -th CY object if and only if  $F(X) \cong X[d]$ , or equivalently,  $F(X)[-d] \cong X$ . Thus, a non-zero  $A$ -module  $M$  without projective direct summands is a  $d$ -th CY module if and only if  $\mathcal{N}(M) \cong \Omega^{-(d+1)}(M)$  in  $\underline{A\text{-mod}}$  (in fact, this isomorphism can be taken in  $A\text{-mod}$ ).

3.2. We have the following basic property.

**Proposition 3.1.** (i) *The Calabi-Yau property for a category or an object, is invariant under triangle-equivalences.*

(ii) *The Calabi-Yau property for a module is “usually” invariant under stable equivalences between self-injective algebras. Precisely, let  $A$  and  $B$  be self-injective algebras,  $G : \underline{A\text{-mod}} \rightarrow \underline{B\text{-mod}}$  a stable equivalence, and  $X$  a CY  $A$ -module of dimension  $d$ . If  $A \not\cong \Lambda(n, 2)$ , or if  $A$  and  $B$  are symmetric algebras, then  $G(X)$  is a CY  $B$ -module of dimension  $d$ .*

**Proof.** (i) Let  $\mathcal{A}$  be a Calabi-Yau category with  $F_{\mathcal{A}} \cong [d]$ , where  $F_{\mathcal{A}}$  is the Serre functor. Clearly  $\bar{F}_{\mathcal{B}} := G \circ F_{\mathcal{A}} \circ G^{-1}$  is a Serre functor of  $\mathcal{B}$  (if  $\mathcal{B}$  has already one, then it is natural isomorphic to  $F_{\mathcal{B}}$ ). By the natural isomorphism  $(\xi^G)^d : G \circ [d] \rightarrow [d] \circ G$ , which is the composition  $G \circ [d] \rightarrow [1] \circ G \circ [d-1] \rightarrow \cdots \rightarrow [d] \circ G$  (A.2 in [B]), we see that  $\mathcal{B}$  is a Calabi-Yau category with  $F_{\mathcal{B}} \cong [d]$ . If  $X$  is a Calabi-Yau object with a natural isomorphism  $\eta$  as in (3.1), then we have natural isomorphism  $\mathrm{Hom}_{\mathcal{B}}(-, (\xi^G)_X^d \circ G \circ \eta \circ G^{-1}) : \mathrm{Hom}_{\mathcal{B}}(G(X), -) \cong D \circ \mathrm{Hom}_{\mathcal{B}}(-, (GX)[d])$ , which implies that  $G(X)$  is a Calabi-Yau object of  $\mathcal{B}$ .

(ii) Recall that an equivalence  $G : \underline{A\text{-mod}} \rightarrow \underline{B\text{-mod}}$  of categories is called a stable equivalence. Note that in general  $G$  is not induced by an exact functor (cf. [ARS], p.339), hence  $G$  may be not a triangle-equivalence (cf. [Hap1], Lemma 2.7, p.22. Note that the converse of Lemma 2.7 is also true). One may assume that  $A$  is connected. If  $A \not\cong \Lambda(n, 2)$ , or if  $A$  and  $B$  are symmetric algebras, then by Corollary 1.7 and Prop. 1.12 in [ARS], p.344, we know that  $G$  commutes with  $\tau$  and  $\Omega$  on

modules, hence we have isomorphism

$$\Omega_B^{-1} \circ \tau_B(G(X)) \cong G(\Omega_A^{-1} \circ \tau_A)(X) \cong G(\Omega_A^{-d}(X)) \cong \Omega_B^{-d}(G(X)),$$

which implies that  $G(X)$  is a Calabi-Yau  $B$ -module of CY dimension  $d$ .  $\blacksquare$

It seems that the Calabi-Yau property for the stable category is also invariant under stable equivalence  $G$  between self-injective algebras. However, this need natural isomorphisms between  $G$  and  $\tau$ , and  $G$  and  $\Omega$ , which are not clear to us.

3.3. The main result of this section is as follows.

**Theorem 3.2.** *Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category, and  $X$  an indecomposable object of  $\mathcal{A}$ . Then  $X$  is a  $d$ -th CY object if and only if there exists an Auslander-Reiten triangle of the form*

$$(3.2) \quad X[d-1] \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} X[d].$$

Moreover,  $Y$  is also a  $d$ -th CY object.

3.4. The proof of the first part of Theorem 3.2 follows an argument of Reiten and Van den Bergh in [RV]. For the convenience we include a complete proof.

**Lemma 3.3.** *Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category, and  $X$  a non-zero object of  $\mathcal{A}$ . Then  $X$  is a  $d$ -th CY object if and only if for any indecomposable  $Z$  there exists a non-degenerate bilinear form*

$$(3.3) \quad (-, -)_Z : \text{Hom}_{\mathcal{A}}(X, Z) \times \text{Hom}_{\mathcal{A}}(Z, X[d]) \longrightarrow k$$

such that for any  $u \in \text{Hom}_{\mathcal{A}}(X, Z)$ ,  $v \in \text{Hom}_{\mathcal{A}}(Z, W)$ , and  $w \in \text{Hom}_{\mathcal{A}}(W, X[d])$ , there holds

$$(3.4) \quad (u, wv)_Z = (vu, w)_W.$$

**Proof.** If  $X$  is a  $d$ -th CY object, then we have  $\eta_Z : \text{Hom}_{\mathcal{A}}(X, Z) \cong D \circ \text{Hom}_{\mathcal{A}}(Z, X[d])$ ,  $\forall Z \in \mathcal{A}$ , which are natural in  $Z$ . Each isomorphism  $\eta_Z$  induces a non-degenerate bilinear form  $(-, -)_Z$  in (3.3) by  $(u, z)_Z := \eta_Z(u)(z)$ , and (3.4) follows from the naturality of  $\eta_Z$  in  $Z$ . Conversely, if we have (3.3) and (3.4) for any indecomposable  $Z$ , then we have isomorphism  $\eta_Z : \text{Hom}_{\mathcal{A}}(X, Z) \cong D \circ \text{Hom}_{\mathcal{A}}(Z, X[d])$  given by  $\eta_Z(u)(z) := (u, z)_Z$ ,  $\forall z \in \text{Hom}_{\mathcal{A}}(Z, X[d])$ . By (3.4)  $\eta_Z$  are natural in  $Z$ . Since  $\mathcal{A}$  is Krull-Schmidt, it follows that we have isomorphisms  $\eta_Z$  for any  $Z$  which are natural in  $Z$ . This means that  $X$  is a  $d$ -th CY object.  $\blacksquare$

The following Lemma in [RV] will be used.

**Lemma 3.4.** ([RV], Sublemma I.2.3) *Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category,  $\tau_{\mathcal{A}}(X) \longrightarrow Y \longrightarrow X \xrightarrow{h} \tau_{\mathcal{A}}(X)[1]$  an Auslander-Reiten triangle of  $\mathcal{A}$ , and  $Z$  an indecomposable in  $\mathcal{A}$ . Then*

- (i) *For any non-zero  $z \in \text{Hom}_{\mathcal{A}}(Z, \tau_{\mathcal{A}}(X)[1])$  there exists  $u \in \text{Hom}_{\mathcal{A}}(X, Z)$  such that  $zu = h$ .*
- (ii) *For any non-zero  $u \in \text{Hom}_{\mathcal{A}}(X, Z)$  there exists  $z \in \text{Hom}_{\mathcal{A}}(Z, \tau_{\mathcal{A}}(X)[1])$  such that  $zu = h$ .*

In a Hom-finite Krull-Schmidt triangulated  $k$ -category without Serre functor (e.g., by [Hap2] and [RV] if  $\text{gl. dim}(A) = \infty$  then  $D^b(A\text{-mod})$  has no Serre functor), one may use *the generalized Serre functor* introduced by Chen [Ch].

**Lemma 3.5.** (Chen [Ch]) *Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category. Consider the full subcategories of  $\mathcal{A}$  given by*

$$\mathcal{A}_r := \{X \in \mathcal{A} \mid D \circ \text{Hom}_{\mathcal{A}}(X, -) \text{ is representable}\}$$

and

$$\mathcal{A}_l := \{X \in \mathcal{A} \mid D \circ \text{Hom}_{\mathcal{A}}(-, X) \text{ is representable}\}.$$

Then both  $\mathcal{A}_r$  and  $\mathcal{A}_l$  are thick triangulated subcategories of  $\mathcal{A}$ . Moreover, one has

(i) *There is a unique  $k$ -functor  $S : \mathcal{A}_r \rightarrow \mathcal{A}_l$  which is an equivalence, such that there are natural isomorphisms*

$$(3.5) \quad \text{Hom}_{\mathcal{A}}(X, -) \simeq D \circ \text{Hom}_{\mathcal{A}}(-, S(X)), \quad \forall X \in \mathcal{A}_r,$$

which are natural in  $X$ .  $S$  is called *the generalized Serre functor*, with range  $\mathcal{A}_r$  and domain  $\mathcal{A}_l$ .

(ii) *There exists a natural isomorphism  $\eta^S : S \circ [1] \rightarrow [1] \circ S$  such that the pair  $(S, \eta^S) : \mathcal{A}_r \rightarrow \mathcal{A}_l$  is an triangle-equivalence.*

In this terminology, a non-zero object  $X$  is a  $d$ -th CY object if and only if  $X \in \mathcal{A}_r$  and  $S(X) \cong X[d]$ , by (3.5) and Yoneda Lemma.

**3.5. Proof of Theorem 3.2.** Let  $X$  be an indecomposable  $d$ -th CY object. By Lemma 3.3 we have a non-degenerate bilinear form  $(-, -)_X : \text{Hom}_{\mathcal{A}}(X, X) \times \text{Hom}_{\mathcal{A}}(X, X[d]) \rightarrow k$ . It follows that there exists  $0 \neq h \in \text{Hom}_{\mathcal{A}}(X, X[d])$  such that  $(\text{radHom}_{\mathcal{A}}(X, X), h)_X = 0$ . Embedding  $h$  into a distinguished triangle as in (3.2). We claim that it is an Auslander-Reiten triangle. For this it remains to prove (AR4) in 2.1. Let  $X'$  be indecomposable and  $t : X' \rightarrow X$  a non-isomorphism. Then by (3.4) for any  $u \in \text{Hom}_{\mathcal{A}}(X, X')$  we have  $(u, ht)_{X'} = (tu, h)_X = 0$ . Since  $(-, -)_{X'}$  is non-degenerate, it follows that  $ht = 0$ .

Conversely, let (3.2) be an Auslander-Reiten triangle. In order to prove that  $X$  is a  $d$ -th CY object, by Lemma 3.3 it suffices to prove that for any indecomposable  $Z$  there exists a non-degenerate bilinear form  $(-, -)_Z$  as in (3.3) satisfying (3.4). For this, choose an arbitrary linear function  $\text{tr} \in D \circ \text{Hom}_{\mathcal{A}}(X, X[d])$  such that  $\text{tr}(h) \neq 0$ , and define  $(u, z)_Z = \text{tr}(zu)$ . Then (3.4) is automatically satisfied. It remains to prove that  $(-, -)_Z$  is non-degenerate. In fact, for any  $0 \neq z \in \text{Hom}_{\mathcal{A}}(Z, X[d])$ , by Lemma 3.4 there exists  $u \in \text{Hom}_{\mathcal{A}}(X, Z)$  such that  $zu = h$ . So  $(u, z)_Z = \text{tr}(zu) = \text{tr}(h) \neq 0$ . Similarly, for any  $0 \neq u \in \text{Hom}_{\mathcal{A}}(X, Z)$  we have  $z \in \text{Hom}_{\mathcal{A}}(Z, X[d])$  such that  $(u, z)_Z \neq 0$ . This proves the non-degenerateness of  $(-, -)_Z$ .

Now we prove that  $Y$  in (3.2) is also a  $d$ -th CY object. We make use of the generalized Serre functor in [Ch] (For the reader prefer Serre functor, one can assume the existence, and use Lemma 2.1). Since  $X, X[d-1] \in \mathcal{A}_r$ , it follows from Lemma 3.5 that  $Y \in \mathcal{A}_r$ . Applying the generalized Serre functor  $(S, \eta^S)$  to (3.2) we get the distinguished triangle (by Lemma 3.5(ii))

$$S(X[d-1]) \xrightarrow{S(f)} S(Y) \xrightarrow{S(g)} S(X) \xrightarrow{\eta_{X[d-1]}^S \circ S(h)} S(X[d-1])[1]. \quad (*)$$



Also, we have the Auslander-Reiten triangle

$$X[2d-1] \xrightarrow{f[d]} Y[d] \xrightarrow{g[d]} X[d] \xrightarrow{(-1)^d h[d]} X[2d].$$

Since  $X$  is a  $d$ -th CY object it follows that we have an isomorphism  $w : S(X) \rightarrow X[d]$ . Note that  $S(h) \neq 0$  means that  $S(g)$  is not a retraction ([Hap1], p.7). Thus, by (AR3) there exists  $v : S(Y) \rightarrow Y[d]$  such that  $w \circ S(g) = g[d] \circ v$ . By the definition of a triangulated category we get  $u : S(X[d-1]) \rightarrow X[2d-1]$  such that the following diagram is commutative

$$\begin{array}{ccccccc} S(X[d-1]) & \xrightarrow{S(f)} & S(Y) & \xrightarrow{S(g)} & S(X) & \xrightarrow{\eta_{X[d-1]}^S \circ S(h)} & S(X[d-1])[1] \\ u \downarrow & & v \downarrow & & w \downarrow & & u[1] \downarrow \\ X[2d-1] & \xrightarrow{f[d]} & Y[d] & \xrightarrow{g[d]} & X[d] & \xrightarrow{(-1)^d h[d]} & X[2d]. \end{array} \quad (**)$$

We claim that  $u : S(X[d-1]) \rightarrow X[2d-1]$  is an isomorphism, hence by the property of a triangulated category we know that  $v : S(Y) \rightarrow Y[d]$  is also an isomorphism, i.e.  $Y$  is a  $d$ -th CY object.

Otherwise  $u : S(X[d-1]) \rightarrow X[2d-1]$  is not an isomorphism. Note that  $S$  is *only* defined on  $\mathcal{A}_r$ , it follows that we do not know if  $(*)$  is an Auslander-Reiten triangle.

Since  $X[d-1]$  is a  $d$ -th CY object, it follows that we have an isomorphism  $\alpha : X[2d-1] \rightarrow S(X[d-1])$ , hence  $\alpha \circ u \in \text{Hom}_{\mathcal{A}}(S(X[d-1]), S(X[d-1]))$ . So we have  $u' \in \text{Hom}_{\mathcal{A}}(X[d-1], X[d-1])$  such that  $\alpha \circ u = S(u')$ , and  $u'$  is also a non-isomorphism. Since  $X[d-1] \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{h} X[d]$  is an Auslander-Reiten triangle it follows from (AR4') that  $u' \circ h[-1] = 0$ , or equivalently,  $S(u') \circ S(h[-1]) = 0$  (note that  $h[-1] \in \mathcal{A}_r$ ). Thus we have

$$u[1] \circ S(h[-1])[1] \circ \eta_{X[-1]}^S = 0$$

where  $\eta_{X[-1]}^S : S(X) \rightarrow S(X[-1])[1]$  is an isomorphism. By the naturality of  $\eta^S$  we have the commutative diagram

$$\begin{array}{ccc} S(X) & \xrightarrow{\eta_{X[-1]}^S} & S(X[-1])[1] \\ S(h) \downarrow & & \downarrow S(h[-1])[1] \\ S(X[d]) & \xrightarrow{\eta_{X[d-1]}^S} & S(X[d-1])[1]. \end{array}$$

It follows that we have  $u[1] \circ \eta_{X[d-1]}^S \circ S(h) = 0$ , and hence by the commutative diagram  $(**)$  we get a contradiction  $(-1)^d h[d] \circ w = u[1] \circ \eta_{X[d-1]}^S \circ S(h) = 0$ . This completes the proof.  $\blacksquare$

**3.6. Remark 3.6.** Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category. If every indecomposable  $X$  in  $\mathcal{A}$  is a  $d_X$ -th CY object, then  $\mathcal{A}$  has a Serre functor  $F$  with  $F(X) \cong X[d_X]$ .

In fact, by Theorem 3.2  $\mathcal{A}$  has right and left Auslander-Reiten triangles, and then by Theorem 1.2.4 in [RV]  $\mathcal{A}$  has Serre functor  $F$ . By Prop.1.2.3 in [RV] and (3.2) we have  $F(X) \cong \tau_{\mathcal{A}}(X)[1] = X[d_X - 1][1] = X[d_X]$  for any indecomposable  $X$ .

However, even if all indecomposables are  $d$ -th CY objects with the same  $d$ , we do not know whether  $\mathcal{A}$  is a Calabi-Yau category, although  $F$  and  $[d]$  coincide on objects. The examples we know have a positive answer to this question.  $\blacksquare$

#### 4. Minimal Calabi-Yau objects

The purpose of this section is to describe all the Calabi-Yau objects of a Hom-finite Krull-Schmidt triangulated  $k$ -category with a Serre functor.

4.1. Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category. A  $d$ -th CY object  $X$  is said to be *minimal* if any *proper* direct summand of  $X$  is *not* a  $d$ -th CY object.

**Lemma 4.1.** *Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category with right Serre functor  $F$ . Then a non-zero object  $X$  is a minimal  $d$ -th CY object if and only if the following are satisfied:*

1. *The indecomposable direct summands of  $X$  can be ordered as  $X = X_1 \oplus \cdots \oplus X_r$  such that*

$$(4.1) \quad F(X_1) \cong X_2[d], F(X_2) \cong X_3[d], \dots, F(X_{r-1}) \cong X_r[d], F(X_r) \cong X_1[d].$$

*We call the cyclic order arising from this property a canonical order of  $X$  (with respect to  $F$  and  $[d]$ ).*

2.  *$X$  is multiplicity-free, i.e. its indecomposable direct summands are pairwise non-isomorphic.*

**Proof.** In the following we often use that a non-zero object  $X$  is a  $d$ -th CY object if and only if  $F(X) \cong X[d]$ .

Let  $X = X_1 \oplus \cdots \oplus X_r$  be a minimal  $d$ -th CY object, with each  $X_i$  indecomposable and  $r \geq 2$ . Then  $F(X_1) \oplus \cdots \oplus F(X_r) \cong X_1[d] \oplus \cdots \oplus X_r[d]$ . Since  $\mathcal{A}$  is Krull-Schmidt, it follows that there exists a permutation  $\sigma$  of  $1, \dots, r$ , such that  $F(X_i) \cong X_{\sigma(i)}[d]$  for each  $i$ . Write  $\sigma$  as a product of disjoint cyclic permutations. Since  $X$  is minimal, it follows that  $\sigma$  has to be a cyclic permutation of length  $r$ . By reordering the indecomposable direct summands of  $X$ , one may assume that  $\sigma = (12 \cdots r)$ . Thus,  $X$  satisfies the condition 1.

Now, we consider a canonical order  $X = X_1 \oplus \cdots \oplus X_r$ . If  $X_i \cong X_j$  for some  $1 \leq i < j \leq r$ , then  $F(X_{j-1}) \cong X_j[d] \cong X_i[d]$ , it follows that  $X_i \oplus \cdots \oplus X_{j-1}$  is already a  $d$ -th CY object, which contradicts the minimality of  $X$ . This proves that  $X$  is multiplicity-free.

Conversely, assume that a multiplicity-free object  $X = X_1 \oplus \cdots \oplus X_r$  is in a canonical order. By (4.1) we have  $F(X) \cong X[d]$ . So  $X$  is a  $d$ -th CY object. It remains to show the minimality. If not, then there exists a proper direct summand  $X_{i_1} \oplus \cdots \oplus X_{i_t}$  of  $X$  which is a minimal  $d$ -th CY object, so  $1 \leq t < r$ . By what we have proved above we may assume that this is a canonical order. Then

$$F(X_{i_1}) \cong X_{i_2}[d], \dots, F(X_{i_{t-1}}) \cong X_{i_t}[d], F(X_{i_t}) \cong X_{i_1}[d].$$

While  $X = X_1 \oplus \cdots \oplus X_r$  is also in a canonical order, it follows that (note that we work on indices modulo  $r$ , e.g. if  $i_1 = r$  then  $i_1 + 1$  is understood to be 1)

$$F(X_{i_1}) \cong X_{i_1+1}[d], \dots, F(X_{i_{t-1}}) \cong X_{i_{t-1}+1}[d], F(X_{i_t}) \cong X_{i_t+1}[d].$$

Since  $X$  is multiplicity-free, it follows that (considering indices modulo  $r$ )

$$i_2 = i_1 + 1, \dots, i_t = i_{t-1} + 1, i_1 = i_t + 1,$$

hence  $i_1 = i_1 + t$ , which means  $r \mid t$ . This is impossible since  $1 \leq t < r$ . ■

4.2. Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category with Serre functor  $F$ . For each  $d \in \mathbb{Z}$ , consider the triangle-equivalence  $G := [-d] \circ F \cong F \circ [-d] : \mathcal{A} \rightarrow \mathcal{A}$ . For each indecomposable  $M \in \mathcal{A}$ , denote by  $o(G_M)$  the relative order of  $G$  respect to  $M$ , that is,  $r := o(G_M)$  is the minimal positive integer such that  $G^r(M) \cong M$ , otherwise  $o(G_M) = \infty$ . Denote by  $\text{Aut}(\mathcal{A})$  the group of the triangle-equivalences of  $\mathcal{A}$ , and by  $\langle G \rangle$  the cyclic subgroup of  $\text{Aut}(\mathcal{A})$  generated by  $G$ . Then  $\langle G \rangle$  acts naturally on  $\text{Ind}(\mathcal{A})$ , the set of the isoclasses of indecomposables of  $\mathcal{A}$ . Denote by  $\mathcal{O}_M$  the  $G$ -orbit of an indecomposable  $M$ . Then  $|\mathcal{O}_M| = o(G_M)$ . If  $|\mathcal{O}_M| < \infty$  then the set  $\mathcal{O}_M$  is a finite  $G$ -orbit.

Denote by  $\text{Fin } \mathcal{O}(\mathcal{A}, d)$  the set of all the finite  $G$ -orbits of  $\text{Ind}(\mathcal{A})$ , and by  $\text{MinCY}(\mathcal{A}, d)$  the set of isoclasses of minimal  $d$ -th CY objects. We have the following

**Theorem 4.2.** *Let  $\mathcal{A}$  be a Hom-finite Krull-Schmidt triangulated  $k$ -category with Serre functor  $F$ . Then*

- (i) *Every  $d$ -th CY object is a direct sum of finitely many minimal  $d$ -th CY objects.*
- (ii) *With the notations above, for each  $d \in \mathbb{Z}$  the map*

$$(4.2) \quad \mathcal{O}_M \mapsto \bigoplus_{X \in \mathcal{O}_M} X = M \oplus G(M) \oplus \cdots \oplus G^{o(G_M)-1}(M)$$

*gives a one-to-one correspondence between the sets  $\text{Fin } \mathcal{O}(\mathcal{A}, d)$  and  $\text{MinCY}(\mathcal{A}, d)$ , where  $G := [-d] \circ F$ .*

*Thus, a minimal  $d$ -th CY object is exactly the direct sum of all the objects in a finite  $G$ -orbit of  $\text{Ind}(\mathcal{A})$ .*

(iii) *Non-isomorphic minimal  $d$ -th CY objects are disjoint, i.e. they have no isomorphic indecomposable direct summands.*

**Proof.** Let  $X$  be a  $d$ -th CY object. If  $X$  is not minimal, then  $X = Y \oplus Z$  with  $Y \neq 0 \neq Z$  such that  $F(Y) \oplus F(Z) \cong Y[d] \oplus Z[d]$  and  $F(Y) \cong Y[d]$ . Since  $\mathcal{A}$  is Krull-Schmidt, it follows that  $F(Z) \cong Z[d]$ , i.e.  $Z$  is also a  $d$ -th CY object. Then (i) follows by induction.

Thanks to Lemma 2.1, (4.1) becomes  $X_i = G^{i-1}(X_1)$ ,  $1 \leq i \leq r$ . Then (ii) is a reformulation of Lemma 4.1. Moreover (iii) follows from (ii).  $\blacksquare$

**Corollary 4.3.** *Let  $A$  be a finite-dimensional self-injective algebra. Then  $X$  is a minimal  $d$ -th CY module if and only if  $X$  is of the form  $X \cong \bigoplus_{0 \leq i \leq r-1} G^i(M)$ , where  $M$  is an indecomposable non-projective  $A$ -module with  $r := o(G_M) < \infty$ , and  $G := \Omega^{d+1} \circ \mathcal{N}$ .*

**Proof.** Note that in this case  $[-d] \circ F = \Omega^{d+1} \circ \mathcal{N}$ .  $\blacksquare$

4.3. As an example, we describe all the Calabi-Yau objects in  $D^b(kQ\text{-mod})$ , the bounded derived category of  $kQ\text{-mod}$ , where  $Q$  is a finite quiver without oriented cycles.

Note that indecomposable objects of  $D^b(kQ\text{-mod})$  are exactly stalk complexes of indecomposable  $kQ$ -modules. The category  $D^b(kQ\text{-mod})$  has Serre functor  $F = [1] \circ \tau_D$ , where  $\tau_D$  is the Auslander-Reiten translation of  $D^b(kQ\text{-mod})$ . Recall that  $\tau_D$  is given by

$$\tau_D(M) = \begin{cases} \tau(M), & \text{if } M \text{ is an indecomposable non-projective;} \\ I[-1], & \text{if } P \text{ is an indecomposable projective,} \end{cases}$$

where  $I$  is the indecomposable injective with  $\text{soc}(I) = P/\text{rad } P$ , and  $\tau$  is the Auslander-Reiten translation of  $kQ\text{-mod}$  ([Hap1], p.51). Note that  $D^b(kQ\text{-mod})$  is *not* a Calabi-Yau category except that  $Q$  is the trivial quiver with one vertex and no arrows. However, the cluster category  $\mathcal{C}_{kQ}$  introduced in [BMRRT], which is the orbit category of  $D^b(kQ\text{-mod})$  respect to the functor  $\tau_D^{-1} \circ [1]$ , is a Calabi-Yau category of CY dimension 2 ([BMRRT]).

Let  $M$  be a minimal Calabi-Yau object of  $D^b(kQ\text{-mod})$  of CY dimension  $d$  (in this case  $o([1]_M) = \infty$ , hence  $d$  is unique). By shifts we may assume that  $M$  is a  $kQ$ -module. By  $F(M) = \tau_D(M[1]) = \tau_D(M)[1] = M[d]$  we see  $d = 1$  or  $0$ . Note that  $kQ$  admits an indecomposable projective-injective module if and only if  $Q$  is of type  $A_n$  with the linear orientation. However, in this case the unique indecomposable projective-injective module  $P = I$  does not satisfy the relation  $\text{soc}(I) = P/\text{rad } P$ . It follows that  $d \neq 0$ . Thus  $d = 1$  and  $\tau(M) = M$ . Consequently,  $Q$  is an affine quiver and  $M$  is a  $\tau$ -periodic (regular) module of period 1. All such modules are well-known, by the classification of representations of affine quivers (see Dlab-Ringel [DR]). Thus we have

**Proposition 4.4.**  *$D^b(kQ\text{-mod})$  admits a Calabi-Yau object  $M$  if and only if  $Q$  is an affine quiver. In this case,  $M$  is minimal if and only if  $M$  is an indecomposable in a homogeneous tube of the Auslander-Reiten quiver of  $kQ\text{-mod}$ , or  $M$  is the direct sum of all the indecomposables of same quasi-length in a non-homogeneous tube of the Auslander-Reiten quiver of  $kQ\text{-mod}$ , up to shifts. Moreover, all such  $M$ 's have CY dimension 1.*

## 5. Calabi-Yau modules of self-injective Nakayama algebras

This purpose of this section is to classify all the  $d$ -th CY modules of self-injective Nakayama algebras  $\Lambda(n, t)$ ,  $n \geq 1$ ,  $t \geq 2$ , where  $d$  is any given integer. By Theorem 4.2(i) it suffices to consider the minimal  $d$ -th CY  $\Lambda$ -modules. By Corollary 4.3 this reduces to computing the relative order  $o(G_M)$  of  $G := \Omega^{d+1} \circ \mathcal{N}$  respect to any indecomposable  $\Lambda(n, t)$ -module  $M = S_i^l$ ,  $1 \leq l \leq t-1$ , for any integer  $d$ .

5.1. Recall that  $\Lambda(n, t)$  is the quotient of the path algebra of the cyclic quiver with  $n$  vertices by the truncated ideal  $J^t$ , where  $J$  is the two-sided ideal generated by the arrows. From now on we write  $\Lambda$  instead of  $\Lambda(n, t)$ .

We keep the notations introduced in 2.5. Note that the indecomposable module  $S_i^l$  ( $i \in \mathbb{Z}/n\mathbb{Z}$ ,  $1 \leq l \leq t-1$ ) has a natural  $k$ -basis, consisting of all the paths of quiver  $\mathbb{Z}_n$  starting at the vertex  $i+l-t$  and of lengths at least  $t-l$ :

$$\gamma_{i+l-t}^{t-l}, \gamma_{i+l-t}^{t-l+1}, \dots, \gamma_{i+l-t}^{t-1}.$$

For  $1 \leq l \leq t-1$  denote by  $\sigma_i^l : S_i^l \rightarrow S_{i-1}^{l+1}$  the inclusion by embedding the basis above; and for  $2 \leq l \leq t$  denote by  $p_i^l : S_i^l \rightarrow S_i^{l-1}$  the  $\Lambda$ -epimorphism given by the right multiplication by arrow  $a_{i+l-t-1}$ . These  $\sigma_i^l$ 's and  $p_i^l$ 's are all the irreducible maps of  $\Lambda\text{-mod}$ , up to scalars.

We need the explicit actions of functors  $\mathcal{N}$  and  $\Omega^{-(d+1)}$  of  $\Lambda\text{-mod}$ . By the exact sequences  $0 \rightarrow S_i^l \rightarrow I(i+l-1) \rightarrow S_{i+l-t}^{t-l} \rightarrow 0$  (with the canonical maps), via the basis above one has the actions of functor  $\Omega^{-1}$  for  $i \in \mathbb{Z}/n\mathbb{Z}$ ,  $1 \leq l \leq t-1$ :

$$\Omega^{-1}(S_i^l) = S_{i+l-t}^{t-l}, \quad \Omega^{-1}(\sigma_i^l) = p_{i+l-t}^{t-l}, \quad \Omega^{-1}(p_i^l) = \sigma_{i+l-t}^{t-l}.$$

By induction one has in  $\Lambda\text{-mod}$  for any integer  $m$  (even negative):

$$(5.1) \quad \Omega^{-(2m-1)}(S_i^l) = S_{i+l-mt}^{t-l}; \quad \Omega^{-2m}(S_i^l) = S_{i-mt}^l;$$

and

$$\Omega^{-(2m-1)}(\sigma_i^l) = p_{i+l-mt}^{t-l}, \quad \Omega^{-(2m-1)}(p_i^l) = \sigma_{i+l-mt}^{t-l}; \quad (*)$$

and

$$\Omega^{-2m}(\sigma_i^l) = \sigma_{i-mt}^l, \quad \Omega^{-2m}(p_i^l) = p_{i-mt}^l. \quad (**)$$

In particular, we have

$$o([1]) = \begin{cases} n, & t = 2; \\ 2m, & t \geq 3, \end{cases} \quad (***)$$

where  $m$  is the minimal positive integer such that  $n \mid mt$ .

5.2. Again using the natural basis of  $S_i^l$  one has the following commutative diagrams in  $\Lambda\text{-mod}$ :

$$\begin{array}{ccc} \mathcal{N}(S_i^l) & \xrightarrow{\mathcal{N}(\sigma_i^l)} & \mathcal{N}(S_{i-1}^{l+1}) \\ \theta_i^l \downarrow \wr & & \theta_{i-1}^{l+1} \downarrow \wr \\ S_{i+1-t}^l & \xrightarrow{\sigma_{i+1-t}^l} & S_{i-t}^{l+1} \end{array} \quad \begin{array}{ccc} \mathcal{N}(S_i^l) & \xrightarrow{\mathcal{N}(p_i^l)} & \mathcal{N}(S_i^{l-1}) \\ \theta_i^l \downarrow \wr & & \theta_i^{l-1} \downarrow \wr \\ S_{i+1-t}^l & \xrightarrow{p_{i+1-t}^l} & S_{i+1-t}^{l-1} \end{array}$$

We justify the commutative diagrams above. Note that for any finite quiver  $Q$  the bimodule structure of  $D(kQ)$  is given by (using the dual basis)

$$p^*a = \begin{cases} b^*, & \text{if } ab = p, \\ 0, & \text{otherwise;} \end{cases} \quad \text{and} \quad ap^* = \begin{cases} b^*, & \text{if } ba = p, \\ 0, & \text{otherwise.} \end{cases}$$

for any paths  $p$  and  $a$ . Note that  $N(S_i^l) = D(\Lambda) \otimes_{\Lambda} S_i^l$  is spanned by  $(\gamma_j^{l'})^* \otimes_{\Lambda} \gamma_{i+l-t}^{t-u}$ , where  $j \in \mathbb{Z}/n\mathbb{Z}$ ,  $1 \leq l' \leq t-1$ ,  $1 \leq u \leq l$ . By  $(\gamma_j^{l'})^* \otimes_{\Lambda} \gamma_{i+l-t}^{t-u} = (\gamma_j^{l'})^* \gamma_{i+l-t}^{t-u} \otimes_{\Lambda} e_{i+l-t}$  we see that if  $(\gamma_j^{l'})^* \otimes_{\Lambda} \gamma_{i+l-t}^{t-u} \neq 0$  then  $j = i+l-l'-u$ ; and in this case we have

$$(\gamma_{i+l-l'-u}^{l'})^* \otimes_{\Lambda} \gamma_{i+l-t}^{t-u} = (\gamma_{i+l-l'-u}^{l'})^* \gamma_{i+l-t}^{t-u} \otimes_{\Lambda} e_{i+l-t} = (\gamma_{i+l-(l'+u)}^{l'+u-t})^* \otimes_{\Lambda} e_{i+l-t}.$$

This makes sense only if  $l' + u \geq t$ . So we have a basis of  $N(S_i^l)$ :

$$(\gamma_{i+l-t-v}^v)^* \otimes_{\Lambda} e_{i+l-t}, \quad 0 \leq v \leq l-1.$$

Using the natural basis of  $S_{i+1-t}^l$  given in 5.1 we have a  $\Lambda$ -isomorphism  $\theta_i^l : \mathcal{N}(S_i^l) \rightarrow S_{i+1-t}^l$  for any  $i \in \mathbb{Z}/n\mathbb{Z}$  and  $1 \leq l \leq t-1$ :

$$\theta_i^l : (\gamma_{i+l-t-v}^v)^* \otimes_{\Lambda} e_{i+l-t} \mapsto \gamma_{i+1+l-2t}^{t-(v+1)}, \quad 0 \leq v \leq l-1.$$

(One checks that this is indeed a left  $\Lambda$ -map.) Note that  $N(\sigma_i^l) : \mathcal{N}(S_i^l) \rightarrow \mathcal{N}(S_{i-1}^{l+1})$  is a natural embedding given by

$$(\gamma_{i+l-t-v}^v)^* \otimes_{\Lambda} e_{i+l-t} \mapsto (\gamma_{i+l-t-v}^v)^* \otimes_{\Lambda} e_{i+l-t}, \quad 0 \leq v \leq l-1;$$

and that  $N(p_i^l) : \mathcal{N}(S_i^l) \longrightarrow \mathcal{N}(S_i^{l-1})$  is a  $\Lambda$ -epimorphism given by

$$(\gamma_{i+l-t-v}^v)^* \otimes_{\Lambda} e_{i+l-t} \mapsto (\gamma_{i+l-t-v}^{v-1})^* \otimes_{\Lambda} e_{i+l-1-t}, \quad 0 \leq v \leq l-1$$

where  $\gamma_{i+l-1-t}^{-1}$  is understood to be 0. Then one easily checks the following

$$\sigma_{i+1-t}^l \circ \theta_i^l = \theta_{i-1}^{l+1} \circ \mathcal{N}(\sigma_i^l); \quad p_{i+1-t}^l \circ \theta_i^l = \theta_i^{l-1} \circ \mathcal{N}(p_i^l).$$

This justifies the commutative diagrams.

Since all these  $\theta_i^l$  depend only on  $i$  and  $l$ , which means that they do not depend on whatever the maps  $\sigma_i^l$  or  $p_i^l$  are (this is important for the bi-naturality of a Calabi-Yau category), it follows, without loss of the generality, that we can specialize these maps to identities. Thus we have

$$(5.2) \quad \mathcal{N}(S_i^l) = S_{i+1-t}^l, \quad \mathcal{N}(\sigma_i^l) = \sigma_{i+1-t}^l, \quad \mathcal{N}(p_i^l) = p_{i+1-t}^l.$$

5.3. By (\*\*\*) we have  $o([1]) < \infty$ , it follows, without loss of generality, that we can assume  $d \geq 0$ . For convenience, set  $d(t) := 1 + \frac{(d-1)t}{2} \in \frac{1}{2}\mathbb{Z}$ , whatever  $d$  is even or odd; denote by  $N = N(d, n, t)$  the minimal positive integer such that

$$(5.3) \quad \begin{cases} n \mid Nd(t), & \text{if } (d-1)t \text{ is even;} \\ n \mid N(2d(t)), & \text{if } (d-1)t \text{ is odd.} \end{cases}$$

(When  $2d(t)$  is odd, we will write it together in the following.)

By (5.1) we have  $\Omega^{(2m-1)}(S_i^l) = S_{i+l+(m-1)t}^{t-l}$  and  $\Omega^{2m}(S_i^l) = S_{i+mt}^l$ , hence by (5.2) we have (remember  $G := \Omega^{d+1} \circ \mathcal{N}$  and  $d \geq 0$ )

$$G(S_i^l) = S_{i+1+(m-1)t}^l, \quad \text{if } d = 2m - 1$$

and

$$G(S_i^l) = S_{i+l+1+(m-1)t}^{t-l}, \quad \text{if } d = 2m.$$

Thus, by induction we have

$$(5.4) \quad G^{m'}(S_i^l) = S_{i+m'(1+(m-1)t)}^l = S_{i+m'd(t)}^l, \quad \text{if } d = 2m - 1;$$

and

$$(5.5) \quad G^{2m'}(S_i^l) = S_{i+m'(2+(2m-1)t)}^l = S_{i+m'(2d(t))}^l, \quad \text{if } d = 2m,$$

and

$$(5.6) \quad G^{2m'+1}(S_i^l) = S_{i+(m'+1)(2d(t))+l-1-mt}^{t-l}, \quad \text{if } d = 2m,$$

for  $m' \geq 0$ .

5.4. If  $d = 2m - 1 \geq 1$ , then by (5.4) we see that  $o(G_{S_i^l}) = N$ , where  $N$  is as given in (5.3), i.e.  $N$  is the minimal positive integer such that  $n \mid N(1 + (m-1)t)$ . It follows from Corollary 4.3 and (5.4) that we have

**Lemma 5.1.** *Let  $d = 2m - 1 \geq 1$ . Then  $M$  is a minimal  $d$ -th CY  $\Lambda$ -module if and only if  $M$  is isomorphic to one of the following*

$$(5.7) \quad S_i^l \oplus S_{i+d(t)}^l \oplus S_{i+2d(t)}^l \oplus \cdots \oplus S_{i+(N-1)d(t)}^l, \quad 1 \leq l \leq t-1, \quad i \in \mathbb{Z}/n\mathbb{Z}.$$

In particular, all the minimal  $d$ -th CY modules have the same number  $N = N(d, n, t)$  of indecomposable direct summands.

5.5. If  $d = 2m \geq 0$  and  $t$  is odd, then by (5.6) we see  $G^{2m'+1}(S_i^l) \neq S_i^l$  for any  $i, l$  (since  $t-l \neq l$ ). Note that in this case  $(d-1)t$  is odd. It follows from (5.5) that  $o(G_{S_i^l}) = 2N$ , where  $N$  is as given in (5.3), i.e.  $N$  is the minimal positive integer such that  $n \mid N(2d(t))$ . It follows from Corollary 4.3, (5.5) and (5.6) that we have

**Lemma 5.2.** *Let  $t \geq 3$  be an odd integer and  $d = 2m \geq 0$ . Then  $M$  is a minimal  $d$ -th CY  $\Lambda$ -module if and only if  $M$  is isomorphic to one of the following*

$$(5.8) \quad S_i^l \oplus S_{i+l'+2d(t)}^{t-l} \oplus S_{i+2d(t)}^l \oplus S_{i+l'+4d(t)}^{t-l} \oplus \cdots \oplus S_{i+2d(t)(N-1)}^l \oplus S_{i+l'+2d(t)N}^{t-l},$$

where  $l' := l - 1 - mt$ ,  $1 \leq l \leq t - 1$  and  $i \in \mathbb{Z}/n\mathbb{Z}$ .

In particular, any minimal  $d$ -th CY modules has  $2N = 2N(d, n, t)$  indecomposable direct summands.

5.6. Let  $d = 2m \geq 0$  and  $t = 2s$ . Then  $d(t) = 1 + (2m - 1)s \in \mathbb{Z}$ .

First, we consider  $o(G_{S_i^s})$ . In this case (5.5) and (5.6) can be written in a unified way:

$$(5.9) \quad G^{m'}(S_i^s) = S_{i+m'd(t)}^s, \quad m' \geq 0, \quad \text{if } d = 2m, \quad t = 2s.$$

So we have  $o(G_{S_i^s}) = N$ , where  $N$  is as given in (5.3), i.e.  $N$  is the minimal positive integer such that  $n \mid N(1 + (2m - 1)s)$ .

Now, we consider  $o(G_{S_i^l})$  with  $l \neq s$ ,  $1 \leq l \leq t - 1$ . In this case (5.5) and (5.6) are written respectively as:

$$(5.10) \quad G^{2m'}(S_i^l) = S_{i+2m'd(t)}^l, \quad \text{if } d = 2m, \quad t = 2s, \quad l \neq s,$$

and

$$(5.11) \quad G^{2m'+1}(S_i^l) = S_{i+(2m'+1)d(t)+l-s}^{t-l}, \quad \text{if } d = 2m, \quad t = 2s, \quad l \neq s$$

for  $m' \geq 0$ . Since  $l \neq t - l$  for  $l \neq s$ , it follows that  $G^{2m'+1}(S_i^l) \neq S_i^l$ . So by (5.10) we see  $o(G_{S_i^l}) = 2N'$ , where  $N'$  is the minimal positive integer such that  $n \mid 2N'd(t)$ . In order to determine  $N'$ , we divided into two cases.

**Case 1.** If  $N = N(d, n, t) = N(2m, n, 2s)$  is even, then  $o(G_{S_i^l}) = N$  for any  $1 \leq l \leq t - 1$ . It follows from Corollary 4.3, (5.9), (5.10), and (5.11) that we have the following (note that in this case (5.9) is exactly (5.10) together with (5.11), by taking  $l = s$ )

**Lemma 5.3.** *Let  $t = 2s$  and  $d = 2m \geq 0$ . Assume that  $N = N(d, n, t)$  is even. Then  $M$  is a minimal  $d$ -th CY  $\Lambda$ -module if and only if  $M$  is isomorphic to one of the following*

$$(5.12) \quad S_i^l \oplus S_{i+d(t)+l-s}^{t-l} \oplus S_{i+2d(t)}^l \oplus S_{i+3d(t)+l-s}^{t-l} \oplus \cdots \oplus S_{i+(N-2)d(t)}^l \oplus S_{i+(N-1)d(t)+l-s}^{t-l}$$

where  $1 \leq l \leq t - 1$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$ .

In particular, all the minimal  $d$ -th CY modules have the same number  $N$  of indecomposable direct summands.

It remains to deal with

Case 2. Let  $N = N(d, n, t) = N(2m, n, 2s)$  be odd. Since by definition  $N$  is the minimal positive integer such that  $n \mid Nd(t)$ , it follows that  $N < o(G_{S_i^t}) = 2N' \leq 2N$ . It is easy to see  $N' = N$ : otherwise  $1 \leq 2N' - N \leq N - 1$  and  $n \mid (2N' - N)d(t)$ , which contradicts the minimality of  $N$ . It follows from Corollary 4.3, (5.9), (5.10), and (5.11) that we have

**Lemma 5.4.** *Let  $t = 2s$  and  $d = 2m \geq 0$ . Assume that  $N = N(d, n, t)$  is odd. Then  $M$  is a minimal  $d$ -th CY  $\Lambda$ -module if and only if  $M$  is isomorphic to one of the following*

$$(5.13) \quad S_i^s \oplus S_{i+d(t)}^s \oplus S_{i+2d(t)}^s \oplus \cdots \oplus S_{i+(N-1)d(t)}^s$$

where  $i \in \mathbb{Z}_n$ , and

$$(5.14) \quad \begin{array}{ccccccc} S_{i+l-s}^l \oplus & S_{i+d(t)+l-s}^{t-l} \oplus & S_{i+2d(t)}^l \oplus & S_{i+3d(t)+l-s}^{t-l} \oplus \cdots \oplus & S_{i+(N-1)d(t)}^l \\ \oplus S_{i+l-s}^{t-l} \oplus & S_{i+d(t)}^l \oplus & S_{i+2d(t)+l-s}^{t-l} \oplus & S_{i+3d(t)}^l \oplus \cdots \oplus & S_{i+(N-1)d(t)+l-s}^{t-l} \end{array}$$

where  $l \neq s$ ,  $1 \leq l \leq t - 1$  and  $i \in \mathbb{Z}_n$ .

In particular, all the minimal  $d$ -th CY modules have either  $N$ , or  $2N$  indecomposable direct summands.

5.7. By Lemmas 5.1-5.4 all the minimal  $d$ -th CY modules of self-injective Nakayama algebras have been classified, where  $d$  is any given integer. The main result of this section is as follows.

**Theorem 5.5.** *For any  $n \geq 1$ ,  $t \geq 2$ ,  $d \geq 0$ , let  $N = N(d, n, t)$  be as in (5.3). Then  $M$  is a minimal  $d$ -th CY  $\Lambda$ -module if and only if  $M$  is isomorphic to one of the following*

- (i) The modules in (5.7), when  $d = 2m - 1$ ;
- (ii) The modules in (5.8), when  $d = 2m$  and  $t$  is odd;
- (iii) The modules in (5.12), when  $d = 2m$ ,  $t = 2s$ , and  $N(d, n, t)$  is even;
- (iv) The modules in (5.13) – (5.14), when  $d = 2m$ ,  $t = 2s$ , and  $N(d, n, t)$  is odd.

In particular, any minimal  $d$ -th CY  $\Lambda$ -module has either  $N$  or  $2N$  indecomposable direct summands; and

$$(5.15) \quad \min\{d \geq 0 \mid N = c(M), \text{ or } 2N = c(M)\} \leq \text{CYdim}(M) < o([1]_M) \leq 2n$$

for any minimal Calabi-Yau  $\Lambda$ -modules  $M$ , where  $c(M)$  is the number of indecomposable direct summands of  $M$ .

**Proof.** By Lemmas 5.1-5.4 the CY dimension  $d$  of any minimal Calabi-Yau module  $M$  satisfies  $N(d, n, t) = c(M)$  or  $2N(d, n, t) = c(M)$ . ■

**Remark 5.6.** The modules in (5.7) – (5.8) and (5.12) – (5.14) have overlaps. This is because  $d$  is not uniquely determined by a minimal  $d$ -th CY module. A general formula of the CY dimensions of the minimal Calabi-Yau modules seems to be difficult to obtain.

Note that the inequality on the left hand side in (5.15) can not be an equality in general. For example, take  $n = 2$ ,  $t = 4$ ,  $m = 2$ ,  $d = 2m - 1 = 3$ . Then  $d(t) = 5$ ,  $N = N(3, 2, 4) = 2$ , and



$S_i^l \oplus S_{i+1}^l$ ,  $1 \leq l \leq 3$ , are all the minimal 3-th CY modules of CY dimension 0 if  $l = 2$ , and 1 if  $l = 1, 3$ . However, the left hand side in (5.15) is 0 since  $N(0, 2, 4) = 2$ .

## 6. Self-injective Nakayama algebras with indecomposable Calabi-Yau modules

In this section we determine all the self-injective Nakayama algebras  $\Lambda = \Lambda(n, t)$ ,  $n \geq 1$ ,  $t \geq 2$ , which admit indecomposable Calabi-Yau modules.

Note that Erdmann and Skowroński have proved in §2 of [ES] that self-injective algebras  $A$  such that  $A\text{-mod}$  is Calabi-Yau of CY dimension 0 (resp. 1) are the algebras Morita equivalent to  $\Lambda(n, 2)$  for some  $n \geq 1$  (resp.  $\Lambda(1, t)$  for some  $t \geq 3$ ). So we assume that  $t \geq 3$ .

**Theorem 6.1.** *Let  $t \geq 3$ . Then  $\Lambda$  has an indecomposable Calabi-Yau module if and only if  $n$  and  $t$  satisfy one of the following conditions*

(i)  *$\text{g.c.d.}(n, t) = 1$ . This is exactly the case where  $\Lambda\text{-mod}$  is a Calabi-Yau category. In this case we have  $\text{CYdim}(\Lambda) = 2m - 1$ , where  $m$  is the minimal positive integer such that  $n \mid (m - 1)t + 1$ .*

(ii)  *$\text{g.c.d.}(n, t) \neq 1$ ,  $t = 2s$ , and  $\text{g.c.d.}(n, s) = 1$ . This is exactly the case where  $\Lambda\text{-mod}$  is not a Calabi-Yau category but admits indecomposable Calabi-Yau modules.*

*In this case, we have  $\text{g.c.d.}(n, t) = 2$  and*

(a)  *$S_i^s$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$ , are all the indecomposable Calabi-Yau modules;*

(b)  *$S_i^l \oplus S_{i+l-s}^{t-l}$ ,  $1 \leq l \leq s - 1$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$ , are all the decomposable minimal  $2m$ -th CY modules, where  $m$  is the minimal non-negative integer such that  $n \mid (2m - 1)s + 1$ ;*

(c) *All of these modules in (a) and (b) have the same CY dimension  $2m$ .*

**Remark.** Bialkowski and Skowroński [BS] have classified representation-finite self-injective algebras whose stable categories are Calabi-Yau. This includes the assertion (i) of Theorem 6.1.

**Proof.** If  $\Lambda$  has an indecomposable  $d$ -th CY module  $S_i^l$ , then  $\mathcal{N}(S_i^l) \cong \Omega^{-(d+1)}(S_i^l)$ . By (5.2) and (5.1) we have

$$(6.1) \quad 1 - t \equiv -mt \pmod{n}, \text{ if } d + 1 = 2m,$$

or

$$(6.2) \quad t = t - l, \quad 1 - t \equiv l - mt \pmod{n}, \text{ if } d + 1 = 2m - 1.$$

In the first case we get  $\text{g.c.d.}(n, t) = 1$ . In the second case we get  $t = 2s$ ,  $l = s$  and  $\text{g.c.d.}(n, s) = 1$ . Excluding the overlap situations we conclude that either  $\text{g.c.d.}(n, t) = 1$ ; or  $\text{g.c.d.}(n, t) \neq 1$ ,  $t = 2s$ ,  $\text{g.c.d.}(n, s) = 1$ .

Assume that  $\text{g.c.d.}(n, t) = 1$ . Then there exists an integer  $m$  such that  $n \mid (m - 1)t + 1$ . We chose a positive (otherwise, add  $n(-m + 2)t$ ), and minimal  $m$ . Set  $d := 2m - 1$ . Then the same computation shows that every indecomposable is a  $d$ -th CY module. We claim that  $\Lambda\text{-mod}$  is a Calabi-Yau category. For this, it remains to show  $\mathcal{N}(f) = \Omega^{-(d+1)}(f) = \Omega^{-2m}(f)$  for any morphism  $f$  between indecomposables of  $\Lambda\text{-mod}$ . Since  $n \mid (m - 1)t + 1$ , it follows from (\*\*\*) in §5

that

$$(6.3) \quad \mathcal{N}(\sigma_i^l) = \Omega^{-2m}(\sigma_i^l), \quad \mathcal{N}(p_i^l) = \Omega^{-2m}(p_i^l).$$

Since  $\Lambda$  is representation-finite, it follows that  $f$  is a  $k$ -combination of compositions of irreducible maps  $\sigma_i^l$ 's and  $p_i^l$ 's, hence  $\mathcal{N}(f) = \Omega^{-2m}(f)$  by (6.3). We stress here that, this argument relies on the fact that all the isomorphisms  $\theta_i^l$  in 5.2 depend only on  $i$  and  $l$ , which means that they do not depend on whatever the maps  $\sigma_i^l$  or  $p_i^l$  are. Otherwise we can not take them as identities, and then we can not get the naturality for the Calabi-Yau category of this case.

This proves the claim, hence  $\Lambda\text{-mod}$  is a Calabi-Yau category of CY dimension  $D$ , with  $0 \leq D \leq d = 2m - 1$ . We claim  $D = d$ . In fact, since every indecomposable is a  $D$ -th CY module, and since we have  $l$  such that  $t \neq t - l$ , it follows from (6.1) and (6.2) that  $D = 2m' - 1$  with  $n \mid (m' - 1)t + 1$ , hence  $D = d$ , by the minimality of  $m$ .

The argument above also proves that if  $\Lambda\text{-mod}$  is a Calabi-Yau category then  $\text{g.c.d.}(n, t) = 1$ .

Assume that  $\text{g.c.d.}(n, t) \neq 1$ ,  $t = 2s$ , and  $\text{g.c.d.}(n, s) = 1$ . Then there exists an integer  $m'$  such that  $n \mid (m' - 1)s + 1$ . We choose a positive  $m'$ . Since  $\text{g.c.d.}(n, t) \neq 1$ , it follows that  $m'$  is even, say  $m' = 2m$  with  $m \geq 1$ . Let  $m$  be the minimal positive integer such that  $n \mid (2m - 1)s + 1$ , and set  $d := 2m$ . Then the same computation shows that  $S_i^s$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$ , are all the indecomposable Calabi-Yau modules. This proves (a).

By applying Lemma 5.4 to  $d$  given above (note that the corresponding  $N = N(n, d, t) = 1$  in this case), we know that  $S_i^l \oplus S_{i+l-s}^{t-l}$ ,  $1 \leq l \leq t - 1$ ,  $l \neq s$ ,  $i \in \mathbb{Z}/n\mathbb{Z}$ , are all the decomposable minimal  $2m$ -th CY modules. By symmetry one can consider  $1 \leq l \leq s - 1$ : if  $l > s$  then one can replace  $l$  by  $t - l$ , since  $i = (i + l - s) + (t - l) - s$ . This proves (b).

It remains to prove (c). Let  $\text{CYdim}(S_i^s) = d'$ . Since  $\text{g.c.d.}(n, t) \neq 1$ , it follows that  $d'$  has to be an even integer  $2m' \geq 0$  with  $n \mid (2m' - 1)s + 1$ . It follows that  $d' = d$ , by the minimality of  $m$ . Let  $\text{CYdim}(S_i^l \oplus S_{i+l-s}^{t-l}) = d'$ . Then we have  $\mathcal{N}(S_i^l) = \Omega^{-(d'+1)}(S_{i+l-s}^{t-l})$ . Since  $l \neq s$  it follows from (5.2) and (5.1) that  $d'$  has to be  $2m' \geq 0$  with  $n \mid (2m' - 1)s + 1$ . Again by the minimality of  $m$  we have  $d' = d$ . This completes the proof.  $\blacksquare$

**Remark 6.2.** (i)  $\text{CYdim}(X)$  usually differs from  $\text{CYdim}(\mathcal{A})$  in a Calabi-Yau category  $\mathcal{A}$ .

For example, if  $n = 3$ ,  $t = 4$ , then  $o([1]) = 6$ ,  $\text{CYdim}(\Lambda) = 5$ , while  $\text{CYdim}(S_i^2) = 2$  and  $o([1]_{S_i^2}) = 3$ ,  $\forall i \in \mathbb{Z}/3\mathbb{Z}$ .

However, for  $t \geq 3$  and  $\text{g.c.d.}(n, t) = 1$ , if  $X$  is indecomposable and  $\text{CYdim}(X)$  is odd then  $\text{CYdim}(X) = \text{CYdim}(\Lambda)$ . In fact, since  $o([1]) < \infty$  it follows that  $\text{CYdim}(X) = 2m' - 1 \geq 1$ , and  $n \mid 1 + (m' - 1)t$ . By Theorem 6.1(i)  $\text{CYdim}(\Lambda) = 2m - 1$ , where  $m$  is the minimal positive integer such that  $n \mid 1 + (m - 1)t$ . It follows that  $m' \geq m$ , hence  $\text{CYdim}(X) \geq \text{CYdim}(\Lambda)$ . On the other hand we have  $\text{CYdim}(X) \leq \text{CYdim}(\Lambda)$  by definition.

(ii) Consider the algebra  $A(t) := kA_\infty^\infty/J^t$  where  $A_\infty^\infty$  is the infinite quiver

$$\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots$$

Then  $A(t)\text{-mod}$  has a Serre functor, and there is a natural covering functor  $A(t)\text{-mod} \longrightarrow \Lambda(n, t)\text{-mod}$  ([Gab], 2.8). But one can prove that in any case  $A(t)\text{-mod}$  is not a Calabi-Yau category.

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