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# ANALYSIS AND COMPARISON OF NONLINEAR FILTERING METHODS

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**Abstract:** This paper deals with the state estimation of a strongly nonlinear system. In a noisy state space representation setting, Central Difference Kalman Filter, Ensemble Kalman Filter and Particle Filter are tested on a second order system. The choice of estimators parameters is then discussed, and their behaviour in relation to noise is studied, in order to compare estimation quality according to noise's variance criteria.

**Keywords:** state estimation, Kalman filters, particle filters, Monte Carlo method, nonlinearity, noise levels.

## 1. INTRODUCTION

Filtering has always taken an important place in automatic control. It can be found in application areas as advanced control, navigation, signal processing and diagnosis.

In a state space representation setting, the most popular tool is the Kalman filter (Kalman, 1960), also known as linear Gaussian optimal filter. But, in reality, most of systems do not respect these hypotheses. Many researchers have attempted to consider non Gaussian cases, with the *Gaussian Sum Filter* (Aspach and Sorenson, 1972), and the nonlinear case, leading to the *Extended Kalman Filter* (EKF). But this state estimator has some well known drawbacks, such as needing to calculate the Jacobians of nonlinear functions, which is not easy, but above all, may cause divergence in some cases (Anderson and Moore, 1979). Re-

cent work partially solved these problems, see for example the *Unscented Kalman Filter* (UKF) (Julier and Uhlmann, 1997), the *Central Difference Kalman Filter* (CDKF) (Nørgaard *et al.*, 2000) and the *Ensemble Kalman Filter* (EnKF) (Burgers *et al.*, 1998). These versions lead to significant results, but are based on empirical developments.

A more general setting is provided by Monte Carlo filters, also called *Particle Filters* (PF) (Arulampalam *et al.*, 2002) (Doucet, 1998). This kind of tool is more powerful, but also more time-consuming and difficult to synthesize.

In this paper, a problem statement is first presented in section 2, and then, different methods of nonlinear filtering are exposed in section 3. The choice of the estimators parameters is discussed in section 4, and the behaviour of these tools in relation to noises is then studied in the next

section. A comparison on a nonlinear second order system is provided in section 6.

## 2. PROBLEM STATEMENT

In a discrete state space setting, the problem of filtering is presented as shown in figure 1. It consists of attempting to make the estimated state  $\hat{\mathbf{x}}_k$  the closest possible to the real value  $\mathbf{x}_k$ . To reach this purpose, the only knowledges we have are :

- the values of input vector  $\mathbf{u}_k$  and output vector  $\mathbf{y}_k$ ,
- the process model, represented by (1), where  $f$  and  $g$  are nonlinear vector fields,
- some statistics of the process noise  $\mathbf{w}_k$  and the measurements noise  $\mathbf{v}_k$ .

$$\begin{cases} \mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}, k-1) \\ \mathbf{y}_k = g(\mathbf{x}_k, \mathbf{u}_k, k) + \mathbf{v}_k \end{cases} \quad (1)$$

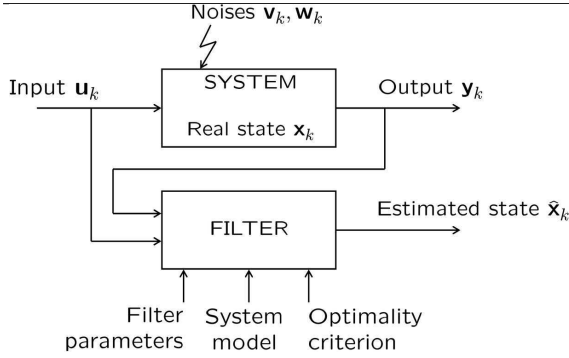


Fig. 1. Filtering problematic

In a stochastic setting,  $\mathbf{x}_k \in \mathbb{R}^{n_x}$ ,  $\mathbf{y}_k \in \mathbb{R}^{n_y}$ ,  $\mathbf{v}_k \in \mathbb{R}^{n_v}$  and  $\mathbf{w}_k \in \mathbb{R}^{n_w}$  are considered as random vectors, and  $\mathbf{u}_k \in \mathbb{R}^{n_u}$  as a deterministic input (Anderson and Moore, 1979).

## 3. NONLINEAR FILTERING METHODS

The optimal filter is described by the probability density  $p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k})$ , which can be recursively calculated by the *optimal Bayesian filtering equation* (2).

$$\begin{aligned} p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k-1}) &= \int_{\mathbb{R}^{n_x}} p(\mathbf{x}_k|\mathbf{x}_{k-1}) \cdot p(\mathbf{x}_{k-1}|\mathbf{y}_{0 \rightarrow k-1}) \cdot d\mathbf{x}_{k-1} \\ p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k}) &= \frac{p(\mathbf{y}_k|\mathbf{x}_k) \cdot p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k-1})}{\int_{\mathbb{R}^{n_x}} p(\mathbf{y}_k|\mathbf{x}_k) \cdot p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k-1}) \cdot d\mathbf{x}_k} \end{aligned} \quad (2)$$

Then, the state can be calculated via the two optimality criterion :

- least square : 
$$\hat{\mathbf{x}}_k = \mathbb{E}(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k}) = \int_{\mathbb{R}^{n_x}} \mathbf{x}_k \cdot p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k}) \cdot d\mathbf{x}_k$$

- maximum likelihood :

$$\hat{\mathbf{x}}_k = \arg \left\{ \max_{\mathbf{x}_k \in \mathbb{R}^{n_x}} [p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k})] \right\}$$

Unfortunately, the equations (2) cannot analytically be solved, excepted in the Gaussian linear<sup>1</sup> case : it leads to the Kalman filter. In the other case, these equations can be computed by Monte Carlo simulation, i.e. the realisation of a particle filter.

The principal nonlinear Kalman filters are the EKF, UKF, CDKF and EnKF. The estimators tested in this article are the last two. Their algorithms are given below:

1) Initialization with  $\mathbf{P}_{0|0}^{xx}$  and  $\hat{\mathbf{x}}_{0|0}$  and :

- $n_a = n_x + n_w$
- $\sigma_x = 2 \cdot n_x + 1$
- $\sigma_a = 2 \cdot n_a + 1$
- $\mathcal{W}_{ma}^{(i)} = \begin{cases} (h^2 - n_a)/(h^2) & \text{if } i = 1 \\ 1/(2 \cdot h^2) & \text{if } 2 \leq i \leq n_a + 1 \end{cases}$
- $\mathcal{W}_{mx}^{(i)} = \begin{cases} (h^2 - n_x)/(h^2) & \text{if } i = 1 \\ 1/(2 \cdot h^2) & \text{if } 2 \leq i \leq n_a + 1 \end{cases}$
- $\mathcal{W}_{c1} = 1/(4 \cdot h^2)$
- $\mathcal{W}_{c2} = (h^2 - 1)/(4 \cdot h^2)$

2) Prediction step :

- $\hat{\mathbf{X}}_{k-1} = \hat{\mathbf{X}}_{k-1|k-1} = [\hat{\mathbf{x}}_{k-1|k-1}^T \quad \mathbf{0}]^T$
- $\mathbf{P}_{k-1}^{aa} = \begin{bmatrix} \mathbf{P}_{k-1|k-1}^{xx} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{ww} \end{bmatrix}$
- $\mathbf{S}_{k-1}^{aa} = \sqrt{\mathbf{P}_{k-1}^{aa}}$
- $\mathbf{X}_{k-1}^{(i)} = \begin{cases} \hat{\mathbf{X}}_{k-1} & \text{if } i = 1 \\ \hat{\mathbf{X}}_{k-1} + h \cdot \mathbf{S}_{k-1}^{aa}(i-1) & \text{if } 2 \leq i \leq n_a + 1 \\ \hat{\mathbf{X}}_{k-1} - h \cdot \mathbf{S}_{k-1}^{aa}(i - n_a - 1) & \text{if } n_a + 2 \leq i \leq \sigma_a \end{cases}$
- $\left[ \left( \mathbf{x}_{k-1}^{(i)} \right)^T \quad \left( \mathbf{w}_{k-1}^{(i)} \right)^T \right]^T = \mathbf{X}_{k-1}^{(i)} \quad i = 1, \dots, \sigma_a$
- $\mathbf{x}_{k|k-1}^{(i)} = f \left( \mathbf{x}_{k-1}^{(i)}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}^{(i)}, k-1 \right) \quad i = 1, \dots, \sigma_a$
- $\hat{\mathbf{x}}_{k|k-1} = \sum_{i=1}^{\sigma_a} \mathcal{W}_{ma}^{(i)} \cdot \mathbf{x}_{k|k-1}^{(i)}$
- $\mathbf{P}_{k|k-1}^{xx} = \sum_{i=2}^{n_a+1} \mathcal{W}_{c1} \left\| \mathbf{x}_{k|k-1}^{(i)} - \hat{\mathbf{x}}_{k|k-1} \right\|^2 + \mathcal{W}_{c2} \left\| \mathbf{x}_{k|k-1}^{(i)} + \mathbf{x}_{k|k-1}^{(i+n_a)} - 2\hat{\mathbf{x}}_{k|k-1} \right\|^2$

3) Correction step :

- $\mathbf{S}_{k|k-1}^{xx} = \sqrt{\mathbf{P}_{k|k-1}^{xx}}$
- $\bar{\mathbf{x}}_{k|k-1}^{(i)} = \begin{cases} \hat{\mathbf{x}}_{k|k-1} & \text{if } i = 1 \\ \hat{\mathbf{x}}_{k|k-1} + h \cdot \mathbf{S}_{k|k-1}^{xx}(i-1) & \text{if } 2 \leq i \leq n_x + 1 \\ \hat{\mathbf{x}}_{k|k-1} - h \cdot \mathbf{S}_{k|k-1}^{xx}(i - n_x - 1) & \text{if } n_x + 2 \leq i \leq \sigma_x \end{cases}$
- $\mathbf{y}_{k|k-1}^{(i)} = g \left( \bar{\mathbf{x}}_{k|k-1}^{(i)}, \mathbf{u}_k, k \right) \quad i = 1, \dots, \sigma_x$
- $\hat{\mathbf{y}}_{k|k-1} = \sum_{i=1}^{\sigma_x} \mathcal{W}_{mx}^{(i)} \cdot \mathbf{y}_{k|k-1}^{(i)}$
- $\mathbf{P}_{k|k-1}^{yy} = \sum_{i=2}^{n_x+1} \mathcal{W}_{c1} \left\| \mathbf{y}_{k|k-1}^{(i)} - \hat{\mathbf{y}}_{k|k-1} \right\|^2 + \mathcal{W}_{c2} \left\| \mathbf{y}_{k|k-1}^{(i)} + \mathbf{y}_{k|k-1}^{(i+n_x)} - 2\hat{\mathbf{y}}_{k|k-1} \right\|^2$

<sup>1</sup> It can also be solved in nonlinear case, when  $p(\mathbf{x}_k|\mathbf{y}_{0 \rightarrow k})$  is Gaussian or close to be Gaussian, and  $f$  and  $g$  keep this Gaussian character : it is the basis of the nonlinear extensions of Kalman filtering. See (Julier and Uhlmann, 1994) for further details.

- $\mathbf{P}_{k|k-1}^{xy} = \sqrt{\mathcal{W}_{k|k-1}} \mathbf{S}_{k|k-1}^{xx} \begin{bmatrix} \mathbf{y}_{k|k-1}^{(1)} \cdots \mathbf{y}_{k|k-1}^{(n_x)} \end{bmatrix}^T$
- $\mathbf{K}_k = \mathbf{P}_{k|k-1}^{xy} \left( \mathbf{P}_{k|k-1}^{yy} \right)^{-1}$
- $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \left( \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1} \right)$
- $\mathbf{P}_{k|k}^{xx} = \mathbf{P}_{k|k-1}^{xx} - \mathbf{K}_k \cdot \mathbf{P}_{k|k-1}^{yy} \cdot \mathbf{K}_k^T$

CDKF algorithm, in the case of additive measurement noise (VanDerMerwe, 2004)

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The nomenclatures is such that :

- in a general way :  
 $\mathbf{P}^{ab} = \mathbb{E} \left[ (\mathbf{a} - \mathbb{E}(\mathbf{a}))(\mathbf{b} - \mathbb{E}(\mathbf{b}))^T \right]$ , where  $T$  denotes the transposition operator,
- the augmented state  $\mathbf{X}$  includes the process noise :  $\mathbf{X} = \begin{bmatrix} \mathbf{x}^T & \mathbf{w}^T \end{bmatrix}^T$ ,
- as covariance matrices are positive and symmetric, the matrices  $\mathbf{S}$  can be computed via the Cholesky decomposition,
- $\mathbf{M}(i)$  denotes the  $i^{\text{th}}$  column of  $\mathbf{M}$ ,
- $\|\mathbf{M}\|^2 = \mathbf{M}\mathbf{M}^T$  and  $\|\mathbf{M}\|_{\mathbf{P}}^2 = \mathbf{M}\mathbf{P}\mathbf{M}^T$ ,
- in a general way :  $\mathbf{x} \sim \mathcal{N}(\mathbf{m}, \mathbf{P})$  means that  $\mathbf{x}$  is normally distributed, with mean  $\mathbf{m}$  and covariance  $\mathbf{P}$ .

- 
- 1) Initialisation with  $\mathbf{P}_{0|0}^{xx}$  and  $\hat{\mathbf{x}}_{0|0}$ .
  - 2) Prediction step :
    - $\mathbf{x}_{k-1}^{(i)} \sim \mathcal{N}(\hat{\mathbf{x}}_{k-1|k-1}, \mathbf{P}_{k-1|k-1}^{xx}) \quad i = 1, \dots, N$
    - $\mathbf{w}_{k-1}^{(i)} \sim \mathcal{N}(\mathbf{0}, \mathbf{P}^{ww}) \quad i = 1, \dots, N$
    - $\mathbf{x}_{k|k-1}^{(i)} = f \left( \mathbf{x}_{k-1}^{(i)}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}^{(i)}, k-1 \right) \quad i = 1, \dots, N$
    - $\hat{\mathbf{x}}_{k|k-1} = \frac{1}{N} \sum_{i=1}^N \mathbf{x}_{k|k-1}^{(i)}$
    - $\tilde{\mathbf{x}}_{k|k-1}^{(i)} = \mathbf{x}_{k|k-1}^{(i)} - \hat{\mathbf{x}}_{k|k-1} \quad i = 1, \dots, N$
    - $\mathbf{P}_{k|k-1}^{xx} = \frac{1}{N} \sum_{i=1}^N \left( \tilde{\mathbf{x}}_{k|k-1}^{(i)} \right) \left( \tilde{\mathbf{x}}_{k|k-1}^{(i)} \right)^T$
  - 3) Correction step :
    - $\mathbf{y}_{k|k-1}^{(i)} = g \left( \mathbf{x}_{k|k-1}^{(i)}, \mathbf{u}_k, k \right)$
    - $\hat{\mathbf{y}}_{k|k-1} = \frac{1}{N} \sum_{i=1}^N \mathbf{y}_{k|k-1}^{(i)}$
    - $\tilde{\mathbf{y}}_{k|k-1}^{(i)} = \mathbf{y}_{k|k-1}^{(i)} - \hat{\mathbf{y}}_{k|k-1}$
    - $\mathbf{P}_{k|k-1}^{yy} = \frac{1}{N} \sum_{i=1}^N \left( \tilde{\mathbf{y}}_{k|k-1}^{(i)} \right) \left( \tilde{\mathbf{y}}_{k|k-1}^{(i)} \right)^T$
    - $\mathbf{P}_{k|k-1}^{xy} = \frac{1}{N} \sum_{i=1}^N \left( \tilde{\mathbf{x}}_{k|k-1}^{(i)} \right) \left( \tilde{\mathbf{y}}_{k|k-1}^{(i)} \right)^T$
    - $\mathbf{K}_k = \mathbf{P}_{k|k-1}^{xy} \left( \mathbf{P}_{k|k-1}^{yy} \right)^{-1}$
    - $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \left( \mathbf{y}_k - \hat{\mathbf{y}}_{k|k-1} \right)$
    - $\mathbf{P}_{k|k}^{xx} = \mathbf{P}_{k|k-1}^{xx} - \mathbf{K}_k \cdot \mathbf{P}_{k|k-1}^{yy} \cdot \mathbf{K}_k^T$

EnKF algorithm (Burgers *et al.*, 1998)

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Concerning the particle filters, we restrict our choice to the simplest, which is also the most popular, i.e. using the transition kernel for importance density. The resulting algorithm is presented below :

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- 1) Initialisation with  $\mathbf{P}_{0|0}^{xx}$  and  $\mathbf{x}_0$  :

- $\mathbf{x}_0^{(i)} \sim \mathcal{N}(\mathbf{x}_0, \mathbf{P}_{0|0}^{xx}) \quad i = 1, \dots, N$
  - $\mathcal{W}_k^{(i)} = \frac{1}{N} \quad i = 1, \dots, N$
- 2) Prediction step :
    - $\mathbf{w}_{k-1}^{(i)} \sim \mathcal{N}(\mathbf{0}, \mathbf{P}^{ww}) \quad i = 1, \dots, N$
    - $\mathbf{x}_k^{(i)} = f \left( \mathbf{x}_{k-1}^{(i)}, \mathbf{u}_{k-1}, \mathbf{w}_{k-1}^{(i)}, k-1 \right)$
  - 3) Correction step :
    - $\tilde{\mathbf{y}}_k^{(i)} = \mathbf{y}_k - g \left( \mathbf{x}_k^{(i)}, \mathbf{u}_k, k \right)$
    - $K_k^{(i)} = \frac{1}{\sqrt{(2\pi)^{n_y} \cdot \det(\mathbf{P}^{vv})}} \exp \left( -\frac{1}{2} \left\| \tilde{\mathbf{y}}_k^{(i)} \right\|_{(\mathbf{P}^{vv})^{-1}}^2 \right)$
    - $\tilde{\mathcal{W}}_k^{(i)} = \mathcal{W}_{k-1}^{(i)} \cdot K_k^{(i)} \quad i = 1, \dots, N$
    - $\mathcal{W}_k^{(i)} = \frac{\tilde{\mathcal{W}}_k^{(i)}}{\sum_{j=1}^N \tilde{\mathcal{W}}_k^{(j)}} \quad i = 1, \dots, N$
  - 4) Resampling step (optional)
  - 5) State estimation :
    - In the case of maximum likelihood :  $\hat{\mathbf{x}}_k = \mathbf{x}_k^{(i)}$  such that  $\mathcal{W}_k^{(i)} \geq \mathcal{W}_k^{(j)} \quad \forall j \in \{1, \dots, N\}, j \neq i$
    - In the case of least squares :  $\hat{\mathbf{x}}_k = \sum_{i=1}^N \mathcal{W}_k^{(i)} \cdot \mathbf{x}_k^{(i)}$

PF's algorithm, in case of Gaussian noises and importance density equal to transition kernel (Doucet, 1998)

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## 4. TUNING OF THE FILTERS PARAMETERS

### 4.1 Description of the system tested

The system we choose to study is commonly used in the particle filtering community (Gordon *et al.*, 1993), (Arulampalam *et al.*, 2002). In order to complicate this further, we added a second non linear state equation, giving the system (3), in a general way as :  $\mathbf{a} = [a^{(1)} \dots a^{(n)}]^T$ .

$$\begin{aligned}
 x_k^{(1)} &= \frac{1}{2} \cdot x_{k-1}^{(1)} + \frac{25 \cdot x_{k-1}^{(1)}}{1 + \left( x_{k-1}^{(1)} \right)^2} + 8 \cos(1.2k) + w_{k-1}^{(1)} \\
 x_k^{(2)} &= 8 \sin \left( x_{k-1}^{(1)} \right) + 8 \sin \left( 1.2x_{k-1}^{(2)} \right) + w_{k-1}^{(2)} \\
 y_k^{(1)} &= \frac{\left( x_k^{(1)} \right)^2}{20} + v_k^{(1)} \\
 y_k^{(2)} &= x_k^{(2)} + v_k^{(2)}
 \end{aligned} \tag{3}$$

where  $\mathbf{v}_k$  and  $\mathbf{w}_k$  are zero mean noises, normally distributed with covariances  $\mathbf{R}^{vv}$  and  $\mathbf{R}^{ww}$ .

The Euclidian norm of the difference between the true measurements and that predicted by the filter ((4), where  $L$  denotes the simulation length) will be termed the filter variance.

$$\begin{aligned}
 \text{variance}(k) &= \left\| \mathbf{y}_k - g(\hat{\mathbf{x}}_k, \mathbf{u}_k, k) \right\|^2 \\
 \text{variance} &= \frac{1}{L} \sum_{k=1}^L \text{variance}(k)
 \end{aligned} \tag{4}$$

### 4.2 Central Difference Kalman Filters parameter

The CDKF present one parameter (excepted the covariance matrices  $\mathbf{P}^{vv}$  and  $\mathbf{P}^{ww}$ ), which is  $h$ .

This parameter is said to be optimally set to  $h = \sqrt{3}$  in the Gaussian case (Nørsgaard *et al.*, 2000). This value has been checked successfully.

#### 4.3 Particle Filters parameters

The parameters of particle filters are the following:

- the number of particles,
- the choice of the state estimator,
- the resampling scheme<sup>2</sup>,
- the resampling indicator.

The state estimator is simple to choose : after a number of running, it is clear that in this case, the estimator using least square gives better results than that based on maximum likelihood.

Concerning the resampling, the different schemes are :

- the multinomial resampling,
- the residual resampling (Liu and Chen, 1995),
- the systematic resampling (Kitagawa, 1996),
- the branching algorithm (Crisan and Grunwald, 1999).

Experimentally, they all provide similar results (Douc and Cappé, 2005). The most natural choice is then to choose the simplest algorithm, which is the systematic resampling. In addition, it is the one whose variance is minimal.

Resampling contribution is real, but it is not advised to use it at each step, because it impoverishes the particles cloud. There are two indicators for deciding when redistribution is necessary : the first calculates the effective number of particles (Kong *et al.*, 1994) and the second the entropy of the particles system (Pham, 2001). The tests done on a filter with  $N = 1000$  particles and different values of threshold give the results presented in tables 1 and 2.

Table 1. Entropy based indicator :

Threshold	$\ln \frac{N}{10}$	$\ln \frac{N}{30}$	$\ln \frac{N}{50}$	$\ln \frac{N}{100}$	$\ln \frac{N}{150}$
Filter's variance	1.46	1.63	1.69	2.01	2.33
MNR*	0.69	0.56	0.50	0.50	0.46

Table 2. Effective particles number based indicator :

Threshold	$\frac{N}{10}$	$\frac{N}{30}$	$\frac{N}{50}$	$\frac{N}{75}$	$\frac{N}{100}$
Filter's variance	1.51	1.69	1.75	1.60	1.66
MNR*	0.7	0.6	0.56	0.5	0.49

\*MNR = Mean Number of Resampling, i.e. the number of resampling during the simulation divided by the running length.

<sup>2</sup> Resampling has been introduced by Gordon (Gordon *et al.*, 1993) for avoiding the divergence problems.

The results provided by these methods are close : we arbitrarily choose the entropy based estimator. The threshold value is a compromise between the filter's variance and the resampling ;  $\ln(N/50)$  has been retained.

The last thing to choose is the number of particles. The method adopted is to run the simulation with different numbers of particles, and to choose the best compromise between computation time and number of particles. By seeing table 3, the number of particles adopted is 1000.

Table 3. PF's variance (PFV) in function of the number of particles ( $N$ ):

$N$	50	100	500	<b>1000</b>	2000	5000	10000
PFV	9.77	4.21	2.15	<b>1.82</b>	1.63	1.11	1.04

#### 4.4 Number of particles of the EnKF

A similar method, as used with the PF, is adopted, providing the table 4. Contrary to PF, the EnKF's variance is quite independent on the number of particles. It can be explained by the fact that PF has a central theorem validating it, contrary to the EnKF. A too poor number of particles (in practice, 350), leads to non positive covariance matrix. The best choice is then to choose a number of 500 particles, in order to keep a security margin.

Table 4. EnKF's variance (EnKFV) in function of the number of particles ( $N$ ):

$N$	350	<b>500</b>	750	1000	2000
EnKFV	1.55	<b>1.52</b>	1.52	1.59	1.56

## 5. FILTERS BEHAVIOUR IN RELATION TO NOISE

Consider the following covariance levels :

$$\begin{aligned} \mathbf{R}^{ww} &= \mathbf{I} \\ \mathbf{R}^{vv} &= a\mathbf{I} \end{aligned} \quad (5)$$

where  $a \in \mathbb{R}^*$  and  $\mathbf{I}$  denotes the identity matrix of appropriate size. The noises covariances are assumed to be known. Consequently, the filters covariance can be set to these values, i.e. :  $\mathbf{P}^{ww} = \mathbf{R}^{ww}$ ,  $\mathbf{P}^{vv} = \mathbf{R}^{vv}$ .

The mean variance of each filter in relation to  $a$  is given in table 5, and traces of the real and estimated states are exposed on figure 2 and 3, for  $a = 10$ .

Table 5. Filters variance as a function of measurement noise covariance :

$a$	1	<b>10</b>	50	100	500	1000
CDKF	9.72	<b>10.55</b>	13.40	16.12	29.24	39.86
EnKF	1.59	<b>3.21</b>	7.74	11.39	26.90	37.96
PF	1.48	<b>3.18</b>	7.65	11.48	26.78	37.86

With low covariance noise, the Monte Carlo sampling based filters (i.e. the EnKF and the PF) outperform clearly the CDKF. However, they cannot be used with a noise covariance smaller than the unity, because particles are not dispatched enough in the state space, leading to filter divergence (LeGland *et al.*, 1998). In addition, in the case of high noise, their performance is severely decreased, providing the same results as the CDKF.

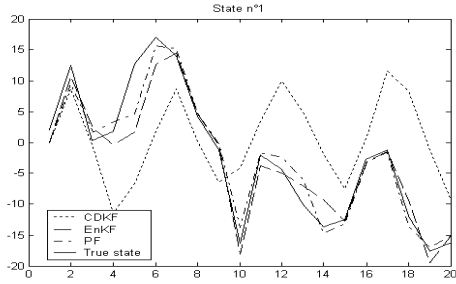


Fig. 2. Real and estimated state : 1<sup>st</sup> component

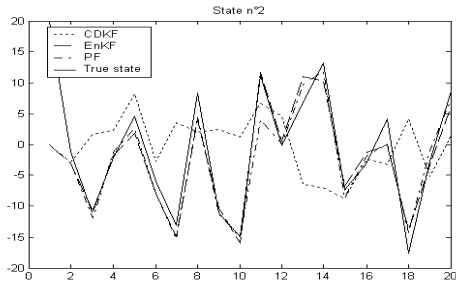


Fig. 3. Real and estimated state : 2<sup>nd</sup> component

## 6. COMPARISON BETWEEN FILTERS PERFORMANCES

As it was exposed in the preceding section, the EnKF and the PF produce better results than the CDKF in the case of small covariance noise, but do not perform better when  $a$  becomes higher than a certain value (in practice,  $a \approx 1000$ ) (figures 4 and 5). But, contrary to the CDKF, these two filters have parameters allowing a better tuning. They are :

- the number of particles,
- the threshold of resampling indicator (for PFs).

The purpose of this section is to study the influence of these values on the estimation quality.

### 6.1 Number of particles of the EnKF

As we can explain, the effect of the number of particles of the EnKF does not affect the variance of the filter (it already was seen in section 4), as illustrated in table 6.

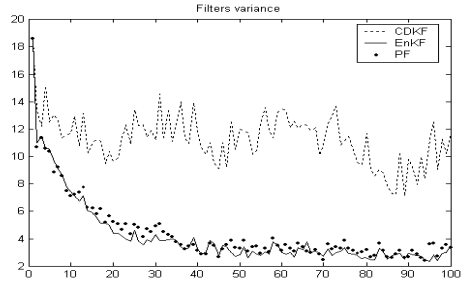


Fig. 4. Filters variance for  $\mathbf{P}^{uv} = 10\mathbf{I}$

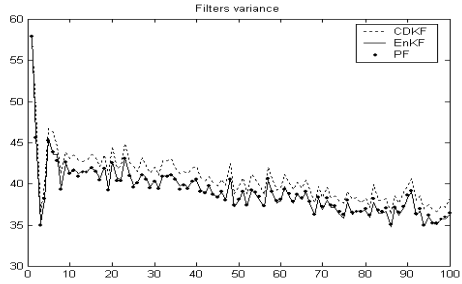


Fig. 5. Filters variance for  $\mathbf{P}^{uv} = 700\mathbf{I}$

Table 6. EnKF's variance (EnKFV) in function of the number of particles ( $N$ ):

$N$	350	500	750	1000	2000
EnKFV	38.83	38.22	38.22	38.11	38.63

### 6.2 Parameters of the PF

With a high covariance measurement noise, the most remarkable fact is that PF does not proceed to resampling. Consequently, it is natural to reduce the threshold of the resampling indicator.

Table 7. PF's variance (PFV) and Mean Number of Resampling (MNR) in function of the threshold of the resampling indicator :

threshold	$\ln \frac{N}{50}$	$\ln \frac{N}{10}$	$\ln \frac{N}{3}$	$\ln \frac{N}{1.5}$	$\ln \frac{N}{1.1}$
MNR	0	0.02	0.03	0.08	0.27
PFV	37.86	36.72	39.12	37.98	39.4

In table 7, it can be seen that reducing the value of the threshold makes the resampling step more active, but it does not affect the PF's variance. It can be explained by the fact that the purpose of this step is to avoid the divergence problem, especially in the case of model uncertainty, which is not what we considered. The reason that the chosen value for this threshold is important with a low covariance noise (section 4, with a unity covariance) is that in this case, the Gaussian distribution is quite fit, making too few particles being likely at the correction step.

Concerning the number of particles, different values have been tested, as exposed in table 8.

Growing the number of particles does not improve the effectiveness of the PF. On the contrary,

Table 8. PF's variance (PFV) in function of the number of particles ( $N$ ) :

$N$	500	1000	2000	5000	10000
PFV	39.95	39.33	39.35	39.34	39.33

reducing it produces an analogue result with less computation time.

## 7. CONCLUSION

On the system tested, the best results are provided by the PF and the EnKF for a measurement covariance noise included between  $1\mathbf{I}$  and  $1000\mathbf{I}$ . Below this interval, the CDKF is the only one which can be used. Above this interval, the performance of the three estimators are equivalent, whatever their tuning : if the computation time consumption is taken into account, the CDKF would be the more efficient.

So, for such a system, when the measurement noise has a low or high covariance, it is advised to use the CDKF. In the other case, the EnKF is proving to be the best choice, because it provides the same results as the PF with half of particles, and furthermore, it is simpler to parameterize.

As an outlook of this work, on the one hand, is to compare with nonlinear robust  $\mathcal{H}_\infty$  filtering. On the other hand, it is to study the behaviour of these estimators with respect to several sensor subsets, in order to develop a tool for the diagnosis of a strongly nonlinear system.

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