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To cite this version:

Aurélien Alfonsi, Benjamin Jourdain. A Call-Put Duality for Perpetual American Options. Nonlinear Differential Equations and Applications, 2009, http://link.springer.com/article/10.1007%2Fs00030-009-0027-8. hal-00121589

HAL Id: hal-00121589

https://hal.science/hal-00121589v1

Submitted on 21 Dec 2006

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hal-00121589, version 1 - 21 Dec 2006

A Call-Put Duality for Perpetual American Options

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December 21, 2006

Abstract

It is well known [5], [1] that in models with time-homogeneous local volatility functions and constant interest and dividend rates, the European Put prices are transformed into European Call prices by the simultaneous exchanges of the interest and dividend rates and of the strike and spot price of the underlying. This paper investigates such a Call Put duality for perpetual American options. It turns out that the perpetual American Put price is equal to the perpetual American Call price in a model where, in addition to the previous exchanges between the spot price and the strike and between the interest and dividend rates, the local volatility function is modified. We prove that equality of the dual volatility functions only holds in the standard Black-Scholes model with constant volatility. Thanks to these duality results, we design a theoretical calibration procedure of the local volatility function from the perpetual Call and Put prices for a fixed spot price x_0. The knowledge of the Put (resp. Call) prices for all strikes enables to recover the local volatility function on the interval $(0, x_0)$ (resp. $(x_0, +\infty)$).

Keywords: Perpetual American options, Dupire's formula, Call-Put Duality, Calibration of volatility, Optimal stopping.

Introduction

In a model with local volatility function $\varsigma(t, x)$, interest rate r and dividend rate δ

$$\begin{cases} dS_t^x = \varsigma(t, S_t^x) S_t^x dW_t + (r - \delta) S_t^x dt, \ t \geq 0 \\ S_0^x = x \end{cases} \tag{1}$$

the initial price

$$h(T, y) = \mathbb{E}\left[e^{-rT}(y - S_T^x)^+\right]$$

of the European Put option considered as a function of the maturity $T > 0$ and the Strike $y > 0$ solves Dupire's partial differential equation [5] :

$$\begin{cases} \partial_T h(T,y) = \frac{\varsigma^2(T,y)y^2}{2}\partial^2_{yy}h(T,y) + (\delta - r)y\partial_y h(T,y) - \delta h(T,y),\ T,y > 0 \\ h(0,y) = (y-x)^+,\ y > 0 \end{cases}$$

One easily deduces that the function $h(T-t,y)$ for $(t,y) \in [0,T] \times \mathbb{R}^*_+$ satisfies the pricing partial differential equation for the Call option with strike x and maturity T in the model

$$\begin{cases} d\bar{S}^{y,T}_t = \varsigma(T-t,\bar{S}^{y,T}_t)\bar{S}^{y,T}_t dW_t + (\delta - r)\bar{S}^{y,T}_t dt,\ t \in [0,T] \\ \bar{S}^{y,T}_0 = y \end{cases} \tag{2}$$

with local volatility function $\varsigma(T-t,y)$, interest rate δ and dividend rate r. Therefore $h(T,y) = \mathbb{E}\left[e^{-\delta T}(\bar{S}^{y,T}_T - x)^+\right]$ and one deduces the following Call-Put duality relation which is also a consequence of [1]

$$\forall T \geq 0,\ \forall x,y > 0,\ \mathbb{E}\left[e^{-rT}(y - S^x_T)^+\right] = \mathbb{E}\left[e^{-\delta T}(\bar{S}^{y,T}_T - x)^+\right].$$

Since it derives from Dupire's formula, this Call-Put duality equality is closely related to calibration issues. One remarks that in the particular case of a time-homogeneous volatility function ($\varsigma(t,x) = \sigma(x)$), then $\bar{S}^{y,T}_t$ also evolves according to the same time-homogeneous volatility function.

In this work, we are interested in deriving such a Call-Put duality relation in the case of American options and in investigating consequences in terms of calibration. In the Black-Scholes model with constant volatility $\varsigma(t,x) = \sigma$, when τ denotes a bounded stopping-time of the natural filtration of the Brownian motion $(W_t)_{t\geq 0}$, one has

$$\begin{aligned} \mathbb{E}\left[e^{-r\tau}\left(y - xe^{\sigma W_\tau + (r-\delta-\frac{\sigma^2}{2})\tau}\right)^+\right] &= \mathbb{E}\left[e^{-\delta\tau}e^{\sigma W_\tau - \frac{\sigma^2}{2}\tau}\left(ye^{-\sigma W_\tau + (\delta - r + \frac{\sigma^2}{2})\tau} - x\right)^+\right] \\ &= \mathbb{E}\left[e^{-\delta\tau}\left(ye^{-\sigma W_\tau + (\delta - r - \frac{\sigma^2}{2})\tau} - x\right)^+\right] \end{aligned}$$

where the second equality follows from Girsanov theorem. Taking the supremum over all stopping-times τ smaller than T one deduces that the price of the American Put option with maturity T is equal to the price of the American Call option with the same maturity up to the simultaneous exchange between the underlying spot price and the strike and between the interest and dividend rates. Extensions of this result when the underlying evolves according to the exponential of a Lévy process have been obtained in [7]. Let us also mention that another kind of duality has been investigated in [13]. But, to our knowledge, no study has been devoted to the case of models with local volatility functions like (1).

In the present paper, we consider the case of perpetual ($T = +\infty$) American options in models with time-homogeneous local volatility functions $\varsigma(t,x) = \sigma(x)$. In the first part,

we recover well-known properties of the perpetual American call and put pricing functions by extending an approach recently developed by Beibel and Lerche [2] in the Black-Scholes case. This makes the paper self-contained.
In the second part, we introduce the framework used in the remaining of the paper.
In the third part of the paper, we consider the exercise boundaries as functions of the strike variable and characterize them as the unique solutions of some non-autonomous ordinary differential equations.
The fourth part is dedicated to our main result. We prove that the perpetual American Put prices are equal to the perpetual American Call prices in a model where, in addition to the exchanges between the spot price of the underlying and the strike and between the interest and dividend rates, the volatility function is modified. We also derive an expression of this modified volatility function. Notice that in the European case presented above, time-homogeneous volatility functions are not modified.
The fifth part addresses calibration issues. It turns out that for a given initial value $x_0 > 0$ of the underlying one recovers the restriction of the time-homogeneous volatility function $\sigma(x)$ to $(0, x_0]$ (resp. $[x_0, +\infty)$) from the perpetual Put (resp. Call) prices for all strikes.
In the last part, we show that at least when $\delta < r$, in the class of volatility functions analytic in a neighbourhood of the origin, the only ones invariant by our duality result are the constants. This means that the case of the standard Black-Scholes model presented above is very specific.

Acknowledgements. We thank Damien Lamberton (Univ. Marne-la-vallée) and Mihail Zervos (King's College) for interesting discussions. We also thank Alexander Schied (TU Berlin) for pointing out the work of Beibel and Lerche [2] to us and Antonino Zanette (University of Udine) for providing us with the routine that calculates American option prices.

1 Perpetual American put and call pricing

We consider a constant interest spot-rate r that is assumed to be nonnegative and an asset S_t which pays a constant dividend rate $\delta \geq 0$ and is driven by a homogeneous volatility function $\sigma : \mathbb{R}_+^* \to \mathbb{R}_+^*$ that satisfies the following hypothesis.

Hypothesis ($\mathcal{H}_{\mathbf{vol}}$)**:** *$\sigma$ is continuous on $\mathbb{R}_+^*$ and there are $0 < \underline{\sigma} < \overline{\sigma} < +\infty$ such that:*

$$\forall x > 0, \underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}.$$

In other words, S_t is assumed to follow under the risk-neutral measure the SDE:

$$dS_t = S_t((r - \delta)dt + \sigma(S_t)dW_t). \tag{3}$$

With the assumption made on σ, we know that for any initial condition $x \in \mathbb{R}_+^*$, there is a unique solution in the sense of probability law (see for example Theorem 5.15 in [10], using a log transformation) denoted by $(S_t^x, t \geq 0)$. Moreover, Theorem 4.20 ensures that the strong Markov property holds for $(S_t^x, t \geq 0)$. Under that model, we denote by

$$P_\sigma(x, y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(y - S_\tau^x)^+\right] \text{ and } C_\sigma(x, y) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-r\tau}(S_\tau^x - y)^+\right]$$

respectively the prices of the American perpetual put and call options with strike $y > 0$ and spot x. Here, $\mathcal{T}_{0,\infty}$ simply denotes the set of the stopping times with respect to the natural filtration of $(S_t^x, t \geq 0)$. Since $e^{-rt}S_t^x = x \exp\left(-\delta t + \int_0^t \sigma(S_u^x)dW_u - \frac{1}{2}\int_0^t \sigma^2(S_u^x)du\right)$ and $\int_0^t \sigma^2(S_u^x)du \geq \underline{\sigma}^2 t \underset{t\to+\infty}{\to} +\infty$, it follows from the Dubins-Schwarz theorem that

$$e^{-rt}S_t^x \underset{t\to+\infty}{\to} 0 \text{ a.s.} \tag{4}$$

As a consequence, $e^{-rt}(y - S_t^x)^+ \underset{t\to+\infty}{\to} \mathbf{1}_{\{r=0\}}y$ and $e^{-rt}(S_t^x - y)^+ \underset{t\to+\infty}{\to} 0$ a.s. On $\{\tau = \infty\}$, we thus set

$$e^{-r\tau}(y - S_\tau^x)^+ = \mathbf{1}_{\{r=0\}}y \text{ and } e^{-r\tau}(S_\tau^x - y)^+ = 0. \tag{5}$$

Let us consider the second-order ordinary differential equation

$$\frac{1}{2}\sigma^2(x)x^2 f''(x) + (r-\delta)xf'(x) - rf(x) = 0,\ x > 0. \tag{6}$$

According to Borodin and Salminen ([3], chap. 2) the functions

$$\forall x > 0,\ f_\uparrow(x) = \begin{cases} \mathbb{E}[e^{-r\tau_1^x}], \text{ if } x \leq 1 \\ 1/\mathbb{E}[e^{-r\tau_x^1}], \text{ if } x > 1 \end{cases} \text{ and } f_\downarrow(x) = \begin{cases} 1/\mathbb{E}[e^{-r\tau_x^1}], \text{ if } x \leq 1 \\ \mathbb{E}[e^{-r\tau_1^x}], \text{ if } x > 1 \end{cases} \tag{7}$$

where for $x, y > 0$, $\tau_y^x = \inf\{t \geq 0, S_t^x = y\}$ ($\inf \emptyset = +\infty$), are the unique solutions (up to a multiplicative constant) that are positive and respectively increasing and decreasing. The volatility function σ being continuous, (7) ensures that these functions are $\mathcal{C}^2$ on $\mathbb{R}_+^*$.

Remark 1.1. *It is easy using the strong Markov property to get:*

$$\forall x, y > 0, \quad \mathbb{E}[e^{-r\tau_y^x}] = \begin{cases} f_\uparrow(x)/f_\uparrow(y), & \text{if } x \leq y \\ f_\downarrow(x)/f_\downarrow(y), & \text{if } x \geq y. \end{cases}$$

Remark 1.2. *Assuming $r > 0$, one has (see Borodin and Salminen [3])*

$$\lim_{x\to 0} f_\downarrow(x) = +\infty, \quad \lim_{x\to+\infty} f_\downarrow(x) = 0$$
$$\lim_{x\to 0} f_\uparrow(x) = 0, \quad \lim_{x\to+\infty} f_\uparrow(x) = +\infty.$$

The function $f_\downarrow$ (resp. $f_\uparrow$) is thus, up to a multiplicative constant, the unique solution to (6) such that $\lim_{x\to+\infty} f(x) = 0$ (resp. $\lim_{x\to 0} f(x) = 0$).

Remark 1.3. *In the case $\delta = 0$, we have the analytical solutions :*

$$f_\downarrow(x) = \frac{\varphi(x)}{\varphi(1)} \text{ where } \varphi(x) = x\int_x^{+\infty}\left(\frac{1}{v^2}\exp\left[-\int_1^v \frac{2r}{u\sigma^2(u)}du\right]\right)dv,\ f_\uparrow(x) = x.$$

Indeed, since $f(x) = x$ *is solution of* $\frac{1}{2}\sigma^2(x)x^2 f''(x) + rxf'(x) - rf(x) = 0$, *we search a general solution that can be written* $f(x) = x\tilde{f}(x)$. *This leads to* $\frac{1}{2}\sigma^2(x)x\tilde{f}''(x) + (r + \sigma^2(x))\tilde{f}'(x) = 0$ *and then* $\tilde{f}'(x) = \frac{C_1}{x^2}\exp\left[-\int_1^x \frac{2r}{u\sigma^2(u)}du\right]$. *Therefore,* $\exists C_1, C_2 \in \mathbb{R},\ f(x) = C_2 x + C_1 x \int_x^{+\infty}\left(\frac{1}{v^2}\exp\left[-\int_1^v \frac{2r}{u\sigma^2(u)}du\right]\right)dv$.

Now, we are in position to show the existence of an optimal stopping time and give the call and put prices. Let us mention here that the problem of perpetual optimal stopping is treated in the paper of Dayanik and Karatzas [4], for a general payoff function and an underlying evolving according to a general one-dimensional time homogeneous diffusion process. Villeneuve [14] considers a model where the constant dividend rate δ in (3) is replaced by a function $\delta(S_t)$ and gives sufficient condition on the payoff function ensuring that a threshold strategy is optimal. Here, we give a direct proof that generalizes the approach developed by Beibel and Lerche [2] in the Black-Scholes case.

Theorem 1.4. *Assume* $r > 0$. *For any strike* $y > 0$, *there is a unique* $x^*_\sigma(y) \in (0, y) \cap (0, \frac{r}{\delta}y]$ *such that* $\tau^P_x = \inf\{t \geq 0, S^x_t \leq x^*_\sigma(y)\}$ *(convention* $\inf\emptyset = +\infty$*) is an optimal stopping time for the put and:*

$$\forall x \leq x^*_\sigma(y), P_\sigma(x, y) = (y - x)^+,\ \forall x > x^*_\sigma(y), P_\sigma(x, y) = \frac{y - x^*_\sigma(y)}{f_\downarrow(x^*_\sigma(y))} f_\downarrow(x) > (y - x)^+. \quad (8)$$

In addition, we have $f'_\downarrow(x^*_\sigma(y)) < 0$ *and:*

$$x^*_\sigma(y) - y = \frac{f_\downarrow(x^*_\sigma(y))}{f'_\downarrow(x^*_\sigma(y))}. \quad (9)$$

Last, the smooth-fit principle holds: $\partial_x P_\sigma(x^*_\sigma(y), y) = -1$.

Remark 1.5. *If* $r = 0$, $\forall x, y > 0, P_\sigma(x, y) = y$ *since for any stopping time* τ, $(y - S^x_\tau)^+ \leq y$ *and equality holds for* $\tau = +\infty$ *by (5).*

Proof. Let us define:

$$\forall z > 0,\ h(z) = \frac{(y - z)^+}{f_\downarrow(z)} \text{ and } h^* = \sup_{z>0} h(z).$$

Since the function h is continuous such that $h(y) = 0$ and $h(0+) = 0$ (Remark 1.2), $x^*_\sigma(y) = \sup\{z > 0,\ h(z) = h^*\}$ belongs to $(0, y)$ and is such that $h(x^*_\sigma(y)) = h^*$. Since the function h is $\mathcal{C}^2$ on $(0, y)$, we have $h'(x^*_\sigma(y)) = 0$ and $h''(x^*_\sigma(y)) \leq 0$. These conditions give easily

$$f_\downarrow(x^*_\sigma(y)) + (y - x^*_\sigma(y))f'_\downarrow(x^*_\sigma(y)) = 0 \text{ and } f''_\downarrow(x^*_\sigma(y)) \geq 0.$$

Since $f_\downarrow$ is positive and $x^*_\sigma(y) < y$, we have $f'_\downarrow(x^*_\sigma(y)) < 0$ and deduce (9). The second order condition and equation (6) then give $x^*_\sigma(y)(r - \delta)f'_\downarrow(x^*_\sigma(y)) - rf_\downarrow(x^*_\sigma(y)) \leq 0$ and so

$$ry - \delta x^*_\sigma(y) \geq 0.$$

Now let us check the optimality of τ_x^P and consider $\tau \in \mathcal{T}_{0,\infty}$. By Fatou's lemma and Doob's optional sampling theorem, we have

$$\begin{aligned}\mathbb{E}[e^{-r\tau}(y-S_\tau^x)^+] &\leq \liminf_{t\to+\infty}\mathbb{E}\left[e^{-r\tau\wedge t}(y-S_{\tau\wedge t}^x)^+\right]\\ &= \liminf_{t\to+\infty}\mathbb{E}\left[e^{-r\tau\wedge t}f_\downarrow(S_{\tau\wedge t}^x)h(S_{\tau\wedge t}^x)\right]\\ &\leq h(x_\sigma^*(y))\liminf_{t\to+\infty}\mathbb{E}\left[e^{-r\tau\wedge t}f_\downarrow(S_{\tau\wedge t}^x)\right]\leq h(x_\sigma^*(y))f_\downarrow(x)\end{aligned}$$

since $e^{-rt}f_\downarrow(S_t^x)=f_\downarrow(x)+\int_0^t e^{-ru}\sigma(S_u^x)S_u^x f_\downarrow'(S_u^x)dW_u$ is a nonnegative local martingale and therefore a supermartingale. If $x\geq x_\sigma^*(y)$, we have using Remark 1.1:

$$\mathbb{E}[e^{-r\tau_x^P}(y-S_{\tau_x^P}^x)^+]=\mathbb{E}[e^{-r\tau_{x_\sigma^*(y)}^x}(y-S_{\tau_{x_\sigma^*(y)}^x}^x)^+]=(y-x_\sigma^*(y))\mathbb{E}[e^{-r\tau_{x_\sigma^*(y)}^x}]=h(x_\sigma^*(y))f_\downarrow(x)$$

and τ_x^P is optimal for $x\geq x_\sigma^*(y)$. Since $x_\sigma^*(y)=\sup\{z>0,\ h(z)=h^*\}$, we have $(y-x)^+=h(x)f_\downarrow(x)<f_\downarrow(x)h(x_\sigma^*(y))$ for $x>x_\sigma^*(y)$, and finally deduces (8) for $x\geq x_\sigma^*(y)$.

We consider now the complementary case $x\in(0,x_\sigma^*(y))$, and set $\tau\in\mathcal{T}_{0,\infty}$. Using the strong Markov property and the optimality result when the initial spot is $x_\sigma^*(y)$, we get

$$\mathbb{E}[e^{-r\tau}(y-S_\tau^x)^+]\leq\mathbb{E}[e^{-r\tau\wedge\tau_{x_\sigma^*(y)}^x}(y-S_{\tau\wedge\tau_{x_\sigma^*(y)}^x}^x)^+].$$

On $\{t<\tau_{x_\sigma^*(y)}^x\}$, we have $S_t^x<x_\sigma^*(y)$, $de^{-rt}(y-S_t^x)=e^{-rt}\underbrace{(\delta S_t^x-ry)}_{\leq 0}dt-e^{-rt}\sigma(S_t^x)S_t^x dW_t$ and so $\mathbb{E}[e^{-r\tau\wedge\tau_{x_\sigma^*(y)}^x}(y-S_{\tau\wedge\tau_{x_\sigma^*(y)}^x}^x)^+]\leq\liminf_{t\to+\infty}\mathbb{E}[e^{-r\tau\wedge\tau_{x_\sigma^*(y)}^x\wedge t}(y-S_{\tau\wedge\tau_{x_\sigma^*(y)}^x\wedge t}^x)^+]\leq(y-x)$. □

Now, we state the similar result for the call prices.

Theorem 1.6. *Assume $\delta>0$. For any strike $y>0$, there is a unique $\Upsilon_\sigma^*(y)\in(y,\infty)\cap[\frac{r}{\delta}y,+\infty)$ such that $\tau_x^C=\inf\{t\geq 0,S_t^x\geq\Upsilon_\sigma^*(y)\}$ is an optimal stopping time for the call and:*

$$\forall x\geq\Upsilon_\sigma^*(y),C_\sigma(x,y)=(x-y)^+,\ \forall x<\Upsilon_\sigma^*(y),C_\sigma(x,y)=\frac{\Upsilon_\sigma^*(y)-y}{f_\uparrow(\Upsilon_\sigma^*(y))}f_\uparrow(x)>(x-y)^+. \quad (10)$$

In addition, we have $f_\uparrow'(\Upsilon_\sigma^*(y))>0$ and:

$$\Upsilon_\sigma^*(y)-y=\frac{f_\uparrow(\Upsilon_\sigma^*(y))}{f_\uparrow'(\Upsilon_\sigma^*(y))}. \quad (11)$$

Last, the smooth-fit principle holds: $\partial_x C_\sigma(\Upsilon^*(y),y)=1$.

Remark 1.7. *If $\delta=0$, $\forall x,y>0,C_\sigma(x,y)=x$. Indeed, the Call-Put parity $\mathbb{E}[e^{-rt}(S_t^x-y)^+]=x-ye^{-rt}+\mathbb{E}[e^{-rt}(y-S_t^x)^+]$ gives the convergence to x in both cases $r>0$ and $r=0$ when $t\to+\infty$. Now, thanks to the Fatou lemma, we have for $\tau\in\mathcal{T}_{0,\infty}$: $\mathbb{E}\left[e^{-r\tau}(S_\tau^x-y)^+\right]\leq\liminf_{t\to+\infty}\mathbb{E}\left[e^{-r\tau\wedge t}(S_{\tau\wedge t}^x-y)^+\right]\leq\liminf_{t\to+\infty}\mathbb{E}\left[e^{-r\tau\wedge t}S_{\tau\wedge t}^x\right]=x$.*

Proof. The proof works as for the put, and we just hint the differences. We define

$$\forall z > 0, h(z) = \frac{(z-y)^+}{f_\uparrow(z)} \text{ and } h^* = \sup_{z>0} h(z).$$

Let us admit for a while that $\lim_{z\to+\infty} h(z) = 0$. Then, since $h(y) = 0$ and h is continuous, h reaches its maximum in $\Upsilon^*_\sigma(y) = \inf\{z > 0,\ h(z) = h^*\}$, and $\Upsilon^*_\sigma(y) \in (y, \infty)$. This gives (11). We then consider the case $x \le \Upsilon^*_\sigma(y)$ and show that $\tau^C_x = \tau^x_{\Upsilon^*_\sigma(y)}$ is optimal. Note that in the special case $r = 0$, we have to use Proposition 1.8 which is stated below. Finally, we prove that τ^C_x is optimal when $x > \Upsilon^*_\sigma(y)$ using that $\delta\Upsilon^*_\sigma(y) - ry \ge 0$.

Now, let us check that $\lim_{z\to+\infty} h(z) = 0$. In the case $r = 0$, it is straightforward using the explicit form given in Proposition 1.8 below that $f_\uparrow(x) \ge \frac{1}{1+\frac{2\delta}{\overline{\sigma}^2}} x^{1+2\delta/\overline{\sigma}^2} - 1$ for $x \ge 1$, and we have then $\lim_{z\to+\infty} h(z) = 0$. When $r > 0$, Itô's Formula gives

$$de^{-rt}(S^1_t)^{1+a} = e^{-rt}(S^1_t)^{1+a}\left\{(a+1)\sigma(S^1_t)dW_t + [a(r + (a+1)\sigma^2(S^1_t)/2) - (a+1)\delta]dt\right\}.$$

When $a > 0$, the drift term is bounded from above by $a(r + (a+1)\overline{\sigma}^2/2) - (a+1)\delta$ and we can find $a > 0$ such that this bound is negative since $a(r + (a+1)\overline{\sigma}^2/2) - (a+1)\delta \underset{a\to 0}{\to} -\delta < 0$. Then, for $x \ge 1$, we have $\mathbb{E}[e^{-r\tau^1_x\wedge t}(S^1_{\tau^1_x\wedge t})^{1+a}] \le 1$ thanks to Doob's optional sampling theorem. The Fatou lemma gives then $\mathbb{E}[e^{-r\tau^1_x}(S^1_{\tau^1_x})^{1+a}] \le 1$, and therefore we get $f_\uparrow(x) = 1/\mathbb{E}[e^{-r\tau^1_x}] \ge x^{1+a}$. This shows $\lim_{z\to+\infty} h(z) = 0$. □

Proposition 1.8. *In the case $r = 0$, the unique nonincreasing and increasing solution of (6) starting from 1 in 1 are respectively:*

$$f_\downarrow(x) = 1,\ f_\uparrow(x) = \frac{\psi(x)}{\psi(1)} \text{ where } \psi(x) = \int_0^x \exp\left[\int_1^v \frac{2\delta}{u\sigma^2(u)}du\right]dv.$$

Moreover, we have $f_\uparrow(x) = 1/\mathbb{P}(\tau^1_x < +\infty)$ for $x \ge 1$ and $f_\uparrow(x) = \mathbb{P}(\tau^x_1 < +\infty)$ for $x \in (0, 1]$.

Proof. When $r = 0$, the differential equation $\frac{1}{2}\sigma^2(y)y^2f''(y) - \delta y f'(y) = 0$ is easy to integrate: $f'(y) = C_3 \exp\left[\int_1^y \frac{2\delta}{u\sigma^2(u)}du\right]$ and then

$$g(y) = C_4 + C_3\int_0^y \exp\left[\int_1^v \frac{2\delta}{u\sigma^2(u)}du\right]dv$$

for $C_3, C_4 \in \mathbb{R}$. For $x \ge 1$, $f_\uparrow(S^1_{\tau^1_x\wedge t})$ is a bounded martingale that converges almost surely to $f_\uparrow(x)\mathbf{1}_{\{\tau^1_x<+\infty\}}$ thanks to (4) and $\lim_{x\to 0} f_\uparrow(x) = 0$. Therefore $f_\uparrow(x)\mathbb{P}(\tau^1_x < +\infty) = 1$, and the proof is the same for $x \in (0, 1]$. □

To conclude this section, we state a comparison result which will enables us to compare $x^*_\sigma(y)$ and $\Upsilon^*_\sigma(y)$ with the exercise boundaries obtained in the Black-Scholes model with constant volatility $\underline{\sigma}$ (resp. $\overline{\sigma}$) where $\underline{\sigma}$ (resp. $\overline{\sigma}$) bounds the function σ from below (resp. above).

Proposition 1.9. *Let us consider two volatility functions σ_1 and σ_2 such that $\forall x > 0,\ \sigma_1(x) \leq \sigma_2(x)$ and that satisfy $(\mathcal{H}_{\mathbf{vol}})$. We also assume that either $f''_{\downarrow,\sigma_1}$ or $f''_{\downarrow,\sigma_2}$ (resp. either $f''_{\uparrow,\sigma_1}$ or $f''_{\uparrow,\sigma_2}$) are nonnegative functions and $r > 0$ (resp. $\delta > 0$). Then, we have*

$$\forall x, y > 0,\ P_{\sigma_1}(x,y) \leq P_{\sigma_2}(x,y) \quad (resp.\ C_{\sigma_1}(x,y) \leq C_{\sigma_2}(x,y))$$

and we can compare the exercise boundaries:

$$\forall y > 0, x^*_{\sigma_1}(y) \geq x^*_{\sigma_2}(y) \quad (resp.\ \Upsilon^*_{\sigma_1}(y) \leq \Upsilon^*_{\sigma_2}(y)).$$

Here and in the proof below, we add in the notation for each mathematical object the volatility function to which it refers. El Karoui and al. [6] and Hobson [9] prove that for a convex payoff function, the price of an American option with finite maturity is a convex function of the underlying spot price. They deduce monotonicity with respect to the local volatility function. Their results imply at the same time the convexity assumption made in the above proposition and its conclusion. In this paper, we prefer to give autonomous proofs of these results in our simple framework. And we will first use proposition 1.9 to compare with the Black-Scholes case where convexity is obvious. We can then deduce (Lemma 3.1) that $f''_\downarrow$ and $f''_\uparrow$ are positive for any σ satisfying $(\mathcal{H}_{\mathbf{vol}})$.

Proof. Let us consider for example the put case with $f''_{\downarrow,\sigma_1} \geq 0$. Let $x \geq z > 0$. Ito's formula gives:

$$\begin{aligned} de^{-rt} f_{\downarrow,\sigma_1}(S_t^{x,\sigma_2}) &= e^{-rt} f'_{\downarrow,\sigma_1}(S_t^{x,\sigma_2})\sigma_2(S_t^{x,\sigma_2})S_t^{x,\sigma_2} dW_t + e^{-rt}\Bigg[\frac{\sigma_2^2(S_t^{x,\sigma_2})}{2}(S_t^{x,\sigma_2})^2 f''_{\downarrow,\sigma_1}(S_t^{x,\sigma_2}) \\ &\quad +(r-\delta)S_t^{x,\sigma_2} f'_{\downarrow,\sigma_1}(S_t^{x,\sigma_2}) - r f_{\downarrow,\sigma_1}(S_t^{x,\sigma_2})\Bigg]dt. \end{aligned}$$

The term between brackets is nonnegative since we have $\sigma_1 \leq \sigma_2$ and $f_{\downarrow,\sigma_1}$ is a convex function solving (6). Therefore we get $\mathbb{E}[e^{-r\nu_n \wedge \tau_z^{x,\sigma_2}} f_{\downarrow,\sigma_1}(S^{x,\sigma_2}_{\nu_n\wedge\tau_z^{x,\sigma_2}})] \geq f_{\downarrow,\sigma_1}(x)$ using the optional sampling theorem, where $\nu_n = \inf\{t \geq 0, S_t^{x,\sigma_2} \geq n\} \wedge n$. Since $\nu_n \underset{a.s.}{\to} +\infty$ and $f_{\downarrow,\sigma_1}(S^{x,\sigma_2}_{\nu_n\wedge\tau_z^{x,\sigma_2}})$ is bounded by $f_{\downarrow,\sigma_1}(z)$, Lebesgue's dominated convergence theorem gives then $\mathbb{E}[e^{-r\tau_z^{x,\sigma_2}}] \geq f_{\downarrow,\sigma_1}(x)/f_{\downarrow,\sigma_1}(z)$ and so $\mathbb{E}[e^{-r\tau_z^{x,\sigma_2}}] \geq \mathbb{E}[e^{-r\tau_z^{x,\sigma_1}}]$ using Remark 1.1. The same conclusion holds when $f_{\downarrow,\sigma_2}$ is convex by estimating $\mathbb{E}[e^{-r\tau_z^{x,\sigma_1}} f_{\downarrow,\sigma_2}(S^{x,\sigma_1}_{\tau_z^{x,\sigma_1}})]$. We then get the result: if $P_{\sigma_1}(x,y) > (x-y)^+$, $P_{\sigma_1}(x,y) = (y - x^*_{\sigma_1}(y))\mathbb{E}[e^{-r\tau^{x,\sigma_1}_{x^*_{\sigma_1}(y)}}] \leq (y - x^*_{\sigma_1}(y))\mathbb{E}[e^{-r\tau^{x,\sigma_2}_{x^*_{\sigma_1}(y)}}] = \mathbb{E}[(y - S^{x,\sigma_2}_{\tau^{x,\sigma_2}_{x^*_{\sigma_1}(y)}})^+ e^{-r\tau^{x,\sigma_2}_{x^*_{\sigma_1}(y)}}] \leq P_{\sigma_2}(x,y)$. Now we just observe that $\{x > 0, P_{\sigma_1}(x,y) > (x-y)^+\} \subset \{x > 0, P_{\sigma_2}(x,y) > (x-y)^+\}$ and thus $x^*_{\sigma_1}(y) = \inf\{x > 0, P_{\sigma_1}(x,y) > (x-y)^+\} \geq \inf\{x > 0, P_{\sigma_2}(x,y) > (x-y)^+\} = x^*_{\sigma_2}(y)$. □

2 Framework and notations

We will present in this section the framework that we will consider in all the paper. To clarify the duality, we will use names that implicitly refer either to the primal (or "real") world, or to the dual world. This denomination has no mathematical meaning since, as we will see, there are no difference between them. On the contrary, from a financial point of view, natural variables such as the interest rate, the dividend rate have their true meaning in the primal world, while in the dual world they interchange their role. This is the reason why we also name the primal world "real" world.

The primal ("real") world

The primal world is the framework we just have described. The spot interest rate r is constant and nonnegative, and S_t is an asset which pays a constant dividend rate $\delta \geq 0$ and is driven by a homogeneous volatility function $\sigma : \mathbb{R}_+^* \to \mathbb{R}_+^*$ that satisfies $(\mathcal{H}_{\mathbf{vol}})$ under a risk-neutral measure. The prices of the perpetual American put and call are respectively denoted by $P_\sigma(x, y)$ and $C_\sigma(x, y)$, and their exercise boundary by x_σ^* and Υ_σ^*.

The dual world

In the dual world, δ plays the role of the interest rate and r of the dividend rate; x plays the role of the strike and y is the spot value of the share. Let $\eta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ be an homogeneous volatility function that is also assumed to satisfy $(\mathcal{H}_{\mathbf{vol}})$. We consider then $(\overline{S}_t^y, t \geq 0)$ the solution of $d\overline{S}_t = \overline{S}_t((\delta - r)dt + \eta(\overline{S}_t)dW_t)$ that starts from y at time 0. Under that model, we denote respectively by

$$p_\eta(y, x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-\delta\tau}(x - \overline{S}_\tau^y)^+\right] \text{ and } c_\eta(y, x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}\left[e^{-\delta\tau}(\overline{S}_\tau^y - x)^+\right]$$

the prices of the perpetual put and call with strike $x > 0$ and spot y. We can define, as in the primal world, $g_\downarrow$ and $g_\uparrow$ as the unique decreasing (non-increasing when $\delta = 0$) and increasing positive solution to

$$\frac{1}{2}\eta^2(x)x^2 g''(x) + (\delta - r)xg'(x) - \delta g(x) = 0, \tag{12}$$

and we name the exercise boundaries $\xi_\eta^*(x) < x$ and $y_\eta^*(x) > x$ that are respectively associated with the put $p_\eta(y, x)$ and the call $c_\eta(y, x)$.

Notations

The aim of this paper is to put in evidence a duality relation and interpret put (resp. call) prices in the primal world as call (resp. put) prices in the dual world for a specific volatility function $\eta = \tilde{\sigma}$ (resp. $\eta = \hat{\sigma}$). When $r = 0$ (resp. $\delta = 0$), this is trivial since $P_\sigma(x, y) = c_\eta(y, x) = y$ (resp. $C_\sigma(x, y) = p_\eta(y, x) = x$) but not really fruitful, and we take thus the following convention in the sequel.

Convention 2.1. *We will always assume $r > 0$ (resp. $\delta > 0$) to state properties on P_σ and c_η (resp. C_σ and p_η).*

Both worlds being mathematically equivalent, *we will work with the put price in the primal world and the call price in the dual world* in order not to do the things twice. Following the Convention 2.1, we will consider a positive interest rate. Let us then denote from now:

$$\boxed{f = f_\downarrow \text{ and } g = g_\uparrow,}$$

and define $\alpha(y) = \frac{y - x^*_\sigma(y)}{f(x^*_\sigma(y))}$ and $\beta(x) = \frac{y^*_\eta(x) - x}{g(y^*_\eta(x))}$. The functions α and β are positive functions and it follows from the present section that :

$$\forall y > 0,\ \forall x \geq x^*_\sigma(y),\ P_\sigma(x, y) = \alpha(y) f(x) \tag{13}$$

$$\forall x > 0,\ \forall y \leq y^*_\eta(x),\ c_\eta(y, x) = \beta(x) g(y). \tag{14}$$

That product form will play an important role for the duality. Let us finally introduce notations relative the the Black-Scholes model. We define for $\varsigma > 0$

$$a(\varsigma) = \frac{\delta - r + \varsigma^2/2 - \sqrt{(\delta - r + \varsigma^2/2)^2 + 2r\varsigma^2}}{\varsigma^2} < 0,$$

$$b(\varsigma) = \frac{r - \delta + \varsigma^2/2 + \sqrt{(\delta - r - \varsigma^2/2)^2 + 2\delta\varsigma^2}}{\varsigma^2} = 1 - a(\varsigma) > 1,$$

and we can easily check that $f(x) = x^{a(\varsigma)}$ (resp. $g = x^{b(\varsigma)}$) when $\sigma(x) = \varsigma$ (resp. $\eta(x) = \varsigma$). In that case, the unique solution to (9) (resp. (11)) is:

$$x^*_\varsigma(y) = \frac{a(\varsigma)}{a(\varsigma) - 1} y \quad (resp.\ y^*_\varsigma(x) = \frac{b(\varsigma)}{b(\varsigma) - 1} x). \tag{15}$$

With Proposition 1.9, we deduce very useful estimations on the exercise boundaries.

Lemma 2.2. *If, $\forall x > 0$, $\underline{\sigma} \leq \sigma(x) \leq \overline{\sigma}$ and $\underline{\sigma} \leq \eta(x) \leq \overline{\sigma}$, then we have :*

$$\frac{a(\overline{\sigma})}{a(\overline{\sigma}) - 1} y \leq x^*_\sigma(y) \leq \frac{a(\underline{\sigma})}{a(\underline{\sigma}) - 1} y \quad with \quad \frac{a(\underline{\sigma})}{a(\underline{\sigma}) - 1} < \min(1, r/\delta) \tag{16}$$

$$\frac{b(\underline{\sigma})}{b(\underline{\sigma}) - 1} x \leq y^*_\eta(x) \leq \frac{b(\overline{\sigma})}{b(\overline{\sigma}) - 1} x \quad with \quad \max(1, \delta/r) < \frac{b(\underline{\sigma})}{b(\underline{\sigma}) - 1}. \tag{17}$$

Proof. It is straightforward from Proposition 1.9 and (15) to get the bound on the exercise boundaries since $x \mapsto x^\gamma$ is convex for $\gamma \notin]0, 1[$. We have to show that for $\varsigma > 0$, $\frac{a(\varsigma)}{a(\varsigma)-1} < \min(1, r/\delta)$ and $\frac{b(\varsigma)}{b(\varsigma)-1} > \min(1, \delta/r)$. Since $a(\varsigma) < 0$ and $b(\varsigma) > 1$, we get that $a(\varsigma)/(a(\varsigma) - 1) \in (0, 1)$ and $b(\varsigma)/(b(\varsigma) - 1) \in (1, +\infty)$. We can also check that $a(\varsigma)/(a(\varsigma) - 1)$ is a root of the polynomial $Q(X) = \delta X^2 - (r + \delta + \varsigma^2/2)X + r$. As $Q(x) = 0 \iff \sigma^2 x = 2(1 - x)(r - \delta x)$ and since $a(\varsigma)/(a(\varsigma) - 1) \in (0, 1)$, we then deduce that $a(\varsigma)/(a(\varsigma) - 1) < r/\delta$. In the same way, we have $b(\varsigma)/(b(\varsigma) - 1) > \delta/r$. □

3 ODE for the exercise boundary

We have seen previously that the exercise boundaries satisfy

$$x^*_\sigma(y) - y = f(x^*_\sigma(y))/f'(x^*_\sigma(y)) \quad (resp.\ y^*_\eta(x) - x = g(y^*_\eta(x))/g'(y^*_\eta(x))). \tag{18}$$

We will soon prove that for fixed $y > 0$ (resp. $x > 0$), (18) admits a unique solution $x^*_\sigma(y)$ (resp. $y^*_\eta(x)$).

Lemma 3.1. *The function f' (resp. g') is negative (resp. positive) and f'' (resp. g'') is positive on $(0, +\infty)$. Moreover, the boundaries $x^*_\sigma(y)$ and $y^*_\eta(x)$ are respectively the unique solution to $y - x + f(x)/f'(x) = 0$ and $y - x - g(x)/g'(x) = 0$. Last, $x^*_\sigma(y)$, $\alpha(y)$, $y^*_\eta(x)$ and $\beta(x)$ are $\mathcal{C}^1$ functions on $\mathbb{R}^*_+$.*

Remark 3.2. *Positivity of f'' and g'' and (13) and (14) imply positivity of $\partial^2_x P_\sigma(x, y)$ and $\partial^2_y c_\eta(y, x)$ in the continuation regions.*

Differentiating (18) with respect to y (resp. x), one obtains $1 = (x^*_\sigma)'(y)\frac{f(x^*_\sigma(y))f''(x^*_\sigma(y))}{f'(x^*_\sigma(y))^2}$ (resp. $1 = (y^*_\eta)'(y)\frac{g(y^*_\eta(y))g''(y^*_\eta(y))}{g'(y^*_\eta(y))^2}$). Using (18) and equation (6) (resp. (12)) one deduces the following result (see equation (22) below).

Proposition 3.3. *Let us assume that the volatility functions σ and η satisfy $(\mathcal{H}_{\mathbf{vol}})$. Then, the boundaries $x^*_\sigma(y)$ and $y^*_\eta(x)$ satisfy the following ODEs:*

$$(x^*_\sigma)'(y) = \frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}, \tag{19}$$

$$(y^*_\eta)'(x) = \frac{\eta^2(y^*_\eta(x))y^*_\eta(x)^2}{2(y^*_\eta(x) - x)(ry^*_\eta(x) - \delta x)}. \tag{20}$$

Proof of Lemma 3.1. We only give the proof in the put case, the argument being similar for the call. By (6), for $x > 0$, $f''(x)$ has the same sign as $h(x) = rf(x) + (\delta - r)xf'(x)$. If for some $x > 0$, $f'(x) = 0$, then since f is positive, $f''(x) > 0$. Therefore x is a local minimum point of f which contradicts the decreasing property of this function. Hence f' is a negative function.
When $\delta \leq r$, h and therefore f'' are positive functions. When $\delta > r$, we remark that if $f''(x) = 0$ then $h'(x) = \delta f'(x) < 0$. Since the continuous function f'' and h have the same sign, this implies that

$$\forall x > \inf\{z > 0 : f''(z) \leq 0\},\ f''(x) < 0. \tag{21}$$

Now for $y > 0$, by (6) then (18) , we have

$$\frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2}\frac{f''(x^*_\sigma(y))}{f'(x^*_\sigma(y))} = r\frac{f(x^*_\sigma(y))}{f'(x^*_\sigma(y))} - (r - \delta)x^*_\sigma(y) = \delta x^*_\sigma(y) - ry. \tag{22}$$

By (16), the right-hand-side is negative and moreover $\lim_{y\to+\infty} x^*_\sigma(y) = +\infty$. Hence $\sup\{z > 0 : f''(z) > 0\} = +\infty$ and with (21), we conclude that $f'' > 0$.

According to (18), $F(x^*_\sigma(y), y) = 0$ where

$$F(x, y) = y - x + f(x)/f'(x).$$

The function F is $\mathcal{C}^1$ on $(0, +\infty) \times (0, +\infty)$ and such that

$$\forall x, y > 0,\ \partial_x F(x, y) = -f(x)f''(x)/f'(x)^2 < 0.$$

Therefore for fixed $y > 0$, $x^*(y)$ is the unique solution to $F(x, y) = 0$. Moreover, $y \to x^*(y)$ is C^1 by the implicit function theorem. Last, one deduces from (18) that $\alpha(y)$ is a C^1 function. □

The positivity of f'' and g'' gives the following result.

Corollary 3.4. *The comparison result stated in Proposition 1.9 holds for any $\sigma_1 \le \sigma_2$ satisfying $(\mathcal{H}_{\mathbf{vol}})$.*

Let us now give a uniqueness result for the ODEs (19) and (20).

Proposition 3.5. *There is only one solution x^*_σ of (19) (resp. y^*_η of (20)) defined on $(0, +\infty)$ that satisfies $\forall y > 0,\ c_1 y \le x^*_\sigma(y) \le c_2 y$ with $0 < c_1 \le c_2 < \min(1, r/\delta)$ (resp. $\forall x > 0,\ d_1 x \le y^*_\eta(x) \le d_2 x$ with $d_1 > \max(1, \delta/r)$).*

Proof. Let us first remark that the uniqueness result for (19) is equivalent to the uniqueness result for (20). Indeed, it is easy to see that $x^*_\sigma(y)$ is solution of (19) if and only if $\hat{y}(x) := 1/(x^*_\sigma(1/x))$ is solution of (20) with the volatility function $\eta(x) = \sigma(1/x)$. This new volatility also satisfies $(\mathcal{H}_{\mathbf{vol}})$. Moreover, $d_1 x \le \hat{y}(x) \le d_2 x$ with $d_1 > \max(1, \delta/r)$ if, and only if $0 \le c_1 y \le x^*_\sigma(y) \le c_2 y$ with $0 < c_1 \le c_2 < \min(1, r/\delta)$.

Let us suppose then that there are two solutions of (20), $y_1(x)$ and $y_2(x)$, that are defined on $\mathbb{R}_+$ and satisfy $d_2 x \ge y_j(x) \ge d_1 x$ for some $d_2 > d_1 > \max(1, \delta/r)$. Since $y_1'(x) > 0$ for $x > 0$, y_1 is invertible and we have:

$$\begin{aligned}
\frac{d}{dx} y_1^{-1}(y_2(x)) &= \frac{y_2(x)^2 \eta(y_2(x))^2}{2(y_2(x) - x)(r y_2(x) - \delta x)} \frac{2(y_2(x) - y_1^{-1}(y_2(x)))(r y_2(x) - \delta y_1^{-1}(y_2(x)))}{y_2(x)^2 \eta(y_2(x))^2} \\
&= \frac{(y_2(x) - y_1^{-1}(y_2(x)))(r y_2(x) - \delta y_1^{-1}(y_2(x))}{(y_2(x) - x)(r y_2(x) - \delta x)}.
\end{aligned}$$

Thus, the function $\psi(x) = y_1^{-1}(y_2(x))/x$ solves

$$\begin{aligned}
\psi'(x) &= \frac{1}{x}\left[\frac{y_2(x) - \psi(x)x}{y_2(x) - x} \times \frac{r y_2(x) - \delta x \psi(x)}{r y_2(x) - \delta x} - \psi(x)\right] \\
&= \frac{1}{x}\left[\left(1 - \frac{\psi(x) - 1}{y_2(x)/x - 1}\right)\left(1 - \frac{\psi(x) - 1}{r y_2(x)/(\delta x) - 1}\right) - \psi(x)\right]. \qquad (23)
\end{aligned}$$

The estimation $d_2x \geq y_j(x) \geq d_1x$ for $j \in \{1,2\}$ with $d_1 > \max(1, \delta/r)$ implies that :

$$\exists A > 0, \forall x > 0, 1/A \leq \psi(x) \leq A, \tag{24}$$

$$\forall x > 0,\ \psi(x) < \min\left(\frac{y_2(x)}{x}, \frac{ry_2(x)}{\delta x}\right),\ \frac{y_2(x)}{x} - 1 > 0 \text{ and } \frac{ry_2(x)}{\delta x} - 1 > 0. \tag{25}$$

Since local uniqueness holds for (23) by the Cauchy Lipschitz theorem, the only solution φ such that $\varphi(1) = 1$ is the constant $\varphi \equiv 1$. Therefore checking that (24) does not hold for solutions φ satisfying (25) and such that $\varphi(1) \neq 1$ is enough to conclude that $\psi \equiv 1$.

Let φ be a solution to (23) satisfying (25). If $\varphi(1) > 1$, by local uniqueness for (23), for all $x \in \mathbb{R}_+^*$, $\varphi(x) > 1$. By (25), one deduces that for all $x \in \mathbb{R}_+^*$, $\varphi'(x) < \frac{1-\varphi(x)}{x} < 0$. Therefore, $\varphi'(x) \leq (1-\varphi(1))/x$ for $x \in (0,1]$, and we have

$$\varphi(x) \geq \varphi(1) + (1-\varphi(1))\ln(x) \underset{x\to 0}{\to} +\infty$$

which is contradictory to (24). In the same manner, if $\varphi(1) < 1$, $\varphi(x) < 1$ for $x \in \mathbb{R}_+^*$ and φ is strictly increasing. In particular, for $x \leq 1$, $\varphi'(x) \geq (1-\varphi(1))/x$ and therefore $\varphi(1) - \varphi(x) \geq (1-\varphi(1))\ln(1/x) \underset{x\to 0}{\to} +\infty$ and this yields another contradiction. □

Corollary 3.6. *Let us denote $\tilde{\mathcal{C}} = \{f \in \mathcal{C}^1(\mathbb{R}_+^*),\ s.t.\ f(0) = 0,\ \exists 0 < a < b, \forall x \geq 0,\ a \leq f'(x) \leq b\}$. The application $\sigma \mapsto x_\sigma^*$ (resp. $\eta \mapsto y_\eta^*$) is one-to-one between the set $\{\sigma \in \mathcal{C}(\mathbb{R}_+^*)$ that satisfies $(\mathcal{H}_{\mathbf{vol}})\}$ and the set of function $\tilde{\mathcal{C}}_x = \{x \in \tilde{\mathcal{C}},\ s.t.\ \exists 0 < c_1 \leq c_2 < \min(1, r/\delta), \forall y > 0,\ c_1 y \leq x(y) \leq c_2 y\}$ (resp. $\tilde{\mathcal{C}}_y = \{y \in \tilde{\mathcal{C}},\ s.t.\ \exists \max(1,\delta/r) < d_1 \leq d_2, \forall x > 0,\ d_1 x \leq y(x) \leq d_2 x\}$.)*

Proof. If σ is a continuous function satisfying $(\mathcal{H}_{\mathbf{vol}})$, by (19) and (16), x_σ^* belongs to $\tilde{\mathcal{C}}_x$. The one to one property is easy to get. If $x_{\sigma_1}^* \equiv x_{\sigma_2}^*$ with σ_1 and σ_2 satisfying $(\mathcal{H}_{\mathbf{vol}})$, the ODE (19) ensures that $\sigma_1^2(x_{\sigma_1}^*(y)) = \sigma_2^2(x_{\sigma_2}^*(y))$ for $y > 0$. Therefore $\sigma_1 \equiv \sigma_2$.
Let us check the onto property and consider $x^*(y) \in \tilde{\mathcal{C}}_x$. The function σ defined by

$$\sigma(x^*(y)) = \frac{\sqrt{2(y - x^*(y))(ry - \delta x^*(y))x^{*\prime}(y)}}{x^*(y)} \tag{26}$$

is well defined thanks to the hypothesis made on x^*. As x_σ^* satisfies (16) and solves the same ODE (19) as x^*, we have $x^* \equiv x_\sigma^*$ using Proposition 3.5.
The proof for $\eta \mapsto y_\eta^*$ is the same and gives incidentally the expression of η in function of the exercise boundary $y^*(x)$:

$$\eta(y^*(x)) = \frac{\sqrt{2(y^*(x) - x)(ry^*(x) - \delta x)y^{*\prime}(x)}}{y^*(x)}. \tag{27}$$

□

4 The call-put duality

This section is devoted to the key result of the paper : for related local volatility functions σ and η, we can interpret a put price in the primal world as a call price in the dual world.

4.1 The main result

Theorem 4.1 (Duality). *The following conditions are equivalent:*

1. $$\forall x, y > 0,\ P_\sigma(x, y) = c_\eta(y, x). \tag{28}$$
2. x^*_σ *and* y^*_η *are reciprocal functions:* $\forall x > 0,\ x^*_\sigma(y^*_\eta(x)) = x$.
3. $\eta \equiv \tilde{\sigma}$ *where*
$$\tilde{\sigma}(y) = \frac{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}{y x^*_\sigma(y)\sigma(x^*_\sigma(y))}. \tag{29}$$
4. $\sigma \equiv \underset{\sim}{\eta}$ *where*
$$\underset{\sim}{\eta}(x) = \frac{2(y^*_\eta(x) - x)(r y^*_\eta(x) - \delta x)}{y^*_\eta(x) x \eta(y^*_\eta(x))}. \tag{30}$$

Remark 4.2. *Thanks to relation (16) (resp. (17)), if* σ *(resp.* η*) satisfies* $(\mathcal{H}_{\mathbf{vol}})$ *then the dual volatility function* $\tilde{\sigma}$ *defined by (29) (resp.* $\underset{\sim}{\eta}$ *defined by (30)) satisfies* $(\mathcal{H}_{\mathbf{vol}})$.

Proof. $1 \implies 2$: We have on the one hand $P_\sigma(x, y) = y - x$ on $\{(x, y),\ x \le x^*_\sigma(y)\}$ and $P_\sigma(x, y) > y - x$ outside, and on the other hand $c_\eta(y, x) = y - x$ on $\{(x, y),\ y \ge y^*_\eta(x)\}$ and $c_\eta(y, x) > y - x$ outside. The duality relation (28) imposes then that $\{(x, y),\ x \le x^*_\sigma(y)\} = \{(x, y),\ y \ge y^*_\eta(x)\}$ and so $y^*_\eta(x^*_\sigma(y)) = y$.

$2 \implies 3, 4$: Taking the derivative of the last relation, we get thanks to (19) and (20) $\frac{x^*_\sigma(y)^2\sigma(x^*_\sigma(y))^2}{2(y-x^*_\sigma(y))(ry-\delta x^*_\sigma(y))}\frac{\eta^2(y)y^2}{2(y-x^*_\sigma(y))(ry-\delta x^*_\sigma(y))} = 1$ and deduce (29) and (30).

$3 \implies 2$ (resp. $4 \implies 2$) : By (19) (resp. (20)) and (29) (resp. (30)), ${x^*_\sigma}^{-1}$ (resp. ${y^*_\eta}^{-1}$) satisfies (20) (resp. (19)). Since by (16) (resp. (17)) this function satisfies (17) (resp. (16)), one concludes by Proposition 3.5.

$2 \implies 1$: The equality (28) is clear in the exercise region since $\{(x, y), x \le x^*_\sigma(y)\} = \{(x, y), y \ge y^*_\eta(x)\}$. Let us check that it also holds in the continuation region. Using the product form (14), and the smooth-fit principle (Theorem 1.6) we get for all $y \in \mathbb{R}^*_+$

$$\begin{cases} y - x^*_\sigma(y) = \beta(x^*_\sigma(y))g(y) \\ 1 = -\beta(x^*_\sigma(y))g'(y). \end{cases}$$

Differentiating the first equality with respect to y, one gets $1 - x^*_\sigma(y)' = x^*_\sigma(y)'\beta'(x^*_\sigma(y))g(y) + \beta(x^*_\sigma(y))g'(y)$, which combined with the second equality gives

$$-1 = \beta'(x^*_\sigma(y))g(y).$$

Dividing by the first equality and using (18), one deduces $\frac{\beta'}{\beta}(x^*_\sigma(y)) = \frac{f'}{f}(x^*_\sigma(y))$. Since $x^*_\sigma : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ is a bijection, there is a constant $C \neq 0$ such that $\beta \equiv Cf$. Since $\forall y > 0, \alpha(y)f(x^*_\sigma(y)) = y - x^*_\sigma(y) = \beta(x^*_\sigma(y))g(y)$, one has $\alpha \equiv g/C$. From (13) and (14), one concludes that (28) holds. □

In this proof, we have shown that α is proportional to g, and so there is a constant $C > 0$ such that $P_\sigma(x, y) = Cf(x)g(y)$ for $x \leq x^*_\sigma(y)$. In the Black-Scholes' case, we have $P_\sigma(x, y) = Cx^{a(\sigma)}y^{b(\sigma)}$ for $x \leq x^*_\sigma(y)$, and we are able to calculate C using the boundary condition $P_\sigma(x^*_\sigma(y), y) = y - x^*_\sigma(y)$. We get that $C = b(\sigma)^{-b(\sigma)}/(-a(\sigma))^{a(\sigma)}$. We retrieve then the already known analytical formulae for the put prices (e.g. Gerber and Shiu [8]).

4.2 An analytic example of dual volatility functions

By (30) and (20), if $y^* \in \tilde{\mathcal{C}}_y$ (where $\tilde{\mathcal{C}}_y$ is defined in Corollary 3.6), then the reciprocal function of y^* is the put exercise boundary x^*_σ associated to the local volatility function

$$\sigma(x) = \frac{\sqrt{2(ry^*(x) - \delta x)(y^*(x) - x)}}{x\sqrt{y^*(x)'}}.$$

Now by (27), y^* is the call exercise boundary associated with the dual volatility function :

$$\tilde{\sigma}(y) = \frac{\sqrt{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))y^{*\prime}(x^*_\sigma(y))}}{y}.$$

Let us consider the family of exercise boundaries

$$y^*(x) = x\frac{x + a}{bx + c}$$

where a, b, c are positive constants such that $\max(c/a, b) < \min(1, r/\delta)$ (condition ensuring $y^* \in \tilde{\mathcal{C}}_y$). Since $y^*(x)' = (bx^2 + 2cx + ac)/(bx + c)^2$, one has

$$\sigma(x) = \sqrt{2\frac{((r - \delta b)x + ra - \delta c)((1 - b)x + a - c)}{bx^2 + 2cx + ac}}, x > 0.$$

Moreover, the function $x^*_\sigma(y)$ is the only positive root of the polynomial function: $X^2 + X(a - by) - cy$, that is:

$$x^*_\sigma(y) = \frac{1}{2}\left(by - a + \sqrt{(by - a)^2 + 4cy}\right)$$

and

$$\forall y > 0,\ \tilde{\sigma}(y) = \frac{\sqrt{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))(bx^*_\sigma(y)^2 + 2cx^*_\sigma(y) + ac)}}{y(bx^*_\sigma(y) + c)}.$$

This example enables us to check numerically the duality. We have plotted in Figure 1, the prices of an American put $P_\sigma(T, x, y)$ in the primal world for the local volatility $\sigma(x)$ and

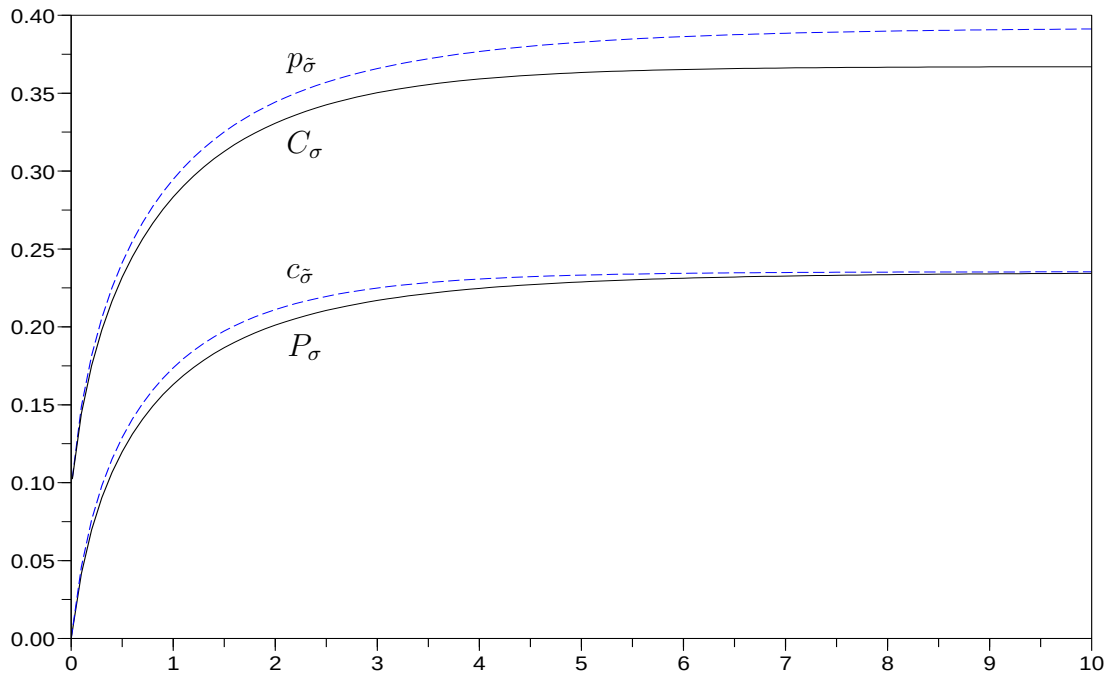

Figure 1: $P_\sigma(T,x,y)$ and $c_{\tilde{\sigma}}(T,y,x)$, and $C_\sigma(T,x,y)$ and $p_{\tilde{\sigma}}(T,y,x)$ as functions of the maturity T for $x=0.5$, $y=0.4$, $r=0.2$, $\delta=0.1$ and the volatility parameters $(a,b,c)=(1,0.4,0.1)$.

an American call $c_{\tilde{\sigma}}(T,y,x)$ in the dual world for the local volatility $\tilde{\sigma}(x)$ as functions of the maturity T. These prices have been computed using a finite difference method. We can see at $T=10$ that the limit value is quite reached and both prices are equal. The plots are nonetheless distinct which means that the duality does not hold for finite maturities. We have also plotted, in function of T, $C_\sigma(T,x,y)$ in the primal world and $p_{\tilde{\sigma}}(T,y,x)$ in the dual world to check numerically whether the volatility function $\tilde{\sigma}$ is such as $C_\sigma(x,y)=p_{\tilde{\sigma}}(y,x)$. As we can see, the curves do not seem to converge toward the same limit when T is large. This means that the volatility function $\hat{\sigma}$ such that $\forall x,y>0,\ C_\sigma(x,y)=p_{\hat{\sigma}}(y,x)$ (obtained from σ as $\underset{\sim}{\eta}$ is obtained from η but with exchange of r and δ) is different from $\tilde{\sigma}$.

5 Consequence of the duality : A (theoretical) method of calibration for the volatility $\sigma(x)$

In that section, we will put in evidence the importance of the duality within the calibration scope. We suppose for this that we are on a (virtual) market where are traded perpetual securities, and where the short interest rate r and the dividend rate δ can be observed.

This means that we know the price of the share x_0, and the market quotes on that share the perpetual American puts and calls for all strikes $K > 0$. We name respectively $p(K)$ and $c(K)$ these prices and denote:

$$X = \sup\{K > 0,\ c(K) = x_0 - K\} \text{ and } Y = \inf\{K > 0,\ p(K) = K - x_0\}. \tag{31}$$

We will first suppose that the put and call prices derive from a time-homogeneous local volatility model before relaxing this assumption.

5.1 The calibration procedure

Let us assume that there is a volatility function σ satisfying $(\mathcal{H}_{\mathbf{vol}})$ such that for all $K > 0$, $p(K) = P_\sigma(x_0, K)$ and $c(K) = C_\sigma(x_0, K)$. The following proposition says that these prices characterize σ and its proof gives a constructive way to retrieve the volatility function from the prices.

Proposition 5.1. *Let us consider $x_0 > 0$. The map*

$$\sigma \mapsto ((P_\sigma(x_0, K), C_\sigma(x_0, K)), K > 0)$$

is one-to-one on the set of volatility functions satisfying $(\mathcal{H}_{\mathbf{vol}})$.

Proof. We first consider the put case. The differential equation satisfied by the put prices in the continuation region makes only appear the values and the derivatives in x, K being fixed. Hence, we cannot exploit directly the prices. But the duality relation enables to get a differential equation in the strike variable. Thanks to the Duality Theorem, we have $P_\sigma(x_0, K) = c_{\tilde{\sigma}}(K, x_0)$ for some $\tilde{\sigma}$ satisfying $(\mathcal{H}_{\mathbf{vol}})$. It is then easy to calibrate $\tilde{\sigma}(.)$. Indeed, one has $\frac{K^2\tilde{\sigma}(K)^2}{2}p''(K) + K(\delta - r)p'(K) - \delta p(K) = 0$ for $K < Y = y^*_\sigma(x_0)$. Since the differential equation is valid only for $K < Y$, we only get $\tilde{\sigma}$ on $(0, Y]$ by continuity:

$$\forall K \leq Y,\ \tilde{\sigma}(K) = \frac{1}{K}\sqrt{\frac{2(\delta p(K) + K(r - \delta)p'(K))}{p''(K)}}$$

which is well defined since $p''(K) = \partial^2_K c_{\tilde{\sigma}}(K, x) > 0$ (Remark 3.2). Then, we can calculate the exercise boundary $y^*_{\tilde{\sigma}}(x)$, for $x \in (0, x_0]$, solving (20) supplemented with the final condition $y^*_{\tilde{\sigma}}(x_0) = Y$ backward. This step only requires the knowledge of $\tilde{\sigma}$ only on the interval $(0, Y]$. Finally, we can recover the desired volatility $\sigma(x)$ for $x \leq x_0$ thanks to (29):

$$\forall x \in (0, x_0],\ \sigma(x) = \frac{2(y^*_{\tilde{\sigma}}(x) - x)(ry^*_{\tilde{\sigma}}(x) - \delta x)}{xy^*_{\tilde{\sigma}}(x)\tilde{\sigma}(y^*_{\tilde{\sigma}}(x))}. \tag{32}$$

Now let us consider the calibration to the call prices. This relies on the same principle, but we have to be careful because the Duality Theorem is stated given to the call interest rate δ and dividend rate r. So we have to interchange these variables when we apply

that theorem. There is a function $\hat{\sigma}$ satisfying $(\mathcal{H}_{\mathbf{vol}})$ such that : $\forall K > 0,\ C_\sigma(x_0, K) = p_{\hat{\sigma}}(K, x_0)$. We have

$$\frac{1}{2}K^2\hat{\sigma}(x)^2 c''(K) + (\delta - r)Kc'(K) - \delta c(K) = 0$$

for $K > X = \xi^*_{\hat{\sigma}}(x_0)$. Thus, we get

$$\forall K \geq X,\ \hat{\sigma}(K) = \frac{1}{K}\sqrt{\frac{2(\delta c(K) + K(r-\delta)c'(K))}{c''(K)}}$$

which is well defined for analogous reasons. We can then obtain as before the exercise boundary solving (19) forward

$$\forall y \geq x_0,\ \xi^*_{\hat{\sigma}}(y)' = \frac{\xi^*_{\hat{\sigma}}(y)^2\hat{\sigma}(\xi^*_{\hat{\sigma}}(y))^2}{2(y - \xi^*_{\hat{\sigma}}(y))(\delta y - r\xi^*_{\hat{\sigma}}(y))},\ \xi^*_{\hat{\sigma}}(x_0) = X$$

and we finally get the volatility $\sigma(y)$ for $y \geq x_0$ using the Duality Theorem. More precisely, we interchange r and δ in (29) to get

$$\sigma(y) = \frac{2(y - \xi^*_{\hat{\sigma}}(y))(\delta y - r\xi^*_{\hat{\sigma}}(y))}{y\xi^*_{\hat{\sigma}}(y)\hat{\sigma}(\xi^*_{\hat{\sigma}}(y))}. \tag{33}$$

□

This calibration method, although being theoretical, sheds light on a striking and interesting result: the perpetual American put prices only give the restriction of $\sigma(x)$ to $(0, x_0]$ and the call prices only the restriction of $\sigma(x)$ to $[x_0, +\infty)$. This has the following economical interpretation : long-term American put prices mainly give information on the downward volatility while long-term American call prices give information on the upward volatility. This dichotomy is remarkable. In comparison, according to Dupire's formula [5], there is no such phenomenon for European options : the knowledge of the call prices gives the whole local volatility surface, not only one part. In other words, the European call and put prices give the same information on the volatility while the perpetual American call and put prices give complementary information.

Thus, one may think that the perpetual American call and put prices only depend on a part of the volatility curve. This is precised by the Proposition below that gives necessary and sufficient conditions on the volatility functions to observe the same put prices (resp. call prices).

Proposition 5.2. *Let us consider $x_0 > 0$ and $\sigma_1(.)$, $\sigma_2(.)$ two volatility functions satisfying $(\mathcal{H}_{\mathbf{vol}})$. Then, the following properties are equivalent:*

(i) $\forall y > 0,\ P_{\sigma_1}(x_0, y) = P_{\sigma_2}(x_0, y)$ *(resp.* $\forall y > 0,\ C_{\sigma_1}(x_0, y) = C_{\sigma_2}(x_0, y)$*)*

(ii) $\forall y \leq y^*_{\hat{\sigma}_2}(x_0), \tilde{\sigma}_1(y) = \tilde{\sigma}_2(y)$. *(resp.* $\forall x \geq \xi^*_{\hat{\sigma}_1}(x_0), \hat{\sigma}_1(x) = \hat{\sigma}_2(x)$ *where* $\hat{\sigma}_j$ *denotes the local volatility function such that* $\forall x, y > 0,\ C_{\sigma_j}(x, y) = p_{\hat{\sigma}_j}(y, x)$*.)*

(iii) $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ *and* $y^*_{\tilde{\sigma}_1}(x_0) = y^*_{\tilde{\sigma}_2}(x_0)$. *(resp.* $\forall x \in [x_0, +\infty),\ \sigma_1(x) = \sigma_2(x)$ *and* $\xi^*_{\hat{\sigma}_1}(x_0) = \xi^*_{\hat{\sigma}_2}(x_0)$*.)*

(iv) $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ *and* $\frac{f'_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_1}(x_0)} = \frac{f'_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_2}(x_0)}$. *(resp.* $\forall x \in [x_0, +\infty),\ \sigma_1(x) = \sigma_2(x)$ *and* $\frac{f'_{\uparrow,\sigma_1}(x_0)}{f_{\uparrow,\sigma_1}(x_0)} = \frac{f'_{\uparrow,\sigma_2}(x_0)}{f_{\uparrow,\sigma_2}(x_0)}$*.)*

(v) $f_{\downarrow,\sigma_1}$ *and* $f_{\downarrow,\sigma_2}$ *(resp.* $f_{\uparrow,\sigma_1}$ *and* $f_{\uparrow,\sigma_2}$*) are proportional on* $(0, x_0]$ *(resp.* $[x_0, +\infty)$*).*

(vi) $\forall x \le x_0, \forall y > 0,\ P_{\sigma_1}(x, y) = P_{\sigma_2}(x, y)$ *(resp.* $\forall x \ge x_0, \forall y > 0,\ C_{\sigma_1}(x, y) = C_{\sigma_2}(x, y)$*).*

Remark 5.3.
- *Among these many conditions, let us remark that condition (ii) on the dual volatility is much simpler than condition (iii) on the primal volatility since the latter requires the equality of the dual exercise boundaries at* x_0*.*
- *When* $\delta = 0$*, according to Remark 1.3, in the put case, condition (iv) also writes* $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ *and*

$$\int_{x_0}^{+\infty} \left(\frac{1}{v^2} \exp\left[-\int_{x_0}^{v} \frac{2r}{u\sigma_1^2(u)} du \right] \right) dv = \int_{x_0}^{+\infty} \left(\frac{1}{v^2} \exp\left[-\int_{x_0}^{v} \frac{2r}{u\sigma_2^2(u)} du \right] \right) dv.$$

- *Since, by definition of* $f_{\downarrow,\sigma_j}$ *(resp.* $f_{\uparrow,\sigma_j}$*) and the strong Markov property* $\forall 0 < z \le x,\ \mathbb{E}[e^{-r\tau^x_{\sigma_j,z}}] = f_{\downarrow,\sigma_j}(x)/f_{\downarrow,\sigma_j}(z)$ *(resp.* $\forall 0 < x \le z,\ \mathbb{E}[e^{-r\tau^x_{\sigma_j,z}}] = f_{\uparrow,\sigma_j}(x)/f_{\uparrow,\sigma_j}(z)$*), the probabilistic counterpart of assertion (v) is* $\forall 0 < z \le x \le x_0,\ \mathbb{E}[e^{-r\tau^x_{\sigma_1,z}}] = \mathbb{E}[e^{-r\tau^x_{\sigma_2,z}}]$ *(resp.* $\forall x_0 \le x \le z,\ \mathbb{E}[e^{-r\tau^x_{\sigma_1,z}}] = \mathbb{E}[e^{-r\tau^x_{\sigma_2,z}}]$*).*

Proof. We consider for example the put case.

$(i) \implies (ii)$: See the proof of Theorem 5.1.

$(ii) \implies (iii)$: Let us define $\psi(x) = (y^*_{\tilde{\sigma}_1})^{-1}(y^*_{\tilde{\sigma}_2}(x))/x$. We can show as in the proof of Proposition 3.5 that $\psi(x_0) = 1$ and then $\psi \equiv 1$ on $(0, x_0]$, otherwise it would go to 0 or $+\infty$ when $x \to 0$, which is not possible thanks to (17). We get then $\forall x \in (0, x_0],\ \sigma_1(x) = \sigma_2(x)$ using (30) that express σ_j in function of $y^*_{\tilde{\sigma}_j}$ and $\tilde{\sigma}_j$, $j \in \{1, 2\}$.

$(iii) \implies (iv)$: Thanks to (18) and Theorem 4.1, we have $\frac{f'_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_1}(x_0)} = \frac{-1}{y^*_{\tilde{\sigma}_1}(x_0) - x_0} = \frac{-1}{y^*_{\tilde{\sigma}_2}(x_0) - x_0} = \frac{f'_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_2}(x_0)}$.

$(iv) \implies (v)$: The set of solutions to $\frac{1}{2}\sigma_1^2(x)x^2 f''(x) + (r - \delta)xf'(x) - rf(x) = 0$ on $(0, x_0]$ is a two-dimensional vector space, but thanks to the relation $\frac{f'_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_1}(x_0)} = \frac{f'_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_2}(x_0)}$, $f_{\downarrow,\sigma_1}$ and $f_{\downarrow,\sigma_2}$ are proportional on $(0, x_0]$:

$$\forall x \le x_0,\ f_{\downarrow,\sigma_1}(x) = \frac{f_{\downarrow,\sigma_1}(x_0)}{f_{\downarrow,\sigma_2}(x_0)} f_{\downarrow,\sigma_2}(x). \tag{34}$$

$(v) \implies (vi)$: The proportionality implies that $\forall x \in (0, x_0],\ \frac{f_{\downarrow,\sigma_1}(x)'}{f_{\downarrow,\sigma_1}(x)} = \frac{f_{\downarrow,\sigma_2}(x)'}{f_{\downarrow,\sigma_2}(x)}$, and then $(y^*_{\tilde{\sigma}_1}(x) - x)^{-1} = (y^*_{\tilde{\sigma}_2}(x) - x)^{-1}$ using (18) and Theorem 4.1. Therefore

$$\forall x \in (0, x_0],\ y^*_{\tilde{\sigma}_1}(x) = y^*_{\tilde{\sigma}_2}(x).$$

We have $\alpha_{\sigma_1}(y^*_{\tilde{\sigma}_1}(x))f_{\downarrow,\sigma_1}(x) = \alpha_{\sigma_2}(y^*_{\tilde{\sigma}_2}(x))f_{\downarrow,\sigma_2}(x)$ using (13), and obtain from (34) that

$$\forall x \leq x_0,\ \forall y \leq y^*_{\tilde{\sigma}_1}(x_0),\ \alpha_{\sigma_1}(y) = \frac{f_{\downarrow,\sigma_2}(x_0)}{f_{\downarrow,\sigma_1}(x_0)}\alpha_{\sigma_2}(y) = \frac{f_{\downarrow,\sigma_2}(x)}{f_{\downarrow,\sigma_1}(x)}\alpha_{\sigma_2}(y). \tag{35}$$

Thus, we deduce from (13), (34) and (35) the equality of the put prices for the low strikes

$$\forall x \leq x_0,\ \forall y \leq y^*_{\tilde{\sigma}_1}(x), P_{\sigma_1}(x,y) = P_{\sigma_2}(x,y).$$

For $y > y^*_{\tilde{\sigma}_1}(x) = y^*_{\tilde{\sigma}_2}(x)$, the equality is clear since both prices are equal to $y - x$.
$(vi) \implies (i)$: clear. □

Let us observe that the point (ii) of the last proposition allows to exhibit different volatility functions with analytic expressions that give the same put (or call) prices. Let us consider the same family as in subsection 4.2 coming from the call exercise boundary $y^*_1(x) = x\frac{x+a}{bx+c}$ (assuming $a, b, c > 0$ and $\max(c/a, b) < \min(1, r/\delta)$). For $x_0 > 0$, we introduce the exercise boundary:

$$y^*_2(x) = y^*_1(x) \text{ for } x \leq x_0 \text{ and } y^*_2(x) = y^*_1(x_0) + (y^*_1)'(x_0)(x - x_0) \text{ for } x \geq x_0.$$

The condition ($y^*_2 \in \tilde{\mathcal{C}}_y$) is satisfied provided that $(y^*_1)'(x_0) > \max(1, \delta/r)$. This is automatically ensured by the assumptions made on a, b, c since $(y^*_1)'(x_0) = (bx_0^2 + 2cx_0 + ac)/(b^2x_0^2 + 2bcx_0 + c^2)$. That family is such that $\tilde{\sigma}_1(y) = \tilde{\sigma}_2(y)$ for $y \leq y^*_2(x_0)$. We can then calculate σ_2 as in subsection 4.2 using the relation $\sigma_2(x) = \frac{\sqrt{2(ry^*_2(x)-\delta x)(y^*_2(x)-x)}}{x\sqrt{y^*_2(x)'}}$. This gives $\sigma_2(x) = \sigma(x)$ for $x \leq x_0$ and for $x \geq x_0$,

$$\sigma_2(x) = \sqrt{2\frac{[(r(y^*_1)'(x_0) - \delta)x + r(y^*_1(x_0) - x_0(y^*_1)'(x_0))][((y^*_1)'(x_0) - 1)x + y^*_1(x_0) - x_0(y^*_1)'(x_0)]}{x^2(y^*_1)'(x_0)}}.$$

In Figure 2, we have plotted the same example as in Figure 1 ($x = 0.5$ and $y = 0.4$), adding the graph of $T \mapsto P_{\sigma_2}(T, x, y)$. The volatility function σ_2 has been calculated with the formula above with $x_0 = 0.5$. According to Proposition 5.2 and the Duality, the three prices are equal when T is large. In the second example ($x = 3$ and $y = 1$), we still observe that $P_\sigma(T, x, y)$ and $c_{\tilde{\sigma}}(T, y, x)$ converge toward the same value when T is large. On the contrary, the limit price of $P_{\sigma_2}(T, x, y)$ is significantly different. To observe the same price, we should have taken, according to Proposition 5.2, $x_0 \geq 3$.

5.2 Calibration to "real" call and put prices

In that subsection, we address some problems that arise if one tries to apply the calibration procedure when the prices $p(K)$ and $c(K)$ do not derive from a time-homogeneous model. We assume however that they are smooth functions of the strike K, and focus for example on the calibration to put prices.

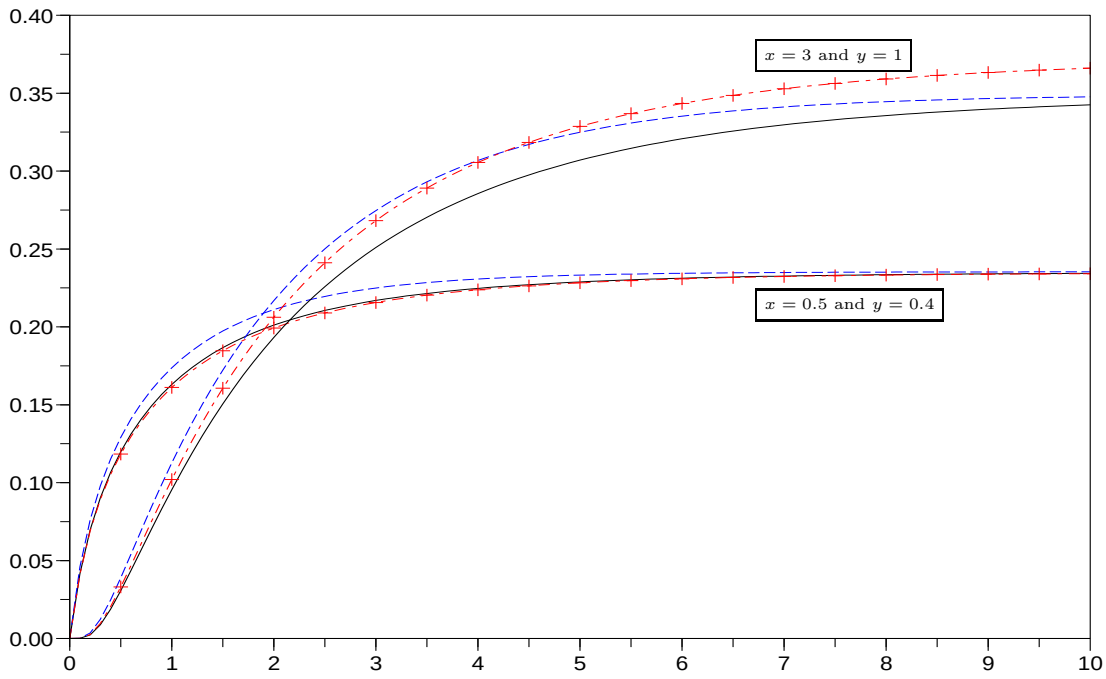

Figure 2: $P_\sigma(T, x, y)$ (solid line), $P_{\sigma_2}(T, x, y)$ (dashed line with crosses) and $c_{\tilde{\sigma}}(T, y, x)$ (dashed line) in function of the time T for $a = 1$, $b = 0.4$, $c = 0.1$, $x_0 = 0.5$, $r = 0.2$ and $\delta = 0.1$.

Firstly, let us observe that the arbitrage-free theory allows to define a dual volatility as previously by $(0, Y]$:

$$\forall K < Y,\ \eta_p(K) = \frac{1}{K}\sqrt{\frac{2(\delta p(K) + K(r-\delta)p'(K))}{p''(K)}}. \tag{36}$$

Indeed, the payoff convexity in K ensures the positivity of $p''(K)$ and the arbitrage-free assumption ensures that $\delta p(K) + K(r-\delta)p(K)$ is nonnegative, so that the square-root is well defined. Let us prove the last point and suppose the contrary (i.e. $\exists y > 0$ such that $\frac{d}{dy}e^{\delta y}p(e^{(r-\delta)y}) < 0$) to exhibit an arbitrage opportunity. In that case, there is $z > y$ such that $e^{\delta y}p(e^{(r-\delta)y}) > e^{\delta z}p(e^{(r-\delta)z})$. We then sell one put with strike $e^{(r-\delta)y}$ and buy $e^{\delta(z-y)}$ puts with strike $e^{(r-\delta)z}$. This initial transaction generates a positive flow. The hedging works as follows: naming τ the time at which the put sold is exercised, we have to pay $e^{(r-\delta)y} - S_\tau$. In other words, we receive one share and borrow $e^{(r-\delta)y}$ in cash. We keep this position until time $\tau + z - y$. At this time, we have exactly $e^{\delta(z-y)}$ shares and puts with strike $e^{(r-\delta)z}$. Thus, we obtain at least $e^{\delta(z-y)}e^{(r-\delta)z} = e^{(r-\delta)y}e^{r(z-y)}$ and we cancel the debt. The next proposition gives sufficient conditions that allow to construct an homogeneous

volatility which is consistent to the observed prices.

Proposition 5.4. *Let us assume that $K \in \mathbb{R}_+^* \mapsto p(K)$ is a $\mathcal{C}^1$ function, $\mathcal{C}^2$ on $\mathbb{R}_+^* - \{Y\}$ with $Y = \inf\{K > 0 : p(K) = K - x_0\} < +\infty$. Let us also assume that η_p defined by (36) is bounded from below and above by two positive constants and admits a left-hand limit in Y. Then, if we extend η_p in any continuous function on $(0, +\infty)$ satisfying ($\mathcal{H}_{\mathbf{vol}}$) still denoted by η_p, we have*

$$\forall K > 0, P_{\underset{\sim}{\eta_p}}(x_0, K) = p(K).$$

Notice that once we choose the extended function η_p, we obtain $\underset{\sim}{\eta_p}$ by first solving (20) on $\mathbb{R}_+^*$ starting from x_0 with the condition $y^*_{\eta_p}(x_0) = Y$ and then using (30).

Proof. The functions $K \mapsto p(K)$ and $K \mapsto c_{\eta_p}(K, x_0)$ solve (12). Since we have $0 \leq p(K) \leq K$ for arbitrage-free reasons, both functions go to 0 when $K \to 0$. Thanks to Remark 1.2, they are proportional to $g_\uparrow$ and therefore there is $\lambda > 0$ such that:

$$\forall K \leq Y,\ p(K) = \lambda c_{\eta_p}(K, x_0).$$

The $\mathcal{C}^1$ assumption made on p ensures $p(Y) = Y - x_0$ and $p'(Y) = 1$. This gives $g_\uparrow(Y)/g'_\uparrow(Y) = Y - x_0$ and therefore $Y = y^*_{\eta_p}(x_0)$ using Lemma 3.1. Thus, $c_{\eta_p}(Y, x_0) = Y - x_0 = p(Y)$ and $\lambda = 1$. One concludes with Theorem 4.1. □

For the call case, everything works in the same manner, but we need to assume moreover that $c(K) \to 0$ when $K \to +\infty$. This is a rather natural hypothesis that plays the same role as $p(K) \to 0$ when $K \to 0$.

Proposition 5.5. *Let us assume that $K \in \mathbb{R}_+^* \mapsto c(K)$ is a $\mathcal{C}^1$ function, $\mathcal{C}^2$ on $\mathbb{R}_+^* - \{X\}$ with $X = \sup\{K > 0,\ c(K) = x_0 - K\} > 0$ and $\lim_{K \to +\infty} c(K) = 0$. Let us also assume that η_c defined by*

$$\forall K > X,\ \eta_c(K) = \frac{1}{K}\sqrt{\frac{2(\delta c(K) + K(r - \delta)c'(K))}{c''(K)}}$$

is bounded from below and above by two positive constants and admits a right-hand limit in X. Then, if we extend η_c in any continuous function on $(0, +\infty)$ satisfying ($\mathcal{H}_{\mathbf{vol}}$) still denoted by η_c, we have

$$\forall K > 0, C_{\underset{\sim}{\eta_c}}(x_0, K) = c(K)$$

where $\underset{\sim}{\eta_c}$ is obtained from η_c like σ from $\hat{\sigma}$ in the end of the proof of Proposition 5.1.

Therefore, we are able to find volatility functions that give exactly the put prices and others that give exactly the call prices. Now, the natural question is whether one can find a volatility function σ that is consistent to both the put and call prices. According to Proposition 5.2, all the volatility functions $\underset{\sim}{\eta_p}$ (resp. $\underset{\sim}{\eta_c}$) giving the put (resp. call) prices

coincide on $(0, x_0)$ (resp. $(x_0, +\infty)$). The only volatility function possibly giving both the put and call prices is

$$\sigma(x) = \begin{cases} \underset{\sim}{\eta_p}(x) \text{ if } x < x_0 \\ \underset{\smile}{\eta_c}(x) \text{ if } x > x_0 \end{cases} .$$

We deduce from Proposition 5.2 :

Proposition 5.6. *Assume that* $\underset{\sim}{\eta_p}(x_0^-) = \underset{\smile}{\eta_c}(x_0^+)$. *Then,*

$$\forall K > 0, p(K) = P_\sigma(x_0, K) \text{ and } c(K) = C_\sigma(x_0, K) \text{ iff } x_\sigma^*(Y) = x_0 \text{ and } \Upsilon_\sigma^*(X) = x_0.$$

6 The Black-Scholes model: the unique model invariant through this duality

The purpose of that section is to put in evidence the particular role played by the Black-Scholes' model for the perpetual American call-put duality. We have recalled in the introduction that constant volatility functions are invariant by the duality. We have also mentioned that for the European case, the call-put duality holds for all maturities without any change of the volatility function. Here, on the contrary, we are going to prove that if the duality holds for the perpetual American options with the same volatility:

$$\forall x, y > 0 \ P_\sigma(x, y) = c_\sigma(y, x) \tag{37}$$

then, under some technical assumptions, necessarily $\sigma(.)$ is a constant function.

Proposition 6.1. *Let us consider a positive interest rate* r *and a nonnegative dividend rate* $\delta < r$. *We suppose that the volatility function* σ *satisfies* $(\mathcal{H}_{\mathbf{vol}})$, *and is analytic in a neighborhood of* 0, *i.e.*

$$\exists \rho > 0, \forall x \in [0, \rho), \ \sigma(x) = \sum_{k=0}^{\infty} \sigma_k x^k. \tag{38}$$

Then, (37) holds if and only if $\forall x \geq 0, \ \sigma(x) = \sigma_0$.

We have already shown in the introduction that (37) holds in the Black-Scholes' case. So we only have to prove the necessary condition. We decompose the proof into the three following lemmas.

Lemma 6.2. *Let us consider a volatility function that satisfies* $(\mathcal{H}_{\mathbf{vol}})$. *If the dual volatility function* $\tilde{\sigma}$ *is analytic in a neighborhood of* 0, *then the boundaries* x_σ^* *and* $y_{\tilde{\sigma}}^*$ *are also analytic in a neighborhood of* 0.

Lemma 6.3. *Let us suppose that σ satisfies $(\mathcal{H}_{\mathbf{vol}})$ and is analytic in a neighborhood of 0. Let us assume moreover that $r > \delta$. If the equality (37) holds, σ is constant in a neighborhood of 0:*

$$\exists \rho > 0, \forall y \in [0, \rho],\ \sigma(y) = \sigma_0.$$

Lemma 6.4. *Let us suppose that σ is a constant function on $[0, \rho]$ for $\rho > 0$ satisfying $(\mathcal{H}_{\mathbf{vol}})$ and (37). Then, σ is constant on $\mathbb{R}_+$ (and x^*_σ and $y^*_{\tilde\sigma}$ are linear functions).*

Proof of Lemma 6.2. Let us first show that x^*_σ is analytic in 0. Thanks to the relation (18), we have $\frac{g(y^*_{\tilde\sigma}(x))}{g'(y^*_{\tilde\sigma}(x))} = y^*_{\tilde\sigma}(x) - x$, and therefore $\frac{g(y)}{g'(y)} = y - x^*_\sigma(y)$. Thus, $x^*_\sigma(y)$ is analytic in 0 iff $\phi(y) = \frac{g(y)}{g'(y)}$ is analytic in 0. Using the relation (12) and $\phi' = 1 - \frac{g''}{g'}\phi$, we get that ϕ is solution of

$$\phi'(y) = 1 + \frac{2}{\tilde\sigma^2(y)}\left((\delta - r)\phi(y)/y - \delta(\phi(y)/y)^2\right). \tag{39}$$

Notice that $\phi(y) = y - x^*_\sigma(y)$ and (16) imply that if ϕ is analytic in 0 then the coefficient of order 0 in its expansion vanishes and the coefficient of order 1 belongs to $(0, 1)$.

To complete the proof we are first going to check that if $\psi(y) = \sum_{k=1}^\infty \phi_k y^k$ with $\phi_1 \in (0, 1)$ solves (39) in a neighborhood of 0 then $\phi \equiv \psi$ in this neighborhood. Then we will prove existence of such an analytic solution ψ. We have $\psi(0) = 0$, and the function ψ being analytic with $\phi_1 \neq 0$, its zeros are isolated points. There is therefore a neighborhood of 0, $(0, 2\epsilon)$ where ψ does not vanish. Let us consider γ a solution of $\gamma' - \frac{1}{\psi}\gamma = 0$ starting from $\gamma(\epsilon) \neq 0$ in ϵ : $\gamma(x) = \gamma(\epsilon)\exp\left(\int_\epsilon^x \frac{1}{\psi(u)}du\right)$. Since ψ solves (39), it is not hard to check that γ is solution of (12) with $\eta = \tilde\sigma$. The limit condition $\gamma(x) \underset{x\to 0}{\to} 0$ (cf. Remark 1.2, still valid for $g_\uparrow$ when $\delta = 0$) is satisfied since we have $\frac{1}{\psi(u)} \underset{u\to 0}{\sim} \frac{1}{\phi_1 u}$ and so $\int_\epsilon^x \frac{1}{\psi(u)}du \underset{x\to 0}{\to} -\infty$. Thus we have $\gamma(y) = cg(y)$ with $c \neq 0$ and $\psi(y) = g(y)/g'(y) = \phi(y) = y - x^*_\sigma(y)$. We can then write $x^*_\sigma(y) = (1 - \phi_1)y - \sum_{k=2}^\infty \phi_k y^k$ in the neighborhood of 0 with $1 - \phi_1 > 0$. It is well-known that in that case, the reciprocal function $y^*_{\tilde\sigma}$ is also analytic in 0.

Let us turn to the existence of ψ. Since $\sigma_0 \geq \underline{\sigma} > 0$, $y \to \frac{2}{\tilde\sigma^2(y)}$ is an analytic function in the neighborhood of 0. Thus, there is $\rho_0 > 0$ and $a_0 > 0$ such that

$$\forall y \in [0, \rho_0],\ \frac{2}{\tilde\sigma^2(y)} = \sum_{k=0}^\infty a_k y^k \text{ and } \sum_{k=0}^\infty |a_k|\rho_0^k < \infty.$$

The analytic function $\sum_{k\geq 1}\phi_k y^k$ solves (39) if and only if

$$\sum_{k=0}^\infty (k+1)\phi_{k+1}y^k = 1 + (\delta - r)\sum_{k=0}^\infty\left(\sum_{i+j=k} a_i\phi_{j+1}\right)y^k - \delta\sum_{k=0}^\infty\left(\sum_{i+j+l=k} a_i\phi_{j+1}\phi_{l+1}\right)y^k.$$

Identifying the terms of order 0, we get that ϕ_1 solves $P(\phi_1) = 0$ where $P(x) = \delta a_0 x^2 + (1 - (\delta - r)a_0)x - 1$. Since $P(0) = -1 < 0$ and $P(1) = ra_0 > 0$, the polynomial P admits

a unique root on $(0,1)$ and we choose ϕ_1 equal to this root. Then, by identification of the terms with order k, we define the sequence $(\phi_k)_{k\geq 1}$ inductively by

$$\phi_{k+1} = \frac{(\delta - r)\sum_{i+j=k, j\neq k} a_i\phi_{j+1} - \delta\sum_{i+j+l=k, j\neq k, l\neq k} a_i\phi_{j+1}\phi_{l+1}}{k+1+(r-\delta)a_0 + 2\delta a_0\phi_1}.$$

This ratio is well defined since $(r-\delta)a_0 + 2\delta a_0\phi_1 = \delta a_0\phi_1 + 1/\phi_1 - 1 > 0$.
We still have to check that the series $\sum_{k\geq 1}\phi_k y^k$ is defined in a neighborhood of 0. To do so, we are going to show that there is $\rho > 0$ such that the sequence $(|\phi_k|\rho^k)_{k\geq 1}$ is bounded. We have for $1 \leq k \leq n$:

$$|\phi_{k+1}|\rho^k \leq \frac{|\delta - r|\sum_{j=0}^{k-1}|a_{k-j}|\rho^{k-j}|\phi_{j+1}|\rho^j + \delta\sum_{i=0}^{k}\left(\sum_{\substack{j+l=k-i\\ j\neq k, l\neq k}}|\phi_{j+1}|\rho^j|\phi_{l+1}|\rho^l\right)|a_i|\rho^i}{k+1}.$$

Let us suppose that for $1 \leq j < k$, $|\phi_{j+1}|\rho^j \leq 1/(j+1)$. Then,

$$|\phi_{k+1}|\rho^k \leq \frac{|\delta - r|\rho\sum_{j=1}^{k}|a_j|\rho^{j-1} + \delta\sum_{i=0}^{k}\left(\sum_{j+l=k-i}\frac{1}{j+1}\frac{1}{l+1}\right)|a_i|\rho^i}{k+1}.$$

We remark that $\sum_{j+l=k-i}\frac{1}{j+1}\frac{1}{l+1} = \frac{1}{k-i+2}\sum_{j+l=k-i}\frac{1}{j+1} + \frac{1}{l+1} \leq 2\frac{\ln(k-i+1)+1}{k-i+2}$, and we finally get:

$$|\phi_{k+1}|\rho^k \leq \frac{2\delta|a_0|\frac{\ln(k+1)+1}{k+2} + \rho(|\delta - r| + 2\delta)\sum_{j=1}^{k}|a_j|\rho^{j-1}}{k+1} \tag{40}$$

since $\frac{\ln(k-i+1)+1}{k-i+2} \leq 1$. Let us now consider k_0 such that $\forall k \geq k_0, 2\delta|a_0|\frac{\ln(k+1)+1}{k+2} < 1/2$. Now, we chose $\rho \in (0,\rho_0)$ small enough such that $\forall k \leq k_0, |\phi_{k+1}|\rho^k \leq 1/(k+1)$ and $\rho(|\delta - r| + 2\delta)\sum_{j=1}^{\infty}|a_j|\rho^{j-1} < 1/2$. Then we get by induction from (40) that $\forall k \geq k_0, |\phi_{k+1}|\rho^k \leq 1/(k+1)$. □

Proof of Lemma 6.3. On the one hand, thanks to the assumption, $\sigma = \tilde{\sigma}$ is analytic in 0, and therefore x^*_σ is analytic in 0 thanks to Lemma 6.2:

$$\exists\rho > 0,\ \forall y \in [0,\rho), x^*_\sigma(y) = \sum_{i=1}^{\infty} x_i y^i \text{ and } \sigma(y) = \sum_{i=0}^{\infty}\sigma_i y^i.$$

On the other hand, it is not hard then to deduce from (29), $\sigma = \tilde{\sigma}$ and the differential equation (19) that

$$x^*_\sigma(y)' = \frac{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}{y^2\sigma(y)^2}. \tag{41}$$

From Corollary 3.6 and (26), we get

$$x_\sigma^*(y)' = \frac{(y - x_\sigma^*(y))(ry - \delta x_\sigma^*(y))((x_\sigma^*)^{-1})'(y)}{((x_\sigma^*)^{-1}(y) - y)(r(x_\sigma^*)^{-1}(y) - \delta y)}. \tag{42}$$

Now, we consider $n = \inf\{i \geq 2, x_i \neq 0\}$ and suppose it finite. We can get easily that:

$x_\sigma^*(y) = x_1 y + x_n y^n + \dots$ $\quad$ $x_\sigma^*(y)' = x_1 + n x_n y^{n-1} + \dots$

$(x_\sigma^*)^{-1}(y) = \frac{1}{x_1} y - \frac{x_n}{x_1^{n+1}} y^n + \dots$ $\quad$ $((x_\sigma^*)^{-1})'(y) = \frac{1}{x_1}(1 - \frac{n x_n}{x_1^n} y^{n-1}) + \dots$

and then

$$\begin{aligned}
(1 - x_\sigma^*(y)/y)(r - \delta x_\sigma^*(y)/y) &= (1 - x_1)(r - \delta x_1) + x_n(2\delta x_1 - (r+\delta))y^{n-1} + \dots \\
\left(\frac{(x_\sigma^*)^{-1}(y)}{y} - 1\right)\left(r\frac{(x_\sigma^*)^{-1}(y)}{y} - \delta\right) &= \frac{1}{x_1^2}\left\{(1 - x_1)(r - \delta x_1) + \frac{x_n}{x_1^n}((r+\delta)x_1 - 2r)y^{n-1}\right\} + \dots
\end{aligned}$$

The right hand side of (42) has then the following expansion:

$$x_1\left\{1 + \frac{x_n}{(1 - x_1)(r - \delta x_1)}\left[2\delta x_1 - (r+\delta) + \frac{2r}{x_1^n} - \frac{r+\delta}{x_1^{n-1}}\right] y^{n-1} - \frac{n x_n}{x_1^n} y^{n-1}\right\} + \dots$$

The equality of the terms of order $n-1$ in (42) then leads to:

$$n x_n x_1^{n-1} = \frac{x_n}{(1 - x_1)(r - \delta x_1)}\left[2\delta x_1^{n+1} - (r+\delta)x_1^n - (r+\delta)x_1 + 2r\right] - n x_n.$$

Since $x_n \neq 0$ and with a simplification we get

$$n(1 + x_1^{n-1}) = \frac{1}{r - \delta x_1}\left[-2\delta x_1^n + (r - \delta)\sum_{k=1}^{n-1} x_1^k + 2r\right]. \tag{43}$$

In the case $\delta = 0$ this gives $n(1 + x_1^{n-1}) = x_1^{n-1} + \dots + x_1 + 2$ which is not possible because $x_1 \in (0,1)$. When $0 < \delta < r$, we denote $\alpha = r/\delta > 1$ and rewrite (43):

$$n(1 + x_1^{n-1})(\alpha - x_1) = -2x_1^n + (\alpha - 1)x_1^{n-1} + \dots + (\alpha - 1)x_1 + 2\alpha = \alpha - x_1^n + (\alpha - x_1)\frac{1 - x_1^n}{1 - x_1}.$$

Therefore, $n(1 + x_1^{n-1}) = \frac{\alpha - x_1^n}{\alpha - x_1} + \frac{1 - x_1^n}{1 - x_1} < 2\frac{1 - x_1^n}{1 - x_1}$ because $\beta \mapsto \frac{\beta - x_1^n}{\beta - x_1}$ is decreasing on $(1, \alpha)$ $(x_1^n < x_1)$. To show that this is impossible, we consider $P_n(x) = n(1 + x^{n-1}) - 2\sum_{k=0}^{n-1} x^k$. We have $P_n(1) = 0$ and for $x < 1$, $P_n'(x) = n(n-1)x^{n-2} - 2\sum_{k=1}^{n-1} k x^{k-1} = 2\sum_{k=1}^{n-1} k(x^{n-2} - x^{k-1}) < 0$. Thus P_n is positive on $[0,1)$ and $P_n(x_1) > 0$ which is a contradiction. □

Proof of Lemma 6.4. It is easy to get from (19) and $\sigma = \tilde{\sigma}$ that

$$x_\sigma^*(y)' = \frac{x_\sigma^*(y)\sigma(x_\sigma^*(y))}{y\sigma(y)}. \tag{44}$$

We have $\sigma(x) = \sigma_0$ for $x \in [0, \rho]$. Since $x^*_\sigma(y)$ solves (44) and $x^*_\sigma(y) \leq y$, $x^*_\sigma(y)' = x^*_\sigma(y)/y$ on $[0, \rho]$. Therefore, $x^*_\sigma(y) = x_1 y$ for $y \in [0, \rho]$. Thanks to (19), x_1 is the unique root in $(0, \min(1, r/\delta))$ of

$$x_1 \sigma_0^2 = 2(1 - x_1)(r - \delta x_1).$$

Now let us observe that (19) gives for $y \in (0, y^*_{\tilde\sigma}(\rho)]$, $x^*_\sigma(y)' = \frac{x^*_\sigma(y)^2 \sigma_0^2}{2(y - x^*_\sigma(y))(ry - \delta x^*_\sigma(y))}$ with $x^*_\sigma(\rho) = x_1 \rho$. Since $y \to x_1 y$ solves this ODE, for which local uniqueness holds thanks to the Cauchy Lipschitz theorem, we then have $x^*_\sigma(y) = x_1 y$ on $[\rho, y^*_{\tilde\sigma}(\rho)]$ and so $y^*_{\tilde\sigma}(\rho) = (x^*_\sigma)^{-1}(\rho) = \rho/x_1$. Then, (44) gives $\sigma_0/\sigma(y) = 1$ on $[\rho, \rho/x_1]$. Thus, we prove by induction on n that $x^*_\sigma(y) = x_1 y$ and $\sigma(y) = \sigma_0$ for $y \in [0, \rho/(x_1)^n]$. This shows the desired result. □

7 Conclusions and further developments

Addressing Call-Put duality for American options with finite maturity in models with time-dependent local volatility functions like (1) would be of great interest. For the perpetual case treated in this paper, we could take advantage of a very nice feature : in the continuation region, the price of the option writes as the product of a function of the underlying spot price by another function of the strike price. Unfortunately, this product property no longer holds in the general case.

Next, according to our numerical experiments (see figure 2), American Put and Call prices computed in infinite maturity dual models may differ for finite maturities. This means that in the case of a time-homogeneous primal local volatility function $\varsigma(t, x) = \sigma(x)$, if there exists a dual local volatility function for some finite maturity T, then this volatility function is either time-dependent or depends on the maturity T. On the contrary, in the European case presented in the introduction, time-homogeneous volatility functions are preserved by the duality.

Let us nevertheless conclude on an encouraging remark. Let $P(T, x, y)$ denote the initial price of the American Put option with maturity T and strike y in the model (1) and $x^*(T, y)$ stand for the corresponding exercise boundary such that $P(T, x, y) = (y - x)^+$ if and only if $x \leq x^*(t, y)$. Then the smooth-fit principle writes

$$\begin{cases} P(T, x^*(T, y), y) = y - x^*(T, y) \\ \partial_x P(T, x^*(T, y), y) = -1 \end{cases}.$$

Differentiating the former equality with respect to y yields

$$\partial_x P(T, x^*(T, y), y) \partial_y x^*(T, y) + \partial_y P(T, x^*(T, y), y) = 1 - \partial_y x^*(T, y).$$

With the second equality, one deduces that $\partial_y P(T, x^*(T, y), y) = 1$. Therefore the smooth-fit principle automatically holds for the dual Call option if there exists any.

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