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Peter Haïssinsky, Kevin Pilgrim

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# Coarse expanding conformal dynamics

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Peter Haissinsky  
LATP/CMI  
Université de Provence  
39, rue Frédéric Joliot-Curie  
13453 Marseille cedex 13  
France  
*E-mail:* phaissin@cmi.univ-mrs.fr

Kevin M. Pilgrim  
Dept. of Mathematics  
Indiana University  
Bloomington  
IN 47405  
USA  
*E-mail:* pilgrim@indiana.edu

### Abstract

Building on the dictionary between Kleinian groups and rational maps, we establish new connections between the theories of hyperbolic groups and certain iterated maps, regarded as dynamical systems. In order to make the exposition self-contained to researchers in many fields, we include detailed proofs and ample background.

*Mathematics Subject Classification 2000:* Primary 53C23, secondary 30C65, 37B99, 37D20, 37F15, 37F20, 37F30, 54E40.

*Keywords:* analysis on metric spaces, quasisymmetric map, conformal gauge, rational map, Kleinian group, dictionary, Gromov hyperbolic, entropy

*This work is dedicated to Adrien Douady.*

“Que s’est-il passé dans ta tête ?  
Tu as pris la poudre d’escampette  
Sans explication est-ce bête  
Sans raison tu m’as planté là  
Ah ah ah ah !”

(Boby Lapointe, *La question ne se pose pas*)



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# Chapter 1

## Introduction

The classical conformal dynamical systems include iterated rational maps and Kleinian groups acting on the Riemann sphere. The development of these two theories was propelled forward in the early 1980's by Sullivan's introduction of quasiconformal methods and of a "dictionary" between the two subjects [Sul4]. Via complex analysis, many basic dynamical objects can be similarly defined and results similarly proven. There is a general deformation theory, which specializes to both subjects and which yields deep finiteness results [MS]. Since then, the dictionary has grown to encompass a guiding heuristic whereby constructions, methods, and results in one subject suggest similar ones in the other. For example, in both subjects there are common themes in the combinatorial classification theories [McM1], [Pil], the fine geometric structure of the associated fractal objects [McM3], [McM5], [SU1], [SU2], and the analysis of certain geometrically infinite systems [McM2]. A Kleinian group uniformizes a hyperbolic three-manifold, and there is now a candidate three-dimensional object associated to a rational map [LM], [KL]. Of course, essential and important differences between the two theories remain.

Other examples of conformal dynamical systems include iteration of smooth maps of the interval to itself and discrete groups of Möbius transformations acting properly discontinuously on higher-dimensional spheres. However, a theorem of Liouville [Ric] asserts that any conformal map in dimensions  $\geq 3$  is the restriction of a Möbius transformation. Thus, there is no nonlinear classical theory of iterated conformal maps in higher dimensions.

Two different generalizations of conformal dynamical systems have been studied. One of these retains the Euclidean metric structure of the underlying space and keeps some regularity of the iterates or group elements, but replaces



their conformality with uniform quasiregularity. Roughly, this means that they are differentiable almost everywhere, and they distort the roundness of balls in the tangent space by a uniformly bounded amount. In dimension two, Sullivan showed that the Measurable Riemann Mapping Theorem and an averaging process imply that each such example is obtained from a rational map or a Kleinian group by a quasiconformal deformation [Sul1]. In higher dimensions, Tukia gave an example to show that this fails [Tuk1]. The systematic study of uniformly quasiconformal groups of homeomorphisms on  $\mathbb{R}^n$  was begun by Gehring and Martin [GM]. They singled out a special class of such groups, the convergence groups, which are characterized by topological properties. The subsequent theory of such quasiconformal groups turns out to be quite rich. The study of iteration of uniformly quasiregular maps on manifolds is somewhat more recent; see e.g. [IM]. At present, examples of chaotic sets of such maps are either spheres or Cantor sets, and it is not yet clear how rich this subject will be in comparison with that of classical rational maps.

A second route to generalizing classical conformal dynamical systems is to replace the underlying Euclidean space with some other metric space, and to replace the condition of conformality with respect to a Riemannian metric with one which makes sense for metrics given as distance functions. Technically, there are many distinct such reformulations—some local, some global, some infinitesimal (quasimöbius, quasisymmetric, quasiconformal). An important source of examples with ties to many other areas of mathematics is the following. A convex compact Kleinian group acting on its limit set in the Riemann sphere generalizes to a negatively curved group (in the sense of Gromov) acting on its boundary at infinity. This boundary carries a natural topology and a natural quasisymmetry class of metrics [GdlH], [BS]. With respect to such a metric, the elements of the group act by uniformly quasimöbius maps. Negatively curved groups acting on their boundaries thus provide a wealth of examples of generalized “conformal” dynamical systems.

Tukia [Tuk3] generalized Gehring and Martin’s notion of a convergence group from spheres to compact Hausdorff spaces, and Bowditch [Bow1] then characterized negatively curved groups acting on their boundaries by purely topological conditions:

**Theorem 1.0.1 (Characterization of boundary actions)** *Let  $\Gamma$  be a group acting on a perfect metrizable compactum  $M$  by homeomorphisms. If the action on the space of triples is properly discontinuous and cocompact, then  $\Gamma$*

is hyperbolic, and there is a  $\Gamma$ -equivariant homeomorphism of  $M$  onto  $\partial\Gamma$ .

Following Bowditch [Bow2] and abusing terminology, we refer to such actions as uniform convergence groups. In addition to providing a topological characterization, the above theorem may be viewed as a uniformization-type result. Since the metric on the boundary is well-defined up to quasimetry, it follows that associated to any uniform convergence group action of  $\Gamma$  on  $M$ , there is a preferred class of metrics on  $M$  in which the dynamics is conformal in a suitable sense: the action is uniformly quasimöbius.

Sullivan referred to convex cocompact Kleinian groups and their map analogs, hyperbolic rational maps, as expanding conformal dynamical systems. Their characteristic feature is the following principle which we may refer to as the *conformal elevator*:

*Arbitrarily small balls can be blown up via the dynamics to nearly round sets of definite size with uniformly bounded distortion, and vice-versa.*

This property is also enjoyed by negatively curved groups acting on their boundaries, and is the basis for many rigidity arguments in dynamics and geometry. Recalling the dictionary, we have then the following table:

<b>Group actions</b>	<b>Iterated maps</b>
Kleinian group	rational map
convex cocompact Kleinian group	hyperbolic rational map
uniform convergence group	?

The principal goal of this work is to fill in the missing entry in the above table. To do this, we introduce *topological* and *metric coarse expanding conformal* (cxc) dynamical systems. We emphasize that topologically cxc systems may be locally non-injective, i.e. branched, on their chaotic sets. Metric cxc systems are topologically cxc by definition. Hyperbolic rational maps on their Julia sets and uniformly quasiregular maps on manifolds with good expanding properties are metric cxc. Thus, our notion includes both the classical and generalized Riemannian examples of expanding conformal dynamical systems mentioned above. As an analog of Bowditch's characterization, viewed as a uniformization result, we have the following result:

**Theorem 1.0.2 (Characterization of metric cxc actions)** *Suppose  $f : X \rightarrow X$  is a continuous map of a compact metrizable space to itself. If  $f$  is topologically cxc, then there exists a metric  $d$  on  $X$ , unique up to quasimetry, such that with respect to this metric,  $f$  is essentially metric cxc.*

(See Corollary 3.5.3). In many cases (e.g. when  $X$  is locally connected) we may drop the qualifier “essentially” from the conclusion of the above theorem. In general, we cannot. It is unclear to us whether this is a shortcoming of our methods, or reflects some key difference between group actions and iterated maps; see §3.5. The naturality of the metric  $d$  implies that quasimetry invariants of  $(X, d)$  then become topological invariants of the dynamical system. Hence, tools from the theory of analysis on metric spaces may be employed. In particular, the conformal dimension (see §3.5) becomes a numerical topological invariant, distinct from the entropy. The existence of the metric  $d$  may be viewed as a generalization of the well-known fact that given a positively expansive map of a compact set to itself, there exists a canonical Hölder class of metrics in which the dynamics is uniformly expanding.

Our class of metric cxc systems  $f : (X, d) \rightarrow (X, d)$  includes a large number of previously studied types of dynamical systems. A rational map is cxc on its Julia set with respect to the standard spherical metric if and only if it is a so-called semihyperbolic map (Theorem 4.2.3). A metric cxc map on the standard two-sphere is quasimetrically conjugate to a semihyperbolic rational map with Julia set the sphere 4.2.7. Using elementary Lie theory, we construct by hand the metric  $d$  in the case when  $X$  is a smooth manifold and  $f$  is a smooth expanding map, and show that in this metric  $f$  becomes locally a homothety (§4.5). Theorems 4.4.4 and 4.4.3 imply that uniformly quasiregular maps on Riemannian manifolds of dimension greater or equal to 3 which are metric cxc are precisely the generalized Lattès examples of Mayer [May].

Just as negatively curved groups provide a wealth of examples of non-classical “conformal” group actions, so our class of metric cxc maps provides a wealth of examples of non-classical “conformal” iterated maps as dynamical systems. The case of the two-sphere is of particular interest. Postcritically finite branched coverings of the two-sphere to itself arising from rational maps were characterized combinatorially by Thurston [DH]. Among such branched coverings, those which are expanding with respect to a suitable orbifold metric give examples of topologically cxc systems on the two-sphere.

Hence by our results, they are uniformized by a metric such that the dynamics becomes conformal. This metric, which is a distance function on the sphere, need not be quasisymmetrically equivalent to the standard one. A special class of such examples are produced from the finite subdivision rules on the sphere considered by Cannon, Floyd and Parry [CFP2], [CFKP]; cf. [Mey]. These provide another source of examples of dynamics on the sphere which are conformal with respect to non-standard metrics. Conjecturally, given a negatively curved group with two-sphere boundary, the visual metric is always quasisymmetrically equivalent to the standard one, hence (by Sullivan's averaging argument and the Measurable Riemann mapping theorem) the action is isomorphic to that of a cocompact Kleinian group acting on the two-sphere. This is Bonk and Kleiner's reformulation of Cannon's Conjecture [BK1].

In Theorem 4.2.11 below, we characterize in several ways when a topologically cxc map on the two-sphere, in its natural metric, is quasisymmetrically conjugate to a rational map. This result was our original motivation. The natural metrics associated to a topologically cxc map  $f : S^2 \rightarrow S^2$  are always linearly locally connected (Cor. 2.6.9). If  $f$  is not quasisymmetrically conjugate to a rational map, e.g. if  $f$  is postcritically finite and has a Thurston obstruction, then Bonk and Kleiner's characterization of the quasisymmetry class of the standard two-sphere [BK1] allows us to conclude indirectly that these natural metrics are never Ahlfors 2-regular. Recent results of Bonk and Meyer [BM], [Bon] suggest that in general, Thurston obstructions manifest themselves directly as metric obstructions to Ahlfors 2-regularity in a specific and natural way. Differences with the group theory emerge: we give an example of a metric cxc map on a  $Q$ -regular two-sphere of Ahlfors regular conformal dimension  $Q$  which is nonetheless not  $Q$ -Loewner. In contrast, for hyperbolic groups, Bonk and Kleiner ([BK3], Theorem 1.3) have shown that if the Ahlfors regular conformal dimension is attained, then the metric is Loewner.

As mentioned above, the dictionary is rather loose in places. From the point of view of combinatorics and finiteness principles, a postcritically finite subhyperbolic rational map  $f$  is a reasonable analog of a cocompact Kleinian group  $G$ . By Mostow rigidity,  $G$  is determined up to Möbius conjugacy by the homotopy type of the associated quotient three-manifold. This in turn is determined by the isomorphism type of  $G$ . Since  $G$  as a group is finitely presented, a finite amount of combinatorial data determines the geometry of Kleinian group  $G$ . For the analogous rational maps, Thurston [DH]

showed that they are determined up to Möbius conjugacy by their homotopy type, suitably defined. Recently, Nekrashevych [Nek] introduced tools from the theory of automaton groups that show that these homotopy types are again determined by a finite amount of group-theoretical data. In a forthcoming work [HP], we introduce a special class of metric cxc systems that enjoy similar finiteness principles. From the point of view of analytic properties, however, our results suggest that another candidate for the analog of a convex cocompact Kleinian group is a so-called semi-hyperbolic rational map, which is somewhat more general (§4.2) and which allows non-recurrent branch points with infinite orbits in the chaotic set.

Our construction of a natural metric associated to a topologically cxc system  $f$  proceeds via identifying the chaotic set  $X$  of the system as the boundary at infinity of a locally finite, negatively curved graph  $\Gamma$  with a preferred basepoint. By metrizing  $\Gamma$  suitably and using the Floyd completion to obtain the metric on the boundary, the dynamics becomes quite regular. The map  $f$  behaves somewhat like a homothety: there exists a constant  $\lambda > 1$  such that if  $f$  is injective on a ball  $B$ , then on the smaller ball  $\frac{1}{4}B$  it multiplies distances by  $\lambda$ . In particular,  $f$  is Lipschitz, and (Theorem 3.2.1 and 3.5.7)  $X$  becomes a BPI-space in the sense of David and Semmes [DaSe]. By imitating the Patterson-Sullivan construction of conformal measures [Pat] as generalized by Coornaert [Coo], we construct a natural measure  $\mu_f$  on the boundary with a perhaps remarkable coincidence of properties. The measure  $\mu_f$  is quasiconformal with constant Jacobian, is the unique measure of maximal entropy  $\log \deg(f)$ , describes the distribution both of backwards orbits and of periodic points, and satisfies Manning's formula relating Hausdorff dimension, entropy, and Lyapunov exponents (§§3.4 and 3.5). Thus, all variation in the distortion of  $f$  is ironed out to produce a metric in which the map is in some sense a piecewise homothety, much like a piecewise linear map of the interval to itself with constant absolute value of slope. In this regard, our results may be viewed as an analog of the Milnor-Thurston theorem asserting that a unimodal map with positive topological entropy is semiconjugate to a tent map whose slope is the exponential of the entropy [MT]. Our estimates generalize those of Misiurewicz-Przytycki [MP] and Gromov [Gro2].

By way of contrast, Zdunik [Zdu] shows that among rational maps, only the usual family of exceptions (critically finite maps with parabolic orbifold) has the property that the measure of maximal entropy is equivalent to the Hausdorff measure in the dimension of the Julia set. Our construction, however, yields a metric with this coincidence for any rational map which is

suitably expanding.

It turns out (Theorem 4.2.3) that  $f$  is semihyperbolic if and only if  $\Gamma$  is quasi-isometric to the convex hull of the Julia set of  $f$  in hyperbolic three-space. Lyubich and Minsky [LM] give a similar three-dimensional characterization of this family of maps using hyperbolic three-manifold laminations. Analogously, convex cocompact Kleinian groups are characterised by the property that their Cayley graphs are quasi-isometric to the convex hull of their limit sets in  $\mathbb{H}^3$ .

In summary, we suggest the following enlargement of the above dictionary:

<b>Group actions</b>	<b>Iterated maps</b>
Kleinian group	rational map
convex cocompact Kleinian group	(semi) hyperbolic rational map
uniform convergence group	topologically cxc map
uniform quasimöbius convergence group	metric cxc map
Cayley graph $\Gamma$	graph $\Gamma$
visual metric	visual metric
quasiconformal measure $\mu$	canonical measure $\mu_f$
Cannon Conjecture on groups with sphere boundary	Thurston's Theorem characterizing rational maps
Cannon's, Bonk-Kleiner's Characterization Theorems of cocompact Kleinian groups	Characterization Theorem for cxc maps on the standard $\mathbb{S}^2$

Our basic method is the following. Since we are dealing with noninvertible mappings whose chaotic sets are possibly disconnected, we imagine the repeller  $X$  embedded in a larger, nice space  $\mathfrak{X}_0$  and we suppose that  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  where  $\overline{\mathfrak{X}_1} \subset \mathfrak{X}_0$ . We require some regularity on  $f$ : it should be a finite branched covering. Our analysis proceeds as follows:

*We suppose that the repellor  $X$  is covered by a finite collection  $\mathcal{U}_0$  of open, connected subsets. We pull back this covering by iterates of  $f$  to obtain a sequence  $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \dots$  of coverings of  $X$ . We then examine the combinatorics and geometry of this sequence.*

The collection of coverings  $\{\mathcal{U}_n\}$  may be viewed as a discretization of Pansu's quasiconformal structures [Pan1]. This motivates our use of the adjective "coarse" to describe our metric dynamical systems.

**Contents.** In Chapter 2, we begin with the topological foundations needed to define topologically cxc mappings. We give the definitions of topologically and metric cxc mappings, prove metric and dynamical regularity properties of the repellor, and prove that topological conjugacies between metric cxc systems are quasisymmetric.

In Chapter 3, we construct the graph  $\Gamma$  associated to topologically cxc maps (and to more general maps as well) and discuss its geometry and the relation of its boundary with the repellor. We construct the natural measure and study its relation to equidistribution, entropy, and Hausdorff dimension. The chapter closes with those properties enjoyed specifically by metric cxc mappings.

Chapter 4 is devoted to a discussion of examples, and contains a proof of the topological characterization of semihyperbolic rational maps among cxc mappings on the two-sphere (Theorem 4.2.11).

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## Chapter 2

# Coarse expanding conformal dynamics

The following setup is quite common in the dynamics of noninvertible maps. One is given a nice, many-to-one map

$$f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$$

where  $\mathfrak{X}_0$  and  $\mathfrak{X}_1$  are nice spaces and  $\mathfrak{X}_1 \subset \mathfrak{X}_0$ . One studies the typically complicated set  $X$  of nonescaping points, i.e. points  $x \in \mathfrak{X}_1$  for which  $f^n(x) \in \mathfrak{X}_1$  for all  $n \geq 0$ . We are particularly interested in maps for which the restriction of  $f$  to  $X$  need not be locally injective. For those readers unused to noninvertible dynamics, we suggest assuming that  $\mathfrak{X}_0 = \mathfrak{X}_1 = X$  upon a first reading.

A basic method for analyzing such systems is to consider the behavior of small open connected sets of  $\mathfrak{X}_0$  under backward, instead of forward, iteration. For this reason, it is important to have some control on restrictions of iterates of the form  $f^k : \tilde{U} \rightarrow U$ , where  $U$  is a small open connected subset of  $\mathfrak{X}_0$ , and  $\tilde{U}$  is a connected component of  $f^{-k}(U)$ . Hence it is reasonable to assume that  $\mathfrak{X}_0, \mathfrak{X}_1$  are at least locally connected. The nonescaping set  $X$  itself, however, may be disconnected and non-locally connected. To rule out topological pathology in taking preimages, we impose some tameness restrictions on  $f$  by assuming that  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is a so-called *branched covering* between suitable topological spaces. When  $\mathfrak{X}_0$  is a metric space it is tempting to ask for control over inverse images of metric balls instead of connected open sets. However, this can be awkward since balls in  $\mathfrak{X}_0$  might not be connected.



We focus on those topological dynamical systems with good expanding properties. However, a map  $f : X \rightarrow X$  which is not locally injective is never positively expansive, and neither is the induced map on the natural extension. Thus, notions of expansiveness in this category need to be defined with some care.

## 2.1 Finite branched coverings

There have been many different definitions of ramified coverings and branched coverings, most of which coincide in the context of manifolds (cf. e.g. [Fox, Edm, DiSi]). We define here the notion of finite branched coverings which suits our purpose: it generalises the topological properties of rational maps of the Riemann sphere, and behaves well for their dynamical study (e.g. pull-backs of Radon measures are well-defined).

Suppose  $X, Y$  are locally compact Hausdorff spaces, and let  $f : X \rightarrow Y$  be a finite-to-one continuous map. The *degree* of  $f$  is

$$\deg(f) = \sup\{\#f^{-1}(y) : y \in Y\}.$$

For  $x \in X$ , the *local degree of  $f$  at  $x$*  is

$$\deg(f; x) = \inf_U \sup\{\#f^{-1}(\{z\}) \cap U : z \in f(U)\}$$

where  $U$  ranges over all neighborhoods of  $x$ .

**Definition 2.1.1 (finite branched covering)** *The map  $f$  is a finite branched covering (abbrev. fbc) provided  $\deg(f) < \infty$  and*

(i)

$$\sum_{x \in f^{-1}(y)} \deg(f; x) = \deg f$$

*holds for each  $y \in Y$ ;*

(ii) *for every  $x_0 \in X$  and any neighborhood  $W$  of  $x_0$  in  $X$ , there is a smaller neighborhood  $U \subset W$  of  $x_0$  in  $X$  such that*

$$\sum_{x \in U, f(x)=y} \deg(f; x) = \deg(f; x_0)$$

*for all  $y \in f(U)$ .*

The composition of fbc's is an fbc, and the degrees of fbc's multiply under compositions. In particular, local degrees of fbc's multiply under compositions.

Given an fbc  $f : X \rightarrow Y$ , a point  $y \in Y$  is a *principal value* if  $\#f^{-1}(y) = \deg(f)$ . Condition (ii) implies that if  $x_n \rightarrow x_0$ , then  $\deg(f; x_n) \leq \deg(f; x_0)$ . It follows that the *branch set*  $B_f = \{x \in X : \deg(f; x) > 1\}$  is closed. The set of *branch values* is defined as  $V_f = f(B_f)$ . Thus  $Y - V_f$  is the set of principal values.

**Lemma 2.1.2** *Let  $X, Y$  be Hausdorff locally compact topological spaces. An fbc  $f : X \rightarrow Y$  of degree  $d$  is open, onto and proper: the inverse image of a compact subset is compact and the image of an open set is open. Furthermore,  $B_f$  and  $V_f$  are nowhere dense.*

Since the spaces involved are not assumed to be metrizable, we are led to use filters instead of sequences in the proof [Bou].

**Proof:** The map is onto by definition.

Since points have finitely many preimages under  $f$  and  $X$  is Hausdorff and locally compact, for any  $x \in X$ , there is some relatively compact neighborhood  $N(x)$  of  $x$  such that  $\overline{N(x)} \cap f^{-1}\{f(x)\} = \{x\}$ . If  $x \notin B_f$ , then we may assume that  $\overline{N(x)} \cap B_f = \emptyset$ .

**Claim.** *For any  $x \in X$ , a relatively compact neighborhood  $V(x) \subset Y$  of  $f(x)$  exist such that, if  $\mathcal{F}$  denotes the set of neighborhoods of  $f(x)$  contained in  $V(x)$ , then  $f^{-1}(\mathcal{F}) \cap N(x)$  is a filterbase converging to  $x$ . Furthermore, for any  $V \in \mathcal{F}$ , the restriction  $f : f^{-1}(V(x)) \cap N(x) \rightarrow V$  is proper.*

**PROOF OF CLAIM.** Let  $y \in Y$  and let us consider the neighborhoods  $(N(x))_{x \in f^{-1}(\{y\})}$  defined above. Fix  $x \in f^{-1}(\{y\})$  and let  $\mathcal{F}_1$  be the set of neighborhoods of  $y$ . Since  $x$  is accumulated by  $f^{-1}(\mathcal{F}_1) \cap N(x)$ , it follows that if  $f^{-1}(\mathcal{F}_1) \cap N(x)$  is not convergent to  $x$ , then there is another accumulation point  $x'$  of  $f^{-1}(\mathcal{F}_1) \cap N(x)$  in  $\overline{N(x)}$  since  $N(x)$  is relatively compact. By continuity of  $f$ , this implies that  $f(x') = y$ , so that  $x' = x$  since  $f^{-1}(\{y\}) \cap \overline{N(x)} = \{x\}$ .

This implies that  $f^{-1}(\mathcal{F}_1) \cap N(x)$  is a filterbase converging to  $x$ . Therefore there is some relatively compact neighborhood  $V(x)$  of  $y$  such that  $f^{-1}(V(x)) \cap N(x) \subset N'(x)$  where  $N'(x) \subset N(x)$  is a compact neighborhood of  $x$ . It follows that  $f^{-1}(\mathcal{F}) \cap N(x)$  is also a filterbase converging to  $x$  where  $\mathcal{F}$  denotes the set of neighborhoods of  $y$  contained in  $V(x)$ .

Let  $K \subset V(x)$  be compact and set  $L = f^{-1}(K) \cap N(x)$ . Then  $L \subset N'(x)$  so that  $L = f^{-1}(K) \cap N'(x)$  is compact.

This ends the proof of the claim.

Let  $\Omega \subset X$  be an open set, and let us consider  $x \in \Omega$  and  $y = f(x)$ . We choose a neighborhood  $N'(x) \subset N(x) \cap \Omega$  as above. It follows from the claim that a neighborhood  $V'(y) \subset V(y)$  exists such that  $f^{-1}(V'(y)) \cap N(x) \subset N'(x)$ . So, for any  $y' \in V'(y)$ , by (ii)

$$\sum_{x' \in f^{-1}(\{y'\}) \cap N(x)} \deg(f; x') = \deg(f; x) \geq 1.$$

Hence,  $y' = f(x')$  for some  $x' \in N(x) \cap \Omega$ . Thus  $V'(y) \subset f(\Omega)$ . This establishes that  $f$  is open.

Let us fix  $y \in Y$  and let us consider  $y' \in V(y)$ . Then

$$\begin{aligned} d = \sum_{f(x)=y} \deg(f; x) &= \sum_{f(x)=y} \left( \sum_{x' \in f^{-1}(\{y'\}) \cap N(x)} \deg(f; x') \right) \\ &= \sum_{x' \in f^{-1}(\{y'\}) \cap (\cup_{f(x)=y} N(x))} \deg(f; x'). \end{aligned}$$

This implies that  $f^{-1}(\{y'\}) \subset \cup_{f(x)=y} N(x)$ . Using the relative compactness and the continuity of  $f$ , it follows that the filterbase  $f^{-1}(\mathcal{F})$  is equivalent to the set of neighborhoods of  $f^{-1}(\{y\})$ , where  $\mathcal{F}$  is any filterbase converging to  $y$ .

Let  $K \subset Y$  be a compact set and set  $L = f^{-1}(K)$ . Let  $\mathcal{F}$  be a filterbase in  $L$ . Since  $f(L)$  is compact, there is some accumulation point  $y$  in  $K$  of  $f(\mathcal{F})$ . We claim that at least one preimage of  $y$  is accumulated by  $\mathcal{F}$ . If it was not the case, then, for any  $x \in f^{-1}(\{y\})$ , there would be some  $F_x \in \mathcal{F}$  with  $x \notin \overline{F_x}$ . Since the fibers are finite,  $\cap_{f(x)=y} F_x \in \mathcal{F}$  and  $f^{-1}(\{y\}) \cap (\cap_{f(x)=y} \overline{F_x}) = \emptyset$ .

Therefore, there is some neighborhood  $V'(y) \subset V(y)$  such that  $f^{-1}(V'(y)) \cap (\cap_{f(x)=y} \overline{F_x}) = \emptyset$ . Hence  $V'(y) \cap \overline{f(\cap_{f(x)=y} F_x)} = \emptyset$ , which contradicts that  $y$  was an accumulation point. Therefore  $f$  is proper.

The set  $V_f$  cannot have interior since  $f$  has bounded multiplicity. Indeed, if  $V_f$  had interior, we could construct a decreasing sequence of open sets  $W(y_n) \subset V(y_n) \cap V(y_{n-1}) \subset V_f$ , so we would have  $p(y_{n+1}) \geq p(y_n) + 1 \geq n + 1$ , where  $p : Y \rightarrow \mathbb{N} \setminus \{0\}$  denotes the map that counts the number of preimages of points in  $Y$ .

Therefore,  $B_f$  cannot have interior either since  $f$  is an open mapping. ■

Many arguments are done using pull-backs of sets and restricting to connected components. It is therefore necessary to work with fbc's defined on sets  $X$  and  $Y$  enjoying more properties. When  $X$  and  $Y$ , in addition to being locally compact and Hausdorff, are assumed locally connected, the following fundamental facts are known (cf. [Edm]).

- If  $V \subset Y$  is open and connected, and  $U \subset X$  is a connected component of  $f^{-1}(V)$ , then  $f|U : U \rightarrow V$  is an fbc as well.
- If  $y \in Y$ , and  $f^{-1}(y) = \{x_1, x_2, \dots, x_k\}$ , then there exist arbitrarily small connected open neighborhoods  $V$  of  $y$  such that

$$f^{-1}(V) = U_1 \sqcup U_2 \sqcup \dots \sqcup U_k$$

is a disjoint union of connected open neighborhoods  $U_i$  of  $x_i$  such that  $f|U_i : U_i \rightarrow V$  is an fbc of degree  $\deg(f; x_i)$ ,  $i = 1, 2, \dots, k$ .

- if  $f(x) = y$ ,  $\{V_n\}$  is sequence of nested open connected sets with  $\bigcap_n V_n = \{y\}$ , and if  $\tilde{V}_n$  is the component of  $f^{-n}(V_n)$  containing  $x$ , then  $\bigcap_n \tilde{V}_n = \{x\}$ .

## 2.2 Topological cxc systems

In this section, we state the topological axioms underlying the definition of a cxc system.

Let  $\mathfrak{X}_0, \mathfrak{X}_1$  be Hausdorff locally compact, locally connected topological spaces, each with finitely many connected components. We further assume that  $\mathfrak{X}_1$  is an open subset of  $\mathfrak{X}_0$  and that  $\overline{\mathfrak{X}_1}$  is compact in  $\mathfrak{X}_0$ . Note that this latter condition implies that if  $\mathfrak{X}_0 = \mathfrak{X}_1$ , then  $\mathfrak{X}_0$  is compact.

Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  be a finite branched covering map of degree  $d \geq 2$ , and for  $n \geq 0$  put

$$\mathfrak{X}_{n+1} = f^{-1}(\mathfrak{X}_n).$$

Then  $f : \mathfrak{X}_{n+1} \rightarrow \mathfrak{X}_n$  is again an fbc of degree  $d$  and since  $f$  is proper,  $\overline{\mathfrak{X}_{n+1}}$  is compact in  $\mathfrak{X}_n$ , hence in  $\mathfrak{X}_0$ .

The *nonescaping set*, or *repellor*, of  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is

$$X = \{x \in \mathfrak{X}_1 \mid f^n(x) \in \mathfrak{X}_1 \ \forall n > 0\} = \bigcap_n \overline{\mathfrak{X}_n}.$$

We make the technical assumption that the restriction  $f|_X : X \rightarrow X$  is also an fbc of degree equal to  $d$ . This implies that  $\#X \geq 2$ . Also,  $X$  is *totally invariant*:  $f^{-1}(X) = \overline{X} = f(X)$ . The definition of the nonescaping set and the compactness of  $\overline{\mathfrak{X}_1}$  implies that given any open set  $Y$  containing  $X$ ,  $\mathfrak{X}_n \subset Y$  for all  $n$  sufficiently large.

The following is the essential ingredient in this work. Let  $\mathcal{U}_0$  be a finite cover of  $X$  by open, connected subsets of  $\mathfrak{X}_1$  whose intersection with  $X$  is nonempty. A *preimage* of a connected set  $A$  is defined as a connected component of  $f^{-1}(A)$ . Inductively, set  $\mathcal{U}_{n+1}$  to be the open cover whose elements  $\tilde{U}$  are preimages of elements of  $\mathcal{U}_n$ . We denote by  $\mathbf{U} = \cup_{n \geq 0} \mathcal{U}_n$  the collection of all such open sets thus obtained.

We say  $f : (\mathfrak{X}_1, X) \rightarrow (\mathfrak{X}_0, X)$  is *topologically coarse expanding conformal with repellor  $X$*  provided there exists a finite covering  $\mathcal{U}_0$  as above, such that the following axioms hold.

1. **[Expansion]** The mesh of the coverings  $\mathcal{U}_n$  tends to zero as  $n \rightarrow \infty$ . That is, for any finite open cover  $\mathcal{Y}$  of  $X$  by open sets of  $\mathfrak{X}_0$ , there exists  $N$  such that for all  $n \geq N$  and all  $U \in \mathcal{U}_n$ , there exists  $Y \in \mathcal{Y}$  with  $U \subset Y$ .
2. **[Irreducibility]** The map  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is *locally eventually onto near  $X$* : for any  $x \in X$  and any neighborhood  $W$  of  $x$  in  $\mathfrak{X}_0$ , there is some  $n$  with  $f^n(W) \supset X$ .
3. **[Degree]** The set of degrees of maps of the form  $f^k|_{\tilde{U}} : \tilde{U} \rightarrow U$ , where  $U \in \mathcal{U}_n$ ,  $\tilde{U} \in \mathcal{U}_{n+k}$ , and  $n$  and  $k$  are arbitrary, has a finite maximum, denoted  $p$ .

Axiom [Expansion] is equivalent to saying that, when  $\mathfrak{X}_0$  is a metric space, the diameters of the elements of  $\mathcal{U}_n$  tend to zero as  $n \rightarrow \infty$ . Axiom [Irreducibility] implies that  $f : X \rightarrow X$  is topologically exact; we give a useful, alternative characterization below.

These axioms are reminiscent of the following properties of a group  $G$  acting on a compact Hausdorff space  $X$ ; see [Bow2]. [Irreducibility] is analogous to  $G$  acting minimally on  $X$ . [Expansion] is analogous to  $G$  acting

properly discontinuously on triples, i.e. that  $G$  is a *convergence group*. Axiom [Degree] is analogous to  $G$  acting cocompactly on triples; we will see later that this condition implies good regularity properties of metrics and measures associated to cxc systems.

Together, a topologically cxc system we view as the analog, for iterated maps, of a *uniform convergence group*.

The elements of  $\mathcal{U}_0$  will be referred to as *level zero good open sets*. While as subsets of  $\mathfrak{X}_0$  they are assumed connected, their intersections with the repeller  $X$  need not be. Also, the elements of  $\mathbf{U}$ , while connected, might nonetheless be quite complicated topologically—in particular they need not be contractible.

If  $\mathfrak{X}_0 = \mathfrak{X}_1 = X$ , then the elements of  $\mathbf{U}$  are connected subsets of  $X$ .

## 2.3 Examples of topological cxc maps

### 2.3.1 Rational maps

Let  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  be a rational function of degree  $d \geq 2$  for which the critical points either converge under iteration to attracting cycles, or land on a repelling periodic cycle (such a function is called *subhyperbolic*). For such maps, every point on the sphere belongs either to the *Fatou set* and converges to an attracting cycle, or belongs to the *Julia set*  $J_f$ . One may find a small closed neighborhood  $V_0$  of the attracting periodic cycles such that  $f(V_0) \subset \text{int}(V_0)$ . Set  $\mathfrak{X}_0 = \widehat{\mathbb{C}} - V_0$  and  $\mathfrak{X}_1 = f^{-1}(\mathfrak{X}_0)$ . Then  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is an fbc of degree  $d$ , the repeller  $X = J_f$ , and  $f|X : X \rightarrow X$  is an fbc of degree  $d$ .

Let  $\mathcal{U}_0$  be a finite cover of  $J_f$  by open spherical balls contained in  $\mathfrak{X}_1$ , chosen so small that each ball contains at most one forward iterated image of a critical point. The absence of periodic critical points in  $J_f$  easily implies that the local degrees of iterates of  $f$  are uniformly bounded at such points, and so Axiom [Degree] holds. Since  $J_f$  can be characterized as the locus of points on which the iterates fail to be locally a normal family, Montel's theorem implies that Axiom [Irreducibility] holds. Finally,  $f$  is uniformly expanding near  $X$  with respect to a suitable orbifold metric, and Axiom [Expansion] holds; see [SL], Thm. 1.1(b).

### 2.3.2 Smooth expanding partial self-covers

Let  $\mathfrak{X}_0$  be a connected complete Riemannian manifold,  $\mathfrak{X}_1 \subset \mathfrak{X}_0$  an open submanifold with finitely many components which is compactly contained in  $\mathfrak{X}_0$ . Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  be a  $C^1$  covering map which is *expanding*, i.e. there are constants  $c > 0, \lambda > 1$  such that whenever  $f^n(x)$  is defined,  $\|Df_x^n(v)\| > c\lambda^n\|v\|$ . If  $X$  denotes the set of nonescaping points, then  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is topologically exc with repeller  $X$ —we may take  $\mathcal{U}_0$  to be a finite collection of small balls centered at points of  $X$ .

One may argue as follows. Since  $X$  is compact, there is a uniform lower bound  $r$  on the injectivity radius of  $\mathfrak{X}_1$  at points  $x \in X$ . Thus, for each  $x \in X$ , the ball  $B(x, r)$  is homeomorphic to an open Euclidean ball; in particular, it is contractible. Let  $\mathcal{U}_0$  be a finite open cover of  $X$  by such balls. Since  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is a covering map, all iterated preimages  $\tilde{U}$  of elements  $U \in \mathcal{U}_0$  map homeomorphically onto their images, so the [Degree] Axiom holds with  $p = 1$ . Since  $f$  is expanding, the diameters of the elements of  $\mathcal{U}_n$  tend to zero exponentially in  $n$ , so the [Expansion] Axiom holds. The restriction  $f|_X : X \rightarrow X$  is clearly an f.b.c. To verify the [Irreducibility] Axiom, we use an alternative characterization given as Proposition 2.4.1(2) below. Suppose  $x \in X$  and  $x_0 \in \mathfrak{X}_0$ . Since  $X \cup \{x_0\}$  is compact, there exists  $L > 0$  such that for all  $n$ , there exists a path  $\gamma_n$  of length at most  $L$  joining  $f^n(x)$  and  $x_0$ . Let  $\tilde{\gamma}_n$  denote the lift of  $\gamma_n$  based at  $x$ . The other endpoint  $\tilde{x}_n$  of  $\tilde{\gamma}_n$  lies in  $f^{-n}(x_0)$ . By expansion, the length of  $\tilde{\gamma}_n$  tends to zero. Hence  $\tilde{x}_n \rightarrow x$  and so  $x \in A(x_0)$ .

Following [Nek], we will refer to the topologically exc system  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  a *smooth expanding partial self-covering*. A common special case is when  $\mathfrak{X}_1, \mathfrak{X}_0$  are connected and the homomorphism  $\iota_* : \pi_1(\mathfrak{X}_1) \rightarrow \pi_1(\mathfrak{X}_0)$  induced by the inclusion map  $\iota : \mathfrak{X}_1 \hookrightarrow \mathfrak{X}_0$  induces a surjection on the fundamental groups. In this case, the preimages of  $\mathfrak{X}_0$  under  $f^{-n}$  are all connected, and the repeller  $X$  itself is connected.

One can generalize the above example so as to allow branching, by e.g. working in the category of orbifolds; see [Nek].

## 2.4 Elementary properties

**Conjugacy.** Suppose  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  and  $g : \mathfrak{Y}_1 \rightarrow \mathfrak{Y}_0$  are f.b.c.'s with repellers  $X, Y$  as in the definition of topologically exc. A homeomorphism

$h : \mathfrak{X}_0 \rightarrow \mathfrak{Y}_0$  is called a *conjugacy* if it makes the diagram

$$\begin{array}{ccc} (\mathfrak{X}_1, X) & \xrightarrow{h} & (\mathfrak{Y}_1, Y) \\ f \downarrow & & \downarrow g \\ (\mathfrak{X}_0, X) & \xrightarrow{h} & (\mathfrak{Y}_0, Y) \end{array}$$

commute. (Strictly speaking, we should require only that  $h$  is defined near  $X$ ; however, we will not need this more general point of view here.)

It is clear that the property of being topologically cxc is closed under conjugation: if  $\mathcal{U}_0$  is a set of good open sets at level zero for  $f$ , then  $\mathcal{V}_0 = \{V = h(U) \mid U \in \mathcal{U}_0\}$  is a set of good open sets at level zero for  $g$ .

Suppose  $\mathfrak{X}_1, \mathfrak{X}_0$  are Hausdorff, locally compact, locally connected topological spaces, each with finitely many connected components,  $\mathfrak{X}_1 \subset \mathfrak{X}_0$  is open, and  $\overline{\mathfrak{X}_1} \subset \mathfrak{X}_0$ .

The proofs of the following assertions are straightforward consequences of the definitions.

**Proposition 2.4.1** *Suppose  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is an fbc of degree  $d \geq 2$  with nonescaping set  $X$  and let  $\mathcal{U}_0$  be a finite open cover of  $X$ .*

1. *The condition that  $f|_X : X \rightarrow X$  is an fbc of degree  $d$  implies that the set  $V_f \cap X$  is nowhere dense in  $X$ .*
2. *Axiom [Expansion] implies*
  - (a)  *$\mathbf{U}$  is a basis for the subspace topology on  $X$ . In particular, if  $U \cap X$  is connected for all  $U \in \mathbf{U}$ , then  $X$  is locally connected.*
  - (b) *For distinct  $x, y \in X$ , there is an  $N$  such that for all  $n > N$ , and all  $U \in \mathcal{U}_n$ ,  $\{x, y\} \not\subset U$ .*
  - (c) *There exists  $N_0$  such that for all  $U'_1, U'_2 \in \mathcal{U}_{N_0}$ ,  $U'_1 \cap U'_2 \neq \emptyset \Rightarrow \exists U \in \mathcal{U}_0$  with  $U'_1 \cup U'_2 \subset U$ .*
  - (d) *Periodic points are dense in  $X$ .*
3. *Axiom [Irreducibility]*
  - (a) *holds if and only if for each  $x_0 \in \mathfrak{X}_0$ , the set  $A(x_0)$  of limit points of  $\cup_{n \geq 0} f^{-n}(x_0)$  equals the nonescaping set  $X$ .*



- (b) implies that either  $X = \mathfrak{X}_0 = \mathfrak{X}_1$ , or  $X$  is nowhere dense in  $\mathfrak{X}_0$ .
- (c) together with  $f|_X : X \rightarrow X$  is an fbc of degree  $d$ , implies that  $X$  is perfect, i.e. contains no isolated points.

4. The class of topologically cxc systems is closed under taking Cartesian products.

In the remainder of this section, we assume  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is topologically cxc with repeller  $X$  and level zero good open sets  $\mathcal{U}_0$ .

To set up the next statement, given  $U \in \mathcal{U}_n$  mapping to  $U \in \mathcal{U}_0$  under  $f^n$ , denote by  $d(U) = \deg(f^n|_U)$  if  $n \geq 1$  and  $d(U) = 1$  if  $n = 0$ .

**Proposition 2.4.2 (Repellers are fractal)** *For every  $x \in X$ , every neighborhood  $W$  of  $x$ , every  $n_0 \in \mathbb{N}$ , and every  $U \in \mathcal{U}_{n_0}$ , there exists a preimage  $\tilde{U} \subset f^{-k}(U)$  with  $\tilde{U} \subset W$  and  $\deg(f^k : \tilde{U} \rightarrow U) \leq \frac{p}{d(U)}$  where  $p$  is the maximal degree obtained in the [Degree] Axiom.*

**Proof:** Let  $\mathcal{Y}$  be an open cover of  $X$  with the property that (i)  $W \in \mathcal{Y}$  and (ii) there exists a neighborhood  $W' \subset W$  of  $x$  such that for all  $Y \neq W$  in  $\mathcal{Y}$ ,  $Y \cap W' = \emptyset$ . Axiom [Expansion] then implies that there exists  $n_1 \in \mathbb{N}$  such that for all  $n \geq n_1$ , any element of  $\mathcal{U}_{n_1+n_0}$  containing  $x$  is contained in  $W$ . Axiom [Irreducibility] implies that there exists  $n_2$  such that  $f^n(W') = X \supset U$  for all  $n \geq n_2$ . Hence for  $k = \max\{n_1, n_2\}$ , there is a preimage  $\tilde{U}$  of  $U$  under  $f^{-k}$  contained in  $W$ . The assertion regarding degrees follows immediately from the multiplicativity of degrees under compositions. ■

**Post-branch set.** The *post-branch set* is defined by

$$P_f = X \cap \overline{\bigcup_{n>0} V_{f^n}}.$$

**Proposition 2.4.3** 1. A point  $x \in X$  belongs to  $X - P_f$  if there exists  $U \in \mathcal{U}$  such that all preimages of  $U$  under iterates of  $f$  map by degree one onto  $U$ .

2. The post-branch set is a possibly empty, closed, forward-invariant, nowhere dense subset of  $X$ .

Without further finiteness hypotheses on the local topology of  $\mathfrak{X}_0$ , we do not know if the converse of (1) holds, i.e. if every point in the complement of the post-branch set has a neighborhood over which all preimages under all iterates map by degree one, as is the case for e.g. rational maps.

**Proof:** **1.** If such a  $U$  exists, then  $U \cap V_{f^n} = \emptyset$  for all  $n$  and so  $x \notin P_f$ .

**2.** All but the last assertion are clear. To show  $P_f$  is nowhere dense, let  $x \in X$  and let  $W$  be any neighborhood of  $x$  in  $\mathfrak{X}_0$ . Let  $\tilde{U} \subset W$  be the element of  $\mathbf{U}$  given by Proposition 2.4.2 applied with a  $U$  chosen so that  $d(U) = p$ . Then all further preimages of  $\tilde{U}$  map by degree one and so  $\tilde{U} \cap V_{f^n} = \emptyset$  for all  $n$ . Hence  $\tilde{U} \cap P_f = \emptyset$ . Finally, since  $\tilde{U} \cap X \neq \emptyset$  we conclude that  $W \cap (X - P_f) \neq \emptyset$  and so  $P_f$  is nowhere dense in  $X$ . ■

## 2.5 Metric cxc systems

In this section, we state the definition of metric cxc systems; we will henceforth drop the adjective, metric.

**Roundness.** Let  $Z$  be a metric space and let  $A$  be a bounded, proper subset of  $Z$  with nonempty interior. Given  $a \in \text{int}(A)$ , let

$$L(A, a) = \sup\{|a - b| : b \in A\}$$

and

$$l(A, a) = \sup\{r : r \leq L(A, a) \text{ and } B(a, r) \subset A\}$$

denote, respectively, the *outradius* and *inradius* of  $A$  about  $a$ . While the outradius is intrinsic, the inradius depends on how  $A$  sits in  $Z$ . The condition  $r \leq L(A, a)$  is necessary to guarantee that the outradius is at least the inradius. The *roundness of  $A$  about  $a$*  is defined as

$$\text{Round}(A, a) = L(A, a)/l(A, a) \in [1, \infty).$$

One says  $A$  is  *$K$ -almost-round* if  $\text{Round}(A, a) \leq K$  for some  $a \in A$ , and this implies that for some  $s > 0$ ,

$$B(a, s) \subset A \subset B(a, Ks).$$

**Metric cxc systems.** A key feature of many conformal dynamical systems is the fact that small balls can be blown up using the dynamics to sets of definite size which are uniformly  $K$ -almost-round and such that ratios of diameters are distorted by controlled amounts. Below, we abstract these properties so as to apply in general metric space settings.

Suppose we are given a topological cxc system  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  with level zero good neighborhoods  $\mathcal{U}_0$ , and that  $\mathfrak{X}_0$  is now endowed with a metric compatible with its topology. The resulting metric dynamical system equipped with the covering  $\mathcal{U}_0$  is called *coarse expanding conformal*, abbreviated cxc, provided there exist

- continuous, increasing embeddings  $\rho_{\pm} : [1, \infty) \rightarrow [1, \infty)$ , the *forward and backward roundness distortion functions*, and
- increasing homeomorphisms  $\delta_{\pm} : [0, 1] \rightarrow [0, 1]$ , the *forward and backward relative diameter distortion functions*

satisfying the following axioms:

4. **[Roundness distortion]**  $(\forall n, k)$  and for all

$$U \in \mathcal{U}_n, \tilde{U} \in \mathcal{U}_{n+k}, \tilde{y} \in \tilde{U}, y \in U$$

if

$$f^{\circ k}(\tilde{U}) = U, f^{\circ k}(\tilde{y}) = y$$

then the *backward roundness bound*

$$\text{Round}(\tilde{U}, \tilde{y}) < \rho_{-}(\text{Round}(U, y)) \tag{2.1}$$

and the *forward roundness bound*

$$\text{Round}(U, y) < \rho_{+}(\text{Round}(\tilde{U}, \tilde{y})). \tag{2.2}$$

hold.

In other words: for a given element of  $\mathbf{U}$ , iterates of  $f$  both forward and backward distorts its roundness by an amount independent of the iterate.

5. **[Diameter distortion]** ( $\forall n_0, n_1, k$ ) and for all

$$U \in \mathcal{U}_{n_0}, U' \in \mathcal{U}_{n_1}, \tilde{U} \in \mathcal{U}_{n_0+k}, \tilde{U}' \in \mathcal{U}_{n_1+k}, \tilde{U}' \subset \tilde{U}, U' \subset U$$

if

$$f^k(\tilde{U}) = U, f^k(\tilde{U}') = U'$$

then

$$\frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} < \delta_- \left( \frac{\text{diam}U'}{\text{diam}U} \right)$$

and

$$\frac{\text{diam}U'}{\text{diam}U} < \delta_+ \left( \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} \right)$$

In other words: given two nested elements of  $\mathbf{U}$ , iterates of  $f$  both forward and backward distort their relative sizes by an amount independent of the iterate.

As a consequence, one has then also the *backward upper and lower relative diameter bounds*:

$$\delta_+^{-1} \left( \frac{\text{diam}U'}{\text{diam}U} \right) < \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} < \delta_- \left( \frac{\text{diam}U'}{\text{diam}U} \right) \quad (2.3)$$

and the *forward upper and lower relative diameter bounds*:

$$\delta_-^{-1} \left( \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} \right) < \frac{\text{diam}U'}{\text{diam}U} < \delta_+ \left( \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} \right). \quad (2.4)$$

The [Expansion] Axiom implies that the maximum diameters of the elements of  $\mathcal{U}_n$  tend to zero uniformly in  $n$ . Since  $\mathcal{U}_0$  is assumed finite, each covering  $\mathcal{U}_n$  is finite, so for each  $n$  there is a minimum diameter of an element of  $\mathcal{U}_n$ . Since  $X$  is perfect and, by assumption, each  $U \in \mathbf{U}$  contains a point of  $X$ , each  $U$  contains many points of  $X$  and so has positive diameter. Hence there exist decreasing positive sequences  $c_n, d_n \rightarrow 0$  such that the *diameter bounds* hold:

$$0 < c_n \leq \inf_{U \in \mathcal{U}_n} \text{diam}U \leq \sup_{U \in \mathcal{U}_n} \text{diam}U \leq d_n. \quad (2.5)$$

## 2.6 Metric regularity of cxc systems

Suppose now  $\mathfrak{X}_0, \mathfrak{X}_1$  are metric spaces. Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  be an f.b.c. as in the previous section with repeller  $X$  and level zero good neighborhoods  $\mathcal{U}_0$ . Throughout this subsection, we assume that the topological axiom [Expansion] and the metric axioms [Roundness distortion] and [Relative diameter distortion] are satisfied. We assume neither axiom [Irreducibility] nor [Degree].

In this section, we derive metric regularity properties of the elements of the coverings  $\mathcal{U}_n$  and the repeller  $X$ .

**A word regarding notation:** In this and the following section,  $U$  will always denote an element of  $\mathbf{U} = \cup_n \mathcal{U}_n$ . Generally,  $\widetilde{(\cdot)}$  denotes an inverse image of  $(\cdot)$  under some iterate of  $f$ . Often, but not always,  $U'$  denotes an element of  $\mathbf{U}$  which is contained in  $U$ . Many of the statements of the propositions below make reference to an element  $U$  of  $\mathbf{U}$ . The typical proof consists of renaming  $U$  as  $\widetilde{U}$ , mapping  $\widetilde{U}$  forward via some iterate to an element  $U$  of definite size, making estimates, and then transporting these estimates to  $\widetilde{U}$  via the distortion functions.

We first resolve a technicality.

**Proposition 2.6.1** *Let  $D_0$  denote the minimum diameter of a connected component of  $\mathfrak{X}_0$ . Then for any ball  $B(a, r)$  in  $\mathfrak{X}_0$  where  $r \leq D_0/2$ , we have  $\text{diam}B(a, r) \geq r$ .*

**Proof:** Fix  $\epsilon > 0$ , and let  $C$  denote the component of  $\mathfrak{X}_0$  containing  $a$ . Pick  $p, q \in C$  with  $|p - q| > D - \epsilon$ . Then at least one of  $|a - p|, |a - q|$  is at least  $(D - \epsilon)/2$ , say  $|a - p|$ . Since  $C$  is connected, the function  $y \mapsto |a - y|$  has an image which contains  $[0, (D - \epsilon)/2]$ . Thus for any  $s \leq (D - \epsilon)/2$ , there exists  $y \in C$  with  $|a - y| = s$ . Letting  $\epsilon \rightarrow 0$  proves that  $B(a, r)$  has diameter at least  $r$ . ■

*When dealing with balls below, we shall always assume that  $r < D_0/2$ .*

**Lebesgue number.** Let  $\mathcal{U}$  be a finite covering of a metric compact space  $X$  by open sets. The *Lebesgue number*  $\delta$  of the covering is the supremum over all radii  $r$  such that, for any point  $x \in X$ , there is some element  $U \in \mathcal{U}$  which contains  $B(x, r)$ . Since the covering is finite,  $\delta$  is positive.

**Proposition 2.6.2 (Uniform roundness)** *There exists  $K > 1$  such that*

1.  $(\forall \tilde{x} \in X)(\forall n)(\exists U \in \mathcal{U}_n)$  such that  $U$  is  $K$ -almost round with respect to  $\tilde{x}$ , i.e.  $(\exists r > 0)$

$$B(\tilde{x}, r) \subset U \subset B(\tilde{x}, Kr).$$

2.  $(\forall n)(\forall U \in \mathcal{U}_n)(\exists \tilde{x} \in X)$  such that  $\text{Round}(U, \tilde{x}) < K$ .

**Proof: 1.** Denote the set we are looking for by  $\tilde{U}$  instead of  $U$ . Let  $\delta$  be the Lebesgue number of the covering  $\mathcal{U}_0$  and  $\Delta = \sup_{U \in \mathcal{U}_0} \text{diam}U$ . Then given any  $x \in X$ , there exists  $U \in \mathcal{U}_0$  such that

$$B(x, \delta) \subset U \subset B(x, \Delta) \Rightarrow \text{Round}(U, x) < K_1 := \frac{\Delta}{\delta}.$$

Now let  $\tilde{x} \in X$  and  $n \in \mathbb{N}$  be arbitrary. Set  $x = f^n(\tilde{x})$  and let  $U \in \mathcal{U}_0$  be the element constructed as in the previous paragraph. Let  $\tilde{U} \in \mathcal{U}_n$  be the component of  $f^{-n}(U)$  containing  $\tilde{x}$ . By the backward roundness bound (2.1),

$$\text{Round}(\tilde{U}, \tilde{x}) < \rho_-(K_1).$$

**2.** Denote the given element of  $\mathcal{U}_n$  by  $\tilde{U}$  instead of  $U$ . For each  $U \in \mathcal{U}_0$ , choose  $x_U \in U$  arbitrarily. Let

$$K_2 = \max_{U \in \mathcal{U}_0} \text{Round}(U, x_U).$$

Given  $\tilde{U} \in \mathcal{U}_n$  arbitrary, let  $U = f^n(\tilde{U})$ , and let  $\tilde{x}_U \in f^{-n}(x_U) \cap \tilde{U}$ . By the backward roundness bound,

$$\text{Round}(\tilde{U}, \tilde{x}_U) < \rho_-(K_2).$$

Thus the conclusions of the lemma are satisfied with

$$K = \max \{ \rho_-(K_1), \rho_-(K_2) \}.$$

■

In the lemma below, let  $K$  denote the constant in Proposition 2.6.2 and  $c_n$  the constants giving the lower bound on the diameters of the elements of  $\mathcal{U}_n$  (2.5).

**Proposition 2.6.3 (Lebesgue numbers)**  $(\forall n)(\forall x \in X)(\forall r < \frac{c_n}{2K})$  there exists  $U \in \mathcal{U}_n$  and  $s > r$  such that

$$B(x, r) \subset B(x, s) \subset U \subset B(x, Ks).$$

In particular, the Lebesgue number of the covering  $\mathcal{U}_n$  is at least  $\delta_n = \frac{c_n}{2K}$ .

**Proof:** Given  $n$  and  $x$ , by Proposition 2.6.2 there is  $s > 0$  and  $U \in \mathcal{U}_n$  with

$$B(x, s) \subset U \subset B(x, Ks).$$

Thus  $c_n < \text{diam}U < 2Ks$  so that  $\frac{c_n}{2K} < s$ , whence  $r < s$ . ■

The next statement says that two elements of covers which intersect over  $X$  have roughly the same diameter as soon as their levels are close.

**Proposition 2.6.4 (Local comparability)** *There exists a constant  $C > 1$  such that  $(\forall x \in X)(\forall n)(\forall U \in \mathcal{U}_n)(\forall U' \in \mathcal{U}_{n+1})$  we have: if  $U \cap U' \cap X \neq \emptyset$  then*

$$\frac{1}{C} < \frac{\text{diam}U'}{\text{diam}U} < C.$$

*That is, two elements of  $\mathbf{U}$  at consecutive levels which intersect at a point of  $X$  are nearly the same size.*

**Proof:** By Axiom [Expansion] there exists  $n_0 \in \mathbb{N}$  such that  $2(d_{n_0} + d_{n_0+1})$  is less than the Lebesgue number of the covering  $\mathcal{U}_0$ . Thus there exist  $r > 0$  and  $n_0$  such that whenever  $U \in \mathcal{U}_{n_0}$  and  $U' \in \mathcal{U}_{n_0+1}$  contain a common point  $x \in X$ , there exists  $V \in \mathcal{U}_0$  depending on the pair  $U, U'$  such that

$$U \cup U' \subset B(x, r) \subset V.$$

By renaming as usual, let  $\tilde{U} \in \mathcal{U}_{n_0+n}$ ,  $\tilde{U}' \in \mathcal{U}_{n_0+n+1}$  denote respectively the sets  $U, U'$  as in the statement of the lemma, and suppose  $\tilde{x} \in \tilde{U} \cap \tilde{U}' \cap X$ . Set  $U = f^{\text{on}}(\tilde{U})$ ,  $U' = f^{\text{on}}(\tilde{U}')$ ,  $x = f^{\text{on}}(\tilde{x})$  and let

$$S = \sup_{U \in \mathcal{U}_{n_0}, U' \in \mathcal{U}_{n_0+1}} \max \left\{ \frac{\text{diam}U}{\text{diam}V}, \frac{\text{diam}U'}{\text{diam}V} \right\}.$$

Note that  $S$  depends only on the integer  $n_0$ . If  $\tilde{V}$  denotes the preimage of  $V$  under  $f^{-n}$  containing  $\tilde{U} \cup \tilde{U}'$ , then the backwards relative diameter bounds (2.3) imply

$$\delta_+^{-1}(S) \leq \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{V}} \leq \delta_-(S)$$

and

$$\delta_+^{-1}(S) \leq \frac{\text{diam}\tilde{U}}{\text{diam}\tilde{V}} \leq \delta_-(S).$$

Dividing yields,

$$\frac{\delta_+^{-1}(S)}{\delta_-(S)} < \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} < \frac{\delta_-(S)}{\delta_+^{-1}(S)}.$$

Since  $S$  and  $n_0$  are independent of  $n$ , property [Local comparability] follows with

$$C = \max \left\{ \left( \frac{\delta_+^{-1}(S)}{\delta_-(S)} \right)^{\pm 1}, \frac{\sup\{\text{diam}U \mid U \in \cup_0^{n_0}\mathcal{U}_n\}}{\inf\{\text{diam}U \mid U \in \cup_0^{n_0}\mathcal{U}_n\}} \right\}.$$

■

The following lemma shows that cxc systems are truly metrically expanding in a natural metric sense, and that the  $\delta_-$  function depends essentially on the relative levels of the sets involved.

**Proposition 2.6.5 (Contraction implies exponential contraction)** *Constants  $C' > 0$  and  $\theta \in (0, 1)$  exist such that, for any  $n, k \geq 0$ , any  $U' \in \mathcal{U}_{n+k}$  and any  $U \in \mathcal{U}_n$ , if  $U' \cap U \cap X \neq \emptyset$ , then*

$$\frac{\text{diam}U'}{\text{diam}U} \leq C'\theta^k.$$

*In particular, in the upper diameter bounds, one may assume  $d_n = C'd_0\theta^n$ .*

**Proof:** The diameters of the elements of  $\mathcal{U}_0$  are bounded from below by the constant  $c_0$ . Since the diameters of the elements of  $\mathcal{U}_n$  tend uniformly to zero (by the [Expansion] Axiom), and the backwards relative diameter distortion function  $\delta_-$  is a homeomorphism, there exists  $N_0 \in \mathbb{N}$  with the following property:

$$(\forall U' \in \mathcal{U}_{N_0})(\exists U \in \mathcal{U}_0) \text{ such that } U' \subset U \text{ and } \delta_- \left( \frac{\text{diam}U'}{\text{diam}U} \right) < \frac{1}{2}.$$



Now let  $k \in \mathbb{N}$  be arbitrary and let  $\tilde{U}' \in \mathcal{U}_{k+N_0}$ . Let  $U' = f^k(\tilde{U}')$ , let  $U \supset U'$  be as above, and let  $\tilde{U}$  be the component of  $f^{-N_0}(U)$  containing  $\tilde{U}'$ . Then by the backwards relative diameter bounds (2.3),

$$\text{diam}\tilde{U}' < \frac{1}{2}\text{diam}\tilde{U}.$$

Thus, for any  $k \in \mathbb{N}$ , for any  $U' \in \mathcal{U}_{N_0+k}$ , there exists  $U \in \mathcal{U}_k$  such that  $U' \subset U$  and  $\text{diam}U' \leq (1/2)\text{diam}U$ .

Let us set  $\theta = 2^{-1/N_0}$  and  $C' = 2C^{N_0-1}$  where  $C$  is given by Proposition 2.6.4.

Let  $n, k \geq 0$ , and let us fix  $U' \in \mathcal{U}_{n+k}$  and  $U \in \mathcal{U}_n$  such that  $U \cap U' \cap X \neq \emptyset$ . There are integers  $a \geq 0$ ,  $b \in \{0, \dots, N_0 - 1\}$  such that  $k = a \cdot N_0 + b$ .

Define inductively  $U_j \in \mathcal{U}_{jN_0+b}$ ,  $j = 0, \dots, a$ , such that  $U_a = U'$ ,  $U_{j+1} \subset U_j$  and  $\text{diam}U_{j+1} \leq (1/2)\text{diam}U_j$ . It follows that

$$\text{diam}U' \leq \left(\frac{1}{2}\right)^a \text{diam}U_0 \leq C^b \left(\frac{1}{2}\right)^a \text{diam}U$$

by Proposition 2.6.4. But

$$2^{-a} = \theta^k 2^{b/N_0} \leq 2\theta^k$$

so the proposition follows. ■

The lemma below shows that in  $\mathfrak{X}_0$ , a possibly disconnected ball  $B(x, r)$  with  $x \in X$  can be both enlarged and shrunk to obtain a pair of elements  $U, U'$  of  $\mathbf{U}$  whose levels are comparable and whose diameters in  $\mathfrak{X}_0$  are comparable to the diameter of  $U$ .

**Proposition 2.6.6 (Balls are like connected sets (BLC))** *There exist constants  $L > 1$  and  $n_0 \in \mathbb{N}$  such that for all  $x \in X$  and  $r < \delta_0$ , there exist levels  $m$  and  $n$  and sets  $U \in \mathcal{U}_n$ ,  $U' \in \mathcal{U}_m$  such that  $|m - n| \leq n_0$  and*

$$B(x, r/L) \subset U' \subset B(x, r) \subset U \subset B(x, Lr).$$

**Proof:** We will first find  $U$  and  $L$  so that  $B(x, r) \subset U \subset B(x, Lr)$ , where  $L = 4KC$ ,  $K$  is the roundness constant from Proposition 2.6.2, and  $C$  is the constant from Proposition 2.6.4.

Let  $\delta_0$  denote the Lebesgue number of  $\mathcal{U}_0$ . Given  $x$  and  $r < \delta_0$ , the number

$$n = \sup\{i : \exists U, \exists i \text{ with } B(x, r) \subset U \in \mathcal{U}_i \text{ and } \text{Round}(U, x) < K\}$$

exists. (The set is nonempty by Proposition 2.6.3 and finite by the [Expansion] Axiom.) Suppose  $U \in \mathcal{U}_n$  and  $B(x, r) \subset U$ . We must bound  $\text{diam}U$  from above.

By Proposition [2.6.2, Uniform roundness], there exists  $V \in \mathcal{U}_{n+1}$  for which  $\text{Round}(V, x) < K$ . Thus,  $B(x, s) \subset V \subset B(x, Ks)$  for some  $s$ . Since  $n$  is maximal,  $s < r$ , and so  $\text{diam}V < 2Ks < 2Kr$ . By Proposition [2.6.4, Local comparability],  $\text{diam}U < C\text{diam}V < C2Ks < 2KCr$  and so  $U \subset B(x, 4KCr)$  as required. Thus, we have found  $U$ .

The same argument applied to  $B(x, r/L)$  produces  $U'$  such that  $B(x, r/L) \subset U' \subset B(x, r)$ . We assume that the level of  $U'$  is  $n+k$ .

Assume  $U' \in \mathcal{U}_m$  and  $U \in \mathcal{U}_n$ . If  $m = n+k$  where  $k \geq 0$ , Proposition 2.6.5 implies  $k \leq -\log(2L^2C')/\log\theta$ . If  $n = m+k$  where  $k \geq 0$ , then another application of the proposition (with the roles of  $U$  and  $U'$  reversed) yields  $k \leq -\log(2C'/\log\theta)$ . The factors of two arise since the diameter of a ball of radius  $r$  is bounded below by  $r$ , not  $2r$  (Proposition 2.6.1). ■

Recall that a metric space is *uniformly perfect* if there is a positive constant  $\lambda < 1$  such that  $B \setminus (\lambda B)$  is non-empty for every ball  $B$  of radius at most the diameter of the space.

**Proposition 2.6.7** *We have  $\text{diam}(U \cap X) \asymp \text{diam}U$  for all  $U \in \mathbf{U}$ . As a consequence, the repellor  $X$  is uniformly perfect.*

**Proof:** Recall that  $X$  is perfect, i.e. contains no isolated points. By Proposition [2.6.6, BLC] it suffices to prove that

$$\text{diam}\tilde{U} \asymp \text{diam}(\tilde{U} \cap X)$$

where  $\tilde{U}$  ranges over any element of  $\mathbf{U}$ .

There exists  $n_0$  large enough such that for each  $U \in \mathcal{U}_0$ , we can choose points  $a, b \in U \cap X$  and neighborhoods  $U'_a, U'_b \in \mathcal{U}_{n_0}$  of  $a$  and  $b$ , respectively, which are disjoint and contained in  $U$ . We now assume such a choice has been fixed.

Given  $\tilde{U} \in \mathcal{U}_k$ , let  $U = f^k(\tilde{U})$  and let  $a, b, U'_a, U'_b$  be as in the previous paragraph. Choose arbitrarily  $\tilde{a} \in \tilde{U} \cap f^{-k}(a)$  and let  $\tilde{U}'_a \in \mathcal{U}_{n_0+k}$  be the unique component of  $f^{-k}(U'_a)$  containing  $\tilde{a}$ . Similarly, define  $\tilde{b}$  and  $\tilde{U}'_b$ . Then  $\tilde{U}'_a$  and  $\tilde{U}'_b$  are disjoint and are contained in  $\tilde{U}$ . Each contains an element of  $X$ , since  $X$  is totally invariant. Thus,  $\text{diam}\tilde{U} \cap X$  is at least as large as the radius  $r$  of the largest ball centered at  $\tilde{a}$  and contained in  $\tilde{U}'_a$ . By the definition of roundness

$$r > \frac{1}{2} \text{diam}\tilde{U}'_a \cdot \text{Round}(\tilde{U}'_a, \tilde{a})^{-1}.$$

The backward relative diameter distortion bounds (2.3) imply

$$\text{diam}\tilde{U}'_a > \text{diam}\tilde{U} \cdot \delta_+^{-1} \left( \frac{\text{diam}U'_a}{\text{diam}U} \right).$$

The backward roundness distortion bound (2.1) implies

$$\text{Round}(\tilde{U}'_a, \tilde{a}) < \rho_-(\text{Round}(U_a, a)).$$

Since  $\mathcal{U}_0$  is finite,  $r/\text{diam}\tilde{U}$  is therefore bounded from below by a constant independent of  $k$ . ■

**Definition 2.6.8 (Linear local connectivity)** *Let  $\lambda \geq 1$ . A metric space  $Z$  is  $\lambda$ -linearly locally connected if the following two conditions hold:*

1. *if  $B(a, r)$  is a ball in  $Z$  and  $x, y \in B(a, r)$ , then there exists a continuum  $E \subset B(a, \lambda r)$  containing  $x$  and  $y$ ;*
2. *if  $B(a, r)$  is a ball in  $Z$  and  $x, y \in Z - B(a, r)$ , then there exists a continuum  $E \subset Z - B(a, r/\lambda)$  containing  $x$  and  $y$ .*

Propositions 2.6.6 and 2.6.7 imply immediately that (i) if  $U \cap X$  is connected for all  $U \in \mathbf{U}$ , then condition (1) above holds, and (ii) if  $X \setminus (U \cap X)$  is connected for all  $U \in \mathbf{U}$ , then condition (2) holds. We obtain immediately

**Corollary 2.6.9** *If, for all  $U \in \mathbf{U}$ , the sets  $U \cap X$  and  $X \setminus (U \cap X)$  are connected, then  $X$  is linearly locally connected.*

Unlike the preceding results in this section, the following lemma uses the [Degree] Axiom. Recall that a metric space is *doubling* if there is a positive constant  $C_d$  such that any set of finite diameter can be covered by  $C_d$  sets of at most half its diameter (cf. [Hei], § 10.13).

**Proposition 2.6.10 (cxc implies doubling)** *If axiom [Degree] is satisfied, then  $X$  is a doubling metric space.*

**Proof:** It follows from Proposition 2.6.5 that an integer  $k_0$  exists such that, for any  $n \geq 0$ , any  $U \in \mathcal{U}_n$ , and any  $U' \in \mathcal{U}_{n+k_0}$ ,  $\text{diam}U' \leq (1/4L)\text{diam}U$  as soon as  $U' \cap U \cap X \neq \emptyset$ .

From the finiteness of  $\mathcal{U}_0$ , it follows that any  $U \in \mathcal{U}_0$  can be covered by  $N$  sets of level  $k_0$ .

Let  $E \subset X$ , and  $x \in E$ . If its diameter is larger than the Lebesgue number of  $\mathcal{U}_0$ , then it can be covered by a uniform number of sets of half its diameter. Otherwise, one can find a level  $n$  and a set  $\tilde{U} \in \mathcal{U}_n$  such that

$$E \subset B(x, \text{diam}E) \subset \tilde{U} \subset B(x, L\text{diam}E)$$

by Proposition 2.6.6.

Let us cover  $f^n(\tilde{U})$  by  $N$  sets  $U'_1, \dots, U'_N$  of level  $k_0$ . Axiom [Degree] implies that  $\tilde{U}$ , so  $E$  as well, is covered by at most  $pN$  sets  $(\tilde{U}'_j)$  of level  $n + k_0$ . Thus,

$$\text{diam}\tilde{U}'_j \leq \frac{1}{4L}\text{diam}\tilde{U} \leq \frac{2L}{4L}\text{diam}E,$$

and  $C_d \leq pN$ . ■

From Assouad's theorem (see [Hei], Thm. 12.1) we obtain

**Corollary 2.6.11** *If axiom [Degree] is satisfied, then  $X$  is quasisymmetrically embeddable in the Euclidean space  $\mathbb{R}^n$ . In particular,  $X$  has finite topological dimension.*

## 2.7 Dynamical regularity

Suppose again that  $\mathfrak{X}_0, \mathfrak{X}_1$  are metric spaces. Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  be an f.b.c. as in the previous section with repeller  $X$  and level zero good neighborhoods

$\mathcal{U}_0$ . Throughout this subsection, we again assume that the topological axiom [Expansion] and the metric axioms [Roundness distortion] and [Relative diameter distortion] are satisfied. We assume neither axiom [Irreducibility] nor [Degree].

Recall that a subset  $A$  of a metric space  $Z$  is  $c$ -porous when every ball of radius  $r < \text{diam}Z$  contains a ball of radius  $cr$  which does not meet  $A$ . A subset is porous if it is  $c$ -porous for some  $c > 0$ .

**Proposition 2.7.1** *If axiom [Degree] is satisfied, then the post-branch set  $P_f = \cup_{n>0} f^n(B_f)$  is porous, and the sets  $B_{f^n} \cap X$ ,  $n = 1, 2, 3, \dots$  are porous with porosity constants independent of  $n$ .*

**Proof:** Axiom [Degree] implies there exists  $n_0$  and  $U_{n_0} \in \mathcal{U}_{n_0}$  so that the degree  $\text{deg}(f^{n_0}|U) = p$ , i.e. is maximal (cf. Proposition 2.4.2). Then all iterated preimages  $\tilde{U}_{n_0}$  of  $U_{n_0}$  map by degree one onto  $U_{n_0}$ . So  $U_{n_0}$  and any iterated preimage  $\tilde{U}_{n_0}$  lie in the complement of the post-branch set. By Proposition 2.4.2, for every element  $U$  of  $\mathcal{U}_0$ , there is a  $k(U) \in \mathbb{N}$  and a preimage  $U'$  of  $U_{n_0}$  under  $f^{-k(U)}$  which is contained in  $U$ . Let

$$c_0 = \min_{U \in \mathcal{U}_0} \frac{\text{diam}U'}{\text{diam}U}.$$

Let  $B(x, r)$  be a small ball in  $\mathfrak{X}_0$  centered at a point  $x \in P_f$ . By Proposition [2.6.6, BLC] there exists some  $n$  and  $\tilde{U} \in \mathcal{U}_n$  such that  $B(x, r/L) \subset \tilde{U} \subset B(x, r)$ . Let  $U = f^n(\tilde{U}) \in \mathcal{U}_0$ . Then by the previous paragraph,  $U \supset U'$  where  $U' \subset X - P_f$ . If  $\tilde{U}'$  is any preimage of  $U'$  under  $f^n$  which is contained in  $\tilde{U}$ , then the forward invariance of  $P_f$  implies  $\tilde{U}' \subset X - P_f$ . By the backward lower relative diameter distortion bounds (2.3),

$$\text{diam}\tilde{U}' > \delta_+^{-1}(c_0)\text{diam}\tilde{U} > \delta_+^{-1}(c_0)r/L = c_1r.$$

Since good open sets are uniformly  $K$ -almost round (Proposition 2.6.2),  $\tilde{U}' \supset B(y, c_1r/K)$  for some  $y \in X$  and so  $X - P_f$  is  $c$ -porous where  $c = c_1/K$ .

We merely sketch the second assertion. Suppose  $B(x, r)$  is a small ball centered at a point  $x \in B_{f^k} \cap X$ . Then for some  $n$ ,  $r \asymp \text{diam}\tilde{U}$  where  $\tilde{U} \in \mathcal{U}_{n+k}$ . Let  $U = f^k(\tilde{U})$ . Since  $f^k(x) \in P_f$  and  $P_f$  is porous, there is some  $U' \subset U$  with  $\text{diam}U' \asymp \text{diam}U$  and  $U' \subset X - P_f$ . If  $\tilde{U}'$  is any preimage of  $U'$  under  $f^k$  which is contained in  $\tilde{U}$ , then  $\tilde{U}' \subset X - B_{f^k} \cap X$ , and the backwards

relative diameter distortion bounds again imply  $\text{diam}\tilde{U}' \asymp \text{diam}\tilde{U} \asymp r$ . Since  $\tilde{U}'$  is  $K$ -almost round this implies that  $X - B_{f^k} \cap X$  is uniformly porous as a subset of  $X$ . ■

The next lemma shows that the roundness distortion control of  $f$ , which was assumed only for the sets in  $\mathbf{U}$ , in fact extends to any iterate of  $f$  and any ball of small enough radius.

**Proposition 2.7.2 (cxc is uniformly weakly quasiregular)** *There is a constant  $H < \infty$  and a sequence of radii  $\{r_n\}_{n=1}^\infty$  decreasing to 0 such that, for any iterate  $n$ , for any  $x \in X$ , and any  $r \in (0, r_n)$ ,*

$$\text{Round}(f^n(B(x, r)), f^n(x)) \leq H.$$

**Proof:** Let  $r_n = \frac{c_n}{2L}$  and fix  $r < r_n$  and  $\tilde{x} \in X$ . By Proposition 2.6.6, there exist  $\tilde{U} \in \mathcal{U}_m$  and  $\tilde{U}' \in \mathcal{U}_{m+n_0}$  such that

$$B(\tilde{x}, r/L) \subset \tilde{U}' \subset B(\tilde{x}, r) \subset \tilde{U} \subset B(\tilde{x}, Lr).$$

Thus  $\text{diam}\tilde{U} \leq 2Lr < c_n$  and so the level  $m > n$  since the sequence  $(c_k)$  is decreasing. Set as usual  $U = f^n(\tilde{U})$ ,  $U' = f^n(\tilde{U}')$ , and  $x = f^n(\tilde{x})$ . Now,

$$\text{Round}(\tilde{U}', \tilde{x}), \text{Round}(\tilde{U}, \tilde{x}) < L$$

and

$$\frac{1}{2L^2} < \frac{\text{diam}\tilde{U}'}{\text{diam}\tilde{U}} \leq 1.$$

By the forward roundness and relative diameter distortion bounds,

$$\text{Round}(U', x), \text{Round}(U, x) < \rho_+(L)$$

and

$$\delta_+ \left( \frac{1}{2L^2} \right) < \frac{\text{diam}U'}{\text{diam}U} \leq 1.$$

Moreover,

$$U' \subset f^n(B(\tilde{x}, r)) \subset U.$$

It follows easily that  $\text{Round}(f^n(B(\tilde{x}, r)), x)$  is bounded by a constant independent of  $x$ ,  $n$ , and  $r$ . ■

## 2.8 Quasisymmetric and topological conjugacies of cxc systems

Recall that a homeomorphism  $h$  between metric spaces is *weakly quasisymmetric* if it distorts the roundness of balls by a uniform factor, i.e.

$$\text{Round}(h(B(x, r)), h(x)) \leq H$$

for all  $x \in X$  and  $r \leq \text{diam}X$ .

We start with a result which will enable us to promote weak quasisymmetry to the usual strong quasisymmetry.

**Theorem 2.8.1** *Let  $X, Y$  be two uniformly perfect doubling metric compact spaces. Let  $h : X \rightarrow Y$  be a homeomorphism. If both  $h$  and  $h^{-1}$  are weakly quasisymmetric, then  $h$  and  $h^{-1}$  are both quantitatively quasisymmetric.*

In the proof, we adapt the argument of Theorem 10.19 of [Hei].

**Proof:** The assumptions imply

1. there is a constant  $\lambda > 1$  such that, for any ball  $B$  in  $X$  or  $Y$  with non-empty complement,  $B \setminus (1/\lambda)B \neq \emptyset$ ;
2. there are constants  $C, \beta > 0$  such that any set of diameter  $d$  in  $X$  or  $Y$  can be covered by at most  $C\epsilon^{-\beta}$  sets of diameter at most  $\epsilon d$ ;
3. there is a constant  $H$  such that

$$\left\{ \begin{array}{l} \text{if } a, b, x \in X, \quad |a - x| \leq |b - x| \quad \text{then } |h(a) - h(x)| \leq H|h(b) - h(x)| \\ \text{if } c, d, y \in Y, \quad |c - y| \leq |d - y| \quad \text{then } |h^{-1}(c) - h^{-1}(y)| \leq H|h^{-1}(d) - h^{-1}(y)|. \end{array} \right.$$

Choose  $t_\epsilon \in (0, 1)$  small enough so that  $t_\epsilon \lambda \leq 1/3$ . Let  $a, b, x \in X$  and set

$$t = \frac{|a - x|}{|b - x|} \quad \text{and} \quad t' = \frac{|h(a) - h(x)|}{|h(b) - h(x)|}.$$

Let us assume that  $t < t_\epsilon$ . Since  $X$  is u.p., there are points  $b_0, \dots, b_s$  such that  $b_j \in B(x, t_\epsilon^j |b - x|) \setminus B(x, (t_\epsilon^j / \lambda) |b - x|)$ , where  $s$  is the least integer such that  $t_\epsilon^s < t$  by 1.

It follows that if  $i < j$  then

$$\frac{|b_i - b_j|}{|b - x|} \geq \frac{|b_i - x|}{|b - x|} - \frac{|x - b_j|}{|b - x|}$$

so that

$$\frac{|b_i - b_j|}{|b - x|} \geq (t_\epsilon^i/\lambda) - t_\epsilon^j \geq (t_\epsilon^i/\lambda)(1 - \lambda t_\epsilon) > 0$$

and these points are all pairwise disjoint.

Furthermore, it follows from the definition of  $s$  that

$$\frac{\log(1/t)}{\log(1/t_\epsilon)} \leq s.$$

Let  $0 \leq i < j \leq s - 1$ ; then  $|a - b_j| \leq 2|x - b_j|$  and

$$|b_i - b_j| \geq (t_\epsilon^{j-1}/\lambda)(1 - \lambda t_\epsilon)|b - x| \geq 2|x - b_j|.$$

Hence  $|a - b_j| \leq |b_i - b_j|$  and it follows that

$$|h(a) - h(b_j)| \leq H|h(b_i) - h(b_j)|$$

by 3. Similarly,  $|x - b_j| \leq |b_i - b_j|$  implies that

$$|h(x) - h(b_j)| \leq H|h(b_i) - h(b_j)|.$$

Therefore

$$|h(a) - h(x)| \leq 2H|h(b_i) - h(b_j)|.$$

It follows that the balls  $B(h(b_j), (1/5H)|h(a) - h(x)|)$  are pairwise disjoint. Indeed, if  $y \in B(h(b_j), (1/5H)|h(a) - h(x)|)$ , then

$$|y - h(b_i)| \geq |h(b_i) - h(b_j)| - |y - h(b_j)| \geq (3/5H)|h(a) - h(x)|$$

so that  $y \notin B(h(b_i), (1/5H)|h(a) - h(x)|)$ . Furthermore they are contained in  $B(h(x), 2H|h(x) - h(b)|)$ , so the doubling property 2. implies

$$s \leq C \left( \frac{t'}{5H} \right)^{-\beta}$$

from which we deduce that  $t'$  is bounded by a function of  $t$  which decreases to 0 with  $t$ .



It follows that there is a homeomorphism  $\eta : [0, 1] \rightarrow [0, \eta(1)]$  such that  $\eta(1) \geq 1$  and if  $|a - x| \leq |b - x|$  then

$$|h(a) - h(x)| \leq \eta \left( \frac{|a - x|}{|b - x|} \right) |h(b) - h(x)|.$$

Similarly, if  $|c - y| \leq |d - y|$  then

$$|h^{-1}(c) - h^{-1}(y)| \leq \eta \left( \frac{|c - y|}{|d - y|} \right) |h^{-1}(d) - h^{-1}(y)|.$$

Let us assume now that  $t \geq 1/\eta^{-1}(1)$ . It follows that

$$|h(b) - h(x)| \leq \eta(1/t)|h(a) - h(x)| \leq |h(a) - h(x)|,$$

whence

$$|b - x| \leq \eta \left( \frac{|h(b) - h(x)|}{|h(a) - h(x)|} \right) |a - x|.$$

It follows that

$$t' \leq 1/\eta^{-1}(1/t).$$

This establishes that  $f$  is quasisymmetric, and  $f^{-1}$  as well. ■

The main result of this section is

**Theorem 2.8.2 (Invariance of cxc)** *Suppose  $f : (\mathfrak{X}_1, X) \rightarrow (\mathfrak{X}_0, X)$  and  $g : (\mathfrak{Y}_1, Y) \rightarrow (\mathfrak{Y}_0, Y)$  are two topological cxc systems which are conjugate via a homeomorphism  $h : \mathfrak{X}_0 \rightarrow \mathfrak{Y}_0$ , where  $\mathfrak{X}_0$  and  $\mathfrak{Y}_0$  are metric spaces.*

1. *If  $f$  is metrically cxc and  $h$  is quasisymmetric, then  $g$  is metrically cxc, quantitatively.*
2. *If  $X$  is connected and  $f, g$  are both metrically cxc, then  $h|_X : X \rightarrow Y$  is quasisymmetric, quantitatively.*

In the proof below, we use subscripts to indicate the dependence of the metric regularity constants on the system, e.g.  $\delta_{\pm f}, \delta_{\pm g}$ , etc.

**Proof: 1.** Suppose first that  $h$  is  $\eta$ -quasisymmetric. Then

- **[Roundness quasi-invariant]**  $h$  sends  $K$ -almost-round sets with respect to  $x$  to  $\eta(K)$ -almost-round sets with respect to  $h(x)$ ;
- **[Relative distance distortion]** for all  $A, B \subset X$  with  $A \subset B$ ,

$$\frac{1}{2\eta\left(\frac{\text{diam}B}{\text{diam}A}\right)} \leq \frac{\text{diam}h(A)}{\text{diam}h(B)} \leq \eta\left(2\frac{\text{diam}A}{\text{diam}B}\right)$$

(see [Hei], Prop. 10.8).

The topological axioms ([Expansion], [Irreducibility], [Degree]) are invariant under topological conjugacies. Axiom [4, Roundness distortion] follows immediately from property [Roundness quasi-invariant] above. Thus, it suffices to check Axiom [5, Diameter distortion]. Let us use small letters and drop “diam” for ease of readability. Let  $\bar{\eta}(t) = 1/(\eta^{-1})(1/t)$ , and notice that  $h^{-1}$  is  $\bar{\eta}$ -quasisymmetric.

We have

$$\begin{aligned} \frac{\tilde{v}'}{\tilde{v}} &< \eta\left(2\frac{\tilde{u}'}{\tilde{u}}\right) && \text{rel. dist. distortion} \\ \frac{\tilde{u}'}{\tilde{u}} &< \delta_{-,f}\left(\frac{u'}{u}\right) && \text{def. } \delta_- \\ \frac{\tilde{v}'}{\tilde{v}} &< \eta\left(2\delta_{-,f}\left(\frac{u'}{u}\right)\right) && \eta \text{ increasing} \\ \frac{u'}{u} &< \bar{\eta}\left(2\frac{v'}{v}\right) && \text{rel. dist. distortion} \end{aligned}$$

Thus,

$$\frac{\tilde{v}'}{\tilde{v}} < \eta\left(2\delta_{-}\left(\bar{\eta}\left(2\frac{v'}{v}\right)\right)\right).$$

Now define

$$\delta_{-g}(t) = \eta(2\delta_{-,f}(\bar{\eta}(2t))).$$

This is a composition of homeomorphisms, hence a homeomorphism, and so it satisfies the requirements. Finding  $\delta_{+g}$  is accomplished similarly:

$$\begin{aligned} \frac{v'}{v} &< \eta\left(2\frac{u'}{u}\right) \\ &< \eta\left(2\delta_{+,f}\left(\frac{\tilde{u}'}{\tilde{u}}\right)\right) \\ &< \eta\left(2\delta_{+,f}\left(\bar{\eta}\left(2\frac{v'}{v}\right)\right)\right). \end{aligned}$$

2. Now suppose  $g$  is metrically exc. By Propositions 2.6.10 and 2.6.7,  $X$  and  $Y$  are doubling and uniformly perfect. Therefore, it suffices to show  $h$  and  $h^{-1}$  are weakly quasimetric (cf. Theorem 2.8.1). Since the setting is symmetric with respect to  $f$  and  $g$ , it is enough to prove that  $h$  is weakly quasimetric. To show this, it suffices to show (since  $h$  and its inverse are uniformly continuous) that if  $B = B(\tilde{x}, r)$  is a sufficiently small ball, then its image  $h(B)$  is almost round with respect to  $\tilde{y} = h(\tilde{x})$ , with roundness constant independent of  $B$ . Our proof below follows the usual method (see [Sul2]): given a small ball  $B$ , we use the dynamics and the distortion axioms to blow it up to a ball of definite size and bounded roundness. By compactness, moving over to  $Y$  via  $h$  distorts roundness by a bounded amount. We then pull back by the dynamics and apply the distortion axioms again.

Our argument is slightly tricky, since we must trap balls, which are possibly disconnected, inside connected sets in order to apply the pullback step and make sense of the “lift” of a ball. We will accomplish this as follows. Let  $\mathbf{U} = \{\mathcal{U}_n\}_{n=0}^\infty$ ,  $\mathbf{V} = \{\mathcal{V}_n\}_{n=0}^\infty$  be the sequences of good open sets for  $f$  and  $g$ , respectively. We are aiming for the following diagram:

$$\begin{array}{ccc}
 \tilde{U}' \subset B \subset \tilde{U} & \xrightarrow{h} & \tilde{V}' \subset h(\tilde{U}') \subset h(B) \subset h(\tilde{U}) \subset \tilde{V} \\
 f^n \downarrow & & \downarrow g^n \\
 U' \subset f^n(B) \subset U & \xrightarrow{h} & V' \subset h(U') \subset h(f^n(B)) = g^n(h(B)) \subset h(U) \subset V
 \end{array} \tag{2.6}$$

Below, we indicate the dependence on the map of the metric regularity constants  $K, C, L, c_n, d_n$ , etc. defined in the previous two sections by subscripts.

The diameters of elements of  $\mathcal{V}_0$  are bounded from below. Since  $\mathfrak{X}_1$  is relatively compact,  $h|_{\mathfrak{X}_1} : \mathfrak{X}_1 \rightarrow \mathfrak{Y}_1$  is uniformly continuous. Hence there exists  $\delta_0 > 0$  such that

$$\text{diam} E < \delta_0 \Rightarrow \text{diam}(h(E)) < \epsilon_0 = \text{Lebesgue } \# \text{ of } \mathcal{V}_0. \tag{2.7}$$

**Finding**  $\tilde{U}, \tilde{U}'$ . The [Expansion] Axiom implies that there exists  $N_0$  such that  $d_{N_0, f} < \delta_0$ . Let  $B = B(\tilde{x}, r)$  where  $r < c_{N_0, f}/(2L_f)$ . By Proposition 2.6.6, there exists  $n_{0, f}$  and  $m \in \mathbb{N}$ ,  $\tilde{U} \in \mathcal{U}_m$ , and  $\tilde{U}' \in \mathcal{U}_{m+n_{0, f}}$  such that

$$B(\tilde{x}, r/L_f) \subset \tilde{U}' \subset B \subset \tilde{U} \subset B(\tilde{x}, L_f r).$$

Thus  $\text{diam} \tilde{U} \leq 2L_f r \leq c_{N_0, f}$  and so  $m = N_0 + n$  where  $n \geq 0$ .

**Finding**  $U', U$ . Let as usual  $U = f^n(\tilde{U})$ ,  $U' = f^n(\tilde{U}')$ ,  $x = f^n(\tilde{x})$ . Then  $U \in \mathcal{U}_{N_0}$  and  $U' \in \mathcal{U}_{N_0+n_0, f}$ .

**Finding**  $V$ . Let  $y = h(x)$ . Since  $U \in \mathcal{U}_{N_0}$  and  $d_{N_0, f} < \delta_0$ , (2.7) implies  $\text{diam}(h(U)) < \epsilon_0$  and so there exists  $V \in \mathcal{V}_0$  with  $h(U) \subset V$ .

**Finding**  $V'$ . The forward roundness bound (2.2) implies that

$$\text{Round}(U', x) < \rho_{+, f}(L_f).$$

Hence

$$U' \supset B(x, s'), \quad \text{where } s' = \frac{c_{N_0+n_0, f}}{2\rho_{+, f}(L_f)}.$$

Since  $X$  is compact,  $h(B(x, s')) \supset B(y, t')$  where

$$t' = \inf\{|h(x) - h(a)| : x \in X, |a - x| = s'\}.$$

The [Expansion] Axiom implies that there exists  $k_0$  such that  $d_{k_0, g} < t'/2$ . Proposition 2.6.2 implies that there exists  $V' \in \mathcal{V}_{k_0}$  such that  $\text{Round}(V', y) < K_g$ . Then

$$V' \subset h(U') \subset h(f^n(B)) \subset h(U) \subset V$$

where

$$\text{Round}(V', y), \text{Round}(V, y) \leq \min\left\{\frac{d_{0, g}}{t'}, K_g\right\} =: R$$

and

$$\frac{\text{diam}V'}{\text{diam}V} < \frac{c_{k_0, g}}{d_{0, g}} =: D.$$

**Finding**  $\tilde{V}, \tilde{V}'$ . Let  $\tilde{V}, \tilde{V}'$  denote the preimages of  $V$  and  $V'$ , respectively, containing  $\tilde{y} = h(\tilde{x})$ . We have now achieved the situation summarized in (2.6).

**Conclusion.** The backwards roundness bound (2.1) and backwards relative diameter distortion bound (2.3) imply

$$\text{Round}(\tilde{V}, \tilde{y}), \text{Round}(\tilde{V}', \tilde{y}) < \tilde{R} = \rho_{-, g}(R)$$

and

$$\frac{\text{diam}\tilde{V}'}{\text{diam}\tilde{V}} > \tilde{D} = \delta_{+, g}(D).$$

Hence  $\text{Round}(h(B), h(\tilde{x})) < 2\tilde{R}^2/\tilde{D}$  and the proof is complete. ■



# Chapter 3

## Geometrization

In this chapter, we assume we are given an fbc  $(\mathfrak{x}_1, X) \xrightarrow{f} (\mathfrak{x}_0, X)$  and a finite cover  $\mathcal{U}$  which satisfy [Expansion]. To this data we will associate a canonical conformal structure on  $X$ , thus promoting our topological dynamical system to a conformal one. This structure appears as the conformal structure at infinity induced by a hyperbolic space upon which  $f$  acts by local isometry. Assuming in addition Axiom [Irreducibility], we study the distribution of preimages of points and of cycles.

When the map is actually cxc, then the conformal structure belongs to the same conformal gauge as the original one, and we prove that there is a unique measure of maximal entropy  $\log d$ .

This approach follows not only Thurston's philosophy that Topology implies a natural Geometry, but also apply Gromov's point of view that coarse notions capture enough information to determine Geometry.

This chapter is organized as follows. In the first section, we review the basic geometric theory of unbounded metric spaces, emphasizing hyperbolicity and compactifications. Section 2 is devoted to the construction of the hyperbolic space  $\Gamma$ , and we establish its first properties. In section 3, the hyperbolicity of  $\Gamma$  is proved, and its naturality is established. Then, we use the hyperbolicity to study measure-theoretic properties: the Patterson-Sullivan procedure produces a geometric measure which describes the distribution of preimages and of cycles. We also estimate the Hausdorff dimension of the repeller with respect to this canonical conformal structure. Finally, in the last section this theory is applied to topological and metric cxc maps.

### 3.1 Compactifications of quasi-starlike spaces

A metric space  $(X, d)$  is said to be *proper* if, for all  $x \in X$ , the function  $y \mapsto d(x, y)$  is proper, meaning that closed balls of finite radius are compact. A *geodesic curve* is a continuous function  $\gamma : I \rightarrow X$  such that  $d(\gamma(t), \gamma(t')) = |t - t'|$  for all  $t, t' \in I$  and where  $I$  is an interval. We will often not distinguish between the function  $\gamma$  and its image in  $X$ . The space  $X$  is said to be *geodesic* if any pair of points can be joined by a geodesic.

Fix a base point  $o \in X$ . A *ray (based at  $o$ )* is a geodesic curve  $\gamma : \mathbb{R}_+ \rightarrow X$  such that  $\gamma(0) = o$ . Let  $\mathcal{R}$  be the set of geodesic curves starting at  $o$ , and let  $\mathcal{R}_\infty$  be the set of rays based at  $o$ . The space  $(X, o)$  is  *$K$ -quasi-starlike (about  $o$ )* if, for any  $x \in X$ , there is a ray  $\gamma \in \mathcal{R}_\infty$  such that  $d(x, \gamma) \leq K$ .

In this section, we assume that  $(X, d)$  is a geodesic proper  $K$ -quasi-starlike space about a point  $o$ . For convenience, we write  $d(x, y) = |x - y|$  and  $|x| = |x - o|$ .

**Hyperbolic spaces.** The *Gromov product* of two points  $x, y \in X$  is defined by  $(x|y) = (1/2)(|x| + |y| - |x - y|)$ . The metric space  $X$  is *Gromov hyperbolic* if there is some constant  $\delta \geq 0$  such that

$$(x|z) \geq \min\{(x|y), (y|z)\} - \delta$$

for any points  $x, y, z \in X$ . (By Proposition 1.2 of [CDP], this definition agrees with the more common one in which the above inequality is required to hold for all  $x, y, z$  and  $o$  instead of just at a single basepoint  $o$ .) Let us note that in such a space,  $(x|y) \sim d(o, [x, y])$  where  $[x, y]$  is any geodesic segment joining  $x$  to  $y$ . We refer to [CDP] and to [GdlH] for more information on Gromov hyperbolic spaces.

**Compactification.** Here, we do not assume  $X$  to be hyperbolic. We propose to compactify  $X$  using the method of W. Floyd [Flo]. Let  $\varepsilon > 0$ , and, for  $x \in X$ , define  $\rho_\varepsilon(x) = \exp(-\varepsilon|x|)$ .

For  $x, y \in X$ , define

$$d_\varepsilon(x, y) = |x - y|_\varepsilon = \inf_\gamma \int_\gamma \rho_\varepsilon$$

where the infimum is taken over all rectifiable curves which join  $x$  to  $y$ . Thus,  $|x - y|_\varepsilon \leq |x - y|$ .

The space  $(X, |\cdot|_\varepsilon)$  is not complete since if  $\gamma \in \mathcal{R}_\infty$  and if  $t' > t$  then

$$|\gamma(t) - \gamma(t')|_\varepsilon \leq \int_t^{t'} e^{-\varepsilon s} ds \leq e^{-\varepsilon t} / \varepsilon.$$

Therefore  $\{\gamma(n)\}$  is a non convergent  $d_\varepsilon$ -Cauchy sequence.

**Definition.** Let  $\overline{X}_\varepsilon$  be the completion of  $(X, d_\varepsilon)$ ,  $\partial X_\varepsilon = \partial_\varepsilon X = \overline{X}_\varepsilon \setminus X$ . Thus,  $\overline{X}_\varepsilon$  is also a length space.

**Remark.** If  $X$  is Gromov hyperbolic, then, for  $\varepsilon$  small enough,  $\partial X_\varepsilon$  coincides with the Gromov boundary of  $X$  and  $d_\varepsilon$  is a visual distance (cf. [CDP, BHK]). That is, we may extend the definition of the Gromov product to the boundary  $\partial X_\varepsilon$  and then,  $|x - y|_\varepsilon \asymp e^{-\varepsilon(x|y)}$  holds if  $0 < \varepsilon \leq \varepsilon_0(\delta)$  for some constant  $\varepsilon_0(\delta) > 0$  which depends only on  $\delta$ . When dealing with a hyperbolic space  $X$ , we will write indifferently

$$\partial_\varepsilon X, \partial_\infty X, \partial X$$

to denote its boundary. In any case and unless specified, metrics on the boundary will always be visual metrics  $d_\varepsilon$  as above for some fixed parameter  $\varepsilon > 0$  small enough.

**Topology on  $\mathcal{R}$ .** If  $\gamma \in \mathcal{R}$  then  $\gamma$  is geodesic for  $d_\varepsilon$ . Indeed, let  $\gamma$  be a curve starting from  $o$  that is parameterised by arclength. It follows that  $|\gamma(t)| \leq t$  for all  $t \in [o, \ell(\gamma)]$ . Therefore,

$$\ell_\varepsilon(\gamma) = \int_0^{\ell(\gamma)} e^{-\varepsilon|\gamma(t)|} dt \geq \int_0^{\ell(\gamma)} e^{-\varepsilon t} dt \geq \frac{1}{\varepsilon} (1 - e^{-\varepsilon \ell(\gamma)}).$$

We have equality when  $\gamma$  is geodesic for  $d_0$ .

For  $\gamma \in \mathcal{R}$ , the limit in  $\overline{X}_\varepsilon$  of  $\gamma(t)$  at  $\ell(\gamma)$  exists since any sequence  $(\gamma(t_n))$  with  $t_n \rightarrow \ell(\gamma)$  is a  $d_\varepsilon$ -Cauchy sequence. Let us define

$$\pi(\gamma) = \lim_{t \rightarrow \ell(\gamma)} \gamma(t).$$

The closure of each element  $\gamma \in \mathcal{R}$  is thus compact in  $\overline{X}_\varepsilon$ . Therefore, we may endow  $\mathcal{R}$  with the Hausdorff topology of non-empty compact subsets of  $\overline{X}_\varepsilon$  with respect to  $d_\varepsilon$ .



**Lemma 3.1.1** *The set  $\mathcal{R}$  is compact and the map  $\pi : \mathcal{R} \rightarrow \overline{X_\varepsilon}$  is continuous and surjective. Furthermore,  $\mathcal{R}_\infty$  is closed in  $\mathcal{R}$  and  $\pi|_{\mathcal{R}_\infty} : \mathcal{R}_\infty \rightarrow \partial X_\varepsilon$  is also surjective.*

**Proof:** Let  $(\gamma_n)$  be a sequence in  $\mathcal{R}$ . If  $\liminf \ell(\gamma_n) < \infty$ , then we may extract a subsequence which converges on any compact subset of  $X$  in the  $d_0$  topology. Therefore, it also converges for the  $d_\varepsilon$  metric.

We might as well assume that each  $\gamma_n$  has infinite length. Then  $(\mathbb{R}_+ \xrightarrow{\gamma_n} X)_n$  is equicontinuous and for any  $t \geq 0$ ,  $|\gamma_n(t)|$  remains in a compact set of  $X$  ( $|\gamma_n(t)| = |t|$ ), so we may apply the Arzela-Ascoli theorem and extract a convergent subsequence to a continuous function  $\gamma : \mathbb{R}_+ \rightarrow X$ . It follows that the convergence is uniform on compact subsets of  $X$  and that  $\gamma \in \mathcal{R}_\infty$ . It remains to prove that  $\gamma$  is also the limit in the  $d_\varepsilon$  topology.

Let  $t_0 > 0$  be large; if  $t > t_0$ , then

$$\begin{aligned} |\gamma_n(t) - \gamma(t)|_\varepsilon &\leq |\gamma_n(t_0) - \gamma_n(t)|_\varepsilon + |\gamma_n(t_0) - \gamma(t_0)|_\varepsilon + |\gamma(t_0) - \gamma(t)|_\varepsilon \\ &\leq |\gamma_n(t_0) - \gamma(t_0)|_\varepsilon + (2/\varepsilon) \exp(-\varepsilon t_0). \end{aligned}$$

Therefore, if  $\eta > 0$ , there is  $t_0$  such that  $(2/\varepsilon) \exp(-\varepsilon t_0) < \eta/2$ , there is  $n_0$  such that, for  $n \geq n_0$ ,  $|\gamma(t) - \gamma_n(t)| \leq \eta/2$  for all  $t \in [0, t_0]$ , and so, for these  $t$ ,

$$|\gamma_n(t) - \gamma(t)|_\varepsilon \leq |\gamma_n(t) - \gamma(t)| \leq \eta/2$$

and for  $t \geq t_0$ ,

$$|\gamma_n(t) - \gamma(t)|_\varepsilon \leq |\gamma_n(t_0) - \gamma(t_0)| + \frac{2}{\varepsilon} e^{-\varepsilon t_0} \leq \eta/2 + \eta/2 \leq \eta.$$

The continuity of  $\pi$  is straightforward. Let us prove that  $\pi$  is surjective. If  $x \in \partial X_\varepsilon$ , then there is a sequence  $(x_n)$  in  $X_\varepsilon$  which converges to  $x$ . Let  $\gamma_n$  be geodesic segments joining  $o$  to  $x_n$ . It follows as above that  $(\gamma_n)$  is a normal family and sub-converges to a geodesic ray  $\gamma \in \mathcal{R}_\infty$ . One obtains, for  $t \leq |x_n|$ ,

$$\begin{aligned} |x - \pi(\gamma)|_\varepsilon &\leq |\pi(\gamma) - \gamma(t)|_\varepsilon + |\gamma(t) - \gamma_n(t)|_\varepsilon + |\gamma_n(t) - x_n|_\varepsilon + |x_n - x|_\varepsilon \\ &\leq (2/\varepsilon) \exp(-\varepsilon t) + |\gamma(t) - \gamma_n(t)|_\varepsilon + |x_n - x|_\varepsilon. \end{aligned}$$

Let  $\eta > 0$ ; if  $t \geq t_0$  then  $(2/\varepsilon) \exp(-\varepsilon t) \leq \eta/3$ . If  $n$  is large enough, then  $|x - x_n|_\varepsilon < \eta/3$ , and  $|\gamma(t_0) - \gamma_n(t_0)|_\varepsilon < \eta/3$  so that  $|x - \pi(\gamma)|_\varepsilon \leq \eta$  and  $x = \pi(\gamma)$ . ■

**Lemma 3.1.2** *The following hold*

$$X_\varepsilon = B_\varepsilon(o, 1/\varepsilon), \partial X_\varepsilon = \{x, |x|_\varepsilon = (1/\varepsilon)\} \quad \text{and} \quad B(o, R) = B_\varepsilon(o, (1/\varepsilon)(1 - e^{-\varepsilon R})).$$

**Proof:** Let  $x \in X$  and  $\gamma \in \mathcal{R}$  be a geodesic segment joining  $o$  to  $x$ . It follows that

$$|x|_\varepsilon \leq \int_\gamma \rho_\varepsilon = \int_0^{|x|} e^{-\varepsilon s} ds = \frac{1}{\varepsilon}(1 - e^{-\varepsilon|x|}) < \frac{1}{\varepsilon}.$$

This implies that  $B(o, R) = B_\varepsilon(o, (1/\varepsilon)(1 - e^{-\varepsilon R}))$  and  $X_\varepsilon \subset B(o, 1/\varepsilon)$ .

Let  $x \in \partial X_\varepsilon$ . There is a sequence  $(x_n)$  of  $X$  such that  $x_n$  converges to  $x$ . Since  $X$  is a proper space, it follows that  $|x_n| \rightarrow \infty$ . Furthermore,

$$|x_n|_\varepsilon \geq (1/\varepsilon)(1 - e^{-\varepsilon|x_n|})$$

so that  $|x|_\varepsilon = 1/\varepsilon$ . This establishes the lemma. ■

**Shadows.** Let  $x \in X$ ,  $R > 0$ . The *shadow*  $\mathcal{U}(x, R)$  of  $B(x, R)$  is the set of points  $y$  in  $\overline{X_\varepsilon}$  for which there is a  $d_0$ -geodesic curve joining  $o$  to  $y$  which intersects  $\overline{B(x, R)}$ . Let  $\mathcal{U}_\infty(x, R) = \mathcal{U}(x, R) \cap \partial X_\varepsilon$  be its trace on  $\partial X_\varepsilon$ . When  $R = 1$  we employ the notation  $\mathcal{U}(x)$  for  $\mathcal{U}(x, 1)$ .

**Lemma 3.1.3** *For any  $x, R$ , there is a constant  $C_R > 0$  such that*

$$\text{diam}_\varepsilon \mathcal{U}(x, R) \leq C_R e^{-\varepsilon|x|}.$$

**Proof:** Let  $y \in \mathcal{U}(x, R)$ . There is a geodesic segment  $[o, y]$  and a point  $p \in B(x, R) \cap [o, y]$ . Therefore,

$$|y - x|_\varepsilon \leq |y - p|_\varepsilon + |x - p|_\varepsilon.$$

Since  $|x| - R \leq |p| \leq |x| + R$ , it follows that  $|x - p|_\varepsilon \leq R e^{\varepsilon R} e^{-\varepsilon|x|}$ . Moreover,

$$|p - y|_\varepsilon \leq e^{\varepsilon R} \int_{|p|}^{|y|} e^{-\varepsilon t} dt \leq \frac{e^{\varepsilon R}}{\varepsilon} e^{-\varepsilon|p|} \leq \frac{e^{2\varepsilon R}}{\varepsilon} e^{-\varepsilon|x|}.$$

This establishes the estimate.

■

**Remark.** Shadows are almost round subsets of the boundary. More precisely: if  $X$  is Gromov hyperbolic and  $K$ -quasi-starlike, then, for a fixed  $R$  which is chosen large enough, there is a constant  $C = C(\varepsilon, R, K)$  such that, for any  $x \in X$ , there is a boundary point  $a \in \partial X$  such that

$$B(a, (1/C)e^{-\varepsilon|x|}) \subset \mathcal{U}_\infty(x, R) \subset B(a, Ce^{-\varepsilon|x|}).$$

A proof of this fact can be found in [Coo]. Furthermore, the family  $\{\text{int}(\mathcal{U}(x, R))\}$  defines a basis of neighborhoods in  $\overline{X}_\varepsilon$  for points at infinity.

**Distance to the boundary.** If  $x \in X_\varepsilon$ , we let  $\delta_\varepsilon(x) = \text{dist}_\varepsilon(x, \partial X_\varepsilon)$ .

**Lemma 3.1.4** *If  $X$  is  $K$ -quasi-starlike, then for all  $x \in X$ ,*

$$\frac{e^{-\varepsilon|x|}}{\varepsilon} \leq \delta_\varepsilon(x) \leq C_{K,\varepsilon} \frac{e^{-\varepsilon|x|}}{\varepsilon}.$$

**Proof:** Let  $x \in X$ . We start with a first coarse estimate:

$$\delta_\varepsilon(x) \geq \int_{|x|}^{\infty} e^{-\varepsilon t} dt = \frac{1}{\varepsilon} e^{-\varepsilon|x|}.$$

If there is a ray  $\gamma \in \mathcal{R}_\infty$  such that  $x \in \gamma$ , then

$$\delta_\varepsilon(x) = \int_{|x|}^{\infty} e^{-\varepsilon t} dt = \frac{e^{-\varepsilon|x|}}{\varepsilon} = \frac{\rho_\varepsilon(x)}{\varepsilon}.$$

In general, since  $X$  is  $K$ -quasi-starlike, there is a ray  $\gamma \in \mathcal{R}_\infty$  and a point  $p \in B(x, K) \cap \gamma$ . Therefore,  $|x| - K \leq |p| \leq |x| + K$  and  $|x - p|_\varepsilon \leq C_K e^{-\varepsilon|x|} |x - p|$ . Then

$$\delta_\varepsilon(x) \leq |x - p|_\varepsilon + \delta_\varepsilon(p) \leq C_K e^{-\varepsilon|x|} + \frac{e^{-\varepsilon(|x|-K)}}{\varepsilon} \leq C_{K,\varepsilon} e^{-\varepsilon|x|}.$$

■

**Quasi-isometries versus quasimetrics.** A *quasi-isometry*  $f : X \rightarrow Y$  between two metric spaces is a map for which there are constants  $\lambda \geq 1$  and  $c > 0$  such that

1. [bi-Lipschitz in the large] for any  $x, x' \in X$ ,

$$\frac{1}{\lambda}|x - x'| - c \leq |f(x) - f(x')| \leq \lambda|x - x'| + c,$$

2. [nearly surjective] for any  $y \in Y$ , there is some  $x \in X$  such that  $|f(x) - y| \leq c$ .

We note that if  $f : X \rightarrow Y$  is a quasi-isometry, then there exists a quasi-isometry  $g : Y \rightarrow X$  such that  $|g \circ f(x) - x| \leq C$  for some constant  $C < \infty$ .

It is well-known that if  $\Phi : X \rightarrow Y$  is a quasi-isometry between two hyperbolic spaces, then it extends as a quasimetric homeomorphism  $\varphi : \partial X \rightarrow \partial Y$ , if we endow the boundaries with visual metrics. For the converse, we have

**Theorem 3.1.5 (M. Bonk & O. Schramm)** *Let  $X, Y$  be two quasi-starlike hyperbolic space. For any quasimetric homeomorphism  $\varphi : \partial X \rightarrow \partial Y$ , there is a quasi-isometric map  $\Phi : X \rightarrow Y$  which extends  $\varphi$ .*

For a proof, see Theorem 7.4 and Theorem 8.2 in [BS].

## 3.2 Spaces associated to finite branched coverings

Suppose  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is an f.b.c. with repeller  $X$ , and all the conditions on  $\mathfrak{X}_0, \mathfrak{X}_1, f$ , and  $X$  stated at the beginning of §2.2 are satisfied. We assume furthermore that we are given a finite open covering  $\mathcal{U} = \mathcal{U}_0$  of  $X$  by connected subsets of  $\mathfrak{X}_0$  which satisfies axiom [Expansion] in the definition of topologically cxc. We assume neither axiom [Irreducibility] nor axiom [Degree].

Under these assumptions, we prove

**Theorem 3.2.1** *The pair  $(f, \mathcal{U})$  defines a proper, geodesic, unbounded, quasi-starlike, metric space  $\Gamma$  together with a continuous map  $F : \Gamma \rightarrow \Gamma$  with the following property. There is a constant  $\varepsilon_0 > 0$  such that, for any  $\varepsilon \in (0, \varepsilon_0)$ , there exists a homeomorphism*

$$\phi_\varepsilon : X \rightarrow \partial_\varepsilon \Gamma$$

such that  $\phi_\varepsilon \circ f = F \circ \phi_\varepsilon$ .

The map  $F$  extends as a Lipschitz map  $F : \overline{\Gamma}_\varepsilon \rightarrow \overline{\Gamma}_\varepsilon$  sending the boundary to the boundary, and,  $F(B(\xi, re^{-\varepsilon})) = B(F(\xi), r)$  holds for any  $\xi \in \overline{\Gamma} \setminus \{o\}$  and any  $r \in (0, |F(\xi)|_\varepsilon)$ .

In the above theorem,  $\Gamma_\varepsilon$  and  $\partial_\varepsilon \Gamma$  are defined as in the previous section, and  $B(\zeta, r)$  denotes the ball of radius  $r$  about  $\zeta$  in  $\overline{\Gamma}_\varepsilon$ .

Note the similarity of this statement with the case of hyperbolic groups.

In the next section, we investigate more closely the geometry of  $\overline{\Gamma}_\varepsilon$ .

**Definition of  $\Gamma$ .** From the map  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  and the cover  $\mathcal{U}$ , we construct a topological space  $\Gamma$  which is a graph equipped with a distinguished basepoint and which we define as follows.

The set  $V(\Gamma)$  of vertices is the union of the elements of  $\mathbf{U} = \cup_{n \geq 0} \mathcal{U}_n$ , together with a base vertex  $o = \{X\}$ . It will be convenient to reindex the levels as follows. For  $n \in \mathbb{N}$  set

$$S(n) = \begin{cases} \mathcal{U}_{n-1} & \text{if } n \geq 1, \\ \{o\} & \text{if } n = 0. \end{cases}$$

For  $n \in \mathbb{N}$  and a vertex  $W \in S(n)$ , we set  $|W| = n$ . Thus,  $V(\Gamma) = \cup_{n \geq 0} S(n)$ .

Two vertices  $W_1, W_2$  are joined by an edge if

$$||W_1| - |W_2|| \leq 1 \quad \text{and} \quad W_1 \cap W_2 \cap X \neq \emptyset.$$

This definition forbids loops from a vertex to itself and multiple edges between vertices, so  $\Gamma$  is indeed a graph as claimed. The graph  $\Gamma$  is turned into a geodesic metric space in the usual way by decreeing that each closed edge is isometric to the Euclidean unit interval  $[0, 1]$ . Since each  $S(n)$  is finite, the valence at each vertex is bounded (though not necessarily uniformly so) and so  $\Gamma$  is proper. Since as subsets of  $\mathfrak{X}_0$ , any vertex  $W \in S(n)$  intersects a set  $W' \in S(n-1)$ , any vertex  $W$  can be joined to the basepoint  $o$  by a geodesic ray in  $\Gamma$ . Hence  $\Gamma$  is connected. It is also  $1/2$ -quasi-starlike, since the midpoint of edges joining vertices at the same level lies at distance  $1/2$  from every geodesic ray emanating from  $o$ . By construction,  $S(n)$  is the sphere of radius  $n$  about the origin  $o$ .

**Action of the finite branched covering.** If  $n \geq 2$  and  $W \in S(n) = \mathcal{U}_{n-1}$ , then as subsets of  $\mathfrak{X}_0$ ,  $f(W) \in S(n-1) = \mathcal{U}_{n-2}$ , so  $f$  induces a map

$F : \cup_{n \geq 2} S(n) \rightarrow \cup_{n \geq 2} S(n-1)$ . Define  $F(W) = o$  for all  $W \in S(1) \cup S(0)$ ; thus  $F$  is defined on the vertex set  $V(\Gamma)$ . To extend  $F$  over edges, observe that if  $n \geq 1$  and if (as subsets of  $\mathfrak{X}_0$ )  $\widetilde{W}, \widetilde{W}'$  are distinct inverse images of  $W$ , then  $\widetilde{W}, \widetilde{W}'$ , being distinct components of the inverse image, cannot intersect. Thus, if  $W_1, W_2$  are joined by an edge, the definition of edges given above then implies that either

1.  $F(W_1) \neq F(W_2)$  and  $F(W_1), F(W_2)$  are joined by an edge, or
2.  $|W_1|, |W_2| \leq 1$  and  $F(W_1) = F(W_2)$ .

Letting  $E$  be the union of edges joining pairs of elements of  $S(1)$ , properties (1) and (2) above imply that  $F$  extends naturally to a continuous map  $F : \Gamma \rightarrow \Gamma$  which collapses  $\overline{B(o, 1)} \cup S(0) \cup E \rightarrow \{o\}$ , and which otherwise sends all edges homeomorphically onto their images.

### Properties of $F$ .

- $F$  is 1-Lipschitz.
- $F$  decreases levels by one:  $|F(\xi)| = |\xi| - 1$  for all  $|\xi| \geq 1$ .
- $F$  sends rays to rays:  $F : (\mathcal{R}, \mathcal{R}_\infty) \rightarrow (\mathcal{R}, \mathcal{R}_\infty)$
- $F$  has the path lifting property for paths which avoid the base point  $o$ : any path  $\gamma$  in  $\Gamma \setminus \{o\}$  can be lifted by  $F^{-1}$ .

Once a basepoint has been chosen, the only ambiguity in defining the lift arises from vertices corresponding to a component on which  $f$  is non-injective. In the sequel of the paper, we will use this property without mentioning it explicitly.

Lifts preserve lengths: if  $\gamma'$  is a lift of a curve  $\gamma$ , then  $\ell(\gamma) = \ell(\gamma')$ .

- $F$  maps shadows onto shadows: for any  $|\xi| \geq 2$ ,  $F(\mathcal{U}(\xi)) = \mathcal{U}(F(\xi))$ .

To see this, note that since  $F$  maps rays to rays, it follows that  $F(\mathcal{U}(\xi)) \subset \mathcal{U}(F(\xi))$ . For the converse, let  $\zeta \in \mathcal{U}(F(\xi))$  and let us consider a geodesic curve  $\gamma$  joining  $F(\xi)$  to  $\zeta$ . The function  $t \mapsto |\gamma(t)|$  is strictly monotone. Since  $F$  has the lifting property, there is a strictly monotonic geodesic curve  $\gamma'$  starting from  $\xi$  such that  $F(\gamma') = \gamma$ . This curve can be extended geodesically to the base point  $o$ . It follows that  $\mathcal{U}(F(\xi)) \subset F(\mathcal{U}(\xi))$ , which proves the claim.

Let  $\xi, \zeta \in \Gamma \setminus \overline{B(o, 1)}$  and let  $\gamma$  be a geodesic segment joining these points. Then

$$d_\varepsilon(F(\xi), F(\zeta)) \leq \int_{F(\gamma)} e^{-\varepsilon|F(\gamma(t))|} dF \leq \int_\gamma e^{-\varepsilon(|\gamma(t)|-1)} dt \leq e^\varepsilon d_\varepsilon(\xi, \zeta).$$

Therefore  $F$  is uniformly continuous so it extends to an  $e^\varepsilon$ -Lipschitz map  $F : \overline{\Gamma}_\varepsilon \rightarrow \overline{\Gamma}_\varepsilon$ .

**Proposition 3.2.2** *For any  $\xi \in \overline{\Gamma}_\varepsilon$ , and  $r < |F(\xi)|_\varepsilon$ ,  $F(B_\varepsilon(\xi, re^{-\varepsilon})) = B_\varepsilon(F(\xi), r)$ .*

Hence  $F$  is an open mapping.

**Proof:** We already know that  $F(B_\varepsilon(\xi, re^{-\varepsilon})) \subset B_\varepsilon(F(\xi), r)$ . Let us consider  $\zeta' \in B_\varepsilon(F(\xi), r)$  and  $\gamma'$  a  $d_\varepsilon$ -geodesic curve joining  $F(\xi)$  to  $\zeta'$ . Since  $r < |F(\xi)|_\varepsilon$ , it follows that  $\gamma'$  avoids  $o$ . We let  $\gamma$  be a lift of  $\gamma'$  which joins  $\xi$  to a point  $\zeta \in \overline{\Gamma}_\varepsilon$ . It follows that

$$|\xi - \zeta|_\varepsilon \leq \ell_\varepsilon(\gamma) = \int_\gamma \rho_\varepsilon(\xi) ds(\xi) = e^{-\varepsilon} \int_\gamma \rho_\varepsilon(F(\xi)) ds(\xi) = e^{-\varepsilon} \int_{\gamma'} \rho_\varepsilon(\xi) ds(\xi)$$

so  $|\xi - \zeta|_\varepsilon \leq e^{-\varepsilon}|F(\xi) - \zeta'|_\varepsilon \leq e^{-\varepsilon}r$  and  $\zeta \in B_\varepsilon(\xi, e^{-\varepsilon}r)$ . ■

The following proposition says that if  $F^n$  is injective on a ball, then it is a similarity on the ball of one-fourth the size.

**Proposition 3.2.3** *Suppose  $B = B(\xi, r) \subset \overline{\Gamma}_\varepsilon$  and  $F^n|_B : B \rightarrow B(F(\xi), e^{n\varepsilon}r)$  is a homeomorphism. Then for all  $\zeta_1, \zeta_2 \in B(\xi, r/4)$ ,*

$$|F^n(\zeta_1) - F^n(\zeta_2)|_\varepsilon = e^{n\varepsilon}|\zeta_1 - \zeta_2|_\varepsilon.$$

**Proof:** We first claim that the above equality holds when  $\zeta_1 = \xi$  and  $\zeta = \zeta_2$  is an arbitrary point in  $B$ . The upper bound is clear. To show the lower bound, notice that  $F^{-n} : B(F^n(\xi), re^{\varepsilon n}) \rightarrow B(\xi, r)$  is well defined, and let  $\gamma \subset B(F^n(\xi), re^{\varepsilon n})$  be a curve joining  $F^n(\xi)$  to  $F^n(\zeta)$ . It follows that  $F^{-n}(\gamma)$  is a curve joining  $\xi$  to  $\zeta$  inside  $B$ , so the proof of Proposition 3.2.2 shows that

$$\ell_\varepsilon(F^{-n}(\gamma)) = e^{-\varepsilon n} \ell_\varepsilon(\gamma).$$

Since  $F^n|_B$  is a homeomorphism, the claim follows.

The proposition follows immediately by applying the claim to the ball centred at  $\zeta_1$  of radius  $|\zeta_1 - \zeta_2|_\varepsilon$ , which by hypothesis is contained in  $B$  and hence maps homeomorphically onto its image under  $F^n$ .

■

**Comparison of  $X$  and  $\partial\Gamma$ .** For any  $x \in X$  and  $n \in \mathbb{N}$ , let  $W_n \in S(n)$  contain  $x$ . The sequence  $(W_n)$  defines a ray  $\gamma_x$  in  $\mathcal{R}_\infty$  such that  $\gamma_x(n) = W_n$ . There is a natural map  $\phi_f : X \rightarrow \partial_\varepsilon\Gamma$  defined by  $\phi_f(x) = \pi(\gamma_x)$ . In other words: the sequence  $(W_n)$  is a Cauchy sequence in  $\overline{\Gamma}_\varepsilon$ , and we let  $\phi_f(x)$  be its limit. This map is well defined: if  $(W'_n)$  is another sequence contained in a ray  $\gamma'_x$ , then  $d(W_n, W'_n) \leq 1$  since  $x \in W_n \cap W'_n \cap X$ , so  $\pi(\gamma_x) = \pi(\gamma'_x)$ . Furthermore,  $F \circ \phi_f = \phi_f \circ f$  on  $X$ .

**Proposition 3.2.4** *The map  $\phi_f : X \rightarrow \partial_\varepsilon\Gamma$  is continuous and onto.*

**Proof:** To prove surjectivity, suppose  $\xi \in \partial\Gamma_\varepsilon$ . By Lemma 3.1.1, there exists a ray  $\gamma \in \mathcal{R}_\infty$  such that  $\pi(\gamma) = \xi$ . For  $k \in \mathbb{N}$  let  $W_k = \gamma(k)$ , so that  $W_k \in S(k)$ . Then  $W_k \in \overline{\Gamma}_\varepsilon$  and  $\xi = \lim W_k$ . But each  $W_k$  is also a subset of  $\mathfrak{X}_0$  whose intersection with the repeller  $X$  contains some point  $w_k$ . Since  $X$  is compact, there exists a limit point  $x$  of  $(w_k)$ .

We claim  $\phi_f(x) = \xi$ . By definition  $\phi_f(x) = \lim V_n$ , where  $V_n$  is an arbitrary element of  $S(n)$  which as a subset of  $X$  contains  $x$  and where the limit is in  $\overline{\Gamma}_\varepsilon$ . Then for each  $n \in \mathbb{N}$ , since  $V_n$  is open and  $w_k \rightarrow x$ , there exists  $k(n) \in \mathbb{N}$  such that  $W_k \cap V_n \cap X \neq \emptyset$  for all  $k \geq k(n)$ . By the definition of shadows,  $W_k \subset \mathcal{U}(V_n)$ . By Lemma 3.1.3,  $|W_{k(n)} - V_n|_\varepsilon \rightarrow 0$  as  $k \rightarrow \infty$ . Hence  $\xi = \lim W_k = \lim W_{k(n)} = \lim V_n = \phi_f(x)$  as required.

To prove continuity, suppose  $x_k \rightarrow x \in X$ . For all  $n \in \mathbb{N}$  choose  $W_n \in S(n)$  containing  $x$ , so that  $\xi = \phi_f(x) = \lim W_n \in \mathcal{U}_\infty(W_n)$ . Then for all  $n \in \mathbb{N}$  there exists  $k(n)$  such that  $x_k \in W_n$  for all  $k \geq k(n)$ . By the definition of  $\phi_f$ ,  $\phi_f(x_k) \in \mathcal{U}_\infty(W_n)$ . By Lemma 3.1.3,  $|\phi_f(x_k) - \xi|_\varepsilon \leq Ce^{-\varepsilon n} \rightarrow 0$  as  $n \rightarrow \infty$  and so  $\phi_f(x_k) \rightarrow \xi = \phi_f(x)$ .

■

We now turn to the proof of Theorem 3.2.1. We first prove the existence of a preliminary metric in which the diameters of the sets  $\phi_f(U)$ ,  $U \in \mathcal{U}_n$  tend to zero exponentially fast in  $n$ .



**Theorem 3.2.5** *Suppose Axiom [Expansion] holds. Then there exists a metric on the repeller  $X$  and constants  $C > 1, \theta < 1$  such that for all  $n \geq 0$ ,*

$$\sup_{U \in \mathcal{U}_n} \text{diam} U \leq C\theta^n.$$

The proof is standard and mimics the proof of a preferred Hölder structure given a uniform structure; see [Bou].

**Proof:** Let  $N_0$  be given by Proposition 2.4.1, 2(c) and put  $g = f^{N_0}$ ,  $\mathcal{V}_0 = \bigcup_{j=0}^{N_0-1} \mathcal{U}_j$ ,  $\mathfrak{Y}_1 = f^{-N_0} \mathfrak{X}_0$ ,  $\mathfrak{Y}_0 = \mathfrak{X}_0$ , and  $\mathcal{V}_n = g^{-n} \mathcal{V}_0$ . Then  $g$  is a finite branched covering, the repeller of  $g$  is  $X$  (by total invariance), and the mesh of  $\mathcal{V}_n$  tends to zero. The conclusion of the above proposition (applied  $U'_1 = U'_2 = V'$ ) and the definition of  $g$  implies

$$\forall V' \in \mathcal{V}_n, \exists V \in \mathcal{V}_{n-1} \text{ with } V' \subset V. \quad (3.1)$$

This and conclusion 2(b) of Proposition 2.4.1 imply immediately that for any distinct  $x, y \in X$ , the quantity

$$[x|y] = \max\{n : \forall 1 \leq i \leq n, \exists V_i \in \mathcal{V}_i \text{ with } \{x, y\} \subset V_i\}$$

is finite. For  $x = y$  set  $[x, y] = \infty$ . The statement (3.1), Proposition 2.4.1 (c) and the definition of  $g$  imply for any triple  $x, y, z \in X$ ,

$$[x|z] \geq \min\{[x|y], [y|z]\} - 1.$$

Fix  $\epsilon > 0$  small, and define

$$\varrho_\epsilon(x, y) = \exp(-\epsilon[x|y]).$$

Then  $\varrho_\epsilon(x, y) = 0$  if and only if  $x = y$ , and indeed  $\varrho_\epsilon$  satisfies all properties of a metric save the triangle inequality. Instead, we have

$$\varrho_\epsilon(x, z) \leq e^\epsilon \max\{\varrho_\epsilon(x, y), \varrho_\epsilon(y, z)\}.$$

There is a standard way to extract a metric bilipschitz equivalent to  $\varrho_\epsilon$ . If  $\epsilon < \frac{1}{2} \log 2$  then Proposition 7.3.10 of [GdlH] implies that there is a metric  $d_\epsilon$  on  $X$  satisfying

$$(1 - 2\sqrt{2})\varrho_\epsilon(x, y) \leq d_\epsilon(x, y) \leq \varrho_\epsilon(x, y).$$

Letting  $\text{diam}_\epsilon$  denote diameter with respect to  $d_\epsilon$ , it is clear from the definitions that  $V \in \mathcal{V}_n \Rightarrow \text{diam}_\epsilon V \leq \exp(-n\epsilon)$ . It is then easy to check that taking  $\theta = \exp(-\epsilon)$  and

$$C = \max\{\text{diam}_\epsilon U : U \in \cup_{i=0}^{N_0-1} \mathcal{U}_i\}$$

will do. ■

We may now prove Theorem 3.2.1.

**Proof:** (Theorem 3.2.1) It follows from Theorem 3.2.5 that there is a metric and constants  $C > 0$  and  $\theta \in (0, 1)$  such that, for any  $W \in S(n)$ ,  $\text{diam} W \leq C\theta^n$ . Let  $x, y \in X$  and let us consider a curve  $\gamma = (W_n)_{n \in \mathbb{Z}}$  joining  $\phi_f(x)$  to  $\phi_f(y)$ . It follows that

$$\ell_\epsilon(\gamma) \asymp \sum e^{-\epsilon|W_n|} = \sum \frac{e^{-\epsilon|W_n|}}{\text{diam} W_n} \text{diam} W_n.$$

If  $\epsilon > 0$  is small enough, then  $e^{-\epsilon|W_n|} \geq (1/C)\theta^{|W_n|} \geq (1/C)\text{diam} W_n$ . Furthermore, there are points  $(z_n)$  such that  $z_n \in W_n \cap W_{n+1}$ . For all  $k \in \mathbb{N}$ , we let  $\gamma_k$  be the subcurve of  $\gamma$  joining  $W_{-k}$  to  $W_k$ . Then

$$\begin{aligned} \ell_\epsilon(\gamma_k) &\geq (1/C) \sum_{|n| \leq k} \text{diam} W_n \\ &\geq (1/C) \sum_{|n| \leq k} d_X(z_n, z_{n+1}) \\ &\geq (1/C) d_X(z_{-k}, z_k). \end{aligned}$$

Since  $\gamma_k$  is a subset of  $\gamma$ ,  $\{z_{-k}, z_k\}$  tends to  $\{x, y\}$ , this implies that  $\ell_\epsilon(\gamma) \geq (1/C)d_X(x, y)$ , where  $C$  is independent of  $\gamma$ . Therefore,  $d_\epsilon(\phi_f(x), \phi_f(y)) \geq (1/C)d_X(x, y)$ .

The fact that  $F$  is Lipschitz and maps balls to balls is the content of Proposition 3.2.2. ■

### 3.3 Geometry of $\Gamma$

Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  and  $\mathcal{U}$  satisfy the conditions listed at the beginning of §3.2. The main result of this section is

**Theorem 3.3.1** *If  $(f, \mathcal{U})$  satisfies [Expansion], then  $\Gamma$  is Gromov hyperbolic. If  $(f, \mathcal{V})$  also satisfies [Expansion], then  $\Gamma(f, \mathcal{U})$  is quasi-isometric to  $\Gamma(f, \mathcal{V})$ . If  $g = f^n : \mathfrak{X}_{n-1} \rightarrow \mathfrak{X}_0$ , then  $\Gamma(g, \mathcal{U})$  is quasi-isometric to  $\Gamma(f, \mathcal{U})$ .*

Hence, as long as the expansion axiom is satisfied, the quasi-isometry class of  $\Gamma$  is an invariant of the conjugacy class of  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$ .

The proof of Thm 3.3.1 will follow from both Proposition 3.3.10 and Proposition 3.3.12.

The naturality results given above are analogous to those enjoyed by Cayley graphs of finitely generated groups.

#### 3.3.1 Metric estimates

We start by gathering information on the geometry of balls, and how they interact with the coverings.

Our main estimates are the following, which assert that the elements  $\phi_f(W)$  enjoy geometric properties with respect to the metric  $d_\varepsilon$  similar to those enjoyed by the sets  $U$  with respect to a metric for a cxc map; compare Propositions 2.6.2 (Uniform roundness) and 2.6.6 (Balls are like connected sets).

**Notation.** For an element  $W \in S(n)$ , regarded as a subset of  $\mathfrak{X}_1$ , we denote by  $\phi_f(W)$  the set  $\phi_f(W \cap X)$ .

**Proposition 3.3.2** 1. *There is some constant  $C > 1$  such that, for all  $W \in V \setminus \{o\}$ , there is a point  $\xi \in \phi_f(W)$  so that*

$$B_\varepsilon(\xi, (1/C)e^{-\varepsilon|W|}) \subset \phi_f(W) \subset \mathcal{U}_\infty(W) \subset B_\varepsilon(\xi, Ce^{-\varepsilon|W|}).$$

2. *There is a radius  $r_1 > 0$  such that, for any  $n \geq 1$  and for any  $\xi \in \partial_\varepsilon \Gamma$ , there is some  $W \in S(n)$  so that  $B_\varepsilon(\xi, r_1 e^{-\varepsilon n}) \subset \phi_f(W)$ .*

3. *A maximal radius  $r_0 > 0$  exists such that, for any  $r \in (0, r_0)$  and any  $\xi \in \partial_\varepsilon \Gamma$ , there exist  $W$  and  $W'$  in  $\mathbf{U}$  such that  $|W - W'| = O(1)$ ,*

$$\phi_f(W') \subset B_\varepsilon(\xi, r) \subset \phi_f(W),$$

and

$$\max\{\text{Round}(\phi_f(W), \xi), \text{Round}(\phi_f(W'), \xi)\} = O(1).$$

Since the set  $\mathfrak{X}_0$  is not assumed to be endowed with a metric, we shall use uniform structures [Bou].

Since  $\mathfrak{X}_1$  has compact closure in  $\mathfrak{X}_0$ , there is a unique uniform structure on  $\overline{\mathfrak{X}_1}$  compatible with its topology. We consider the uniform structure on  $\mathfrak{X}_1$  induced by the one on  $\overline{\mathfrak{X}_1}$ . Let us recall that an *entourage*  $\Omega$  is a neighborhood of the diagonal of  $\mathfrak{X}_1 \times \mathfrak{X}_1$ . If  $x \in \mathfrak{X}_1$ , then  $\Omega(x) = \{y \in \mathfrak{X}_1, (x, y) \in \Omega\}$  and

$$\Omega \circ \Omega = \Omega^2 = \{(x, z) \in \mathfrak{X}_1 \times \mathfrak{X}_1, \exists y \in \mathfrak{X}_1, (x, y) \in \Omega, (y, z) \in \Omega\}.$$

**Proposition 3.3.3** *Given an entourage  $\Omega$ , there is some constant  $r = r(\Omega) > 0$  such that, whenever  $U \in S(1)$ ,  $u \in U \cap X$  and  $\Omega(u) \subset U$ , then, for any  $n \geq 1$ , any  $\tilde{U} \in S(n)$  such that  $f^{n-1}(\tilde{U}) = U$ , and any preimage  $\tilde{u} \in \tilde{U} \cap f^{-(n-1)}(\{u\})$ , the ball  $B_\varepsilon(\phi_f(\tilde{u}), r\varepsilon^{-n})$  is contained in  $\phi_f(\tilde{U})$ .*

Let us first prove some lemmata.

**Lemma 3.3.4** *Let  $\gamma : \mathbb{R} \rightarrow \Gamma$  be a curve such that  $\gamma(\mathbb{Z}) \subset V \setminus \{o\}$  and which connects two points  $u$  and  $v$  from the boundary. Let  $r \in (0, 1/\varepsilon)$ . If  $\ell_\varepsilon(\gamma) < r$  then*

$$\overline{\cup_{n \in \mathbb{Z}} \phi_f(\gamma(n))} \subset B_\varepsilon(u, r).$$

**Proof:** For any fixed  $n$  and any  $z \in \phi_f(\gamma(n))$ ,

$$|z - u|_\varepsilon \leq |z - \gamma(n)|_\varepsilon + |\gamma(n) - u|_\varepsilon.$$

But Lemma 3.1.4 implies that

$$|z - \gamma(n)|_\varepsilon = \text{dist}_\varepsilon(\gamma(n), \partial_\varepsilon \Gamma) \leq |\gamma(n) - v|_\varepsilon \leq \ell_\varepsilon(\gamma|_{[n, \infty[})$$

and  $|\gamma(n) - u|_\varepsilon \leq \ell_\varepsilon(\gamma|_{]-\infty, n])}$  so that  $|z - u|_\varepsilon \leq \ell_\varepsilon(\gamma) < r$ . Therefore  $z \in \overline{B_\varepsilon(u, \ell_\varepsilon(\gamma))}$  and  $\phi_f(\gamma(n)) \subset \overline{B_\varepsilon(u, \ell_\varepsilon(\gamma))}$  for all  $n \in \mathbb{Z}$ . Hence

$$\overline{\cup_{n \in \mathbb{Z}} \phi_f(\gamma(n))} \subset B_\varepsilon(u, r).$$

■

**Definition.** Given  $u \in \partial_\varepsilon \Gamma$  and  $r \in (0, 1/\varepsilon)$ , we let  $V(u, r)$  be the set of all vertices of  $\Gamma$  contained in curves of  $d_\varepsilon$ -length less than  $r$  joining  $u$  to another boundary point.

It follows from the lemma above that  $\phi_f(U) \subset B_\varepsilon(u, r)$  for any  $U \in V(u, r)$ .

**Lemma 3.3.5** *There is a function  $n : (0, 1/\varepsilon) \rightarrow \mathbb{N}$  tending to infinity as  $r$  goes to 0 such that, the level of any  $U \in V(u, r)$  is at least  $n(r)$ .*

**Proof:** Let  $U \in V(u, r)$  and denote by  $\gamma$  a curve defining  $U$ . One has

$$\frac{1}{\varepsilon} e^{-\varepsilon|U|} = \text{dist}_\varepsilon(U, \partial\Gamma) \leq \ell_\varepsilon(\gamma) < r$$

so that  $|U| \geq (1/\varepsilon) \log 1/(\varepsilon r)$ . ■

**Lemma 3.3.6** *Let  $\Omega$  be an entourage of  $\mathfrak{X}_1$ . There is a radius  $r > 0$  which depends only on  $\Omega$  such that, for any  $u \in \partial\Gamma$ ,  $W \subset \Omega(\phi_f^{-1}(u))$  whenever  $W \in V(u, r)$ .*

**Proof:** Let us consider an entourage  $\Omega_0$  such that  $\Omega_0^2 \subset \Omega$ .

Since [Expansion] holds, an integer  $n_0$  exists so that, given any  $n \geq n_0$ , any  $x \in X$  and any  $W \in S(n)$  with  $x \in W$ , the inclusion  $W \subset \Omega_0(x)$  holds.

The uniform continuity of  $\phi_f^{-1}$  provides us with a first radius  $r_1 > 0$  (independent from  $u$ ) such that, for any  $u \in \partial\Gamma$ ,  $\phi_f^{-1}(B_\varepsilon(u, r_1)) \subset \Omega_0(\phi_f^{-1}(u))$ .

We choose  $r \in (0, r_1)$  as large as possible so that  $n(r) \geq n_0$ . Note that  $r$  depends only on  $\Omega$ . Thus, if  $W \in V(u, r)$  and  $v \in \phi_f(W)$  then  $\phi_f^{-1}(v) \in \Omega_0(\phi_f^{-1}(u))$  by Lemma 3.3.4. Besides,  $|W| \geq n_0$ , so that  $W \subset \Omega_0(\phi_f^{-1}(v))$ . Therefore,  $W \subset \Omega_0^2(\phi_f^{-1}(u)) \subset \Omega(\phi_f^{-1}(u))$ . ■

We are now ready for the proofs of the Propositions.

**Proof:** (Proposition 3.3.3). Let  $\Omega$  be an entourage of  $\mathfrak{X}_1$ ,  $U \in S(1)$ ,  $u \in U \cap X$ , satisfy  $\Omega(u) \subset U$ . Let us choose another entourage  $\Omega_0$  such that  $\overline{\Omega_0} \subset \Omega$ .

Choose  $n \geq 1$ ,  $\tilde{U} \in S(n)$  and  $\tilde{u} \in f^{-(n-1)}(\{u\}) \cap \tilde{U}$  such that  $f^{n-1}(\tilde{U}) = U$ . Let us also consider the constant  $r > 0$  given by Lemma 3.3.6 applied to  $\Omega_0$ .

Let  $\tilde{v} \in \phi_f^{-1}(B_\varepsilon(\phi_f(\tilde{u}), re^{-\varepsilon(n-1)}))$  and  $\gamma$  be a curve joining  $\phi_f(\tilde{u})$  to  $\phi_f(\tilde{v})$  of  $d_\varepsilon$ -length less than  $re^{-\varepsilon(n-1)}$ . Set

$$K = \overline{\cup_{n \in \mathbb{Z}} \gamma(n)} \subset \mathfrak{X}_0.$$

Then  $K$  is a continuum by definition which joins  $\tilde{u}$  to  $\tilde{v}$ . Therefore,  $f^{n-1}(K)$  joins  $u$  to  $f^{n-1}(\tilde{v}) = v$ , and  $F^{n-1}(\phi_f(K)) \subset B_\varepsilon(\phi_f(u), r)$ . By Lemma 3.3.6,  $f^{n-1}(\gamma(k))$  is in  $\Omega_0(\phi_f^{-1}(u))$  for any  $k \in \mathbb{Z}$ , so that  $f^{n-1}(K) \subset U$ . It follows that  $K \subset \tilde{U}$  since  $f^{n-1} : \tilde{U} \rightarrow U$  is proper and  $K$  is connected. ■

**Proof:** (Proposition 3.3.2) Let  $\Omega$  be an entourage such that, for any  $x \in X$ , there is some  $U \in S(1)$  such that  $\Omega(x) \subset U$ .

1. Let  $n$  be the level of  $W$  and pick some  $x' \in (X \cap f^{n-1}(W))$ .

Let  $x \in f^{-(n-1)}(\{x'\}) \cap W$ ; it follows from Proposition 3.3.3 that  $\phi_f(W)$  will contain the ball  $B_\varepsilon(\xi, re^{-\varepsilon n})$  where  $r = r(\Omega)$  and  $\xi = \phi_f(x)$ .

Furthermore, Lemma 3.1.3 implies that  $\text{diam}_\varepsilon \phi_f(W) \asymp \text{diam}_\varepsilon \mathcal{U}(W) \asymp e^{-\varepsilon|W|}$ . It follows that there is some constant  $C > 1$  such that, for all  $W \in V$ , there is a point  $\xi \in \phi_f(W)$  so that

$$B_\varepsilon(\xi, (1/C)e^{-\varepsilon|W|}) \subset \phi_f(W) \subset \mathcal{U}_\infty(W) \subset B_\varepsilon(\xi, Ce^{-\varepsilon|W|}).$$

2. Similarly, Proposition 3.3.3 implies that, for any  $n \geq 1$ , there is some  $W \in S(n)$  such that  $\phi_f(W)$  will contain the ball  $B_\varepsilon(\xi, r_1 e^{-\varepsilon n})$  where  $r_1 = r(\Omega)$  is given by the proposition.
3. Fix  $r \in (0, \delta)$  and  $\xi \in \partial\Gamma$ , where  $\delta$  is the Lebesgue number of  $S(1)$  in  $\partial_\varepsilon \Gamma$ .

It follows from point 1. above that, for any  $n$  and any  $W \in S(n)$ ,  $\text{diam}_\varepsilon \phi_f(W) \asymp e^{-\varepsilon n}$ .

Moreover, from point 2., there is some  $W$  such that  $\text{Round}(\phi_f(W), \xi) = O(1)$  and  $B_\varepsilon(\xi, r_1 e^{-\varepsilon n}) \subset \phi_f(W)$ . Let  $m \geq n$  be so that the diameter of any element of  $S(m)$  is at most  $r$ . It follows from the diameter control

above that  $m$  may be chosen so that  $|m - n| = O(1)$ . Point 2. provides us with an element  $W' \in S(m)$  so that  $\text{Round}(\phi_f(W'), \xi) = O(1)$  and  $\phi_f(W') \subset B_\varepsilon(\xi, r)$ .

■

As a consequence of Proposition 3.3.2 and its proof, we obtain the following. For  $n \in \mathbb{N}$ , let  $\mathcal{V}_n = \{\phi_f(U) : U \in \mathcal{U}_n\}$ . Thus for each  $n$ ,  $\mathcal{V}_n$  is a covering of  $\partial_\varepsilon \Gamma$  by open sets which, in general, need not be connected.

**Proposition 3.3.7** *The map  $F : \partial_\varepsilon \Gamma \rightarrow \partial_\varepsilon \Gamma$  and the sequence of coverings  $\mathcal{V}_n, n = 0, 1, 2, \dots$  together satisfy Axioms [Roundness distortion] and [Relative diameter distortion].*

**Remark:** If in addition axiom [Degree] is satisfied, it would be tempting to assert that  $F : \partial_\varepsilon \Gamma \rightarrow \partial_\varepsilon \Gamma$  is also metrically cxc. However, even though  $\bar{\Gamma}$  is locally connected (since shadows define connected neighborhoods of points at infinity), the boundary  $\partial_\varepsilon \Gamma$  need not be (locally) connected. Our definition of metrically cxc is not purely intrinsic to the dynamics on the repeller  $X$  since we require that the covering  $\mathcal{U}_0$  consists of connected sets which are contained in an a priori larger space  $\mathfrak{X}_1$ . Unfortunately, in general we do not know how to modify the definition of  $\Gamma$  so that  $F : \bar{\Gamma}_\varepsilon \rightarrow \bar{\Gamma}_\varepsilon$  becomes a finite branched covering map on an open connected neighborhood of  $\partial_\varepsilon \Gamma$ . If this were possible, it seems likely that one could then establish a variant of Proposition 3.3.7 in which the conclusion asserted that the model dynamics was indeed metrically cxc.

**Proof:** The forward and backward relative diameter distortion bounds follow immediately from Proposition 3.3.2. Since  $F$  maps round balls in the metric  $d_\varepsilon$  to round balls, the forward roundness function  $\rho_+$  may be taken to be the identity. We claim that we may take the backward roundness distortion function to be linear.

First, suppose  $F^n : (\tilde{V}, \tilde{\xi}) \rightarrow (V, \xi)$  where  $V = \phi_f(W)$  and  $W \in S(k)$ . Suppose  $B(\xi, r) \subset V \subset B(\xi, Kr)$ . Then  $Kr \asymp e^{-\varepsilon k}$ . Proposition 3.3.3 shows that  $B(\tilde{\xi}, ce^{-\varepsilon n} r) \subset \tilde{V}$  for some uniform constant  $c > 0$ . By Proposition 3.3.2,  $\text{diam}_\varepsilon(\tilde{V}) \asymp e^{-(n+k)\varepsilon}$ . Hence

$$\text{Round}(\tilde{V}, \tilde{\xi}) \lesssim \frac{e^{-(n+k)\varepsilon}}{e^{-n\varepsilon} r} \asymp K \asymp \text{Round}(V, \xi).$$

■

**Proposition 3.3.8** *Suppose axiom [Expansion] holds. Let  $Y$  denote the set of points  $y$  in  $X$  such that there exists an element  $U'$  of  $\mathbf{U}$  containing  $y$  such that all iterated preimages  $\tilde{U}'$  of  $U'$  map by degree one onto  $U'$ .*

*If Axiom [Degree] fails, and if  $Y \cap X$  is dense in  $X$ , then  $\partial_\varepsilon \Gamma$  fails to be doubling.*

**Remarks:**

1. We have always  $f^{-1}(Y) \subset Y$ . If Axiom [Irreducibility] holds and  $Y$  is nonempty then  $Y$  is dense in  $X$ , so the above proposition implies that  $\partial_\varepsilon \Gamma$  fails to be doubling.
2. It is reasonable to surmise that  $Y = X - P_f$ —this is the case e.g. for rational maps. However, we are neither able to prove this assertion nor find counterexamples.

**Proof:** Suppose Axiom [Degree] fails. It follows easily that then there exists some  $U \in \mathcal{U}_0$  such that for all  $p \in \mathbb{N}$ , there exists  $n \in \mathbb{N}$  and a preimage  $\tilde{U} \in \mathcal{U}_n$  of  $U$  such that  $f^n : \tilde{U} \rightarrow U$  has degree  $\geq p$ . The assumption and axiom [Expansion] imply that there exists  $U' \subset U$ ,  $U' \in \mathcal{U}_N$  independent of  $p$  and of  $\tilde{U}$  such that  $\tilde{U}$  contains at least  $p$  disjoint preimages  $\tilde{U}'$  of  $U'$ .

By Proposition 3.3.2,  $\phi_f(\tilde{U})$  and  $\phi_f(\tilde{U}')$  are uniformly almost round,  $\text{diam}_\varepsilon(\phi_f(\tilde{U})) \asymp \exp(-\varepsilon n)$ , and  $\text{diam}_\varepsilon(\phi_f(\tilde{U}')) \asymp \exp(-\varepsilon(n + N))$ . So at least  $p$  balls of radius  $C' \cdot \exp(-\varepsilon(n + N))$  are needed to cover a ball of radius  $C \exp(-\varepsilon n)$ , where  $C', C$  are independent of  $n$ . Therefore  $\partial_\varepsilon \Gamma$  fails to be doubling.

■

We close this section with the following consequence of Proposition 3.3.2 which will be useful in our characterization of rational maps; cf. Definition 2.6.8 and Corollary 2.6.9.

**Corollary 3.3.9** *If for each  $W \in \cup S(n)$ , the sets  $\phi(W \cap X)$  and  $X \setminus \phi(W \cap X)$  are connected, then  $\partial \Gamma$  is linearly locally connected.*



**Proof:** Let us fix  $B_\varepsilon(\xi, r)$ . Proposition 3.3.2, point 3, implies the existence of vertices  $W, W'$  such that  $||W| - |W'|| \leq C_1$  for some universal constant  $C_1$  and such that

$$\phi(W) \subset B_\varepsilon(\xi, r) \subset \phi(W').$$

Therefore,  $\text{diam}_\varepsilon \phi(W) \asymp \text{diam}_\varepsilon B_\varepsilon(\xi, r) \asymp \text{diam}_\varepsilon \phi(W')$ .

If  $\zeta, \zeta' \in B_\varepsilon(\xi, r)$ , then they are connected by  $\phi(W')$  which is connected by assumption. Similarly, if  $\zeta, \zeta' \notin B_\varepsilon(\xi, r)$  then they are joined within  $X \setminus \phi(W)$ . ■

### 3.3.2 Hyperbolicity

We are now ready to prove the first part of Theorem 3.3.1.

**Proposition 3.3.10** *If  $\phi_f : X \rightarrow \Gamma$  is a homeomorphism, then  $\Gamma$  is hyperbolic.*

The proof is an adaptation of Proposition 2.1 in [BP] and its main step is given by the following lemma.

**Lemma 3.3.11** *For any  $W, W' \in V$ ,*

$$\text{diam}_\varepsilon(\phi_f(W) \cup \phi_f(W')) \asymp e^{-\varepsilon(|W|W')}.$$

**Proof:** We assume that  $|W'| \geq |W|$ . We let  $n \in \mathbb{N} \cup \{\infty\}$  be the smallest integer such that

$$\text{dist}_\varepsilon(\phi_f(W), \phi_f(W')) \geq r_1 e^{-\varepsilon n}$$

where  $r_1$  is the constant given by Proposition 3.3.2. Let  $(\xi, \xi') \in \overline{\phi_f(W)} \times \overline{\phi_f(W')}$  satisfy  $\text{dist}_\varepsilon(\phi_f(W), \phi_f(W')) = |\xi - \xi'|_\varepsilon$ . Let  $m = \min\{|W|, n\} - 1$ ; there is some  $C \in S(m)$  such that  $B_\varepsilon(\xi, r_1 e^{-\varepsilon n}) \subset \phi_f(C)$ , so that  $W, W' \in \mathcal{U}(C)$  and

$$\text{diam}_\varepsilon(\phi_f(W) \cup \phi_f(W')) \geq \max\{\text{dist}_\varepsilon(\phi_f(W), \phi_f(W')), \text{diam}_\varepsilon \phi_f(W)\}.$$

The maximum of  $\text{dist}_\varepsilon(\phi_f(W), \phi_f(W'))$  and of  $\text{diam}_\varepsilon \phi_f(W)$  is at least of order  $e^{-\varepsilon m}$ . Hence

$$\text{diam}_\varepsilon(\phi_f(W) \cup \phi_f(W')) \gtrsim e^{-\varepsilon |C|}.$$

Since  $(W|W') \geq |C|$ , it follows that

$$\text{diam}_\varepsilon(\phi_f(W) \cup \phi_f(W')) \gtrsim e^{-\varepsilon(W|W')}.$$

For the other inequality, we let  $\{W_j\}_{0 \leq j \leq |W-W'|}$  be a geodesic chain which joins  $W$  to  $W'$ . For convenience, set  $m = |W|$ ,  $m' = |W'|$  and  $D = |W - W'|$ . Then

$$\begin{aligned} \text{diam}_\varepsilon(\phi_f(W) \cup \phi_f(W')) &\leq \sum_{0 \leq j \leq D} \text{diam}_\varepsilon(\phi_f(W_j)) \\ &\leq \sum_{0 \leq j \leq k} \text{diam}_\varepsilon(\phi_f(W_j)) + \sum_{k+1 \leq j \leq D} \text{diam}_\varepsilon(\phi_f(W_j)) \\ &\lesssim \sum_{0 \leq j \leq k} e^{-\varepsilon(m-j)} + \sum_{0 \leq j \leq D-(k+1)} e^{-\varepsilon(m'-j)} \\ &\lesssim e^{-\varepsilon(m-k-1)} + e^{-\varepsilon(m'-D+k)} \end{aligned}$$

Choosing  $k = (1/2)(D + m - m')$ , one gets

$$\text{diam}_\varepsilon(\phi_f(W) \cup \phi_f(W')) \lesssim e^{-\varepsilon(1/2)(m'+m-D)} \lesssim e^{-\varepsilon(W|W')}.$$

The lemma is established. ■

**Proof:** It follows from Lemma 3.3.11 that if  $W_1, W_2, W_3$  are three vertices, then

$$\begin{aligned} e^{-\varepsilon(W_1|W_3)} &\lesssim \text{diam}_\varepsilon(\phi_f(W_1) \cup \phi_f(W_3)) \\ &\lesssim \text{diam}_\varepsilon(\phi_f(W_1) \cup \phi_f(W_2)) + \text{diam}_\varepsilon(\phi_f(W_2) \cup \phi_f(W_3)) \\ &\lesssim e^{-\varepsilon(W_1|W_2)} + e^{-\varepsilon(W_1|W_3)} \\ &\lesssim \max\{e^{-\varepsilon(W_1|W_2)}, e^{-\varepsilon(W_2|W_3)}\} \end{aligned}$$

so that there is a constant  $c$  such that

$$(W_1|W_3) \geq \min\{(W_1|W_2), (W_2|W_3)\} - c.$$

This proves the hyperbolicity of  $X$ . ■

The hyperbolicity of  $\Gamma$  implies that the homeomorphism and quasimetry type of  $\partial\Gamma$  does not depend on an  $\varepsilon > 0$  provided that  $\varepsilon$  is small enough, and that, for such a parameter,

$$|\xi - \zeta|_\varepsilon \asymp e^{-\varepsilon(\xi|\zeta)}$$

for points on the boundary.

We turn now to the second part of Theorem 3.3.1—the uniqueness of the quasi-isometry type of  $\Gamma = \Gamma(f, \mathcal{U})$ .

**Proposition 3.3.12** *Assume that  $f : (\mathfrak{X}_1, X) \rightarrow (\mathfrak{X}_0, X)$  is a finite branched covering of degree  $d$ . Let  $S_j(1)$ ,  $j = 1, 2$ , be finite coverings. We denote by  $\Gamma_j$ ,  $F_j$ ,  $\varepsilon_j$  and  $\phi_j : X \rightarrow \partial\Gamma_j$  the graph, dynamics, weight and projection map associated to  $S_j(1)$ . If both coverings satisfy [Expansion] and if  $\phi_1$  and  $\phi_2$  are both homeomorphisms, then  $\Gamma_1$  is quasi-isometric to  $\Gamma_2$ .*

By Theorem 3.1.5, it is enough to show that  $\partial\Gamma_j$ ,  $j = 1, 2$  are quasimetrically equivalent, and this is what we actually prove. Without further combinatorial finiteness or uniformity properties, it seems difficult to work directly with the graphs  $\Gamma_j$  to show that they are quasi-isometric.

We start with some lemmata.

Axiom [Expansion] and Proposition 2.4.1(2)(a) imply the following result.

**Lemma 3.3.13** *For  $j = 1, 2$ , integers  $n_j$  exist such that*

1. *for any  $U_j \in S_j(n_j)$ , there is  $U_{3-j} \in S_{3-j}(1)$  which contains  $U_j$ .*
2. *for any  $U_{3-j} \in S_{3-j}(1)$ , there is  $U_j \in S_j(n_j)$  contained in  $U_{3-j}$ .*

**Proof:** The finiteness of  $S_1(1) \cup S_2(1)$  implies that there is some entourage  $\Omega$  of  $\mathfrak{X}_1$  such that any  $U \in S_1(1) \cup S_2(1)$  contains  $\Omega(x)$  for some  $x \in X$ , and, for any  $x \in X$ ,  $\Omega(x)$  is contained in some element of  $S_1(1)$  and of  $S_2(1)$ .

We treat the case  $j = 1$ . Since [Expansion] holds, there is some  $n_1$  so that any  $W \in S_1(n_1)$  is contained in  $\Omega(x)$  for any  $x \in W \cap X$ .

1. If  $U_1 \in S_1(n_1)$ , then consider  $x \in U_1$  so that  $U_1 \subset \Omega(x)$ . There is some  $U_2 \in S_2(1)$  such that  $\Omega(x) \subset U_2$ . Therefore  $U_1 \subset U_2$ .
2. If  $U_2 \in S_2(1)$ , let  $x \in U_2$  such that  $\Omega(x) \subset U_2$ . Let  $U_1 \in S_1(n_1)$  contain  $x$ . Thus,

$$U_1 \subset \Omega(x) \subset U_2.$$

■

**Lemma 3.3.14** *A constant  $K \geq 1$  exists such that, for  $j = 1, 2$ , for any  $x \in X$  and any  $n \geq n_j$ , there are some  $U \in S_j(n)$ ,  $W' \in S_{3-j}(n + n_{3-j} - 1)$  and  $W \in S_{3-j}(n - n_j + 1)$  such that*

1.  $x \in W' \subset U \subset W$ .
2.  $\text{Round}_{\varepsilon_j}(\phi_j(U), \phi_j(x)) \leq K$ ,  $\text{Round}_{\varepsilon_{3-j}}(\phi_{3-j}(W'), \phi_{3-j}(x)) \leq K$  and  $\text{Round}_{\varepsilon_{3-j}}(\phi_{3-j}(W), \phi_{3-j}(x)) \leq K$ .

*It follows that*

$$\text{diam}_{\varepsilon_{3-j}} \phi_{3-j}(W') \asymp \text{diam}_{\varepsilon_{3-j}} \phi_{3-j}(U) \asymp \text{diam}_{\varepsilon_{3-j}} \phi_{3-j}(W).$$

**Proof:** We let  $j = 1$ . Proceeding as usual, let us rename  $x = \tilde{x}$ .

Let  $\tilde{x} \in X$ . Using the fact that  $S_1(n_1)$  is finite, there is some  $U \in S_1(n_1)$  such that  $\text{Round}_{\varepsilon_1}(\phi_1(U), \phi_1(f^{n-n_1}(\tilde{x}))) \leq K'_1$  for some constant  $K'_1 \geq 1$ .

Let  $\tilde{U}$  be the component of  $f^{-(n-n_1)}(U)$  which contains  $\tilde{x}$ . Then  $\tilde{U} \in S_1(n)$  and Propositions 3.3.3 and 3.3.2(1) imply  $\text{Round}_{\varepsilon_1}(\phi_1(\tilde{U}), \phi_1(\tilde{x})) \leq K_1$  for some constant  $K_1 \geq 1$ .

By Lemma 3.3.13, there is some  $W \in S_2(1)$  which contains  $U$ . It follows from compactness that there exists a constant  $K'_2$  independent of  $\tilde{x}$  such that

$$\text{Round}_{\varepsilon_2}(\phi_2(W), \phi_2(f^{n-n_1}(\tilde{x}))) \leq K'_2;$$

see the proof of Proposition 2.6.2(1).

Let  $\widetilde{W}$  be the component of  $f^{-(n-n_1)}(W)$  which contains  $\tilde{x}$ . Then  $\widetilde{W} \in S_1(n - n_1 + 1)$  and Proposition 3.3.3 implies  $\text{Round}_{\varepsilon_2}(\phi_2(\widetilde{W}), \phi_2(\tilde{x})) \leq K_2$  for some constant  $K_2 \geq 1$ .

By Lemma 3.3.13, the point  $f^{n-1}(\tilde{x})$  belongs to some  $W' \in S_2(n_2)$  contained in  $f^{n-1}(\widetilde{U})$ . Since  $S_2(n_2)$  is finite, one can assume that

$$\text{Round}_{\varepsilon_2}(\phi_2(W'), \phi_2(f^{n-1}(\tilde{x}))) \leq K'_3$$

for some constant  $K'_3 \geq 1$ .

Let  $\widetilde{W}'$  be the component of  $f^{-(n-1)}(W')$  which contains  $\tilde{x}$ . Then  $\widetilde{W}' \in S_1(n + n_2 - 1)$  and Proposition 3.3.3 implies  $\text{Round}_{\varepsilon_2}(\phi_2(\widetilde{W}'), \phi_2(\tilde{x})) \leq K_3$  for some constant  $K_3 \geq 1$ .

Let  $K = \max\{K_1, K_2, K_3\}$ . The lemma follows from Proposition 3.3.2 once we have noticed that  $|W - W'| = n_1 + n_2$ . ■

We now give the proof of Proposition 3.3.12.

**Proof:** (Proposition 3.3.12)

Let  $\mathbf{U}_j$ ,  $j = 1, 2$  denote the corresponding collections of open sets defined by two different coverings at level zero. For  $j = 1, 2$  let  $|\cdot|_j$  denote the metric on the repeller  $X$  of  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  obtained by pulling back the metric  $d_{\varepsilon_j}$  on  $\partial_{\varepsilon_j}\Gamma$  via the homeomorphism  $\phi_j$ . Roundness and diameters in these metrics will be denoted with subscripts. We will show that the identity map is quasimetric: we want to find a homeomorphism  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that, given any  $x, y, z \in X$ ,

$$\frac{|x - z|_2}{|x - y|_2} \leq \eta \left( \frac{|x - z|_1}{|x - y|_1} \right).$$

Let  $\Omega$  be an entourage such that, for any  $x \in X$  any  $j = 1, 2$ , there is some  $U_j \in S_j(1)$ , such that  $\Omega(x) \subset U_j$ .

By the uniform continuity of  $\phi_1$ ,  $\phi_2$  and their inverses, it is enough to consider  $x, y, z \in X$  such that  $y, z \in \Omega(x)$ .

The strategy is the following. Let us assume that  $z$  is closer to  $x$  than  $y$ . Then, we may find neighborhoods  $U_y, U_x \in \mathbf{U}$  of  $x$  and  $y$  respectively such

that the “ring”  $U_y \setminus U_x$  separates the set  $\{x, z\}$  from  $y$ . The 3-point condition will follow from a straightforward argument using what is known about the sizes of the neighborhoods in each of the two metrics.

By Proposition 3.3.2, there exists a neighborhood  $U_y \in \mathbf{U}_1$  of  $x$  not containing  $y$  such that  $|x - y|_1 \asymp \text{diam}_1(U_y)$ , and  $\text{Round}_1(U_y, x) \leq K$ , where  $K$  is a uniform constant.

Again by Proposition 3.3.2, there exists a neighborhood  $U_z \in \mathbf{U}_1$  of  $x$  containing  $z$  such that  $|x - z|_1 \asymp \text{diam}_1(U_z)$  and  $\text{Round}_1(U_z, z) \leq K$ .

Therefore, Lemma 3.3.14 implies the existence of  $W'_y \in S_2(|U_y|_1 + n_2 - 1)$  which contains  $x$  but is contained in  $U_y$  such that  $\text{Round}_2(W'_y, x) \leq K$  and  $\text{diam}_2(W'_y) \asymp \text{diam}_2(U_y)$ .

Similarly, a vertex  $W_z \in S_2(|U_z|_1 + n_1 + 1)$  which contains  $U_z$  exists such that  $\text{Round}_2(W_z, x) \leq K$  and  $\text{diam}_2(W_z) \asymp \text{diam}_2(U_z)$ .

Since  $\text{Round}_2(W'_y, x) \leq K$ , it follows that

$$\frac{|x - z|_2}{|x - y|_2} \lesssim \frac{\text{diam}_2(W_z)}{\text{diam}_2(W'_y)} \asymp e^{-\varepsilon_2(|W_z|_2 - |W'_y|_2)}$$

But since  $x \in W_z \cap W'_y$ ,

$$|W_z|_2 - |W'_y|_2 = (|U_z|_1 - |U_y|_1) + (n_2 - n_1)$$

one obtains

$$\frac{|x - z|_2}{|x - y|_2} \lesssim \left( \frac{\text{diam}_1(U_z)}{\text{diam}_1(U_y)} \right)^{\varepsilon_2/\varepsilon_1} \lesssim \left( \frac{|x - z|_1}{|x - y|_1} \right)^{\varepsilon_2/\varepsilon_1}$$

and so the identity map is a quasisymmetry. ■

This concludes the proof of the second conclusion of Theorem 3.3.1.

The last conclusion of Theorem 3.3.1 is easily proved along the following lines. There is a canonical inclusion  $\iota$  from the vertices of  $\Gamma(f^n, \mathcal{U})$  to those of  $\Gamma(f, \mathcal{U})$  which sends a vertex of  $\Gamma(f^n, \mathcal{U})$ , say  $V \in \mathcal{U}_{nk}$  with  $|V| = k$ , to the vertex in  $\Gamma(f, \mathcal{U})$  called again  $V \in \mathcal{U}_{nk}$  with now  $|V| = nk$ . The image of  $\iota$  is clearly  $n$ -cobounded, an isometry on horizontal paths, and multiplies the lengths of vertical paths by a factor of  $n$ . Hence  $\iota$  is  $n$ -Lipschitz. Using these facts one proves easily that  $\iota$  is in fact a quasi-isometry, and the proof of Theorem 3.3.1 is complete. ■

### 3.4 Measure theory

In this section, we assume that  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is a degree  $d$  fbc with repellor  $X$  as in §3.2. As in the previous section, we assume that we are given a covering  $\mathcal{U}$  of  $X$  by connected open subsets of  $\mathfrak{X}_1$  which satisfies [Expansion]. Let  $\Gamma = \Gamma(f, \mathcal{U})$  be the Gromov hyperbolic graph associated to  $f$  and  $\mathcal{U}$  as in the previous section. Fix  $\varepsilon > 0$  small enough so that  $\phi_f : X \rightarrow \partial_\varepsilon \Gamma$  is a homeomorphism.

We now assume that axiom [Irreducibility] holds as well.

The main result of this section is the following result.

**Theorem 3.4.1** *Assume that Axioms [Expansion] and [Irreducibility] hold. Then there is a unique invariant mixing and ergodic measure  $\mu_f$  which is quasiconformal of dimension  $(1/\varepsilon) \log d$  on  $(\partial \Gamma, d_\varepsilon)$ . This measure describes the distribution of preimages of points and of periodic points. Furthermore, the metric entropy and topological entropy satisfy the following bounds*

$$\log d - \int \log d_F d\mu_f \leq h_\mu(F) \leq h_{top}(F) \leq v \leq \log d$$

and

$$\frac{h_\mu(F)}{\varepsilon} \leq \dim \mu_f \leq \dim \partial_\varepsilon \Gamma \leq \frac{v}{\varepsilon} \leq \frac{\log d}{\varepsilon},$$

where

$$v = \lim_{n \rightarrow \infty} \frac{1}{n} \log |S(n)|.$$

Precise statements and definitions are given in the next few subsections.

In the top chain of inequalities, the first one is a consequence of Rokhlin's formula, which we will show applies. The second follows from the Variational Principle, the third from generalities since  $F$  is Lipschitz, and the last since  $F$  is a degree  $d$  fbc.

In the remainder of this section, we dispense completely with the topological spaces  $X, \mathfrak{X}_0, \mathfrak{X}_1$  and deal exclusively with  $F : \bar{\Gamma}_\varepsilon \rightarrow \bar{\Gamma}_\varepsilon$ .

#### 3.4.1 Quasiconformal measures

Recall that  $f$  induces a continuous surjective Lipschitz map  $F : \bar{\Gamma}_\varepsilon \rightarrow \bar{\Gamma}_\varepsilon$  which maps vertices to vertices and edges (outside  $\bar{B}_\varepsilon(o, 2)$ ) homeomorphically onto edges.

**Multiplicity function for  $F$ .** Let  $d_f(x)$  denote the local degree of  $f$  at a point  $x \in X$ .

- If  $\xi \in \partial_\varepsilon \Gamma$ , let  $d_F(\xi) = d_f(\phi_f^{-1}(\xi))$ .
- If  $W \in V$ ,  $|W| \geq 2$ , let  $d_F(W) = \deg(f|_W)$ .
- For each (open) edge  $e = (W, W')$  with  $|W|, |W'| \geq 1$ , choose a point  $x_e \in W \cap W' \cap X$ .

If  $e \subset \bar{\Gamma}_\varepsilon \setminus B(o, 2)$ , set, for all  $\xi \in e$ ,

$$d_F(\xi) = \sum_{y \in (W \cap W' \cap f^{-1}(x_{F(e)}))} d_f(y).$$

**Remarks:**

1. The definition depends on the choices of points  $x_e$ , but this is irrelevant for our purposes.
2. The function  $d_F$  may vanish on certain edges. For example, let  $X = \mathfrak{X}_1 = \mathfrak{X} + 0 = \mathbb{R}/\mathbb{Z}$ , let  $f(x) = 2x$  modulo 1, and let  $\mathcal{U}_0 = \{U, V\}$  where  $U = X - \{1/4 + \mathbb{Z}\}$  and  $v = X - \{3/4 + \mathbb{Z}\}$ . Note that  $0 + \mathbb{Z} \subset U \cap V$  but that  $U \cap V$  is not connected.  $S(1)$  consists of the two vertices  $U, V$  joined by a single edge  $e$ . Choose  $x_e = 0 + \mathbb{Z}$ . The four elements of  $S(2)$  are the two preimages of  $U$  given by the intervals (mod  $\mathbb{Z}$ )  $(-3/8, 1/8)$  and  $(1/8, 5/8)$  and the two preimages of  $V$  are  $(-1/8, 3/8)$  and  $(3/8, 7/8)$ . According to the definition, the edge joining  $(1/8, 5/8)$  and  $(-1/8, 3/8)$  is given weight zero by  $d_F$  since the intersection of these two intervals contains neither 0 nor  $1/2$ , the preimages of the origin.
3. If  $d_F(\xi) \geq 2$  at a point  $\xi$  in the interior of an edge  $e$  (such as when the chosen point  $x_e \in \partial_\varepsilon \Gamma$  is a branch point of  $F$  on the boundary), then  $F$  is never a branched covering with degree function  $d_F$ , since  $d_F$  is constant on interiors of edges, and an honest f.b.c. is unramified on a dense open set. Conversely, if  $d_F \equiv 1$  on  $\partial \Gamma_\varepsilon$ , then  $F$  is an fbc in a neighborhood of  $\partial_\varepsilon \Gamma$ .

The following properties hold.



**Lemma 3.4.2** *The multiplicity function behaves as a local degree function. More precisely,*

(i) *for any  $\xi \in \bar{\Gamma}_\varepsilon \setminus B(o, 1)$ ,*

$$\sum_{F(\zeta)=\xi} d_F(\zeta) = d;$$

(ii) *for any  $\xi \in \bar{\Gamma}_\varepsilon \setminus B(o, 2)$ , there is a neighborhood  $N$  such that, for any  $\zeta \in N$ ,*

$$d_F(\xi) = \sum_{\zeta' \in F^{-1}(\{F(\zeta)\}) \cap N} d_F(\zeta').$$

**Proof:**

(i) The statement is clear for vertices and points from the boundary. Let  $e = (W, W')$  be an edge, and let us denote by  $\widetilde{W}_1, \dots, \widetilde{W}_k$  the components of  $f^{-1}(W)$ , and by  $\widetilde{W}'_1, \dots, \widetilde{W}'_{k'}$  the components of  $f^{-1}(W')$ .

If  $f(y) = x_e$ , then  $y$  there exists a unique edge  $\tilde{e} = (\widetilde{W}_y, \widetilde{W}'_{y'})$  such that  $y \in \widetilde{W}_y \cap \widetilde{W}'_{y'}$ . Therefore

$$\sum_{F(\tilde{e})=e} d_F(\tilde{e}) = \sum_{F(\tilde{e})=e} \sum_{y \in (\widetilde{W}_y \cap \widetilde{W}'_{y'}) \cap f^{-1}(x_e)} d_f(y) = \sum_{f(y)=x_e} d_f(y) = d.$$

(ii) The statement is clear on  $\Gamma \setminus B(o, 2)$ . Let  $\xi \in \partial\Gamma$ . There is some vertex  $W_0$  such that  $\phi_f(W_0) \ni \xi$ , and  $d_F(W_0) = d_F(\xi)$ . Let  $W_1 \subset W_0$  small enough so that  $\mathcal{U}_\infty(F(W_1)) \subset \phi_f(F(W_0))$ . Thus, for any  $U \in \text{int}(\mathcal{U}(F(W_1)))$ ,  $U \subset F(W_0)$ , so that

$$\sum_{F(\tilde{U})=U, \tilde{U} \subset W_0} d_F(\tilde{U}) = d_f(W_0) = d_F(\xi).$$

■

Note that if we set  $d_{F^n}(\xi) = d_F(\xi) \dots d_F(F^{n-1}(\xi))$ , then the lemma remains true for  $d_{F^n}$  as well.

**Action of  $F$  on measures.** If  $\varphi$  is a continuous test function defined on  $\overline{\Gamma}_\varepsilon \setminus B_\varepsilon(o, 1)$ , then its *pullback* under  $F$ , given by the formula  $F^*\varphi(\xi) = \varphi \circ F(\xi)$ , defines a continuous function on  $\overline{\Gamma}_\varepsilon \setminus B(o, 2)$ . By duality, one may define for Borel probability measures  $\nu$  with support in  $\overline{\Gamma}_\varepsilon \setminus B(o, 2)$  its *pushforward* by  $\langle F_*\nu, \varphi \rangle = \langle \nu, F^*\varphi \rangle$ . Thus in particular,  $(F_*\nu)(E) = \nu(F^{-1}(E))$  for all Borel sets  $E$ .

The point of the construction of the multiplicity function  $d_F$  is the following. If  $\varphi$  is a continuous test function on  $\overline{\Gamma}_\varepsilon \setminus B_\varepsilon(o, 1)$ , its *push-forward* under  $F$

$$F_*\varphi(\xi) = \sum_{F(\zeta)=\xi} d_F(\zeta)\varphi(\zeta)$$

is again a continuous function on  $\overline{\Gamma} \setminus B(o, 1)$ . By duality, we define the *pullback* of a Borel measure  $\nu$  by the formula  $\langle F^*\nu, \varphi \rangle = \langle \nu, F_*\varphi \rangle$  (cf. [DiSi], §2).

**Quasiconformal measures.** If  $\mu, \nu$  are measures we write  $\mu \ll \nu$  if  $\nu$  is absolutely continuous with respect to  $\mu$ . Let  $\mu$  be a regular Borel probability measure on  $\partial_\varepsilon\Gamma$ . Inspired by the group setting [Coo], we say  $\mu$  is *quasiconformal measure of dimension  $\alpha$*  if, for all  $n \geq 1$ ,  $(F^n)^*\mu \ll \mu$  and the Radon-Nikodym derivative satisfies

$$\frac{d(F^n)^*\mu}{d\mu} \asymp (e^{n\varepsilon})^\alpha \quad \mu - a.e..$$

The quantity  $e^{n\varepsilon}$  stands for the derivative of  $F^n$  (cf. Proposition 3.2.2).

Let  $\mu$  be a quasiconformal measure on  $\partial_\varepsilon\Gamma$ . Fix  $n \in \mathbb{N}$ . Suppose  $E \subset \partial_\varepsilon\Gamma$  is a Borel subset of positive measure,  $F^n|_E$  is injective, and the local degree of  $F^n$  is constant on  $E$ , i.e. for all  $\xi \in E$ ,  $d_{F^n}(\xi) = d_E$ . Then, it follows from the regularity of the measure that

$$\langle (F^n)^*\mu, \chi_E \rangle = \langle \mu, (F^n)_*\chi_E \rangle = \int \sum_{F^n(\zeta)=\xi} d_F(\zeta)\chi_E(\zeta)d\mu(\xi) = d_E\mu(F^n(E)),$$

and quasiconformality of the measure implies  $\langle (F^n)^*\mu, \chi_E \rangle \asymp e^{n\alpha\varepsilon}\mu(E)$ . Hence

$$\mu(F^n(E)) \asymp \frac{e^{n\alpha\varepsilon}}{d_E}\mu(E). \quad (3.2)$$

Axiom [Irreducibility] implies that the support of a quasiconformal measure is the whole set  $\partial_\varepsilon\Gamma$ . Therefore, there is some  $m > 0$  such that, for all  $x \in S(1)$ ,  $\mu(\phi_f(W(x))) \geq m$ .

We let  $d(W)$  be the degree of  $f^{n-1}|_W$  for  $W \in S(n)$ . Since  $\mu$  is a quasiconformal measure, it follows that

$$0 < m \leq \mu(\phi_f(f^{n-1}(W))) = \mu(F^{n-1}\phi_f(W)) \asymp \frac{e^{n\alpha\varepsilon}}{d(W)}\mu(\phi_f(W)).$$

This proves

**Lemma 3.4.3 of the shadow.** *For any  $W \in V$ ,*

$$\mu(\phi_f(W)) \asymp d(W)e^{-\alpha\varepsilon|W|}.$$

We use this lemma for the classification of quasiconformal measures.

**Theorem 3.4.4** *Let  $\mu$  be a quasiconformal measure of dimension  $\alpha$ . The following are equivalent.*

(i)  $\mu$  is atomic.

(ii)  $\alpha = 0$ .

(iii)  $\partial_\varepsilon\Gamma$  is a point.

*If  $\alpha > 0$  then  $\alpha = \frac{1}{\varepsilon} \log d$ , and any two such quasiconformal measures are equivalent.*

**Proof:** Consider first the constant function  $\varphi = 1$  on  $\partial_\varepsilon\Gamma$ . Then  $F_*\varphi = d\varphi$  so that

$$\langle (F^n)^*\mu, \varphi \rangle = \langle \mu, F_*^n\varphi \rangle = \langle \mu, d^n\varphi \rangle = d^n \asymp e^{n\alpha\varepsilon}.$$

Thus,

$$\alpha = \frac{1}{\varepsilon} \log d.$$

It follows that  $\alpha = 0$  if and only if  $d = 1$ , so that  $\partial_\varepsilon\Gamma$  is a point since  $f$  satisfies [Expansion]. Hence (ii) implies (iii).

If  $\mu$  is atomic, then there is some  $\xi \in \partial_\varepsilon\Gamma$  such that  $\mu\{\xi\} > 0$ . By definition, for all  $n \geq 0$  and any  $x \in \partial_\varepsilon\Gamma$ ,

$$((F^n)^*\mu)(\{x\}) = d_{F^n}(x)\mu\{F^n(x)\}$$

and since  $\mu$  is quasiconformal then by Equation (3.2)

$$d_{F^n}(\xi)\mu\{F^n(\xi)\} \asymp e^{n\alpha\varepsilon}\mu\{\xi\} = d^n\mu\{\xi\}$$

But  $d_{F^n}(\xi) \leq d^n$  so  $\mu\{F^n(\xi)\} \gtrsim \mu\{\xi\}$ . Since the total mass of  $\mu$  is finite, the orbit of  $\xi$  has to be finite. Let  $\zeta = F^\ell(\xi)$  be periodic and let  $k$  be its period. Then  $\mu\{\zeta\} > 0$  and

$$(d_{F^k}(\zeta))^n \mu\{\zeta\} = ((F^{kn})^* \mu)(\{\zeta\}) \asymp d^{nk} \mu\{\zeta\}$$

from which we deduce that  $d_{F^k}(\zeta) = d^k$ . This means that the local degree at every point in its orbit is maximal, so that its grand orbit is finite. Since  $f$  satisfies [Irreducibility], the grand orbit of any point is dense in  $X$  (Proposition 2.4.13(a)) and so  $\partial_\varepsilon \Gamma$  is a point,  $d = 1$  and  $\alpha = 0$ . So (i) implies (ii) and (iii).

The last implication (iii) implies (i) is obvious.

The Lemma of the Shadow (Lemma 3.4.3) and the assumption that quasiconformal measures are regular imply that two measures of the same dimension are equivalent. ■

We will now construct a quasiconformal measure using the Patterson-Sullivan procedure [Coo]. It turns out that this measure will be invariant.

**Poincaré series.** In this part, we construct an invariant quasiconformal measure. Let

$$P(s) = |S(1)| \sum_{n \geq 1} d^{n-1} e^{-ns} = |S(1)| \frac{1}{e^s - d}.$$

It follows that  $P(s) < \infty$  if and only if  $s > \log d$ . Let, for  $s > \log d$ ,

$$\mu_s = \frac{1}{P(s)} \sum_{n \geq 1} \sum_{\xi \in S(n)} e^{-ns} d(\xi) \delta_\xi.$$

For every  $n \geq 1$ ,  $F^n : S(n+1) \rightarrow S(1)$  has degree  $d^n$ . Recall that for  $\xi \in S(n)$ , we denoted by  $d(\xi) = d_{F^{n-1}}(\xi)$ . So

$$|S(n+1)| = d^n |S(1)| - \sum_{\xi \in S(n+1)} [d(\xi) - 1]$$

and  $\sum_{\xi \in S(n+1)} d(\xi) = d^n |S(1)|$ . Therefore

$$\mu_s(\bar{\Gamma}_\varepsilon) = \frac{1}{P(s)} \sum_{n \geq 1} e^{-ns} \sum_{\xi \in S(n)} d(\xi) = \frac{1}{P(s)} \sum_{n \geq 1} e^{-ns} d^{n-1} |S(1)| = 1.$$

Hence  $\{\mu_s\}_{s>\log d}$  is a family of probability measures on  $\bar{\Gamma}_\varepsilon$ . Let  $\mu_f$  be any weak limit of this family as  $s$  decreases to  $\log d$ . Since the Poincaré series diverges at  $\log d$ , it follows that its support is contained in  $\partial\Gamma$ .

If  $\varphi$  is a continuous function with support close to  $\partial\Gamma$ , then

$$\begin{aligned} \langle F^*\mu_s, \varphi \rangle &= \frac{1}{P(s)} \sum_{n \geq 1} e^{-ns} \sum_{\xi \in S(n)} d(\xi)(F_*\varphi)(\xi) \\ &= \frac{1}{P(s)} \sum_{n \geq 1} e^{-ns} \sum_{\xi \in S(n)} d(\xi) \sum_{F(\zeta)=\xi} d_F(\zeta)\varphi(\zeta) \\ &= \frac{1}{P(s)} \sum_{n \geq 1} e^{-ns} \sum_{\zeta \in S(n+1)} d(\zeta)\varphi(\zeta) \\ &= e^s \langle \mu_s, \varphi \rangle + O(1/P(s)) \end{aligned}$$

where we have used that  $d(\zeta) = d(F(\zeta))d_F(\zeta)$ .

It follows that, as  $s$  decreases to  $\log d$ ,

$$\langle F^*\mu_f, \varphi \rangle = \langle d\mu_f, \varphi \rangle$$

and so  $F^*\mu_f = d\mu_f$ . In other words,  $\mu_f$  is a quasiconformal measure of dimension  $(1/\varepsilon)\log d$ .

Let us look at  $F_*\mu_f$ :

$$\langle F_*\mu_f, \varphi \rangle = \langle \mu_f, F^*\varphi \rangle = (1/d)\langle F^*\mu_f, F^*\varphi \rangle = (1/d)\langle \mu_f, F_*(F^*\varphi) \rangle.$$

But

$$F_*(F^*\varphi)(\xi) = \left( \sum_{F(\zeta)=\xi} d_F(\zeta)(F^*\varphi)(\zeta) \right) = d\varphi(\xi).$$

Therefore,  $F_*\mu_f = \mu_f$  so  $\mu_f$  is an invariant measure.

Let us prove that  $\mu_f$  is ergodic. Let  $E$  be an invariant subset of  $\partial_\varepsilon\Gamma$  with positive measure. Let  $\nu = \mu_f|_E/\mu_f(E)$ . It follows that  $\nu$  is also an invariant quasiconformal measure. The Lemma of the shadow (Lemma 3.4.3) implies that  $\mu_f(W) \asymp \nu(W)$  for all  $W \in V$ . This implies that  $\mu_f$  and  $\nu$  are equivalent. Hence  $\mu_f(E) = 1$ .

Since  $\mu_f$  is an ergodic invariant measure, it follows that  $\mu_s$  converges to  $\mu_f$  in the weak-\* topology when  $s$  decreases to  $\log d$ .

**Remark.** On  $\partial_\varepsilon\Gamma$ , the local degree function  $d_F$  is multiplicative:  $d_{F^n}(\xi) = \prod_{i=0}^{n-1} d_F(f^i(\xi))$ . From the Birkhoff Ergodic theorem and the ergodicity of  $F$  with respect to  $\mu_f$ , it follows that for  $\mu_f$ -almost any  $\xi \in \partial_\varepsilon\Gamma$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log d_{F^n}(\xi) = \int \log d_F d\mu_f.$$

Thus, either the critical set has measure 0 and the Jacobian of  $F$  with respect to  $\mu_f$  is constant and equal to  $d$  almost everywhere, or almost every point visits the branch set so frequently that the local degrees increase exponentially fast. Unfortunately, given the assumptions under which we are currently working, we have neither a proof that this latter possibility cannot occur, nor an example showing that it can occur.

### 3.4.2 Entropy

We refer to [KH], [Mañ] and [PU], chap. 1, for background on entropy.

**Topological entropy.** Let  $T : Z \rightarrow Z$  be a continuous map of a compact metric space to itself. The *dynamical distance* and the corresponding *dynamical balls at level  $n$*  are defined as

$$d_n(\xi, \zeta) = \max_{0 \leq j \leq n} \{d_\varepsilon(T^j(\xi), T^j(\zeta))\} \quad \text{and} \quad S(\xi, n, r) = \{\zeta \in Z, d_n(\xi, \zeta) \leq r\}.$$

Let  $c_n(r)$  be the minimal number of dynamical balls  $S(\cdot, n, r)$  at level  $n$  needed to cover  $Z$  and  $s_n(r)$  the maximal number of disjoint dynamical balls  $S(\cdot, n, r)$ . The *topological entropy of  $T$*  may be defined as

$$h_{top}(T) = \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log c_n(r) = \lim_{r \rightarrow 0} \liminf_{n \rightarrow \infty} \frac{1}{n} \log s_n(r).$$

We now estimate  $h_{top}(F)$ , where  $F$  denotes the restriction of  $F : \bar{\Gamma}_\varepsilon \rightarrow \bar{\Gamma}_\varepsilon$  to the boundary  $\partial_\varepsilon\Gamma$ .

Since  $F$  is  $e^\varepsilon$ -Lipschitz, we have  $d_n(\xi, \zeta) \leq e^{n\varepsilon} d_\varepsilon(\xi, \zeta)$  and hence  $S(\xi, n, r) \supset B_\varepsilon(\xi, r e^{-\varepsilon n})$ . For any  $n \geq 1$ ,  $\{\mathcal{U}_\infty(\xi)\}_{\xi \in S(n)}$  is a covering of  $\partial\Gamma$  by at most  $|S(n)|$  sets. For any  $\xi \in S(n)$ ,  $\text{diam}\mathcal{U}_\infty(\xi) \leq C e^{-\varepsilon|\xi|}$ . So,  $\mathcal{U}_\infty(\xi) \subset$

$S(\xi', p, Ce^{-\varepsilon(n-p)})$ , for any  $p \in \mathbb{N}$  and for any  $\xi' \in \mathcal{U}_\infty(\xi)$ . Hence  $c_p(Ce^{-\varepsilon(n-p)}) \leq |S(n)|$ .

Recall that by definition  $v = \lim \frac{1}{n} \log |S(n)|$ ; the limit exists since  $|S(n+1)| \leq d|S(n)|$ . Let  $\eta > 0$  be small. For any  $p \geq 1$ , there is some  $n \in \mathbb{N}$  such that  $\eta \asymp e^{-\varepsilon(n-p)}$ , meaning that  $n \sim p + (1/\varepsilon) \log 1/\eta$ . The discussion above implies that  $\partial_\varepsilon \Gamma \subset \cup_{\xi \in S(n)} S(\xi, p, Ce^{-\varepsilon(n-p)})$ . Since for any  $\eta' > 0$  and any  $n$  large enough,  $\log |S(n)| \leq n(v + \eta')$  holds, we have  $\log c_p(\eta) \leq n(v + \eta')$  and

$$h_{top}(F) \leq \lim_{\eta \rightarrow 0} \limsup_{p \rightarrow \infty} \frac{p + (1/\varepsilon) \log 1/\eta}{p} (v + \eta')$$

from which  $h_{top}(F) \leq v$  follows. Since  $|S(n)| \leq d^n$ , one has  $v \leq \log d$ .

**Measure-theoretic entropy.** We recall first the definition of measure-theoretic entropy. Suppose  $(Z, \nu)$  is a probability space,  $T : Z \rightarrow Z$  preserves  $\nu$ . If  $\mathcal{P}$  is a measurable partition of  $Z$ , define its entropy with respect to  $\nu$  by

$$H(\mathcal{P}, \nu) = \sum_{A \in \mathcal{P}} \nu(A) \log(1/\nu(A)).$$

For  $n \in \mathbb{N}$  set

$$\mathcal{P}_n = \mathcal{P} \vee T^{-1}(\mathcal{P}) \dots \vee T^{-n}(\mathcal{P}).$$

Then

$$h(\mathcal{P}, \nu) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\mathcal{P}_n, \nu)$$

exists. The supremum of  $h(\mathcal{P}, \nu)$  over all partitions with finite entropy defines the metric entropy  $h_\nu(T)$ . A partition  $\mathcal{P}$  with finite entropy is called a generator if it separates points i.e., for any distinct  $z, z' \in Z$ , there is some  $n \geq 0$ , disjoint sets  $A, A' \in \mathcal{P}_n$  such that  $z \in A$  and  $z' \in A'$ ; equivalently, there is some  $n \geq 0$ , disjoint pieces  $A, A' \in \mathcal{P}$  such that  $T^n(z) \in A$  and  $T^n(z') \in A'$ . For a generating partition,  $h(\mathcal{P}, \nu)$  achieves the maximum of  $h(\mathcal{P}', \mu)$  over all measurable partitions with finite entropy so that  $h(\mathcal{P}, \nu) = h_\nu(T)$  holds.

The *variational principle* (see [Wal], Thm. 8.6) asserts that, when  $T$  is continuous, then  $h_{top}(T) = \sup_\mu h_\mu(T)$  where  $\mu$  varies over all invariant ergodic Borel measures.

**Jacobian.** Let  $T : Z \rightarrow Z$  be a continuous map countable to 1, and  $\nu$  an invariant regular Borel probability measure on  $Z$ . A *special set*  $A$  is a measurable subset of  $Z$  such that  $T|_A$  is injective. A *weak Jacobian* is a

measurable function  $J_\nu : Z \rightarrow \mathbb{R}_+$  such that there is some set  $Y$  such that  $\nu(Y) = 0$  and, for any special set  $A$  disjoint from  $Y$ ,

$$\nu(T(A)) = \int_A J_\nu d\nu$$

holds. The function  $J_\nu$  is a (*strong*) *Jacobian* if one can choose  $Y = \emptyset$ . Weak Jacobians always exist for finite branched coverings defined on a compact set  $Z$ , and there are well-defined mod 0 sets. Let us sketch their construction in this case.

Let  $\mathcal{A}$  be a finite measurable partition of  $Z$  by special sets. Let us first fix  $A \in \mathcal{A}$ . Since  $T|_A$  is injective, the formula  $\nu_A(B) = \nu(TB)$  defined on measurable subsets  $B \subset A$  defines a measure on  $A$ . By the invariance of  $\nu$ , it follows that  $\nu|_A$  is absolutely continuous with respect to  $\nu_A$  so that the Radon-Nikodym theorem implies the existence of a measurable non-negative function  $h_A$  defined on  $A$  such that  $d\nu = h_A d\nu_A$ .

Let  $Y_A = \{h_A = 0\}$ , and let us define  $J_\nu$  on  $A$  by

$$J_\nu = \begin{cases} 0 & \text{on } Y_A \\ 1/h_A & \text{on } A \setminus Y_A \end{cases}$$

It follows that  $\nu(Y_A) = 0$  and that for any special set  $B \subset A$  disjoint from  $Y_A$ ,

$$\nu(TB) = \int_B d\nu_A = \int_B J_\nu(h_A d\nu_A) = \int_B J_\nu d\nu.$$

Let us define  $Y = \cup_A Y_A$ ; it follows that  $\nu(Y) = 0$  and for any special set  $B$  disjoint from  $Y$ ,

$$\nu(TB) = \sum_A \nu(T(B \cap A)) = \sum_A \int_{B \cap A} J_\nu d\nu = \int_B J_\nu d\nu.$$

If  $\nu$  is ergodic, and if it admits a countable generator of finite entropy, then the so-called following *Rohlin formula* holds:

$$h_\nu(T) = \int \log J_\nu d\nu$$

where  $J_\nu$  is the Jacobian.



We now estimate  $h_{\mu_f}(F)$  where  $F : \partial\Gamma \rightarrow \partial\Gamma$  and  $\mu_f$  is the quasiconformal measure constructed in the previous section. To do so, we first prove the existence of a *finite generator*:

**Proposition 3.4.5** *Let  $(\mathfrak{X}_1, \mathfrak{X}_0, X, f)$  be a finite branched covering of degree  $d \geq 2$ . Let us assume that there is a finite cover  $\mathcal{U}$  of  $X$  by open connected sets that satisfies [Expansion]. We let  $\Gamma$  be the hyperbolic graph associated to  $(f, \mathcal{U})$ . For any  $n_0 \geq 1$ , a finite cover by Borelean sets  $\mathcal{Q}$  of  $\partial\Gamma$  of diameter at most  $e^{-\varepsilon n_0}$ , a constant  $\lambda > 1$  and an iterate  $k \geq 1$  exist such that, for any  $Q \in \mathcal{Q}$ , for  $\xi, \zeta \in Q$ ,*

$$|F^k(\xi) - F^k(\zeta)|_\varepsilon \geq \lambda|\xi - \zeta|_\varepsilon$$

*holds. Furthermore, for any  $Q \in \mathcal{Q}$ , there is some  $d_Q$  such that, for any  $\xi \in Q$ ,  $d_{F^k}(\xi) = d_Q$ .*

The proof will be achieved in 12 steps. The first four steps use the hyperbolicity of the graph  $\Gamma$  to construct *normal curves* joining pair of points, from which we can read their distance, and use the definite contraction coming from path-lifting. The fifth step provides us with a first finite cover by open sets which isolates different inverse branches. The sixth step shows that pieces of degree 1 already satisfy the proposition. The remaining steps are devoted to splitting further components which contain branch points. Step **7.** defines building blocks for the splitting, and some of their properties are established in **8.** and **9.**. Steps **10.** and **11.** define the splitting of a component with branch points and step **12.** concludes the proof.

### Preliminary steps.

**1.** For each  $\xi \in \partial\Gamma$ , fix once and for all a sequence  $(W_n(\xi))_n \in \prod S(n)$  such that  $\text{Round}(W_n, \xi) \leq K$  (cf. Proposition [3.3.2, 3.]).

**2.** If  $\zeta \in \partial\Gamma$  is another point, let  $c = c(\xi, \zeta)$  be the largest level  $n$  such that  $W_n(\xi) \cap W_n(\zeta) \neq \emptyset$ . Note that  $c$  is well-defined, since all the rays meet at  $o$ , and since  $\text{diam}_\varepsilon W_n(\xi) \lesssim e^{-\varepsilon n}$ .

It follows from hyperbolicity that

$$|c(\xi, \zeta) - (\xi|\zeta)| \leq C(\delta)$$

so that

$$|\xi - \zeta|_\varepsilon \asymp e^{-\varepsilon c}$$

where the implicit constants depend only on  $K$ ,  $\delta$  and  $\varepsilon$ . We write  $\xi = \phi_f(x)$  and  $\zeta = \phi_f(y)$ .

Fix  $k \geq 1$ , which will be defined later. For each  $\xi \in \partial_\varepsilon \Gamma$  and  $n \geq 1$ , let  $W'_n(\xi) \in S(n)$  be the component of  $f^{-k}(W_{n+k}(F^k(\xi)))$  that contains  $x$ .

It follows from Proposition 3.3.3 that

$$\text{Round}(W'_n(\xi), \xi) \leq K'$$

for some universal  $K'$ .

Let  $c' = c'(\xi, \zeta)$  be the largest level  $n$  such that  $W'_n(\xi) \cap W'_n(\zeta) \neq \emptyset$ . Then

$$|c'(\xi, \zeta) - (\xi|\zeta)| \leq C'(\delta)$$

and

$$|\xi - \zeta|_\varepsilon \asymp e^{-\varepsilon c'}$$

hold, and once  $k$  has been chosen the implicit constants depend only on  $K'$ ,  $\delta$  and  $\varepsilon$ .

If  $F^k(\xi) = F^k(\zeta)$ , then either  $W'_n(\xi) \cap W'_n(\zeta) = \emptyset$  or  $W'_n(\xi) = W'_n(\zeta)$ . In this last case,  $\deg W'_n(\xi) > 1$ .

**3.** Let  $\xi, \zeta \in \partial\Gamma$  be such that  $c(\xi, \zeta) \geq 1$ ; let us consider the *normal curve*

$$\gamma(\xi, \zeta) = \gamma = \{W_n(\xi), n \geq c(\xi, \zeta)\} \cup \{W_n(\zeta), n \geq c(\xi, \zeta)\}$$

which joins  $\xi$  to  $\zeta$ . Similarly, we let

$$\gamma'(\xi, \zeta) = \gamma' = \{W'_n(\xi), n \geq c'(\xi, \zeta)\} \cup \{W'_n(\zeta), n \geq c'(\xi, \zeta)\}.$$

Let us define  $K(\gamma) = \overline{\cup_{n \in \mathbb{Z}} \gamma(n)}$  which is a compact connected subset of  $\mathfrak{X}_1$ . It follows that  $f^{-k}(K(\gamma))$  is a disjoint union of continua which join preimages of  $\xi$  to preimages of  $\zeta$ . These continua induce a relation between the preimages of  $\xi$  and  $\zeta$ : to belong to a common component of the set  $f^{-k}(K(\gamma))$ . If  $\xi', \zeta'$  are in relation,  $F^k(\xi') = \xi$  and  $F^k(\zeta') = \zeta$ , then the component of  $F^{-k}(K(\gamma))$  which contains  $\xi'$  also contains the curve  $\gamma'(\xi', \zeta')$  joining  $\xi'$  to  $\zeta'$ . Note that if  $\xi', \zeta'$  are related, then  $F^k(\gamma'(\xi', \zeta')) = \gamma(\xi, \zeta)$ , so that

$$\ell_\varepsilon(\gamma) = e^{\varepsilon k} \ell_\varepsilon(\gamma') \asymp |\xi' - \zeta'|_\varepsilon e^{\varepsilon k} \asymp e^{-\varepsilon[(\xi'|\zeta') - k]}.$$

**4.** Let  $\xi_1, \dots, \xi_4 \in \partial\Gamma$  be such that  $c(\xi_i, \xi_j) \geq 1$ . Define

$$\Gamma_0 = \cup_{i \neq j} \gamma(\xi_i, \xi_j).$$

From hyperbolicity (“Approximation par les arbres” in [GdlH]), there is a tree  $T$  and a  $(1, C)$ -quasi-isometry

$$\varphi : \Gamma_0 \rightarrow T.$$

Let  $\tilde{\Gamma}_0$  be a connected component of  $F^{-k}(\Gamma_0) \cap \Gamma$ , so we can also define an approximate tree  $T'$ , and the map  $F^k : T' \rightarrow T$  is an isometry along normal curves between endpoints with distinct images.

If we consider preimages  $\xi'_1, \dots, \xi'_4$  of  $\xi_1, \dots, \xi_4$  in  $\tilde{\Gamma}$ , then

$$c'(\xi'_i, \xi'_j) = c(\xi_i, \xi_j) + k.$$

Therefore, given  $\lambda > 1$ , for any iterate  $k$  large enough,

$$|\xi_j - \xi_i|_\varepsilon \geq \lambda \cdot |\xi'_j - \xi'_i|_\varepsilon.$$

The implicit constants are swallowed by  $k$  to keep the expansion.

We fix such an iterate  $k$ , and write  $G = F^k$ .

**5.** Let us fix  $n_0 \geq 1$ . The facts presented at the end of §2.1 imply that for any  $x \in X$ , there is some  $U_0(x) \in \mathbf{U}$  with  $|U_0(x)| \geq n_0$  such that the following hold:

- for any component  $\tilde{U}$  of  $f^{-k}(U_0(x))$ ,  $\tilde{U} \cap f^{-k}(\{x\})$  is a single point  $\tilde{x}$ ,
- $\deg(f^k, \tilde{U}) = \deg(f^k, \tilde{x})$ , and
- if  $\tilde{U}$  and  $\tilde{U}'$  are two components of  $f^{-k}(U_0(x))$ , then  $\text{dist}_\varepsilon(\phi_f \tilde{U}, \phi_f \tilde{U}') \geq 4 \max\{\text{diam}_\varepsilon \phi_f \tilde{U}, \text{diam}_\varepsilon \phi_f \tilde{U}'\}$ , so that each component  $\tilde{U}$  is independent from the other preimages.

We let  $U_1(x) \in \mathbf{U}$  be another neighborhood of  $x$  so that, for any  $\xi, \zeta \in \phi_f(U_1(x))$ ,  $K(\gamma(\xi, \zeta)) \subset U_0(x)$ , where  $K(\gamma(\xi, \zeta))$  is defined in **3.**

This forces an even better independence. It will also allow us to work with normal curves. So, if  $\zeta, \xi \in \phi_f(U_1(x))$ , and if  $\tilde{\xi}, \tilde{\zeta}$  are two preimages, then some lift of a normal curve under  $G$  joins  $\tilde{\xi}$  to  $\tilde{\zeta}$  only if  $\tilde{\xi}, \tilde{\zeta}$  belong to the same component of  $G^{-1}(\phi_f U_1(x))$ .

We let  $\mathcal{V}$  be the family of sets  $\{U_1(x)\}$ ,  $\mathcal{W}'$  be a finite cover of  $X$  provided by  $\mathcal{V}$ , and we let  $\mathcal{W}$  be the cover of  $\partial\Gamma$  obtained by pulling back  $\mathcal{W}'$  with  $f^k$  and by mapping it into  $\partial\Gamma$  under  $\phi_f$ .

**The actual proof.**

**6.** Let  $W \in \mathcal{W}$ , and let us assume that  $d_G(W) = 1$ . Then, according to **5**, for any  $\xi, \zeta \in W$ , there is a unique pull-back of  $\gamma(G(\xi), G(\zeta))$  ending at  $\xi$  and  $\zeta$ , so that

$$|G(\xi) - G(\zeta)|_\varepsilon \geq \lambda \cdot |\xi - \zeta|_\varepsilon.$$

Therefore,  $W$  will be one of the pieces of the cover we are looking for.

Let  $W \in \mathcal{W}$ , and let us assume that  $d_G(W) = p > 1$ . We need to split  $W$  into finitely many pieces; the local degree of  $G$  on each piece will be constant. We start with degree 1, and then proceed to higher ones. Step **7**. fixes further notation.

**7.** If  $\xi \in W$ , we let

$$\mathcal{C}_0(\xi) = \{\zeta \in W, c'(\xi, \zeta) > \max\{c'(\xi, \zeta'), \zeta' \in G^{-1}\{G(\zeta)\} \setminus \{\zeta\}\},$$

$$\mathcal{C}(\xi) = \{\zeta \in W, c'(\xi, \zeta) > \max\{c'(\xi, \zeta'), \zeta' \in G^{-1}\{G(\zeta)\} \setminus \{\zeta\}\} \cap \{d_G = d_G(\xi)\},$$

$$\widehat{\mathcal{C}}_0(\xi) = \{\zeta \in W, c'(\xi, \zeta) \geq \max\{c'(\xi, \zeta'), \zeta' \in G^{-1}\{G(\zeta)\} \setminus \{\zeta\}\}.$$

and

$$\widehat{\mathcal{C}}(\xi) = \{\zeta \in W, c'(\xi, \zeta) \geq \max\{c'(\xi, \zeta'), \zeta' \in G^{-1}\{G(\zeta)\} \setminus \{\zeta\}\} \cap \{d_G = d_G(\xi)\}.$$

We note that any point in  $\widehat{\mathcal{C}}_0(\xi)$  is related to  $\xi$ , and that points from  $\mathcal{C}_0(\xi)$  are uniquely related to  $\xi$ , since the inequality is strict in the definition of  $\mathcal{C}_0(\xi)$ .

For any  $\zeta, \zeta' \in \mathcal{C}_0(\xi)$ , the set

$$\gamma'(\xi, \zeta) \cup \gamma'(\xi, \zeta')$$

is connected, and the structure of its tree is provided by both of these curves, so that we can read from it the distance  $|\zeta - \zeta'|_\varepsilon$ , hence

$$|G(\zeta) - G(\zeta')|_\varepsilon \geq \lambda \cdot |\zeta - \zeta'|_\varepsilon$$

according to **3** and **4**.

We now begin the procedure of splitting  $W$ .

**8.** We let  $\xi' \in G(W)$  be a regular value for  $G|_W$ . Therefore,  $\xi'$  has  $p$  distinct preimages  $\xi_1, \dots, \xi_p$  under  $G$  in  $W$ . We note that  $\cup_j \widehat{\mathcal{C}}_0(\xi_j) = W$ , and  $\cup_j \widehat{\mathcal{C}}(\xi_j) = W \cap \{d_G = 1\}$ .

We let  $Q_0(\xi_j) = \mathcal{C}(\xi_j)$ .

We need to make choices in order to split the remaining points. This will be done inductively on the sets from  $\mathcal{V}$ .

It follows that if  $\zeta \in W$  and  $\zeta \notin \cup Q_0(\xi_j)$ , and if  $d_G(\zeta) = 1$ , then  $\zeta$  belongs to some  $\widehat{\mathcal{C}}_0(\xi_j)$  for some  $j \in \{1, \dots, p\}$ , and there is some  $\zeta' \in \widehat{\mathcal{C}}_0(\xi_j)$  such that  $G(\zeta) = G(\zeta')$  and

$$|\zeta - \xi_j|_\varepsilon = |\zeta' - \xi_j|_\varepsilon.$$

Roughly speaking,  $\zeta$  and  $\zeta'$  are beyond a critical point as seen from  $\xi_j$ .

**9.** Let  $\zeta = \phi_f(y) \in W$  with  $d_G(\zeta) = 1$ , then it follows from **5.** and **6.** that  $\phi_f(U_1(y)) \subset \mathcal{C}_0(\zeta)$ .

**10.** The set  $G(W)$  is covered by countably many pieces from  $\mathcal{V}$  of higher level which we may arrange into a sequence. We will first construct a finite cover of  $W \cap \{d_G = 1\}$  inductively. We assume that  $Q_n(\xi_j) \subset \widehat{\mathcal{C}}(\xi_j)$ ,  $j = 1, \dots, p$ , has been constructed so that for any  $\zeta, \zeta' \in Q_n(\xi_j)$ ,

$$G(\gamma'(\zeta, \zeta')) = \gamma'(G(\zeta), G(\zeta'))$$

and

$$|G(\zeta) - G(\zeta')|_\varepsilon \geq \lambda \cdot |\zeta - \zeta'|_\varepsilon.$$

We let  $U' \in \mathcal{V}$  be the vertex  $\mathcal{V}$  which arises next in the sequence, which is not contained in any  $G(Q_n(\xi_j))$ , and such that there exists a preimage of  $U'$  under  $G$  meeting  $W$  and having local degree 1. If no such  $U'$  exists, then we have split  $W \cap \{d_G = 1\}$  into finitely sets.

Let  $\zeta' \in U' \setminus (\cap G(Q_n(\xi_j)))$  be a regular value. It follows that each component of  $G^{-1}(\gamma(\xi', \zeta'))$  has as many points of  $G^{-1}(\zeta')$  than  $G^{-1}(\xi')$  at its ends (recall that  $\xi'$  has been defined in **8.**). So we may consider a splitting of  $G^{-1}(\zeta')$  associated to  $G^{-1}(\xi')$  such that, if  $\zeta \in Q_n(\xi_j)$ , then  $\zeta$  is paired with  $\xi_j$ . We label  $G^{-1}(\zeta')$  so that  $\zeta_j$  is paired with  $\xi_j$ .

Let us define

$$Q_{n+1}(\xi_j) = Q_n(\xi_j) \cup (\mathcal{C}(\zeta_j) \cap \widehat{\mathcal{C}}(\xi_j)).$$

Let  $\zeta \in \mathcal{C}(\zeta_j) \cap \widehat{\mathcal{C}}(\xi_j)$ ,  $\xi \in Q_n(\xi_j)$ , and let us consider

$$\Gamma_0 = \gamma(G(\xi), G(\xi_j)) \cup \gamma(G(\xi_j), G(\zeta_j)) \cup \gamma(G(\zeta_j), G(\zeta)).$$

We note that the rays

$$\begin{aligned} & \{W_n(G(\xi_j)), n \geq \min\{c(G(\xi_j), G(\xi)), c(G(\xi_j), G(\zeta_j))\}\} \\ & \{W_n(G(\xi)), n \geq c(G(\xi), G(\xi_j))\} \\ & \{W_n(G(\zeta_j)), n \geq \min\{c(G(\zeta_j), G(\zeta)), c(G(\zeta_j), G(\xi_j))\}\} \\ & \{W_n(G(\zeta)), n \geq c(G(\zeta), G(\zeta_j))\} \end{aligned}$$

force the structure of the tree, so that it provides an approximate value for  $|\zeta - \xi|_\varepsilon$ .

The connected set

$$\tilde{\Gamma}_0 = \gamma'(\xi, \xi_j) \cup \gamma'(\xi_j, \zeta_j) \cup \gamma'(\zeta_j, \zeta)$$

provides the pull-back of  $\Gamma_0$  which contains respectively  $\xi_j$ ,  $\xi$ ,  $\zeta$  and  $\zeta_j$  since  $\zeta_j, \xi \in \widehat{\mathcal{C}}(\xi_j)$  and  $\zeta \in \mathcal{C}(\zeta_j)$ . It follows from **4.** that

$$|G(\xi) - G(\zeta)|_\varepsilon \geq \lambda \cdot |\xi - \zeta|_\varepsilon.$$

We may then proceed to the next vertex. We note that every non-branched point is contained in some vertex with local degree 1. So, by **9.**, in the end, we have split  $W \cap \{d_G = 1\}$  into  $p$  Borel sets on which  $G$  is  $\lambda$ -expanding.

**11.** We assume that we have already split  $W \cap \{d_G < p_0\}$  into finitely many sets on which  $G$  is  $\lambda$ -expanding. We proceed inductively as above: we look at the first vertex in  $G^{-1}(\mathcal{V})$  of degree  $p_0$  which contains a point  $\xi$  of local degree  $p_0$  too. We then define  $Q_0(\xi) = \mathcal{C}(\xi)$ , and similarly for the other preimages of  $G(\xi)$  with the same local degree (if any). Inductively, we make choices according to the natural splitting for points with local degree  $p_0$  which we distribute so that they are associated to  $\xi$  only if it belongs to  $\widehat{\mathcal{C}}(\xi)$ .

In the end, we have split  $W \cap \{d_G = p_0\}$  into finitely many Borel sets on which  $G$  is  $\lambda$ -expanding.

**12.** Finally, steps **10.** and **11.** have split  $W$  into finitely many Borel sets as required. We proceed as above for all non-injective vertices to prove the proposition.

■

**Corollary 3.4.6** *For any ergodic invariant probability measure,*

$$h_\nu(F) = \int \log J_\nu d\nu.$$

**Proof:** It follows from the proposition above that we can define a finite measurable partition which separates points by taking finite intersections of the cover  $\mathcal{Q}$ . Therefore, since it is finite, it will have finite entropy for any probability measure. Rohlin formula then applies.

■

For  $\nu = \mu_f$ , one has  $J_{\mu_f}(F) = d/d_F$ , so that

$$h_{\mu_f}(F) = \int_{\partial_\varepsilon \Gamma} \log J_{\mu_f}(F) d\mu_f = \log d - \int_{\partial_\varepsilon \Gamma} \log d_F d\mu_f.$$

The variational principle applied to  $F$  then implies

$$\log d - \int \log d_F d\mu_f = h_{\mu_f}(F) \leq h_{top}(F) \leq v \leq \log d.$$

We obtain

**Proposition 3.4.7** *If the branch set  $B_F$  has measure zero, then  $\mu_f$  has maximal entropy  $\log d$ .*

### 3.4.3 Thermodynamic formalism

In the section, we introduce a couple of notions from the thermodynamic formalism and apply them to the study of fbc's satisfying [Expansion].

Let  $(Z, T)$  be a continuous finite-to-one map defined on a compact set. Given a potential  $\varphi : Z \rightarrow \mathbb{R}$ , one can define its topological pressure via the formula

$$P(\varphi) = \sup \left\{ h_\nu(T) + \int \varphi d\nu \right\}$$

where the supremum is taken over all ergodic invariant probability measures  $\nu$ . Let us remark that the topological entropy appears as the pressure

function of the potential 0. An ergodic invariant measure  $\nu$  is called an equilibrium measure associated to a potential  $\varphi$  if

$$P(\varphi) = h_\nu(T) + \int \varphi d\nu.$$

In this section, we prove:

**Proposition 3.4.8** *Let  $(\mathfrak{X}_1, \mathfrak{X}_0, f, X)$  be an fbc satisfying [Expansion]. The measure  $\mu_f$  is the unique equilibrium measure associated to the potential  $\log(d_F/d)$ , the pressure of which is 0.*

This will imply the following result.

**Corollary 3.4.9** *If there is an invariant measure  $\nu$  of entropy  $\log d$ , then  $\nu = \mu_f$ . In particular, the measure  $\mu_f$  is the unique measure of maximal entropy as soon as the critical set carries no  $\mu_f$ -mass.*

It is known that the corollary applies at least in the following situations, assuming [Expansion] as usual:

- $\mathfrak{X}_0$  is a surface (the critical points form a discrete set, and  $\mu_f$  has no atoms according to Theorem 3.4.4 as soon as  $d \geq 2$ );
- no critical point is recurrent (by Poincaré recurrence theorem, it then follows that the critical set carries no  $\mu_f$ -mass);
- the postcritical set is not dense (as above);
- the map is topological cxc (cf. Theorem 3.5.6).

**Transfer operator.** We are grateful to Guillaume Havard for this exposition. We specialise the definitions to our setting. Let  $T : Z \rightarrow Z$  be an fbc defined on a compact space, and let us assume that  $\nu$  is an invariant Borel probability measure.

Let  $L^1(\nu)$  denote the Banach space of real-valued  $\nu$ -integrable functions. In this subsection, we show the existence of an induced *transfer operator*

$$\mathcal{L}_\nu : L^1(\nu) \rightarrow L^1(\nu)$$



satisfying the following property:  $\forall \psi \in L^\infty(\nu), \forall \phi \in L^1(\nu)$ ,

$$\int \psi \cdot \mathcal{L}_\nu(\phi) \, d\nu = \int (\psi \circ T) \cdot \phi \, d\nu = \int \psi(x) \left( \sum_{T(y)=x} \frac{\phi(y)}{J_\nu(y)} \right) \, d\nu(x) \quad (3.3)$$

where  $J_\nu$  is the Jacobian of  $T$ . Since our transformation may be locally non-injective, the role played by the Jacobian is crucial. We have not found a general detailed treatment in the extant literature, so we provide full details here.

Given a function  $\phi \in L^1(\nu)$ , define a signed measure  $\nu_\phi$  on the collection of  $\nu$ -measurable sets  $A$  by

$$\nu_\phi(A) = \int_A \phi \, d\nu = \int (\chi_A \cdot \phi) \, d\nu$$

where  $\chi_A$  denotes the characteristic function of  $A$ . The *pushforward* of this measure under  $T$ , which we denote here by  $\nu_\phi \circ T^{-1}$ , is defined by

$$(\nu_\phi \circ T^{-1})(A) = \nu_\phi(T^{-1}(A)) = \int_{T^{-1}A} \phi \, d\nu = \int (\chi_{T^{-1}(A)} \cdot \phi) \, d\nu.$$

Since  $\chi_A \circ T = \chi_{T^{-1}A}$  we have

$$(\nu_\phi \circ T^{-1})(A) = \int ((\chi_A \circ T) \cdot \phi) \, d\nu.$$

If  $A$  is such that  $\nu(A) = 0$  then, by the invariance of  $\nu$ , we have  $\nu(T^{-1}(A)) = 0$ . It follows that the pushforward measure  $\nu_\phi \circ T^{-1}$  is absolutely continuous with respect to  $\nu$ . By the Radon-Nikodym theorem, the Radon-Nikodym derivative  $\frac{d\nu_\phi}{d\nu}$  is an element of  $L^1(\nu)$ , which we will denote by  $\mathcal{L}_\nu(\phi)$ , such that for each  $\nu$ -measurable set  $A$ ,

$$(\nu_\phi \circ T^{-1})(A) = \int ((\chi_A \circ T) \cdot \phi) \, d\nu = \int (\chi_A \cdot \mathcal{L}_\nu(\phi)) \, d\nu.$$

The assignment  $\phi \mapsto \frac{d\nu_\phi}{d\nu}$  yields a linear operator  $\mathcal{L}_\nu : L^1(\nu) \rightarrow L^1(\nu)$ , and the preceding equalities show that  $\mathcal{L}_\nu$  has norm bounded by 1.

It also follows from the previous computation that for every  $\psi \in L^\infty(\nu)$ ,

$$\int (\psi \cdot \mathcal{L}_\nu(\phi)) \, d\nu = \int ((\psi \circ T) \cdot \phi) \, d\nu.$$

In other words: the dual of  $\mathcal{L}_\nu$ , acting on  $L^\infty(\nu)$  which is the dual of  $L^1(\nu)$ , is the composition with  $T$ .

We derive now the expression for  $\mathcal{L}_\nu(\phi)$  at  $\nu$ -almost every point given by Equation (3.3). Since  $T$  is assumed to be an fbc on a compact set, there exists a finite measurable partition  $\mathcal{A}$  of  $Z$  such that  $T|_A$  is injective for each  $A \in \mathcal{A}$ . From the definition of the Jacobian we may assume that for each  $A \in \mathcal{A}$ ,

$$\nu(T(A)) = \int_A J_\nu d\nu.$$

Let  $t_A : T(A) \rightarrow A$  denote the inverse of  $T|_A : A \rightarrow T(A)$ . For  $B \subset A$  we have then

$$\int_{T(A)} \chi_B \circ t_A d\nu = \int_{T(A)} \chi_{T(B)} d\nu = \int_A \chi_B \cdot J_\nu d\nu.$$

We obtain that for every  $h \in L^1(\nu)$ ,

$$\int_{T(A)} h \circ t_A d\nu = \int_A h \cdot J_\nu d\nu.$$

Since  $\nu$  is invariant,  $J_\nu \geq 1$  almost everywhere with respect to  $\nu$ , hence  $\frac{h}{J_\nu} \in L^1(\nu)$ . Applying the preceding equality (with  $\frac{h}{J_\nu}$  in place of  $h$ ) shows that

$$\int_{T(A)} \frac{h \circ t_A}{J_\nu \circ t_A} d\nu = \int_A h d\nu.$$

Letting  $\psi \in L^\infty(\nu)$ ,  $\phi \in L^1(\nu)$ , and  $h = (\psi \circ T) \cdot \phi$  in the above equality then yields

$$\begin{aligned} \int \psi \cdot \mathcal{L}_\nu(\phi) d\nu &= \int (\psi \circ T) \cdot \phi d\nu \\ &= \sum_A \int_A (\psi \circ T) \cdot \phi d\nu \\ &= \sum_A \int_{T(A)} \frac{\psi \cdot (\phi \circ t_A)}{J_\nu \circ t_A} d\nu \\ &= \int \psi \cdot \left( \sum_A \chi_{T(A)} \frac{\phi \circ t_A}{J_\nu \circ t_A} \right) d\nu \\ &= \int \psi(x) \cdot \left( \sum_{T(y)=x} \frac{\phi(y)}{J_\nu(y)} \right) d\nu(x). \end{aligned}$$

This operator is extended to any measurable non-negative functions using the above formula, so that the identity

$$\int \mathcal{L}_\nu(\phi) d\nu = \int \phi d\nu$$

still holds (by truncation and the monotone convergence theorem).

The invariance of the measure implies that  $\mathcal{L}_\nu(1) = 1$  a.e.

**Transfer operators with respect to potentials.** Given a measurable real-valued potential function  $\varphi : Z \rightarrow \mathbb{R}$ , the *transfer operator with potential*  $\varphi$  is defined by

$$\mathcal{L}_\varphi(\psi)(z) = \sum_{T(z')=z} e^{-\varphi(z')} \psi(z').$$

**Remark.** In the case of our main interest—when  $(Z, T, \nu) = (\partial_\varepsilon \Gamma, F, \mu_f)$ —Equation 3.3 implies that

$$\mathcal{L}_{\mu_f}(\phi) = (1/d)F_*\phi = \mathcal{L}_{\log d/d_F}(\phi)$$

for every continuous function  $\phi : \partial_\varepsilon(\Gamma) \rightarrow \mathbb{R}$ .

We will use the following result which is proved using the concavity of the logarithm [Hav]:

**Proposition 3.4.10 (G. Havard)** *Let  $\varphi > 0$  be a measurable function defined on a backward invariant set of  $\nu$ -full measure. There is a set  $Z' \subset Z$  of  $\nu$ -full measure such that, for any  $z \in Z'$ ,*

$$\mathcal{L}_\nu \left( \log \frac{J_\nu}{\varphi} \right) \leq \log(\mathcal{L}_{\log \varphi}(1)(z)).$$

*Moreover, if the equality holds on a set of full  $\nu$ -measure, then  $J_\nu$  is a strong Jacobian and  $J_\nu = \varphi \mathcal{L}_{\log \varphi}(1) \circ T$ .*

**Proof:** (Proposition 3.4.8) We apply Proposition 3.4.10 to  $Z = \partial\Gamma$ ,  $T = F$  and  $\varphi = d/d_F$ , and to any ergodic invariant probability measure  $\nu$  on  $\partial\Gamma$ : one obtains

$$\mathcal{L}_\nu \left( \log \frac{J_\nu}{\varphi} \right) (\xi) \leq \log \mathcal{L}_{\log \varphi}(1)(\xi).$$

But

$$\mathcal{L}_{\log \varphi(1)}(\xi) = \sum_{F(\zeta)=\xi} \frac{d_F(\zeta)}{d} = 1$$

so that we get

$$\mathcal{L}_\nu \left( \log J_\nu - \log \frac{d}{d_F} \right) (\xi) \leq 0.$$

Integrating against  $\nu$ ,

$$\int \mathcal{L}_\nu \left( \log J_\nu - \log \frac{d}{d_F} \right) d\nu \leq 0.$$

From the definition of the transfer operator, it follows that

$$\int \log J_\nu d\nu - \int \log \frac{d}{d_F} d\nu \leq 0.$$

By Rohlin formula,

$$h_\nu(F) + \int \log \frac{d_F}{d} d\nu \leq 0.$$

This shows that the pressure of  $\log(d_F/d)$  is non-positive.

Now, if we take  $\nu = \mu_f$ , then

$$h_{\mu_f}(F) + \int \log \frac{d_F}{d} d\mu_f = \int \log J_{\mu_f} d\mu_f - \int \log J_{\mu_f} d\mu_f = 0$$

so that  $\mu_f$  is an equilibrium measure for  $\log d_F/d$ , and  $P(\log d_F/d) = 0$ .

Let us prove the uniqueness of the equilibrium measure. Let  $\nu$  be a measure such that

$$\int \log J_\nu d\nu - \int \log \frac{d}{d_F} d\nu = 0.$$

It follows that

$$\int \mathcal{L}_\nu \left( \log J_\nu - \log \frac{d}{d_F} \right) d\nu = 0,$$

but since the integrand is non-positive, we can deduce that

$$\mathcal{L}_\nu \left( \log J_\nu - \log \frac{d}{d_F} \right) = 0 \quad a.e.$$

Therefore, the case of equality in Proposition 3.4.10 applies, so that  $J_\nu$  is a strong Jacobian, equal to  $d/d_F$ . This implies that  $\nu$  is a quasiconformal measure and that  $\nu = \mu_f$ .

■

**Proof:** (Corollary 3.4.9). If the entropy of an ergodic invariant probability measure  $\nu$  is  $\log d$ , then, it follows that

$$\int \log d_F d\nu \leq P(\log d_F/d) = 0$$

so that  $d_F = 1$  for  $\nu$  a.e.  $\xi$ . Thus,  $\nu$  is an equilibrium measure for the potential  $\log d_F/d$ , and, by uniqueness,  $\nu = \mu_f$ .

If the critical set has  $\mu_f$ -measure 0, then  $J_{\mu_f} = d$  a.e., and  $h_{\mu_f} = \log d$ , so the previous argument applies.

■

### 3.4.4 Equidistribution

In this subsection, we prove that iterated preimages of points and periodic points are equidistributed according to  $\mu_f$ .

Let us note that since  $F_*F^*\varphi = d\varphi$ , the operator  $\nu \mapsto (1/d)F^*\nu$  has norm equal to 1.

**Theorem 3.4.11 (Equidistribution of preimages)** *For any probability measure  $\nu$  whose support is disjoint from  $o \in \overline{\Gamma}_\varepsilon$ , the sequence  $(1/d^n)(F^n)^*\nu$  converges to  $\mu_f$  in the weak-\* topology. In particular, if we set*

$$\mu_n^\xi = (1/d^n) \sum_{F^n(\zeta)=\xi} d_{F^n}(\zeta)\delta_\zeta = (1/d^n)(F^n)^*\delta_\xi$$

for any  $\xi \in \overline{\Gamma}_\varepsilon \setminus \{o\}$  and  $n \geq 1$ , then  $\mu_n^\xi$  converges to  $\mu_f$  in the weak-\* topology.

We may then deduce the following.

**Theorem 3.4.12 (Equidistribution of periodic points)** *The sequence of measures supported on  $\partial\overline{\Gamma}_\varepsilon$*

$$\hat{\mu}_n = \frac{1}{d^n} \sum_{F^n(\xi)=\xi} d_{F^n}(\xi)\delta_\xi$$

converges to  $\mu_f$  in the weak-\* topology.

**Remark:** Since the number of cycles of period  $n$  is not known, the measures  $\hat{\mu}_n$  need not be probability measures.

We start with a lemma (compare with the theory of primitive almost periodic operators, e.g. Theorem 3.9 in [EL]).

**Lemma 3.4.13** *For any continuous function  $\varphi : \bar{\Gamma} \setminus B(o, 1) \rightarrow \mathbb{R}$ , the sequence of functions  $(1/d^n)(F^n)_*\varphi$  is uniformly convergent towards the constant function*

$$\int \varphi d\mu_f.$$

**Proof:** Let us define  $A(\varphi) = (1/d)F_*\varphi$ . Let us consider two points  $\xi$  and  $\zeta$  close enough so that there exists a curve  $\gamma$  joining them and avoiding  $o$ . It follows that the points of  $F^{-n}(\{\xi\})$  and  $F^{-n}(\{\zeta\})$  are joined together by subcurves of  $F^{-n}(\gamma)$  of length bounded by  $\ell_\varepsilon(\gamma) \cdot e^{-\varepsilon n}$ .

If  $\varphi$  is a continuous function on  $\bar{\Gamma} \setminus B(0, 1)$  with modulus continuity  $\omega_\varphi$ , it follows that

$$|A^n\varphi(\xi) - A^n\varphi(\zeta)|_\varepsilon \leq \frac{1}{d^n} \sum_{F^n(\xi')=\xi} d_{F^n}(\xi')\omega_\varphi(\ell_\varepsilon(\gamma)e^{-\varepsilon n}) \leq \omega_\varphi(\ell_\varepsilon(\gamma)e^{-\varepsilon n}).$$

This shows that the sequence  $(A^n\varphi)$  is uniformly equicontinuous and that any limit is locally constant. Thus, if  $\bar{\Gamma} \setminus B(0, 1)$  is connected, then any limit is constant. Furthermore, since  $F^*\mu_f = d\mu_f$ , it follows that, for any  $n$ ,

$$\int A^n\varphi d\mu_f = \int \varphi d\mu_f$$

so that any constant limit has to be  $\int \varphi d\mu_f$ .

If  $\bar{\Gamma} \setminus B(0, 1)$  is not connected, one can argue as follows. Adding a constant if necessary, we can assume that  $\varphi \geq 0$ . Then  $(A^n\varphi)$  is a sequence of non-negative functions, and

$$\|A(\varphi)\|_\infty \leq \|\varphi\|_\infty,$$

so that the norms of  $(A^n(\varphi))$  form a decreasing convergent sequence. Let  $\varphi_\infty$  be any limit. One knows that it is locally constant; let us assume that it is not constant. We let  $k$  be any iterate large enough so that, for any

maximal open set  $E$  such that  $\varphi_\infty$  is constant,  $F^k(E \cap \partial\Gamma) = \partial\Gamma$ . Then, for any  $\xi \in \partial\Gamma$ ,

$$|(F^k)_*\varphi_\infty(\xi)| = \sum_{F^k(\zeta)=\xi} \frac{d_{F^k}}{d^k} \varphi_\infty(\zeta) < \|\varphi_\infty\|_\infty$$

since  $\varphi_\infty$  is not locally constant, but non-negative. This contradicts the fact that

$$\|\varphi_\infty\|_\infty = \inf_n \|A^n(\varphi)\|_\infty.$$

Thus  $\varphi_\infty$  is constant. ■

**Corollary 3.4.14** *The measure  $\mu_f$  is mixing.*

**Proof:** For any continuous function  $\varphi$ , and almost every  $\xi \in \partial_\varepsilon\Gamma$ , the sequence  $(1/d^n)(F^n)_*\varphi(\xi)$  tends to the value  $\mu_f(\varphi)$  by the above lemma. The operator  $A$  has norm one, so for all  $\xi$ ,  $|(1/d^n)(F^n)_*\varphi(\xi)| \leq \|\varphi\|_\infty$ . Hence

$$\left| \frac{1}{d^n}(F^n)_*\varphi - \mu_f(\varphi) \right|^2 \leq 4\|\varphi\|_\infty^2$$

and the dominated convergence theorem implies that  $F_*^n\varphi \rightarrow \mu_f(\varphi)$  in  $L^2(\partial_\varepsilon\Gamma, \mu_f)$ . It follows from Proposition 2.2.2 in [DiSi] that  $\mu_f$  is mixing. ■

**Proof:** (Theorem 3.4.11) Let  $\nu$  be a measure supported off the origin in  $\Gamma$ . For any continuous function  $\varphi$ , one has

$$\langle (1/d^n)(F^n)_*\nu, \varphi \rangle = \langle \nu, (1/d^n)(F^n)_*\varphi \rangle = \int A^n\varphi d\nu.$$

It follows from dominated convergence that this sequence tends to

$$\int \left( \int \varphi d\mu_f \right) d\nu = \int \varphi d\mu_f = \langle \mu_f, \varphi \rangle$$

so that  $(1/d^n)(F^n)_*\nu$  tends to  $\mu_f$ . ■

We begin the proof of Theorem 3.4.12 by a couple of intermediate results concerning periodic points, beginning with an index-type result.

**Proposition 3.4.15** *Let  $U$  be a Hausdorff connected, locally connected and locally compact open set,  $U'$  a relatively compact connected subset of  $U$ , and  $g : U' \rightarrow U$  a finite branched covering of degree  $d \geq 1$  which satisfies Axiom [Expansion] with respect to the covering  $\mathcal{U}_0 = \{U\}$ . Then*

$$d = \sum_{g(x)=x} d_g(x).$$

**Proof:** If, for every  $n$ ,  $g^{-n}(U)$  is connected, then Axiom [Expansion] implies that  $\cap f^{-n}(U)$  is a single point  $x$ , which is fixed: thus  $d = d_g(x)$ .

Otherwise, let  $k_0$  be the maximal integer such that  $g^{-k_0}(U)$  is connected. Then  $g^{-(k_0+1)}(U)$  is a finite union of connected open sets  $U_1^0, \dots, U_{m_0}^0$  where  $m_0 > 1$ . Each restriction  $g_j : U_j^0 \rightarrow g^{-k_0}(U)$  is a finite branched covering of degree  $d_j < d$ , and  $d = \sum d_j$ .

For each  $g_j$ , one may repeat this procedure until it stops. The proposition follows easily. ■

**Lemma 3.4.16** *Let  $U$  be a vertex, and let us consider open subsets  $W_1$  and  $W_2$  of  $\bar{\Gamma}_\varepsilon \setminus \{o\}$  which intersect  $\partial_\varepsilon \Gamma$  such that  $\bar{W}_1 \subset W_2$  and such that  $\bar{W}_2 \cap \partial_\varepsilon \Gamma \subset \phi_f(U)$ . For  $n$  large enough,*

$$\mu_n^U(W_1) \leq \hat{\mu}_n(W_2) \quad \text{and} \quad \hat{\mu}_n(W_1) \leq \mu_n^U(W_2).$$

**Proof:** See Figure 3.4.4.

Suppose  $\tilde{U} \in S(n+|U|)$  and, as a vertex point in  $\bar{\Gamma}_\varepsilon$ , belongs to  $W_1$ . Then  $\phi_f(\tilde{U}) \subset W_2$  if  $n$  is large enough, since  $d_\varepsilon(\tilde{U}, \phi_f(\tilde{U})) = (1/\varepsilon)e^{-|\tilde{U}|} \rightarrow 0$ . Hence  $\tilde{U} \subset \phi_f^{-1}(\bar{W}_2 \cap \partial_\varepsilon \Gamma) \subset U$ . So, if moreover  $f^n(\tilde{U}) = U$  then  $f^n|_{\tilde{U}} : \tilde{U} \rightarrow U$  satisfies the hypotheses of Proposition 3.4.15 and so

$$d_{f^n}(\tilde{U}) = \sum_{f^n(x)=x, x \in \tilde{U}} d_{f^n}(x).$$

For the periodic points  $x$  appearing in the sum,  $\phi_f(x) \in W_2$ .



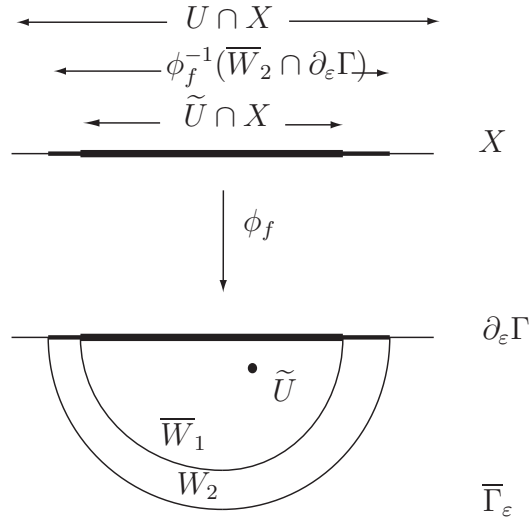


Figure 3.4.4

Therefore,

$$\begin{aligned}
 \mu_n^U(W_1) &= \frac{1}{d^n} \sum_{f^n(\tilde{U})=U, \tilde{U} \in W_1} d_{f^n}(\tilde{U}) \\
 &= \frac{1}{d^n} \sum_{f^n(\tilde{U})=U, \tilde{U} \in W_1} \sum_{x \in \tilde{U}, f^n(x)=x} d_{f^n}(x).
 \end{aligned}$$

Since for each such  $\tilde{U}$  appearing in the sum, (i)  $\phi_f(\tilde{U}) \subset W_2$ , and (ii) for fixed  $n$ , the  $\tilde{U}$ 's and their images under  $\phi_f$  are pairwise disjoint, we have

$$\mu_n^U(W_1) \leq \frac{1}{d^n} \sum_{x \in W_2, f^n(x)=x} d_{f^n}(x) = \hat{\mu}_n(W_2).$$

Similarly, if  $f^n(x) = x$  and  $x \in W_1$ , then there is a unique  $\tilde{U} \in f^{-n}(U)$  such that  $x \in \tilde{U}$ . Therefore, for  $n$  large enough,  $\phi_f(\tilde{U})$  has compact closure in  $W_2$  and  $\tilde{U} \in W_2$ . Thus

$$\hat{\mu}_n(W_1) \leq \frac{1}{d^n} \sum_{f^n(\tilde{U})=U, \tilde{U} \in W_2} \sum_{f^n(x)=x, \phi_f x \in \tilde{U}} d_{f^n}(x) \leq \frac{1}{d^n} \sum_{f^n(\tilde{U})=U, \tilde{U} \in W_1} d_{f^n}(\tilde{U}) = \mu_n(W_2).$$

■

**Proof:** (Theorem 3.4.12) Note that the number of cycles is unknown. Nevertheless, it follows from Lemma 3.4.16 that  $\{\hat{\mu}_n\}_n$  is relatively compact in the weak topology. Let  $\hat{\mu}$  be an accumulation point. We will prove that  $\hat{\mu} = \mu_f$  using their Borel regularity.

Let  $U$  be a vertex, and let us consider a compact subset  $K$  of  $\partial\Gamma$ , open subsets  $W_1, W_2$  and  $W_3$  of  $\bar{\Gamma} \setminus \{o\}$  such that

$$K \subset W_1 \cap \partial\Gamma \subset \overline{W_1} \subset W_2 \subset \overline{W_2} \subset W_3$$

and such that  $\overline{W_3} \cap \partial\Gamma \subset \phi_f(U)$ .

Let  $\varphi_1$  and  $\varphi_2$  be two continuous functions such that

$$\chi_K \leq \varphi_1 \leq \chi_{W_1} \leq \chi_{W_2} \leq \varphi_2 \leq \chi_{W_3}.$$

Let us fix  $\eta > 0$ ; if  $n$  is large enough then

$$\left\{ \begin{array}{l} |\hat{\mu}(\varphi_j) - \hat{\mu}_n(\varphi_j)| \leq \eta \\ |\mu_f(\varphi_j) - \mu_n^U(\varphi_j)| \leq \eta \end{array} \right.$$

for  $j = 1, 2$ .

Therefore, by the preceding Lemma 3.4.16 and the regularity of the mea-

sures,

$$\begin{aligned}
\hat{\mu}(K) &\leq \hat{\mu}(\varphi_1) \leq \hat{\mu}_n(\varphi_1) + \eta \leq \hat{\mu}_n(W_1) + \eta \\
&\leq \mu_n^U(W_2) + \eta \leq \mu_n^U(\varphi_2) + \eta \leq \mu_f(\varphi_2) + 2\eta \\
&\leq \mu_f(U) + 2\eta.
\end{aligned}$$

Since this is true for any compact subset of  $U$ , the regularity of the measures imply  $\hat{\mu}(U) \leq \mu_f(U)$ .

Similarly,

$$\begin{aligned}
\mu_f(K) &\leq \mu_f(\varphi_1) \leq \mu_n^U(\varphi_1) + \eta \leq \mu_n^U(W_1) + \eta \\
&\leq \hat{\mu}_n(W_2) + \eta \leq \hat{\mu}_n(\varphi_2) + \eta \leq \hat{\mu}(\varphi_2) + 2\eta \\
&\leq \mu_f(\phi_f(U)) + 2\eta
\end{aligned}$$

from which we deduce  $\hat{\mu}(\phi_f(U)) \geq \mu_f(\phi_f(U))$ , so that  $\hat{\mu} = \mu_f$ . ■

### 3.4.5 Hausdorff dimension

For a nonempty subset  $U$  of a metric space  $Z$  let  $|U|$  denote the diameter of  $U$ . Given  $\delta > 0$ , a  $\delta$ -cover of  $U$  is a covering of  $U$  by sets of diameter at most  $\delta$ . For  $s \geq 0$ , set

$$\mathcal{H}_\delta^s(U) = \inf \sum_i |U_i|^s$$

where the infimum is over all  $\delta$ -coverings of  $U$  by sets  $U_i$ . As  $\delta$  decreases,  $\mathcal{H}_\delta^s$  increases and so the  $s$ -dimensional Hausdorff measure of  $U$

$$\mathcal{H}^s(U) = \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^s(U) \in [0, \infty]$$

exists. The *Hausdorff dimension* of  $U$  is given by

$$\dim_H(U) = \inf\{s : \mathcal{H}^s(U) = 0\} = \sup\{s : \mathcal{H}^s(U) = \infty\}.$$

Using balls instead of arbitrary sets in the definition leaves the dimension unchanged. See for instance [Mat].

We now compute the Hausdorff dimension of the boundary  $\partial_\varepsilon\Gamma$ . Fix  $s > 0$ . By Lemma 3.1.3, for any vertex  $\xi \in S(n)$ ,  $\text{diam}_\varepsilon \mathcal{U}_\infty(\xi) \leq Ce^{-\varepsilon n}$ . Therefore  $\partial_\varepsilon\Gamma$  is covered by at most  $|S(n)|$  sets of diameter  $\delta_n = Ce^{-\varepsilon n}$  and so

$$\mathcal{H}_{\delta_n}^s(\partial_\varepsilon\Gamma) \leq |S(n)|e^{-\varepsilon ns}.$$

Suppose now that  $s > \frac{v}{\varepsilon}$ . Recall that by definition,  $v = \lim \frac{1}{n} \log |S(n)|$ . There exists  $\eta > 0$  with  $v + \eta - \varepsilon s < 0$ . It follows that for all  $n$  sufficiently large,

$$\mathcal{H}_{\delta_n}^s(\partial_\varepsilon\Gamma) \leq |S(n)|e^{-\varepsilon ns} < e^{n(v+\eta-\varepsilon s)} < \infty.$$

Hence

$$\mathcal{H}^s(\partial_\varepsilon\Gamma) < \infty \quad \text{for all } s > \frac{v}{\varepsilon}$$

and therefore  $\dim_H(\partial_\varepsilon\Gamma) \leq \frac{v}{\varepsilon}$ .

We now investigate lower bounds by appealing to the following result, which is similar to R. Mañé's dimension formula [Mañ]:

**Theorem 3.4.17** *If  $\mu$  is an ergodic invariant measure with positive entropy, then*

$$\liminf \frac{\log \mu(B_\varepsilon(\xi, r))}{\log r} \geq h_\mu(F)/\varepsilon,$$

for  $\mu$ -almost every  $\xi$ .

**Proof:** Since  $F$  is  $e^\varepsilon$ -Lipschitz, it follows that  $B_\varepsilon(\xi, r) \subset S(\xi, n, re^{n\varepsilon})$ . Since  $\mu$  is invariant and ergodic, it follows from a formula of Brin and Katok [BK] that an equivalent definition of metric entropy is

$$h_\mu(F) = \lim_{r \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{n} \log \mu(S(\xi, n, r)),$$

for  $\mu$ -a.e.  $\xi$ .

Choose a generic point  $\xi$  for  $\mu$  and let  $\eta > 0$ ; we will write  $B_\varepsilon(r) = B_\varepsilon(\xi, r)$ . There are some  $r_0 > 0$  and  $n_0 \in \mathbb{N}$  such that, if  $r \leq r_0$  and  $n \geq n_0$  then

$$\left| -\frac{1}{n} \log \mu S(n, r) - h_\mu(F) \right| \leq 2\eta.$$

We choose  $r_n = r_0 e^{-\varepsilon n}$  and we obtain

$$\frac{\log \mu(B_\varepsilon(r_n))}{\log r_n} \geq -\frac{\log \mu(S(n, r_0))}{n(\varepsilon - \log(r_0)/n)} \geq \frac{h_\mu(F) - 2\eta}{\varepsilon - \log(r_0)/n},$$

so

$$\liminf \frac{\log \mu(B_\varepsilon(r_n))}{\log r_n} \geq \frac{h_\mu(F) - 2\eta}{\varepsilon}.$$

Given  $r > 0$ , fix  $n$  so that  $B_\varepsilon(r_{n+1}) \subset B_\varepsilon(r) \subset B_\varepsilon(r_n)$  and

$$\frac{\log \mu(B_\varepsilon(r))}{\log r} \geq \frac{\log \mu(B_\varepsilon(r_n))}{\left(\log \frac{r_{n+1}}{r_n}\right) + \log r_n}.$$

Thus,

$$\liminf \frac{\log \mu(B_\varepsilon(r))}{\log r} \geq h_\mu(F)/\varepsilon. \quad \blacksquare$$

It follows that for the measure we have constructed and for any  $\eta > 0$  and  $r$  small enough,

$$\mu_f(B_\varepsilon(r)) \leq r^{(1/\varepsilon)h_\mu(F) - \eta}.$$

This implies that the local upper pointwise dimension of  $\mu$  satisfies  $\dim \mu_f \geq (1/\varepsilon)h_\mu(F)$ . Therefore

$$\frac{h_\mu(F)}{\varepsilon} \leq \dim \mu_f \leq \dim \partial\Gamma_\varepsilon \leq \frac{v}{\varepsilon} \leq \frac{1}{\varepsilon} \log d.$$

**Proof:** (Theorem 3.4.1) By ergodicity and uniqueness of the class of quasisconformal measures of given dimension, it follows that  $\mu_f$  is unique (cf. Theorem 3.4.4). Theorem 3.4.11 and Theorem 3.4.12 prove the equidistribution of preimages and periodic points according to  $\mu_f$ . The mixing property has also been proved (Corollary 3.4.14). The claimed entropy and dimension estimates were proven in §3.4.2 and §3.4.5, respectively. \blacksquare

### 3.5 Properties for cxc maps following hyperbolicity

In this section we assume that  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  has repeller  $X$  and is topologically cxc with respect to some open cover  $\mathcal{U}_0$ . Thus, the topological Axioms [Expansion], [Irreducibility], and [Degree] hold.

Let  $\Gamma = \Gamma(f, \mathcal{U}_0)$  be the associated Gromov hyperbolic graph as in Section 3.2. Recall that by Theorem 3.2.1, for  $\varepsilon > 0$  small enough, there is a homeomorphism  $\phi_f : X \rightarrow \partial_\varepsilon \Gamma$  conjugating  $f$  on  $X$  to the Lipschitz map  $F : \partial_\varepsilon \Gamma \rightarrow \partial_\varepsilon \Gamma$ .

**Theorem 3.5.1** *If  $f$  is topological cxc, then  $\Gamma$  is hyperbolic for any covering with a mesh small enough and  $F : \partial \Gamma \rightarrow \partial \Gamma$  satisfies axioms [Roundness distortion] and [Relative diameter distortion] with respect to the covering  $\mathcal{V}_0 = \{\phi_f(\mathcal{U}_0 \cap X)\}_{U \in \mathcal{U}_0}$ . If  $f$  is furthermore metric cxc, then  $\partial_\varepsilon \Gamma$  is quasimetrically equivalent to  $X$ .*

So, if  $f$  is topologically cxc, dynamics on  $X$  equipped with the metric pulled back from  $\Gamma$  via  $\phi_f$  is essentially metrically cxc.

**Remark:** Since  $F : \partial_\varepsilon \Gamma \rightarrow \partial_\varepsilon \Gamma$  is not metrically cxc, Theorem 2.8.2 does not apply.

**Proof:** Since  $f$  is topological cxc, a metric exists so that the mesh of  $S(n)$  has exponential decay (cf. Theorem 3.2.5) and  $\phi_f : X \rightarrow \partial_\varepsilon \Gamma$  is a homeomorphism as soon as  $\varepsilon$  is small enough. Therefore Theorem 3.3.1 implies that  $\Gamma$  is hyperbolic and that its quasi-isometry is well-defined.

We let  $\mathbf{V} = \phi_f(\mathbf{U})$ . Axioms [Irreducibility], [Expansion] and [Degree] hold through the conjugation. The [Diameter distortion] and [Roundness distortion] Axioms follow from Proposition 3.3.7.

Let us assume from now on that  $f$  is cxc. Our strategy is as follows. We will first establish that  $\phi_f$  is weakly quasimetric by the blowing up/down argument given in the proof of Theorem 2.8.2. The proof concludes by arguments similar to those given in the proof of Proposition 3.3.12.

Let  $\delta$  be the Lebesgue number of  $\mathcal{U}_0 = S(1)$ . Let  $x \in X$  and let  $r \in (0, \delta/L)$  where  $L$  is given by Proposition 2.6.6.

It follows from Proposition 2.6.6 that we may find vertices  $W', W$  such that

$$B(x, r/L) \subset W' \subset B(x, r) \subset W \subset B(x, Lr).$$

It follows from  $\text{diam}W' \asymp \text{diam}W$  that  $|W - W'| = ||W| - |W'|| \leq N$  for some constant  $N$ . Let  $n = \min\{|W|, |W'|\} - 1$ . It follows that  $f^n(W') \subset f^n(B(x, r)) \subset f^n(W)$  and that the roundness of  $f^n(W')$ ,  $f^n(B(x, r))$  and  $f^n(W)$  at  $f^n(x)$  is bounded by  $\rho_+(L)$ .

It follows from the uniform continuity of the conjugacy  $\phi_f$  and its inverse, and the fact that all these sets have a definite size, that a constant  $K$  exists such that the roundness of  $\phi_f(f^n(W'))$ ,  $\phi_f(f^n(B(x, r)))$  and  $\phi_f(f^n(W))$  at  $\phi_f(f^n(x))$  is bounded by  $K$ .

Therefore, radii  $r$  and  $r'$  exist such that

$$B(F^n \phi_f(x), r' e^{\varepsilon p}) \subset \phi_f(f^n(W')) \subset B(F^n \phi_f(x), K r' e^{\varepsilon n})$$

and

$$B(F^n \phi_f(x), r e^{\varepsilon p}) \subset \phi_f(f^n(W)) \subset B(F^n \phi_f(x), K r e^{\varepsilon n})$$

From Proposition 3.3.3, it follows that there is some finite constant  $H$  such that

$$\text{Round}(\phi_f(B(x, r)), \phi_f(x)) \lesssim \frac{\text{diam}_\varepsilon \phi_f(W)}{\text{diam}_\varepsilon \phi_f(W')} \asymp e^{-\varepsilon(|W - W'|)} \leq H.$$

Therefore  $\phi_f$  is weakly quasimetric.

By the uniform continuity of  $\phi_f$  and its inverse, it is enough to consider  $x, y, z \in X$  such that  $|x - y|_X, |x - z|_X \leq \delta/L$ . We argue as for Proposition 3.3.12.

It follows from above that we may find  $W'_y$  and  $W_z$  in  $\Gamma$  such that

1.  $y \notin W'_y$ ,  $\text{diam}W'_y \asymp |x - y|_X$  and  $\text{Round}(W'_y, x) \leq K$ ,
2.  $z \in W_z$ ,  $\text{diam}W_z \asymp |x - z|_X$  and  $\text{Round}(W_z, x) \leq K$ ,

for some universal  $K$ .

It follows that

$$\frac{|\phi_f(x) - \phi_f(y)|_\varepsilon}{|\phi_f(x) - \phi_f(z)|_\varepsilon} \asymp \frac{\text{diam}_\varepsilon W'_y}{\text{diam}_\varepsilon W_z}.$$

If  $|x - y|_X$  and  $|x - z|_X$  are equivalent, then Proposition 2.6.4 implies the bounds.

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If  $|x - y|_X$  is small in front of  $|x - z|_X$ , then  $W'_y \subset W_z$ . Therefore

$$\frac{|x - y|_X}{|x - z|_X} \asymp \frac{\text{diam}W'_y}{\text{diam}W_z} \geq \delta_+^{-1} \left( \frac{c_{|W'_y| - |W_z| + 1}}{d_1} \right),$$

where we recall that  $c_n$  denotes the smaller diameter of sets in  $S(n)$ .

This implies that

$$\frac{|\phi_f(x) - \phi_f(y)|_\varepsilon}{|\phi_f(x) - \phi_f(z)|_\varepsilon}$$

is bounded by a function of

$$\frac{|x - y|_X}{|x - z|_X}$$

which goes to zero as the ratio tends to zero.

If  $|x - y|_X$  is large in front of  $|x - z|_X$ , then  $W'_y \supset W_z$ . Therefore

$$\frac{|x - z|_X}{|x - y|_X} \asymp \frac{\text{diam}W_z}{\text{diam}W'_y} \geq \delta_- \left( \frac{d_{|W_z| - |W'_y| + 1}}{c_1} \right).$$

We may conclude as above.

This proves that  $\phi_f$  is quasisymmetric. ■

**Remark.** Theorem 2.8.2 can be recovered with Theorem 3.3.1 and Theorem 3.5.1.

As an application, we obtain the following result.

**Definition 3.5.2** *Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  have repellor  $X$  and be topologically cxc with respect to some open covering  $\mathcal{U}_0$ . The associated conformal gauge  $\mathcal{G}$  is the set of all metrics on  $X$  which are quasisymmetrically equivalent to a metric of the form  $\phi_f^*(d_\varepsilon)$ , where  $d_\varepsilon$  is the metric on  $\partial_\varepsilon\Gamma$  and  $\phi_f : X \rightarrow \partial_\varepsilon\Gamma$  is as above.*

**Theorem 3.5.3 (Canonical gauge)** *1. Let  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  have repellor  $X$  and be topologically cxc with respect to some open covering  $\mathcal{U}_0$ . Then the conformal gauge  $\mathcal{G}$  is nonempty and depends only on its topological conjugacy class.*



2. If  $U \cap X$  is connected for every  $U \in \mathcal{U}_0$ , then the conformal gauges of  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  and  $f|_X : X \rightarrow X$  agree.
3. If the system is in addition metrically cxc with respect to some metric  $d$  on  $\mathfrak{X}_0$ , the conformal gauge  $\mathcal{G}$  of  $f$  agrees with the conformal gauge of the metric space  $(X, d|_X)$ .

**Proof:** (1) follows from the uniqueness result Theorem 3.3.12 and (3) from the preceding Theorem 3.5.1. The graph constructed using  $f$  and  $\mathcal{U}_0$  is naturally identified with that constructed using  $f|_X$  and  $\mathcal{V}_0 = \{U \cap X : U \in \mathcal{U}_0\}$  and the induced conjugacies respect this identification. Therefore the metrics on  $X$  obtained by pulling back the metrics on the boundaries of the two graphs coincide and (2) follows. ■

**Remark.** The preceding theorem implies that the gauge depends only on the dynamics *near* the repellor. One may surmise that it should really depend only on the dynamics *on* the repellor itself. Conclusion (2) implies that this is true once  $X$  is locally connected. In the non-connected case, however, a proof remains elusive.

To illustrate the subtleties, fix  $d \geq 2$ , let  $X = \{1, 2, \dots, d\}^{\mathbb{N}}$  be equipped with the metric  $|x - y| = 2^{-(x|y)}$  where  $(x|y) = \min_i \{x_i \neq y_i\}$ , and suppose  $h : X \rightarrow X$  is a topological conjugacy, i.e. an automorphism of the one-sided shift on  $d$  symbols. If the gauge of  $f$  depends only on the dynamics on  $X$ , then every such  $h$  should be quasisymmetric. This is indeed the case, and a proof may be given along the following lines.

1. Start with a round closed disk  $D \subset \mathbb{C}$ . Fix  $0 < \lambda \ll 1$ . For each  $i = 1, \dots, d$  choose a similarity  $g_i : \mathbb{C} \rightarrow \mathbb{C}$  such that  $g_i(D) \cap g_j(D) = \emptyset$  whenever  $i \neq j$ . This defines a conformal iterated function system (IFS). There is a unique nonempty compact set  $K \subset D$  for which  $K = \cup_{i=1}^d g_i(K)$ . Using a blowing up/down argument, one shows that the attractor of this IFS is quasisymmetrically equivalent to  $X$ .
2. Using quasiconformal surgery, one builds a uniformly quasiregular map  $G : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $G|_{g_i(D)} = g_i^{-1}$  for each  $i$ , and such that  $G = z^d + O(z^{d-1})$  as  $z \rightarrow \infty$ . By Sullivan's Theorem 4.4.1,  $G$  is quasiconformally conjugate to a degree  $d$  polynomial  $p(z)$ , and  $K$  is quasiconformally (hence quasisymmetrically) equivalent to the Julia set  $J$  of  $p$ .

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3. By results of Blanchard, Devaney, and Keen [BDK], every automorphism of the shift on  $d$  symbols is realized as monodromy in the shift locus of degree  $d$  polynomials. (The proof depends on the existence of a nice set of generators.)
4. As a polynomial varies in the shift locus, its Julia sets varies holomorphically [MS]. Hence the induced monodromy is quasiconformal, hence quasimetric. In conclusion, we see that every automorphism is realized by a quasimetric map.

There seems to be a combinatorial obstacle to promoting topological conjugacies to quasimetric conjugacies for noninvertible expanding conformal dynamical systems with disconnected repellers. Even for hyperbolic rational maps  $f, g$  with disconnected Julia sets  $J_f, J_g$ , it is not known if every topological conjugacy  $h : J_f \rightarrow J_g$  is quasimetric.

This is known in the following special cases. First, if  $h$  extends to a conjugacy on a neighborhood of  $J_f, J_g$  then Theorem 2.8.2 applies and  $h$  is quasimetric. However, even for maps with connected Julia set, such an extension need not exist. Second, if  $f$  and  $g$  are merely combinatorially equivalent in the sense of McMullen [McM4] on a neighborhood of their Julia sets, then there is a quasiconformal conjugacy between  $f$  and  $g$  near their Julia sets. In both cases, conditions on the dynamics near, not just on, the Julia sets are assumed.

In contrast, we have the following result in the setting of hyperbolic groups. Suppose  $G_1, G_2$  are two hyperbolic groups, and suppose  $h : \partial G_1 \rightarrow \partial G_2$  conjugates the action of  $G_1$  to the action of  $G_2$ . By definition, this implies that there is some isomorphism  $\Phi : G_1 \rightarrow G_2$  for which  $h(g(x)) = \Phi(g)(h(x))$  for all  $x \in \partial G_1$  and all  $g \in G_1$ . One has necessarily that  $h$  arises as the boundary values of  $\Phi$ . To see this, note that it is enough to verify that  $h = \partial\Phi$  on the dense set of fixed points of hyperbolic elements. Suppose  $g_1 \in G_1$  is hyperbolic with attracting fixed point  $\omega_1$  and  $g_2 = \Phi(g_1)$  has attracting fixed point  $\omega_2$ . Since  $h$  is a continuous conjugacy we have  $h(\omega_1) = \omega_2$ . But  $\omega_i = \lim_n g_i^n$  and this forces  $\partial\Phi(\omega_1) = \omega_2 = h(\omega_1)$ . Thus, every topological conjugacy on the boundary is induced from a combinatorial equivalence, i.e. from an isomorphism of the groups.

This suggests that perhaps there is yet another essential difference between the setting of noninvertible cxc maps and of hyperbolic groups.

**Corollary 3.5.4** *If  $f : X \rightarrow X$  is a topological cxc map, where  $\mathfrak{X}_1 = \mathfrak{X}_0 = X$ , then  $F : \partial\Gamma \rightarrow \partial\Gamma$  is metrically cxc. Therefore  $X$  admits a metric, unique up to quasisymmetry, for which the dynamics is metrically cxc.*

**Proof:** The assumptions imply that  $X$  is locally connected, and that  $(S(n))_n$  is a basis of the topology by connected open sets. Proposition 3.3.2 implies the cxc property. ■

**Corollary 3.5.5** *If  $f : (\mathfrak{X}_1, X) \rightarrow (\mathfrak{X}_0, X)$  is a topological cxc map with  $f$  a non ramified covering, then there is some  $R > 0$ , such that, if we set  $\mathfrak{Y}_0 = \overline{\Gamma}_\varepsilon \setminus B_\varepsilon(o, R)$  and  $\mathfrak{Y}_1 = F^{-1}(\mathfrak{Y}_0)$ , then  $F : (\mathfrak{Y}_1, \partial\Gamma) \rightarrow (\mathfrak{Y}_0, \Gamma)$  is cxc.*

**Proof:** Since  $f$  is a cover, there is some level  $n_0$  such that, for any  $n \geq n_0$ , any  $U \in S(n)$ , the restriction of  $f$  to  $U$  is injective. This implies that the local degree function for  $F$  is 1 at any point close enough to  $\partial\Gamma$ .

Furthermore, if  $n_0$  is large enough, then  $F^{-1}(\mathfrak{U}(U))$  will be a disjoint union of  $d$  shadows based at  $F^{-1}(\{U\})$ .

Therefore, if we set  $\mathfrak{Y}_0 = \overline{\Gamma}_\varepsilon \setminus B_\varepsilon(o, e^{-\varepsilon n_0})$  and  $\mathfrak{Y}_1 = F^{-1}(\mathfrak{Y}_0)$ , then  $F : \mathfrak{Y}_1 \rightarrow \mathfrak{Y}_0$  is a degree  $d$  covering.

For any  $\xi \in \partial\Gamma$ , let  $V(\xi)$  be the connected component of the interior of  $\mathfrak{U}(W)$  for some  $W \in S(n_0)$  containing  $\phi_f^{-1}(\xi)$ . Note that the interior of  $\mathfrak{U}(W)$  is not empty since it contains  $\phi_f(W)$ . Since  $\overline{\Gamma}$  is locally connected,  $V(\xi)$  is open, and we may extract a finite subcover  $\mathcal{V}$ . Proposition 3.3.2 implies that  $F$  is cxc. ■

We may now state and prove the following theorem.

**Theorem 3.5.6** *Let  $f : (\mathfrak{X}_1, X) \rightarrow (\mathfrak{X}_0, X)$  be a topological cxc map. Then the measure  $\mu_f$  is the unique measure of maximal entropy  $\log d$ , and is Ahlfors regular of dimension  $\alpha = (1/\varepsilon) \log d$ .*

**Proof:** Ahlfors regularity follows from Axiom [Degree] and the Lemma of the shadow. Let us fix a ball  $B_\varepsilon(\xi, r) \subset \partial\Gamma$ . First, Proposition 3.3.2 implies

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that we may find two vertices  $W_1, W_2$  such that  $\phi_f(W_1) \subset B_\varepsilon(\xi, r) \subset \phi_f(W_2)$  and

$$r \asymp e^{-\varepsilon|W_1|} \asymp e^{-\varepsilon|W_2|}.$$

From the Lemma of the shadow and Axiom [Degree] follows

$$r^\alpha \asymp e^{-\varepsilon\alpha|W_1|} \lesssim \mu_f(B_\varepsilon(\xi, r)) \lesssim e^{-\varepsilon\alpha|W_2|} \asymp r^\alpha.$$

The fact that the entropy is  $\log d$  follows from Theorem 3.4.1 and Axiom [Degree]: since the degree is bounded along any pull-back, it follows that, for any  $\xi \in \partial\Gamma$ ,

$$\lim \frac{1}{n} \log d_{F^n}(\xi) = 0.$$

Hence Birkhoff's ergodic theorem implies that

$$\int \log d_F d\mu_f = 0$$

so that  $d_F(\xi) = 1$  for  $\mu_f$ -a.e. every  $\xi$ .

The uniqueness of  $\mu_f$  follows from Corollary 3.4.9.

■

**Remark.** Since  $\phi_f$  is a homeomorphism, one can choose to see  $\mu_f$  on  $X$  rather than on  $\partial\Gamma$ . In that case, the measure is just doubling (there is some constant  $C > 0$  such that  $\mu_f(2B) \leq C\mu_f(B)$ ), since this property is preserved under quasisymmetric mappings (cf. [Hei]), and since any Ahlfors regular measure is doubling.

**BPI-spaces.** Following David and Semmes [DaSe], a bounded space  $(X, d, \mu)$  is called BPI (“Big pieces of itself”) if  $X$  is Ahlfors regular of dimension  $\alpha$ , and if the following homogeneity condition holds. There are constants  $\theta < 1$  and  $C > 1$  such that, given any balls  $B(x_1, r_1)$  and  $B(x_2, r_2)$  with  $r_1, r_2 \leq \text{diam}X$ , there exists a closed set  $A \subset B(x_1, r_1)$  with  $\mu(A) \geq \theta r_1^\alpha$  and a homeomorphism  $h : A \rightarrow B(x_2, r_2)$  such that  $h$  is a  $(C, r_2/r_1)$ -quasisimilarity, i.e.

$$C^{-1} \leq \frac{|h(a) - h(b)|}{(r_2/r_1)|a - b|} \leq C$$

for all  $a, b \in A$ .

**Theorem 3.5.7** *Suppose  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  has repellor  $X$  and is topologically cxc with respect to some open cover  $\mathcal{U}_0$ . Let  $\Gamma$  be the hyperbolic graph and  $\varepsilon_0$  the constant given by Theorem 3.2.1. Then for all  $\varepsilon < \varepsilon_0$ , the metric space  $\partial_\varepsilon \Gamma$  is BPI.*

**Proof:** We start with a preliminary step.

Suppose  $\phi_f : X \rightarrow \partial_\varepsilon \Gamma$  is the conjugacy given by Theorem 3.2.1 and  $d_\varepsilon$  is the metric on  $\partial_\varepsilon X$ . Let  $d_{\varepsilon, X} = \phi_f^*(d_\varepsilon)$ . For convenience of notation, we will show  $(X, d_{\varepsilon, X})$  is BPI.

Recall that since  $f$  is topologically cxc, there is a uniform (in  $n$ ) upper bound  $p$  on the degree  $m(U)$  by which an element of  $U \in \mathcal{U}_n$  maps under  $f^n$ . Choose  $W \in \mathcal{U}_{n_0}$  arbitrarily so that the multiplicity  $m(W)$  is maximal, so that any further preimages  $\widetilde{W}$  of  $W$  map onto  $W$  by degree one i.e., are homeomorphisms. It follows from Proposition 3.3.2 that its image under  $\phi_f$  contains some ball  $B_\varepsilon(\xi, 4r)$ , such that, for any iterate  $n$ , any  $\xi \in F^{-n}(\xi)$ ,  $F^n : B_\varepsilon(\xi, 4re^{-\varepsilon n}) \rightarrow B_\varepsilon(\xi, 4r)$  is a homeomorphism. Therefore, Proposition 3.2.3 shows that  $F^n : B_\varepsilon(\xi, re^{-\varepsilon n}) \rightarrow B_\varepsilon(\xi, r)$  is a  $(1, e^{\varepsilon n})$  quasisisimilarity.

By Proposition 2.4.2, for each  $U_0 \in \mathcal{U}_0$ , there exists  $k \in \mathbb{N}$  and some  $\widetilde{W} \in \mathcal{U}_k$  such that

- $\widetilde{W} \subset U_0$ ,
- $\widetilde{W}$  is a preimage of  $W$  under  $f^k$ , and
- $\deg(f^k : \widetilde{W} \rightarrow W) = 1$ .

Since  $\mathcal{U}_0$  is finite, the  $\widetilde{W}$ 's considered above have a level bounded by some  $n_0 + k_0$ .

Furthermore, for any  $n$  and any  $U \in \mathcal{U}_n$ , one has  $f^n(U) \in \mathcal{U}_0$ , so one may find a preimage  $W_U$  of  $W$  so that  $\overline{W_U} \subset U$ , and  $|W_U| = n + O(1)$ . Thus, one can find a ball  $B(\xi', re^{-\varepsilon(n+k)}) \subset W_U$  so that  $f^{n+k} : B(\xi', re^{-\varepsilon(n+k)}) \rightarrow B(\xi, r)$  is a  $(1, e^{-\varepsilon(n+k)})$  quasisisimilarity. Let us note that  $re^{-\varepsilon(n+k)} \asymp \text{diam}_\varepsilon U$ .

Now suppose we are given  $d_{\varepsilon, X}$  balls  $B_i = B(\xi_i, r_i) \subset X$ ,  $i = 1, 2$ . By Proposition 3.3.2, there exist  $U'_i, U_i \in \mathbf{U}$  with

$$U'_i \cap X \subset B_i \subset U_i \cap X$$

such that  $n_i = |U'_i| = \frac{1}{\varepsilon} \log \frac{1}{r_i} + O(1)$ . For each  $i = 1, 2$ , let  $W_i$  be a preimage of  $W$  so that  $\overline{W_i} \subset U'_i$ , and  $|W_i| = n_i + k_i$  as in the previous

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paragraph. Moreover, we consider balls  $B(\xi'_i, re^{-\varepsilon(n_i+k_i)}) \subset W_i$  as above. It follows from Ahlfors-regularity that  $\mu_f(B(\xi'_i, re^{-\varepsilon(n_i+k_i)})) \asymp \mu_f(B_i)$ . Let  $h_i = f^{n_i+k_i}|_{B(\xi'_i, re^{-\varepsilon(n_i+k_i)})}$ , for  $i = 1, 2$ ; the map  $h = h_2^{-1} \circ h_1$  is a quasimilarity between big pieces of  $B_1$  and  $B_2$ .

■



# Chapter 4

## Examples of cxc systems

### 4.1 No exotic cxc systems on $S^1$

Metric cxc systems on the Euclidean circle include the covering maps  $z \mapsto z^d$ ,  $|d| \geq 2$ , and essentially nothing else.

**Theorem 4.1.1 (Cxc on  $S^1$  implies qs conjugate to  $z^d$ )** *Suppose  $f : X \rightarrow X$  is a metric cxc dynamical system where  $X$  is homeomorphic to  $\mathbb{S}^1$ . Then there exists a quasisymmetric homeomorphism  $h : X \rightarrow \mathbb{S}^1$  conjugating  $f$  on  $X$  to the map  $z \mapsto z^{\deg f}$  on the Euclidean circle  $\mathbb{S}^1$ .*

**Proof:** An open connected subset of  $\mathbb{S}^1$  is an interval. Since  $f$  is open, it sends small open intervals onto small open intervals. Moreover, if these intervals are small enough,  $f$  must be injective on such intervals, else there is a turning point in the graph and openness fails. Hence  $f$  is a local homeomorphism. A local homeomorphism on a compact space is a covering map [see [AH], Thm. 2.1.1]. In particular  $f$  is strictly monotone.

Such a map admits a monotone factor map  $\pi$  onto  $g(z) = z^d$  where  $d = \deg f$  [[KH], Prop. 2.4.9]. If  $\pi$  is not a homeomorphism, then there is an interval  $I \subset \pi^{-1}(x)$  for some  $x \in \mathbb{S}^1$ . Axiom [Irreducibility] implies  $f^N(I) = \mathbb{S}^1$  for some  $N$ . Then

$$g^N(x) = g^n(\pi(I)) = \pi(f^N(I)) = \pi(\mathbb{S}^1) = \mathbb{S}^1$$

which is impossible. Thus  $\pi$  is a homeomorphism and  $f$  is topologically conjugate to  $g$ . Since  $g$  is cxc with respect to the Euclidean metric,  $\pi$  is quasisymmetric, by Theorem 2.8.2. ■



## 4.2 Semi-hyperbolic rational maps

We endow the Riemann sphere  $\widehat{\mathbb{C}}$  with the spherical metric, and we will talk of disks  $D(x, r)$  rather than balls  $B(x, r)$  in this context.

If  $g$  is a rational map, the Fatou set  $F(g)$  of  $g$  is the set of points  $z \in \widehat{\mathbb{C}}$  which admit a neighborhood  $N(z)$  such that the restrictions  $\{(f^n)|_{N(z)}\}_n$  forms a normal family. The Julia set  $J(g)$  of  $g$  is the complement of  $F(g)$ . We shall say that  $g$  is chaotic when  $J(g) = \widehat{\mathbb{C}}$ .

The class of semi-hyperbolic rational maps has been introduced by L. Carleson, P. Jones and J.-C. Yoccoz in [CJY]. In their paper, they provide several different characterizations, some of which we recall now.

**Theorem 4.2.1 (definition of semi-hyperbolic rational maps)** *Let  $g$  be a rational map. The following conditions are equivalent and define the class of semi-hyperbolic rational maps.*

1. *A radius  $r > 0$  and a maximal degree  $p < \infty$  exist, such that, for any  $z \in J(g)$ , for any iterate  $n \geq 1$  and any connected component  $W$  of  $g^{-n}(D(x, r))$ , the degree of  $g^n|_W$  is at most  $p$ .*
2. *A radius  $r > 0$ , a maximal degree  $p < \infty$ , and constants  $c > 0$  and  $\theta < 1$  exist such that, for any  $x \in J(g)$ , any iterate  $n \geq 1$ , and any component  $W$  of  $g^{-n}(D(x, r))$ , the degree of the restriction of  $g^n$  to  $W$  is at most  $p$ , and the diameter of  $W_n$  is at most  $c\theta^n$ .*
3. *The map  $g$  has no recurrent critical point in the Julia set nor parabolic cycles.*
4. *A maximal degree  $p_0$  exists such that, for any  $r > 0$  and any  $x \in J(g)$ , if we let  $n$  be the least iterate such  $g^n(D(x, r) \cap J(g)) = J(g)$  then  $g^n|_{D(x, 2r)}$  has degree at most  $p_0$ .*

We refer to Theorem 2.1 in [CJY] for the proofs of the equivalence above.

**Corollary 4.2.2 (Topological cxc rational maps are semi-hyperbolic)** *A rational map is topological cxc if and only if it is semi-hyperbolic.*

**Proof:** Let us assume that  $f$  is a topological cxc rational map. Then it satisfies conclusion 1(1) of Theorem 4.2.1 with radius the Lebesgue number of the cover  $\mathcal{U}$ .

Conversely, the classification of stable domains implies that the complement of the Julia set  $J(f)$  consists of points which converge to attracting cycles under iteration. Thus if  $\mathfrak{X}_0$  is the complement of a suitable neighborhood of attracting cycles and their preimages, then  $\mathfrak{X}_1 = f^{-1}(\mathfrak{X}_0)$  has closure in  $\mathfrak{X}_0$  and the branch points of  $f : \mathfrak{X}_0 \rightarrow \mathfrak{X}_1$  lie in  $J(f)$ .

The axiom [Irreducibility] holds for any rational map in a neighborhood of its Julia set.

If  $f$  is semihyperbolic, Theorem 4.2.1 asserts that there is an  $r > 0$ ,  $p < \infty$ ,  $c > 0$ , and  $\theta < 1$  such that, for any  $x \in J(f)$ , any iterate  $n \geq 1$ , and any component  $W'_n$  of  $f^{-n}(D(x, 3r))$ , the degree of the restriction of  $f^n$  to  $W'_n$  is at most  $p$ , and the diameter of  $W'_n$  is at most  $c\theta^n$ . We let  $\mathcal{U}_0$  be a finite subcovering of  $J(f)$  of  $\{D(x, r), x \in J(f)\}$ , and  $\mathcal{U}_n$  be the set of components of  $f^{-n}(U)$  when  $U$  ranges over  $\mathcal{U}_0$ . Therefore, Axioms [Degree] and [Expansion] hold, so that  $f$  is a topological cxc map. ■

Note that the last item shows that any point in the Julia set is conical (as for convex cocompact groups), meaning that we may use the dynamics to go from small scales to large scales and vice versa with bounded distortion (cf. Lemma 4.2.6 below). Indeed, Lyubich and Minsky [LM] call semihyperbolic maps *convex cocompact* and show that such maps are characterized by the following property: the quotient (by the induced invertible dynamics of  $f$ ) of the convex hull of the “Julia set” (the hull taken in their affine hyperbolic three-dimensional lamination associated to  $f$ ) is compact.

The aim of this section is first to prove that these maps are cxc (Theorem 4.2.4) and also to strengthen their relationship to convex cocompact Kleinian groups within the dictionary by establishing new characterisations of this class. Theorem 4.2.4, Theorem 4.2.7 and Theorem 4.2.8 below imply the following.

**Theorem 4.2.3 (Characterizations of semi-hyperbolic rational maps)**

*Let  $g$  be a rational map. The following propositions are equivalent.*

1.  $g$  is semi-hyperbolic.
2.  $g$  is metric cxc on its Julia set, with respect to the spherical metric.
3. There is a covering  $\mathcal{U}$  of  $J(g)$  such that the associated graph  $\Gamma$  is quasi-isometric to the convex hull of  $J(g)$  in  $\mathbb{H}^3$  by a quasi-isometry which extends to  $\phi_g : J(g) \rightarrow \partial\Gamma$ .

The map  $\phi_g$  in the statement above is the one defined by Theorem 3.2.1.

The last paragraphs deal with the topological characterization of semi-hyperbolic maps in the spirit of Cannon's conjecture for hyperbolic groups, which claims that a hyperbolic group  $G$  with a topological 2-sphere as boundary admits a faithful cocompact Kleinian action.

### 4.2.1 Characterization of cxc mappings on the standard 2-sphere

**Theorem 4.2.4 (semi-hyperbolic rational maps are cxc)** *Let  $f$  be a semi-hyperbolic rational map with Julia set  $J$ . Then there are closed neighborhoods  $\mathfrak{X}_0, \mathfrak{X}_1$  of  $J$  in the sphere such that  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is cxc with repeller  $J$  with good open sets given by a finite collection  $\mathcal{U}_0$  of open spherical balls.*

**Notation.** Let  $\sigma = |dz|/(1 + |z|^2)$  denote the spherical Riemannian metric on  $\widehat{\mathbb{C}}$ . For a simply-connected hyperbolic domain  $V$  in  $\widehat{\mathbb{C}}$ , let  $\rho_V$  denote the hyperbolic metric on  $V$  and  $V^c$  its complement in  $\widehat{\mathbb{C}}$ . Given a metric  $g$ ,  $B(a, r; g)$  denotes the ball of radius  $r$  about  $a$  and  $\text{diam}(A; g)$  the diameter of a set  $A$ .

**Lemma 4.2.5 (Comparing metrics)** *There exists a universal constant  $C$  such that the following holds. Let  $W \subset \widehat{\mathbb{C}}$  be a simply-connected hyperbolic domain of spherical diameter  $< \pi/4$ ,  $x \in W$ , and  $D = B(x, 2; \rho_W)$ . Then, restricted to the domain  $D$ , the metrics  $\rho_W$  and  $\sigma/\text{diam}(D, \sigma)$  are  $C$ -bilipschitz equivalent.*

**Proof:** By applying a rigid spherical rotation we may assume  $W$  is contained in the Euclidean unit disk  $\Delta$  about the origin. For such domains,  $\sigma$  and the Euclidean metric  $|dw|$  are bilipschitz equivalent. By the Koebe 1/4 theorem,  $\rho_W$  is 4-bilipschitz equivalent to the metric

$$\eta = \frac{1}{\text{dist}(w, W^c)} |dw|$$

where  $\text{dist}(w, W^c)$  denotes the Euclidean distance from  $w$  to the complement  $W^c$  of  $W$  in  $\mathbb{C}$ . Suppose  $\phi : (\Delta, 0) \rightarrow (W, x)$  is a holomorphic isomorphism. By compactness of the space of Schlicht functions, and the fact that  $D$  is a hyperbolic ball of radius 2,

$$\text{diam}(D; \sigma) \asymp \text{diam}(D; |dw|) \asymp |\phi'(0)|. \quad (4.1)$$

Let  $w \in D$ . The Schwarz lemma implies  $\text{dist}(w, W^c) \leq \text{const} \cdot |\phi'(0)|$ , and the Koebe 1/4-theorem implies  $\text{dist}(w, W^c) \geq \text{const} \cdot |\phi'(0)|$ , so that

$$\text{dist}(w, W^c) \asymp |\phi'(0)|. \quad (4.2)$$

Dividing (1) by (2) yields

$$\frac{\text{diam}(D; \sigma)}{\text{dist}(w, W^c)} \asymp 1$$

and so

$$\frac{\rho_W(w)}{\sigma(w)/\text{diam}(D; \sigma)} \asymp \frac{\text{diam}(D; \sigma)}{\text{dist}(w, W^c)} \asymp 1.$$

■

**Lemma 4.2.6 (Distortion of  $p$ -valent maps)** *For  $p \in \mathbb{N}$  and  $\tilde{r}, r > 0$ , there exist real-valued functions  $C_h(p, r)$  and  $C_h^{-1}(p, r)$ , tending to zero as  $r, \tilde{r}$  tend to zero, with the following property. Suppose  $\tilde{W}, W \subset \hat{\mathbb{C}}$  are hyperbolic simply-connected domains,  $f : \tilde{W} \rightarrow W$  is a proper, holomorphic map such that  $\#f^{-1}(w) \leq p$  for all  $w \in W$ , and  $f(\tilde{w}) = w$ .*

1. Let  $B = B(w, r; \rho_W) \subset W$  and let  $\tilde{B}$  be the component of  $f^{-1}(B)$  containing  $\tilde{w}$ . Then

(a)  $B(\tilde{w}, r; \rho_{\tilde{W}}) \subset \tilde{B} \subset B(\tilde{w}, C_h(p, r); \rho_{\tilde{W}})$ .

(b) If  $B$  is replaced by an open connected set, then

$$\text{diam}(B; \rho_W) \leq \text{diam}(\tilde{B}; \rho_{\tilde{W}}) \leq C_h(p, \text{diam} B).$$

2. Given  $\tilde{r} > 0$ ,

$$B(w, C_h^{-1}(p, \tilde{r}); \rho_W) \subset f(B(\tilde{w}, \tilde{r}; \rho_{\tilde{W}})) \subset B(w, \tilde{r}; \rho_W).$$

Basically, the above lemma says that for connected sets of a fixed size, preimages cannot be too large or too small, and images cannot be too large or too small.

**Proof:** 1(a) is the content of Lemma 2.2 in [CJY] and implies the lower containment in (2) and the upper bound in 1(b). The lower bound in 1(a) and the upper containment in (2) follow from the Schwarz-Pick lemma.

■

**Proof:** (Theorem 4.2.4) Suppose  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  is semihyperbolic. Let  $r, p$  be the constants as in Theorem 4.2.1. Let  $\mathfrak{X}_0$  be the complement of a forward-invariant neighborhood of the attractors as in the proof of Corollary 4.2.2 and  $\mathfrak{X}_1$  its preimage, so that  $f : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$  is an fbc. To define the level zero good open sets  $\mathcal{U}_0$  we proceed as follows.

**Definition of  $\mathcal{U}_0$ .** For  $x \in J(f)$ , let  $W(x)$  be the spherical ball whose radius is  $r/2$ . By Lemma 4.2.6 there exists  $r_0$  so small that  $C_h(p, r_0) < 1/2$  and let  $U(x) = B(x, r_0; \rho_{W(x)})$ . Let  $\mathcal{U}_0$  be a finite open cover of  $J(f)$  by pointed sets of the form  $(U(x), x)$ . Then we have a finite set of triples  $(W(x), U(x), x)$ . By taking preimages, we obtain for each  $n \in \mathbb{N}$  a covering  $\mathcal{U}_n$  of  $J(f)$  by Jordan domains  $\widetilde{U}$  such that each has a preferred basepoint  $\tilde{x}$  and is compactly contained in a larger domain  $\widetilde{W}$ . Moreover,

$$f^k : (\widetilde{W}, \widetilde{U}, \tilde{x}) \rightarrow (W, U, x)$$

whenever  $U \in \mathcal{U}_n$ ,  $\widetilde{U} \in \mathcal{U}_{n+k}$ , and  $f^k(\tilde{x}) = x$ . Note that by construction and Lemma 4.2.6, for all  $n$  and all  $U \in \mathbf{U} = \cup_n \mathcal{U}_n$  with basepoint  $x$ ,

$$B(x, r_0; \rho_W) \subset U \subset B(x, 1/2; \rho_W).$$

In particular,  $2r_0 \leq \text{diam}(U; \rho_W) \leq 1$ .

**Diameter distortion.** Suppose  $f^k : (\widetilde{U}, \widetilde{U}') \rightarrow (U, U')$ , and let  $\widetilde{W}$  and  $W$  be the larger sets given with  $\widetilde{U}$  and  $U$ . We have by Lemma 4.2.5

$$\frac{\text{diam}(U'; \sigma)}{\text{diam}(U; \sigma)} = \text{diam}(U'; \sigma / \text{diam}(U; \sigma)) \asymp \text{diam}(U'; \rho_W) \quad (4.3)$$

and similarly

$$\frac{\text{diam}(\widetilde{U}'; \sigma)}{\text{diam}(\widetilde{U}; \sigma)} \asymp \text{diam}(\widetilde{U}'; \rho_{\widetilde{W}}). \quad (4.4)$$

By Lemma 4.2.6 1(b), we have

$$\text{diam}(U'; \rho_W) \leq \text{diam}(\widetilde{U}'; \rho_{\widetilde{W}}) \leq C_h(p, \text{diam}(U'; \rho_W)). \quad (4.5)$$

Together, (4.3), (4.4), and (4.5) imply

$$\frac{\text{diam}(U'; \sigma)}{\text{diam}(U; \sigma)} \leq \text{const} \cdot \frac{\text{diam}(\widetilde{U}'; \sigma)}{\text{diam}(\widetilde{U}; \sigma)}$$

and

$$\frac{\text{diam}(\tilde{U}'; \sigma)}{\text{diam}(\tilde{U}; \sigma)} \leq \text{const} \cdot C_h \left( p, \text{const} \frac{\text{diam}(U'; \sigma)}{\text{diam}(U; \sigma)} \right)$$

as required.

**Roundness distortion.** We first estimate the distortion of roundness with respect to hyperbolic metrics, and then relate the hyperbolic to the spherical metric.

Suppose  $U \in \mathbf{U} = \cup_n \mathcal{U}_n$ ,  $a \in U$ , and  $\text{Round}(U, a) = K$  in the hyperbolic metric of  $W$ . By the definition of roundness, there exists  $s > 0$  such that with respect to the hyperbolic metric on  $W$ ,

$$B(a, s) \subset U \subset B(a, Ks)$$

and no smaller  $K$  will do. Thus

$$\frac{1}{2} \text{diam}(U) \leq Ks \leq \text{diam}(U).$$

Since by construction  $2r_0 \leq \text{diam}(U) \leq 1$  we have  $K \asymp 1/s$ .

Now suppose  $f^k : (\tilde{U}, \tilde{a}) \rightarrow (U, a)$ .

**Backward roundness distortion.** By Lemma 4.2.6 1(a), with respect to the hyperbolic metric on  $\tilde{W}$ ,

$$B(\tilde{a}, s) \subset \tilde{U} \subset B(\tilde{a}, C_h(p, Ks)) \subset B(\tilde{a}, C_h(p, 1))$$

and so

$$\text{Round}(\tilde{U}, \tilde{a}) \leq \frac{C_h(p, 1)}{s} \leq \text{const} \cdot K$$

since  $K \asymp 1/s$ . Hence we obtain a linear backwards roundness distortion function.

**Forward roundness distortion.** Suppose now  $\text{Round}(\tilde{U}, \tilde{a}) = \tilde{K}$ . Then with respect to the hyperbolic metric on  $\tilde{W}$ , there exists  $\tilde{s} > 0$  such that

$$B(\tilde{a}, \tilde{s}) \subset \tilde{U} \subset B(\tilde{a}, \tilde{K}\tilde{s}) \subset B(\tilde{a}, 1)$$

so that  $\tilde{K} \asymp 1/\tilde{s}$ . Hence by Lemma 4.2.6(2), with respect to the hyperbolic metric on  $W$ ,

$$B(a, C_h^{-1}(p, \tilde{s})) \subset U \subset B(a, 1)$$

and so

$$\text{Round}(U, a) \leq 1/C_h^{-1}(p, \tilde{s}) \leq \text{a function of } \tilde{K}$$

since  $\tilde{s} \asymp 1/\tilde{K}$ .

It remains only to transfer the roundness estimates from the hyperbolic to the spherical metric. Suppose  $U \in \mathbf{U}$  has basepoint  $x$ ,  $a \in U$ , and  $\text{Round}(U, a) = K$  with respect to the hyperbolic metric on  $W$ . By construction,

$$U = B(x, r_0; \rho_W) \subset B(x, 1; \rho_W)$$

and we have already shown

$$B(a, Ks; \rho_W) \subset B(a, 1; \rho_W).$$

Therefore,  $U$ , its largest inscribed ball  $B(a, s)$  about  $a$ , and smallest circumscribing ball  $B(a, Ks)$  about  $a$  are all contained in the hyperbolic ball  $D = B(x, 2)$ . On this set, Lemma 4.2.5 implies that the hyperbolic metric  $\rho_W$  is bilipschitz equivalent to the metric  $\sigma/\text{diam}(D; \sigma)$ . Since roundness is invariant under constant scalings of the metric, the factor  $1/\text{diam}(D; \sigma)$  is irrelevant. It follows easily that the roundness computed with respect to the hyperbolic metric on  $W$  is comparable to that computed with respect to the spherical metric  $\sigma$ .

This completes the proof of Theorem 4.2.4. ■

We may now provide a converse statement :

**Theorem 4.2.7 (cxc on the Euclidean  $\mathbb{S}^2$  implies uniformly quasiregular)**

*Suppose  $f : \mathbb{S}^2 \rightarrow \mathbb{S}^2$  is an orientation-preserving cxc with respect to the standard spherical metric. Then  $f$  is quasisymmetrically, hence quasiconformally conjugate to a chaotic semi-hyperbolic rational map.*

This result is reminiscent of D. Sullivan's theorem which says that a uniform convergence group which acts as a uniform quasiconformal group on the Euclidean standard 2-sphere is quasiconformally conjugate to a cocompact Kleinian group [Sul1].

**Proof:** Any iterate  $f^n$  is also a finite covering from the sphere to itself, hence has exactly  $2d^n - 2$  critical points counted with multiplicity. We will first prove that a constant  $K$  exists such that  $f^n$  is  $K$ -quasiregular for all  $n \geq 1$ .

Fix  $n$  and consider a small disk  $2D$  disjoint from the critical set of  $f^n$ . It follows from Lemma 2.7.2 that there is a constant  $H < \infty$  such that, for all  $x \in 2D$ ,

$$\limsup_{r \rightarrow 0} \frac{\max\{|f^n(x) - f^n(y)| : |x - y| = r\}}{\min\{|f^n(x) - f^n(y)| : |x - y| = r\}} \leq H.$$

Therefore F. Gehring's metric characterisation of quasiconformal maps implies that there is a universal  $K = K(H)$  such that  $f^n$  is  $K$ -quasiconformal on  $D$  [Geh].

This implies that  $f^n$  is  $K$ -quasiregular off the critical set of  $f^n$ . But finitely many points are removable for quasiregularity, hence  $f^n$  is  $K$ -quasiregular. It follows from Theorem 4.4.1 that  $f$  is quasiconformally conjugate to a rational map. The semi-hyperbolicity follows from the property of bounded degree along pull-backs (cf. Theorem 4.2.1). ■

### 4.2.2 Convex Hull of Julia sets

In this paragraph, we prove the following theorem.

**Theorem 4.2.8** *Let  $f$  be a rational map. There is a finite cover  $\mathcal{U}$  of  $J(f)$  such that the space  $\Gamma = \Gamma(f, \mathcal{U})$  is quasi-isometric to the convex hull of the Julia set in  $\mathbb{H}^3$  by a quasi-isometry which extends as  $\phi_f : J(f) \rightarrow \partial\Gamma$  if and only if  $f$  is semi-hyperbolic.*

We note that if  $\Gamma$  is quasi-isometric to the convex hull of  $J(f)$  then the elements of  $S(n)$  have their diameter exponentially small, which forces  $f$  to be a so-called *topologically Collet-Eckmann* map, cf. [PR] and [PRLS].

**Proposition 4.2.9** *Let  $f$  be a rational map, and  $\Gamma$  the graph constructed from a finite covering  $\mathcal{U}$ . If  $\Gamma$  is hyperbolic and  $\phi_f : J(f) \rightarrow \partial_\varepsilon\Gamma$  is quasimetric for some  $\varepsilon > 0$  then  $f$  is semi-hyperbolic. Furthermore, the measure of maximal entropy is doubling on  $\widehat{\mathbb{C}}$ .*

Under the assumptions of Proposition 4.2.9, and since  $\phi_f$  is a homeomorphism and the mesh of  $(S(n))_n$  tends to 0 with respect to the metric  $d_\varepsilon$ , it follows that  $(f, \mathcal{U})$  satisfies the Axiom [Expansion].

We start with a lemma.



**Lemma 4.2.10** *Let  $K$  be a compact subset of  $J(f) - \{\infty\}$ ; for any  $\alpha = (w_0, W)$ , where  $W \in \mathbf{U}$  and  $w_0 \in W \cap K$ , let  $A_\alpha(z) = w_0 + \text{diam}(W)z$ . Under the assumptions of Proposition 4.2.9, the family of maps  $\{f^{|W|} \circ A_\alpha\}_\alpha$  is normal on  $\mathbb{C}$ . Furthermore, any limit is open.*

The collection  $\{W_n\}$  of neighborhoods of  $w_0$  is a local basis of neighborhoods, so eventually each  $W_n$  is contained in some fixed Euclidean disk about  $w_0$  on which the Euclidean and spherical metrics are comparable. Changing metrics does not affect normality, so we use whichever is most convenient.

The notation  $\text{diam}W$  will denote the diameter with respect to the spherical metric in  $\widehat{\mathbb{C}}$ , otherwise, we will write  $\text{diam}_\varepsilon$  for the metric  $d_\varepsilon$  on  $\partial\Gamma$ .

**Proof:** Let us consider a sequence  $(\alpha_n)$ ; if the sequence of levels is bounded, then the lemma is clearly true. So we might as well assume that  $\alpha_n = (w_n, W_n)$  with  $|W_n| = n$ . We shall then write  $A_{\alpha_n} = A_n$ .

We note that  $F^n$  is  $e^{\varepsilon n}$ -Lipschitz in  $\partial_\varepsilon\Gamma$ . But Proposition 3.3.2 asserts that  $\text{diam}_\varepsilon W \asymp e^{-\varepsilon|W|}$  for all  $W \in V$ . Therefore, if  $W \in S(n)$ , then the Lipschitz constant of

$$F^n : \left( \partial\Gamma, \frac{d_\varepsilon}{\text{diam}_\varepsilon W} \right) \rightarrow (\partial\Gamma, d_\varepsilon)$$

does not depend on  $W$  nor  $n$ .

We assume that  $\phi_f$  is  $\eta$ -quasisymmetric. We observe that the family

$$\phi_f \circ A_n : A_n^{-1}(J(f)) \rightarrow \left( \partial\Gamma, \frac{d_\varepsilon}{\text{diam}W} \right)$$

is equicontinuous. This follows from Proposition 10.26 in [Hei]: all these maps are  $\eta$ -quasisymmetric, and normalized: for any  $z, w \in W$ ,

$$|A_n^{-1}(z) - A_n^{-1}(w)| \leq 1$$

and

$$\frac{|\phi_f(z) - \phi_f(w)|_\varepsilon}{\text{diam}_\varepsilon \phi_f(W)} \leq \eta \left( 2 \frac{|z - w|}{\text{diam}W} \right) \leq \eta(2|A_n^{-1}(z) - A_n^{-1}(w)|) \leq \eta(2).$$

This implies that, for all  $R > 0$ , all the  $(f^n \circ A_n)|_{A_n^{-1}(J(f)) \cap D(0, R)}$  share a common modulus of continuity  $\omega_R$  since

$$f^n \circ A_n = \phi_f^{-1} \circ F^n \circ (\phi_f \circ A_n)$$

and

$$F^n : \left( \partial\Gamma, \frac{d_\varepsilon}{\text{diam}_\varepsilon W} \right) \rightarrow (\partial\Gamma, d_\varepsilon)$$

is uniformly Lipschitz.

Let us now prove the lemma. If  $(f^n \circ A_n)_{n \geq 0}$  was not a normal family at a point  $z_\infty \in \mathbb{C}$ , then Zalcman's lemma would imply the existence of a convergent sequence of points  $(z_k)$  with  $z_\infty$  as a limit, a decreasing sequence to 0 of positive numbers  $\rho_k$  and a subsequence  $(n_k)$  such that  $f^{n_k} \circ A_{n_k} \circ B_k$  tends to an open map  $g : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$  where  $B_k(z) = z_k + \rho_k z$  [Zal].

Let  $R = 2|z_\infty|$ , and let us choose  $R' > 2d(0, g^{-1}(J(f)))$ . Then, for  $k$  large enough, it follows that  $B_k(D(0, R')) \subset D(0, R)$  and

$$\text{diam}(f^{n_k} \circ A_{n_k}) \circ B_k(D(0, R') \cap (A_{n_k} \circ B_k)^{-1}(J(f))) \leq \omega_R(2\rho_k R')$$

which tends to 0. This contradicts the fact that  $g$  is open since  $\text{diam}(g^{-1}(J(f)) \cap D(0, R')) > 0$ .

Therefore,  $(f^n \circ A_n)_{n \geq 0}$  is a normal family on  $\mathbb{C}$ .

By construction, for all  $k$  the domains  $(f^{n_k} \circ A_{n_k})^{-1}(W_0)$  have diameter one, contain the origin, and map onto the domain  $W_0$ . Therefore  $h$  is nonconstant, hence open. ■

**Proof:** (Proposition 4.2.9) We will prove that the condition 1. in Theorem 4.2.1 follows from Lemma 4.2.10: let  $r > 0$  be such that any disk of radius  $r$  centered at a point of  $J(f)$  is contained in some open set defining the cover  $\mathcal{U}$ . If the condition was not satisfied, we would find a sequence of points  $z_k \in J(f)$ , connected components  $W_k$  of  $f^{-n_k}(D(z_k, r))$  such that the degree of  $f^{n_k}|_{W_k}$  would tend to infinity. Let  $w_k \in W_k$  be such that  $f^{n_k}(w_k) = z_k$ .

It follows from Lemma 4.2.10 that  $(z \mapsto f^n(w_k + \text{diam}W_k z))_k$  is a normal sequence on  $\mathbb{C}$  with open limits. So the degree has to be eventually bounded. Therefore,  $f$  is semi-hyperbolic.

The statement on the measure follows from the following argument. Since  $f$  is semi-hyperbolic,  $f$  is also cxc (Theorem 4.2.4), so Theorem 3.5.6 implies that  $\mu_f$  is the unique measure of maximal entropy and that  $\mu_f$  is also Ahlfors-regular of dimension  $(1/\varepsilon) \log d$ .

In particular  $\mu_f$  is doubling: there is a constant  $C > 0$  such that, for any ball  $B(x, r)$ , with  $r \leq \text{diam}_\varepsilon \partial\Gamma$ ,  $\mu_f(B(x, 2r)) \leq 2\mu_f(B(x, r))$ . Since this condition is preserved under the application of quasisymmetric mappings, the same is true for  $\phi_f^* \mu_f$  (cf. Cor. 4.15 from [Hei]). Furthermore, metric entropy is invariant under Borelian isomorphisms (Proposition 4.3.16 in [KH]), and in particular under homeomorphisms, we recover the fact that  $f$  admits a unique measure of maximal entropy: the pull-back under  $\phi_f$  of  $\mu_f$ , and this measure is doubling. ■

We may now prove Theorem 4.2.8:

**Proof:** (Theorem 4.2.8) Suppose  $\Gamma$  is quasi-isometric to the convex hull of  $J(f)$  via a map which extends as  $\phi_f$ . First, since quasi-isometries between proper geodesic spaces preserve hyperbolicity, it follows at once that  $\Gamma$  is hyperbolic. Alternatively, the comment following the statement of Proposition 4.2.9 shows that one may apply Theorem 3.3.1 to get the hyperbolicity of  $\Gamma$ . Second, since quasi-isometries extend as quasisymmetric maps,  $\phi_f$  is quasisymmetric.

Therefore, Proposition 4.2.9 applies and shows that  $f$  is semi-hyperbolic.

Conversely, if  $f$  is semi-hyperbolic, then Theorem 3.5.1 shows that  $\phi_f$  is quasisymmetric and that  $\Gamma$  is hyperbolic. Since both sets  $\Gamma$  and the convex hull of  $J(f)$  are quasi-starlike Gromov spaces, then the quasisymmetry extends as a quasi-isometry between  $\Gamma$  and the convex hull of  $J(f)$  (Theorem 3.1.5). ■

### 4.2.3 Topological characterizations of chaotic semi-hyperbolic rational maps

In this paragraph, we prove the following theorem.

**Theorem 4.2.11 (Characterization of chaotic semi-hyperbolic rational maps)**

*Let  $f : S^2 \rightarrow S^2$  be an orientation-preserving finite branched covering map defined on a topological 2-sphere which satisfies [Expansion] with respect to some covering  $\mathcal{U}$ , and suppose  $\varepsilon$  is small enough so that Theorem 3.2.1 applies. Then the following are equivalent:*

1.  $f$  is topologically conjugate to a semi-hyperbolic rational function  $R : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  with  $J_R = \widehat{\mathbb{C}}$ .
2.  $\partial_\infty \Gamma$  is quasimetrically equivalent to the standard Euclidean  $\mathbb{S}^2$ .
3.  $\Gamma$  is quasi-isometric to hyperbolic three-space  $\mathbb{H}^3$ .
4. The gauge of  $\partial_\infty \Gamma$  contains a 2-Ahlfors regular metric.
5. The map  $f$  is topological cxc and the sequence  $\{\mathcal{U}_n\}_n$  of coverings of  $S^2$  is conformal in the sense of Cannon.

Let us recall that the conformal gauge of a metric space  $(X, d)$  is the set of metrics  $\hat{d}$  on  $X$  such that the identity  $Id : (X, d) \rightarrow (X, \hat{d})$  is quasimetric (see Chapter 15 in [Hei]).

The equivalence with **5** will be proved after developing some needed background. This corresponds to a theorem of Cannon and Swenson for hyperbolic groups whose boundary is homeomorphic to the two-sphere [CS].

**Proof:** Suppose  $\mathcal{U} = \mathcal{U}_0$  satisfies [Expansion]. Then there exists some  $N$  such that for all  $n \geq N$ , elements of  $\mathcal{U}_n$  contain at most one branch value of  $f$ . Let  $\mathcal{V}$  be a finite covering of the sphere by Jordan domains  $V$  which is finer than  $\mathcal{U}_N$  and such that  $\partial V$  avoids the countable set of forward orbits of critical points. The elements of  $\mathbf{V} = \cup_n \mathcal{V}_n$  are then homeomorphic to Jordan domains, since they are coverings of disks ramified over at most one point and their boundaries are unramified covers of the Jordan curve boundaries of elements of  $\mathcal{V}$ . Since the quasi-isometry class of  $\Gamma$  does not depend on the cover (Theorem 3.3.1), we may assume at the outset that  $\mathcal{U} = \mathcal{V}$  and hence that elements  $U$  of  $\mathbf{U}$  and their complements in the sphere are connected.

**1.  $\Rightarrow$  2.** Suppose  $h_1 : S^2 \rightarrow \widehat{\mathbb{C}}$  conjugates  $f$  to a semi-hyperbolic rational function  $R$  and  $h_2 : S^2 \rightarrow \partial_\infty \Gamma$  conjugates  $f$  to the dynamics  $F$  on the boundary of  $\Gamma$ . Since  $R$  is semi-hyperbolic,  $f$  is topologically cxc (Corollary 4.2.2) and so  $F$  is topologically cxc as well. By Corollary 3.5.4,  $F$  is metric cxc. The rigidity theorem, Theorem 2.8.2, implies that  $h_2 \circ h_1^{-1} : \widehat{\mathbb{C}} \rightarrow \partial \Gamma$  is quasimetric.

**2.  $\Rightarrow$  1.** Suppose  $h : \partial_\infty \Gamma \rightarrow \mathbb{S}^2$  is a quasimetric map, where  $\mathbb{S}^2$  is the standard Euclidean two-sphere. By Propositions 3.3.7 and 2.7.2,  $F$  is uniformly weakly quasiregular. Since this condition is preserved under quasimetric conjugacies, so is  $G = hFh^{-1}$ . By Theorem II.6.2 in [Ric],  $G$  is

uniformly quasiregular. D. Sullivan's Theorem 4.4.1 implies that  $G$  is conjugate to a rational map  $R$ . But, since  $h$  is a quasismetry, Theorem 4.2.8 and Theorem 3.1.5 imply that  $R$  has to be semi-hyperbolic.

**3.**  $\iff$  **2.** This follows from Theorem 3.1.5: boundary values of quasismetries are quasismetries and, conversely, quasismetric maps of boundaries extend to quasismetries.

**4.**  $\iff$  **2.** The fact that **2** implies **4.** follows from the fact that  $\widehat{\mathbb{C}}$  is naturally a 2-Ahlfors regular metric space.

For the converse, since all elements of  $\mathbf{U}$  are Jordan domains, Proposition 2.6.6 shows that  $\partial\Gamma$  is linearly locally connected. Since linear local connectivity is a quasismetry invariant [Hei], there exists a metric in the gauge of  $\partial\Gamma$  which is both linearly locally connected and, by hypothesis, Ahlfors 2-regular. By M. Bonk and B. Kleiner's characterization of the standard two-sphere [BK1], this implies that this metric is quasismetrically equivalent to the standard Euclidean two-sphere.

■

These statements mimic similar theorems for Gromov hyperbolic groups in the context of Cannon's conjecture. Statement **2** is concerned with Sullivan/Tukia's straightening theorem of quasiconformal groups [Sul1, Tuk2]; statement **3** is due to J. Cannon and D. Cooper [CC] in the context of groups; statement **4.** is due to M. Bonk and B. Kleiner, and can be deduced either from [BK2], or from [BK1] and [BK3].

In [BK3], M. Bonk and B. Kleiner also prove that a Gromov hyperbolic group admits a cocompact Kleinian action on  $\widehat{\mathbb{C}}$  if the Ahlfors-regular conformal dimension of the gauge of its boundary is attained. The Ahlfors-regular conformal dimension of  $(X, d)$  is the infimum of the Hausdorff dimensions over all Ahlfors-regular metrics in its gauge. In our context of non-invertible dynamical systems, this statement does not hold:

**Proposition 4.2.12** *There is a metric  $d$  on the 2-sphere  $\mathbb{S}^2$  and a cxc map  $f : (\mathbb{S}^2, d) \rightarrow (\mathbb{S}^2, d)$  such that the Ahlfors-regular conformal dimension is given by  $d$ , but  $f$  is not conjugate to a rational map.*

**Proof:** Let us consider  $F : \mathbb{C} \rightarrow \mathbb{C}$  be defined by  $F(x + iy) = 2x + 3iy$ . Let us consider the metric  $\hat{d}(x + iy, x' + iy') = |x - x'| + |y - y'|^\alpha$  where  $\alpha = \log 2 / \log 3$ . One may check that  $(\mathbb{C}, \hat{d})$  is Ahlfors regular of dimension  $1 + 1/\alpha$

and that this dimension is also its Ahlfors-regular conformal dimension since the  $(1 + 1/\alpha)$ -modulus of the family of horizontal curves is clearly positive (cf. Theorem 15.10 in [Hei]).

Since  $F(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$  and  $F(-z) = -z$ , this map descends to a map  $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ , on which one can push down the metric  $\hat{d}$  to a metric  $d$ . It follows that this metric satisfies the same properties as  $\hat{d}$ . Furthermore, since  $\hat{d}(F(z), F(z')) = 2\hat{d}(z, z')$  clearly holds for any  $z, z' \in \mathbb{C}$ , it follows that  $f$  is cxc with the metric  $d$ .

But since the conformal dimension of  $(\widehat{\mathbb{C}}, d)$  is strictly larger than 2, Theorem 4.2.11 shows that  $f$  is not equivalent to a rational map. ■

This implies that the sphere need not be a Loewner space even if the Ahlfors-regular conformal dimension is attained. But, if  $\partial\Gamma$  admits an Ahlfors regular Loewner metric in its gauge, then the theorem above with [BK1] imply  $f$  is conjugate to a rational map.

#### 4.2.4 Cannon's combinatorial Riemann mapping theorem

Before we prove the equivalence with (5), we will first review the notions that are needed to understand the statement and the proof.

**Combinatorial moduli.** Let  $\mathcal{S}$  be a covering of a topological surface  $X$ . Denote by  $\mathcal{M}(\mathcal{S})$  the set of maps  $\rho : \mathcal{S} \rightarrow \mathbb{R}_+$  such that  $0 < \sum \rho(s)^2 < \infty$  which we call *admissible metrics*. Let  $K \subset X$ ; the  $\rho$ -length of  $K$  is by definition

$$\ell_\rho(K) = \sum_{s \cap K \neq \emptyset} \rho(s)$$

and its  $\rho$ -area is

$$A_\rho(K) = \sum_{s \cap K \neq \emptyset} \rho(s)^2.$$

If  $\Gamma$  is a family of curves in  $X$  and if  $\rho \in \mathcal{M}(\mathcal{S})$ , we define

$$L_\rho(\Gamma, \mathcal{S}) = \inf_{\gamma \in \Gamma} \ell_\rho(\gamma),$$

and its *combinatorial modulus* by

$$\text{mod}(\Gamma, \mathcal{S}) = \inf_{\rho \in \mathcal{M}(\mathcal{S})} \frac{A_\rho(X)}{L_\rho(\Gamma, \mathcal{S})^2} = \inf_{\rho \in \mathcal{M}(\mathcal{S})} \text{mod}(\Gamma, \rho, \mathcal{S}).$$

Let  $A$  be an annulus in  $X$ . Let  $\Gamma_t$  be the set of curves in  $A$  which join the boundary components of  $A$ , and  $\Gamma_s$  those which separate the boundary components of  $A$ . Define

$$\text{mod}_{\text{sup}}(A, \mathcal{S}) = \frac{1}{\text{mod}(\Gamma_t, \mathcal{S})} \quad \text{and} \quad \text{mod}_{\text{inf}}(A, \mathcal{S}) = \text{mod}(\Gamma_s, \mathcal{S}).$$

The classical moduli of  $\Gamma_s, \Gamma_t$  are mutually reciprocal. In the combinatorial setting, this is no longer quite true. However J. Cannon, W. Floyd and W. Parry have proved that always  $\text{mod}_{\text{inf}}(A, \mathcal{S}) \leq \text{mod}_{\text{sup}}(A, \mathcal{S})$  [CFP1].

A covering  $\mathcal{S}$  has  *$N$ -bounded overlap* if, for all  $x \in X$ ,

$$\sum_{s \in \mathcal{S}} \chi_s(x) \leq N$$

where  $\chi_s$  denotes the characteristic function of  $s$ . Two coverings are said to be  *$N$ -equivalent* if each piece of one intersect at most  $N$  pieces of the other, and vice-versa.

**Sequence of coverings.** In order to state J.W. Cannon's combinatorial Riemann mapping theorem, we introduce a couple of new notions.

**Definition.** A *shingle* is connected compact subset of  $X$ , and a *shingling* is a covering of  $X$  by shingles.

**Definition.** A sequence of coverings  $(\mathcal{S}_n)$  is  *$K$ -conformal* ( $K \geq 1$ ) if

1. the mesh of  $(\mathcal{S}_n)$  tends to zero;
2. for any annulus  $A$  in  $X$ , there exist an integer  $n_0$  and a positive constant  $m = m(A) > 0$  such that, for all  $n \geq n_0$ ,

$$\text{mod}_{\text{sup}}(A, \mathcal{S}_n), \text{mod}_{\text{inf}}(A, \mathcal{S}_n) \in [m/K, Km];$$

3. for any  $x \in X$ , any  $m > 0$  and any neighborhood  $V$ , there is an annulus  $A \subset V$  which separates  $x$  from  $X \setminus V$  such that  $\text{mod}_*(A, \mathcal{S}_n) \geq m$  for all large  $n$ , where  $*$   $\in$   $\{\text{inf}, \text{sup}\}$ .

The quantity  $m(A)$  will be referred to as the *combinatorial modulus* of  $A$  with respect to the sequence  $(\mathcal{S}_n)$ . If  $\mathcal{S}' = (\mathcal{S}'_n)$  is another sequence of shinglings whose elements  $\mathcal{S}'_n$  are  $N$ -equivalent with  $\mathcal{S}_n$ , where  $N$  is independent of  $n$ , then the combinatorial moduli computed with respect to  $\mathcal{S}$  and  $\mathcal{S}'$  are known to be comparable (Theorem 4.3.1 in [CS]). Hence  $\mathcal{S}$  is conformal iff  $\mathcal{S}'$  is conformal.

**Theorem 4.2.13 (combinatorial Riemann mapping theorem [Can])**

*If  $(\mathcal{S}_n)$  is a conformal sequence of shinglings, on a topological surface  $X$ , then  $X$  admits a complex structure such that the analytical moduli of annuli are comparable with their combinatorial moduli.*

There is also a converse:

**Theorem 4.2.14** *If  $(\mathcal{S}_n)$  is a sequence of shinglings on the Riemann sphere such that the mesh goes to zero, each  $\mathcal{S}_n$  has overlap bounded by some universal constant  $N$ , and such that a constant  $K > 1$  exists such that, for any  $n$ , any  $s \in \mathcal{S}_n$ , there are two concentric disks  $D_s \subset s \subset \Delta_s$  with  $\text{diam}\Delta_s \leq K \text{diam}D_s$ , then  $(\mathcal{S}_n)$  is conformal.*

This is slightly different from Theorem 7.1 in [Can]. There, the smaller disks  $D_s$  are required to be pairwise disjoint. There is no such assumption here, so we provide a proof. We will use the following lemma of J. Strömberg and A. Torchinsky [ST]. Below, disks are spherical, and integrals over the whole sphere.

**Lemma 4.2.15** *Let us consider a family of disks  $B \in \mathcal{B}$  to which we assign a positive weight  $a_B > 0$ . For any  $p > 1$  and any  $\lambda \in (0, 1)$ , a constant  $C = C(p, \lambda) > 0$  independent from  $\mathcal{B}$  and from the weights exists such that*

$$\int \left( \sum a_B \chi_B \right)^p \leq C \int \left( \sum a_B \chi_{\lambda B} \right)^p .$$

**Proof:** (Theorem 4.2.14) It suffices to prove that there is some constant  $C > 0$  such that, for any annulus  $A$ , there is some  $n(A)$  such that, if  $n \geq n(A)$ , then  $\text{mod}_{\text{inf}}(A, \mathcal{S}_n) \geq (1/C)\text{mod}A$  and  $\text{mod}_{\text{sup}}(A, \mathcal{S}_n) \leq C\text{mod}A$ .

Fix an annulus  $A$ . Since the mesh of  $\mathcal{S}_n$  tends to 0, we may find some  $n(A)$  and  $\kappa > 0$  ( $\kappa$  independent from  $A$  and  $n$ ) such that, for any  $n \geq n(A)$ , any piece  $s \in \mathcal{S}_n$  which intersects  $A$  and any curve  $\gamma \in \Gamma_t \cup \Gamma_s$  which intersects  $s$ , the length of  $\gamma \cap 2\Delta_s$  is at least  $\kappa \text{diam}s$ .



Let  $\Gamma$  denote  $\Gamma_s$  or  $\Gamma_t$  and  $\mathcal{S} = \mathcal{S}_n$  for some  $n \geq n(A)$ . If  $\gamma \in \Gamma$ , the family of pieces  $s \in \mathcal{S}$  which intersects  $\gamma$  is denoted by  $\mathcal{S}(\gamma)$ .

If  $\rho : \mathcal{S} \rightarrow \mathbb{R}_+$  is an admissible metric for  $\Gamma$ , we define a classical test metric

$$\hat{\rho} = \sum_{s \in \mathcal{S}} \frac{\rho(s)}{\text{diam}_s} \chi_{2\Delta_s},$$

where  $\chi_{2\Delta_s}$  denotes the characteristic function of  $2\Delta_s$ . Therefore, if  $\gamma \in \Gamma$ , then the definitions of  $\hat{\rho}$  and  $\kappa$  imply

$$\ell_{\hat{\rho}}(\gamma) \geq \sum_{s \in \mathcal{S}(\gamma)} \frac{\rho(s)}{\text{diam}_s} \ell(\gamma \cap 2\Delta_s)$$

$$\geq \kappa \sum_{s \in \mathcal{S}(\gamma)} \rho(s)$$

$$\geq \kappa L_{\rho}(\Gamma, \mathcal{S})$$

and so

$$L_{\hat{\rho}}(\Gamma) \geq \kappa L_{\rho}(\Gamma, \mathcal{S}). \tag{4.6}$$

On the other hand,

$$\begin{aligned} \text{Area}(\widehat{\mathbb{C}}, \hat{\rho}) &= \int_{\widehat{\mathbb{C}}} \left( \sum_{s \in \mathcal{S}} \frac{\rho(s)}{\text{diam}_s} \chi_{2\Delta_s} \right)^2 \\ &\leq C \int_{\widehat{\mathbb{C}}} \left( \sum_{s \in \mathcal{S}} \frac{\rho(s)}{\text{diam}_s} \chi_{D_s} \right)^2 \end{aligned}$$

by Lemma 4.2.15. Since  $\mathcal{S}$  has bounded overlap,

$$\begin{aligned} \left( \sum_{s \in \mathcal{S}} \frac{\rho(s)}{\text{diam}_s} \chi_{D_s} \right)^2 &\leq N^2 \left( \max \left\{ \frac{\rho(s)}{\text{diam}_s} \chi_{D_s} \right\} \right)^2 \\ &\leq N^2 \sum_{s \in \mathcal{S}} \left( \frac{\rho(s)}{\text{diam}_s} \chi_{D_s} \right)^2. \end{aligned}$$

Therefore

$$\begin{aligned} \text{Area}(\widehat{\mathbb{C}}, \hat{\rho}) &\leq CN^2 \sum_{s \in \mathcal{S}} \int_s \left( \frac{\rho(s)}{\text{diam}_s} \right)^2 \\ &\leq CN^2 K^2 \pi \sum_{s \in \mathcal{S}} \rho(s)^2. \end{aligned}$$

Hence

$$\text{Area}(\widehat{\mathbb{C}}, \hat{\rho}) \leq CN^2 K^2 \pi A_\rho(\widehat{\mathbb{C}}). \quad (4.7)$$

Combining (4.6) and (4.7) yields

$$\text{mod}(\Gamma) \leq c \text{mod}(\Gamma, \mathcal{S})$$

where  $c = C(NK)^2\pi/\kappa^2$  is independent of the level  $n$  of  $\mathcal{S} = \mathcal{S}_n$ . Taking  $\Gamma = \Gamma_s$ , we obtain

$$\text{mod}_{\text{inf}}(A, \mathcal{S}) = \text{mod}(\Gamma_s, \mathcal{S}) \geq \frac{1}{c} \text{mod}(\Gamma_s) = \frac{2\pi}{c} \text{mod}(A).$$

Taking  $\Gamma = \Gamma_t$ , we obtain

$$\text{mod}_{\text{sup}}(A, \mathcal{S}) = \frac{1}{\text{mod}(\Gamma_t, \mathcal{S})} \leq \frac{c}{\text{mod}(\Gamma_t)} = \frac{c}{2\pi} \text{mod}(A).$$

■

### 4.2.5 Proof of rational iff Cannon-conformal

**Proof:** [5 $\Rightarrow$  1] We assume that  $f$  is topological cxc, given by covers  $(\mathcal{U}_n)$ , and that this sequence is conformal. Considering a cover  $S(0)$  by Jordan domains fine enough, the existence of some constant  $N$  independent from  $n$  so that the covers  $\mathcal{U}_n$  and  $S(n)$  are  $N$ -equivalent follows from Axiom [Degree]. Thus,  $(S(n))_n$  is conformal, and we may now apply the combinatorial Riemann mapping theorem:  $\partial_\infty\Gamma$  has a complex structure compatible with its combinatorial structure. In other words, there is a homeomorphism  $h : \partial_\infty\Gamma \rightarrow \widehat{\mathbb{C}}$  such that, for any annulus  $A$  and for  $n$  large enough,

$$\text{mod}_*(A, \mathcal{S}_n) \asymp \text{mod}h(A).$$

Hence the map  $G = h \circ F \circ h^{-1}$  is a ramified covering of  $\widehat{\mathbb{C}}$ . Let us prove that it is uniformly quasiregular: this will establish **1.** by D. Sullivan's straightening theorem (Theorem 4.4.1), and the fact that  $G$  is topological cxc (Corollary 4.2.2). Fix  $k$  and  $z \in \widehat{\mathbb{C}}$  off the (finite) branching set  $B(G^k)$  of  $G^k$ . Let  $V$  be a neighborhood of  $z$  disjoint from  $B(G^k)$ . Therefore,  $G^k|_V$  is injective so, if  $A \subset V$  is an annulus, then, for all  $n$  large enough

$$\text{mod}G^k(A) \asymp \text{mod}_{\text{sup}}(F^k(h^{-1}(A)), S(n)) = \text{mod}_{\text{sup}}(h^{-1}(A), S(n+k)) \asymp \text{mod}A$$

hence  $G^k|_V$  is  $K$ -quasiconformal for some universal  $K$ . Therefore,  $G^k$  is  $K$ -quasiregular since  $B(G^k)$  is finite hence removable.

■

**Proof:** [2 $\Rightarrow$  5] Let  $\phi : S^2 \rightarrow \partial\Gamma$  be the conjugacy given by Theorem 3.2.1. Let us fix a quasisymmetric homeomorphism  $h : \partial_\infty\Sigma \rightarrow \widehat{\mathbb{C}}$ .

It suffices to prove that  $(h \circ \phi(S(n)))_n$  satisfies the assumption of Theorem 4.2.14. Since  $h$  is quasisymmetric, Proposition 3.3.2 implies that the roundness of  $h \circ \phi(W)$  is uniformly bounded for any  $W \in \cup S(n)$ . Furthermore, each covering has bounded overlap, so Theorem 4.2.14 applies.

Furthermore, since [2 $\Rightarrow$  1], it follows that  $f$  is also topological cxc. ■

### 4.3 Finite subdivision rules

Finite subdivision rules have been intensively studied since they give natural concrete examples with which to study Cannon's problem of determining when a sequence of combinatorial structures yields a compatible conformal structure; see [CFP2], [CFKP], [CFP3] and the discussion in the preceding section.

A finite subdivision rule (f. s. r.)  $\mathcal{R}$  consists of a finite 2-dimensional CW complex  $S_{\mathcal{R}}$ , a subdivision  $\mathcal{R}(S_{\mathcal{R}})$  of  $S_{\mathcal{R}}$ , and a continuous cellular map  $\phi_{\mathcal{R}} : \mathcal{R}(S_{\mathcal{R}}) \rightarrow S_{\mathcal{R}}$  whose restriction to each open cell is a homeomorphism. When the underlying space of  $S_{\mathcal{R}}$  is homeomorphic to the two-sphere  $S^2$  (for concreteness, we consider only this case) and  $\phi_{\mathcal{R}}$  is orientation-preserving,  $\phi_{\mathcal{R}}$  is a postcritically finite branched covering of the sphere with the property that pulling back the tiles effects a recursive subdivision of the sphere. That is, for each  $n \in \mathbb{N}$ , there is a subdivision  $\mathcal{R}^n(S_{\mathcal{R}})$  of the sphere such that  $f$  is a cellular map from the  $n$ th to the  $(n - 1)$ st subdivisions. Thus, we may speak of *tiles* (which are closed 2-cells), *faces* (which are the interiors of tiles), *edges*, *vertices*, etc. at level  $n$ . It is important to note that formally, an f. s. r. is *not* a combinatorial object, since the map  $\phi_{\mathcal{R}}$ , which is part of the data, is assumed given. In other words: as a dynamical system on the sphere, the topological conjugacy class of  $\phi$  is well-defined.

Let  $\mathcal{R}$  be a finite subdivision rule on the sphere such that  $\phi_{\mathcal{R}}$  is orientation-preserving.  $\mathcal{R}$  has *mesh going to zero* if for every open cover of  $S_{\mathcal{R}}$ , there is some integer  $n$  for which each tile at level  $n$  is contained in an element of the cover. A *tile type* is a tile at level zero equipped with the cell structure induced by the first subdivision.  $\mathcal{R}$  is *irreducible* if, given any pair of tile types, an iterated subdivision of the first contains an isomorphic copy of the

second. If  $\mathcal{R}$  has mesh going to zero, then it is easy to see that  $\mathcal{R}$  is irreducible: any two tile types are joined by a path of edges of some bounded length.  $\mathcal{R}$  is of *bounded valence* if there is a uniform upper bound on the valence of any vertex at any level.

If in addition  $\mathcal{R}$  has bounded valence, then  $\mathcal{R}$  yields a topologically cxc dynamical system on the sphere with respect to a naturally arising family of open sets at level zero. To define these sets, we recall a few notions from [CFP3]. Given a subcomplex  $Y$  of a CW complex  $X$  the *star*  $\text{star}(Y, X)$  of  $Y$  in  $X$  is the union of all closed tiles intersecting  $Y$ . Let  $X$  denote the CW structure on the sphere at level zero, and set  $X_n = \mathcal{R}^n X$ .

**Lemma 4.3.1** *Suppose  $\mathcal{R}$  is a finite subdivision rule on the two-sphere which is orientation-preserving and has mesh going to zero. Then there exist  $n_0, n_1 \in \mathbb{N}$  with the following property. For each closed 2-cell  $t \in X_{n_0}$ , the set  $D_t = \text{star}(t, X_{n_0+n_1})$  is a closed disk which, if it meets the postcritical set  $P$  of  $\phi_{\mathcal{R}}$ , does so in at most one point, and this point lies in the interior  $U_t$  of  $D$ .*

**Proof:** Mesh going to zero implies that for some  $n_0$ , each 2-cell  $t$  of  $X_{n_0}$  meets  $P$  in at most one point. It also implies that for some  $n_1$ , for any 2-cell  $s$  of  $X_0$ , and any two 0-cells  $x, y$  of  $s$ , no 2-cell of  $X_{n_1}$  contains both  $x$  and  $y$ . Together, these two observations imply that for any 2-cell  $t$  of  $X_{n_0}$ ,  $D_t = \text{star}(t, X_{n_0+n_1})$  is a cell complex which contains  $t$  in its interior  $U_t$ , and which, if it intersects  $P$ , does so in its interior. Since  $D_t$  is the closure of  $U_t$  and its boundary is a simple closed curve,  $D_t$  is a disk. ■

Let  $\mathcal{U}_0$  be the finite open covering of the two-sphere underlying  $X$  given by the Jordan domains  $U_t$  constructed above, and consider the topological dynamical system  $f = \phi_{\mathcal{R}} : X \rightarrow X$  together with  $\mathcal{U}_0$ . Since  $\mathcal{R}$  is irreducible, Proposition 2.4.1 3(a) implies that axiom [Irreducibility] in the definition of topologically cxc holds. For any  $k \in \mathbb{N}$ , the restriction of  $f^k$  to an element  $\tilde{U}$  of  $\mathcal{U}_k$  is a branched covering onto its image  $U$  which is ramified at at most one 0-cell  $c$  which maps onto some 0-cell  $v$ . Let  $\tilde{w} \in \tilde{U}$  be a 0-cell and put  $w = f^k(\tilde{w})$ . Then  $w$  is joined by an edge-path (i.e. a union of 1-cells) to  $v$  whose interior avoids  $v$ , and the length of this edge path (i.e. the number of 1-cells comprising it) is at most some constant  $q$ . Since  $f^k : \tilde{U} \rightarrow U$  is ramified only at  $c$ , this edge-path lifts to an edge path of length at most  $q$

joining  $\tilde{w}$  to  $c$ . It follows that the combinatorial diameter of the zero-skeleton of  $\overline{U}$  is uniformly bounded. Since  $\mathcal{R}$  has mesh going to zero, it follows that axiom [Expansion] holds.

Moreover, if in addition  $\mathcal{R}$  has bounded valence, then the ramification of  $f^k$  at  $c$  is uniformly bounded. This implies that  $\overline{U}$  comprises a uniformly bounded number of cells and hence that the degree of  $f^k : \tilde{U} \rightarrow U$  is uniformly bounded, so that axiom [Degree] holds.

Hence,  $f : X \rightarrow X$  together with  $\mathcal{U}_0$  yields a topologically cxc system on the sphere. By Corollary 3.5.4, there is a preferred metric on  $X$  for which the dynamics is metrically cxc. By Theorem 4.2.11,  $X$  is quasisymmetrically equivalent to the standard two-sphere if and only if  $f$  is conjugate to a postcritically finite rational map whose Julia set is the whole sphere; compare [Mey].

## 4.4 Uniformly quasiregular dynamics

Let  $M$  be a compact Riemannian manifold of dimension  $n \geq 2$ . Given  $K < \infty$ , a non-invertible continuous map  $f : M \rightarrow M$  is said to be  $K$ -quasiregular if  $f \in W_{loc}^{1,n}(M)$  and if  $|D_x f|^n \leq K \cdot Jf$  for almost every  $x$ . References on quasiregular mappings include S. Rickman's monograph [Ric].

When  $f$  is not constant, then  $f$  is a finite branched covering (see § I.4 in [Ric]). We say that  $f$  is *uniformly quasiregular* if all its iterates are  $K$ -quasiregular for a fixed  $K$ .

When  $n = 2$ , D. Sullivan proved the following theorem [Sul3] in parallel with a similar statement for quasiconformal groups on the 2-sphere [Sul1]:

**Theorem 4.4.1 (D. Sullivan)** *A uniformly quasiregular map of the standard Euclidean two-sphere to itself is quasiconformally conjugate to a rational map.*

The iteration of uniformly quasiregular on the standard two-sphere boils down to the iteration of rational maps. In higher dimension  $n \geq 3$ , these maps generalize one-dimensional holomorphic dynamics, and have been introduced in this setting by T. Iwaniec and G. Martin in [IM]. Uniformly quasiregular maps on space-forms have been classified in [MMP]. They can be seen as analogs of quasiconformal groups.

For such maps, Fatou sets are defined as the set of normality, and Julia sets as the set of non-normality.

In [May], V. Mayer proposes a generalization of Lattès examples to higher dimensions. They are uniformly quasiregular maps of finite degree  $f : M \rightarrow M$ , where  $M$  is a compact Riemannian manifold, which are defined as follows.

There are a crystallographic group  $\Gamma$  and an onto  $\Gamma$ -automorphic quasiregular map  $h : \mathbb{R}^n \rightarrow M$  such that  $h(x) = h(y)$  if and only if there is some element  $\gamma \in \Gamma$  so that  $y = \gamma(x)$ , and there are a matrix  $U \in \text{SO}_n(\mathbb{R})$  and a constant  $\lambda > 1$  such that, if we set  $A = \lambda U$ , then  $A\Gamma A^{-1} \subset \Gamma$  and such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{A} & \mathbb{R}^n \\ & & \\ h \downarrow & & \downarrow h \\ & & \\ M & \xrightarrow{f} & M \end{array}$$

For more precise statements, we refer to V. Mayer's article [May].

Let us recall the following compactness result (cf. Theorem 2.4 [MSV]):

**Theorem 4.4.2 (normality of qr mappings)** *Suppose that  $0 < r < R \leq \infty$ ,  $0 < r' < \infty$ ,  $1 \leq K < \infty$ ,  $N \geq 1$ , and that  $\mathcal{F}$  is a family of  $K$ -quasiregular mappings  $f : B^n(R) \rightarrow \mathbb{R}^n$  such that every point has at most  $N$  preimages,  $f(0) = 0$ , and such that for each  $f \in \mathcal{F}$  there is a continuum  $A(f)$  with the properties*

$$0 \in A(f), \quad \max\{|x|, x \in A(f)\} = r, \quad \max\{|f(x)|, x \in A(f)\} = r'.$$

*Then  $\mathcal{F}$  is a normal family and any limit map is  $K$ -quasiregular, and any point in the range has at most  $N$  preimages.*

This implies that, under the assumptions of Theorem 4.4.2, assuming  $R = 1$ , there are functions  $d_+$  and  $d_-$  such that  $d_{\pm}(t) \rightarrow 0$  with  $t$  and such that, for any  $f \in \mathcal{F}$ , and any set  $U \subset B^n(1)$ ,  $\text{diam}f(U) \leq d_+(\text{diam}U)$  and, for any compact connected subset  $V$  of the image of  $f$  which contains the origin,  $\text{diam}W \leq d_-(\text{diam}V)$  where  $W$  denotes the component of  $f^{-1}(V)$  which contains the origin.

**Theorem 4.4.3** *Lattès examples are cxc.*

**Proof:** Axiom [Irreducibility] clearly holds.

Fix  $r_0 > 0$ ; for any  $x \in \mathbb{R}^n$ , we denote by  $W(x)$  the connected component of  $h^{-1}(B(h(x), r_0))$  which contains  $x$ . It follows from the quasiregularity and the fact that  $h$  is automorphic with respect to a cocompact group of Euclidean motions that we may choose  $r_0 > 0$  such that a constant  $N < \infty$  exists so that, for all  $x \in \mathbb{R}^n$ , the degree of  $h|_{W(x)}$  is bounded by  $N$  (Lemma III.4.1 in [Ric]).

We fix some size  $r_1 > 0$  small enough so that, for any  $x, y \in \mathbb{R}^n$ , if  $x$  belongs to the component  $V(y)$  of  $h^{-1}(B(h(y), r_1))$  containing  $y$ , then  $B(x, 2\text{diam}V(y)) \subset W(x)$ .

We define  $\mathcal{U}_0$  as a finite subcover of  $\{B(x, r_1), x \in M\}$ . Then  $\mathcal{U}_0$  satisfies [Degree] and [Expansion]. It remains to prove [Roundness distortion] and [Diameter distortion]. Since  $f$  is semi-conjugate to a conformal map, one then needs only verify (i)  $h$  distorts the roundness of (small) sets by a controlled amount, and (ii)  $h$  distorts ratios of diameters of nested sets by a controlled amount.

We note that since  $M$  is compact, one may find uniformly quasiconformal charts which map balls of radius  $3r_1$  in  $M$  onto the unit ball of  $\mathbb{R}^n$ . Therefore, we may assume that  $h$  takes its image into  $\mathbb{R}^n$  in the sequel.

For each  $x_0 \in \mathbb{R}^n$  and each connected open set  $V$  contained in some  $V(y)$  which contains  $x_0$ , we consider the map

$$h_{x_0, V} : x \in B^n(0, 1) \mapsto \frac{1}{\text{diam}h(V)} \cdot h(x_0 + \text{diam}Vx).$$

All these maps define a compact family  $\mathcal{F}$  of degree at most  $N$  according to Theorem 4.4.2 since  $h_{x_0, V}(B(0, 1))$  contains at least one point at distance  $1/2$  from the origin.

If  $W \subset V \subset \mathbb{R}^n$ , then it follows that  $W \subset B(x_0, \text{diam}V)$  and

$$\frac{\text{diam}h(W)}{\text{diam}h(V)} \leq d_+ \left( \frac{\text{diam}W}{\text{diam}V} \right).$$

Similarly, if  $V' \subset M$  is small enough, if  $W' \subset V'$  and if  $V$  and  $W$  denote connected components of  $h^{-1}(V)$  and  $h^{-1}(W')$  such that  $W \subset V$ , then

$$\frac{\text{diam}W}{\text{diam}V} \leq d_- \left( \frac{\text{diam}W'}{\text{diam}V'} \right).$$



This establishes (ii).

Let  $V \subset \mathbb{R}^n$  contained in some  $V(y)$ , and let  $x_0 \in V$ . Denote by  $K = \text{Round}(V, x_0)$  its roundness. Then  $B(x_0, \text{diam}V/(2K)) \subset V$  so

$$B(h(x_0), d_+(1/(2K))\text{diam}h(V)) \subset h(V).$$

This proves that  $\text{Round}(h(V), x_0) \leq 1/d_+(1/(2K))$ .

Let us denote by  $\mathcal{F}(K')$  the subset of  $\mathcal{F}$  obtained from pairs  $(V, x_0)$  such that  $\text{diam}h(V) \leq r_1$  and  $\text{Round}(h(V), h(x_0)) \leq K'$ . This family is also compact, so the roundness of  $V$  at  $x_0$  depends only on  $K'$ . Hence (i) holds.

This ends the proof that a Lattès example is cxc. ■

Conversely, one has :

**Theorem 4.4.4** *Let  $f : M \rightarrow M$  be an orientation preserving metric cxc mapping, where  $M$  is a compact Riemannian manifold of dimension at least 3. Then  $f$  is a Lattès map.*

**Proof:** It follows from Proposition 2.7.2 and Theorem II.6.2 in [Ric] that  $f$  is uniformly quasiregular. Furthermore, it follows from compactness arguments of qr mappings that every point is conical: for any  $x_0 \in M$ , a sequence of sizes  $r_n \rightarrow 0$  and a sequence of iterates  $k_n$  exist such that  $x \in B(0, 1) \mapsto f^{k_n}(x_0 + r_n x)$  defines a convergent sequence to a non constant map. Therefore, Theorem 1.3 in [MM] implies that  $f$  is a Lattès map. ■

Let us note that V. Mayer has also generalized the notion of power maps in [May]: these are uniformly quasiregular maps  $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ ,  $n \geq 3$ , such that the Fatou set consists of two totally invariant attracting basins, the Julia set is a sphere  $\mathbb{S}^{n-1}$ , and the dynamics on the Julia set is of Lattès type. These maps are also clearly cxc, if one restricts the dynamics to suitable neighborhoods of the Julia sets.

For all other known examples of uniformly quasiregular maps, the Julia set is a Cantor set, and the Fatou set is the basin of an attracting or of a parabolic fixed point [IM, Mar1, Pel, HMM, Mar2, MMP]. In the former case, when  $f$  has degree  $d$ , then there are  $d + 1$  embedded balls  $B_0, \dots, B_d$ ,

such that  $B_1, \dots, B_d$  have pairwise disjoint closures, all of them contained in  $B_0$ , and the restriction to each  $B_j$ ,  $j = 1, \dots, d$ , is a homeomorphism onto  $B_0$ : the Julia set is contained in these balls, and the restriction of  $f$  to these balls is clearly cxc.

## 4.5 Expanding maps on manifolds

**A baby example.** Let  $X = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$  be the two-torus and  $f : X \rightarrow X$  the degree twelve covering map induced by  $v \mapsto \phi v$  where  $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the linear map given by  $\Phi(x, y) = (3x, 4y)$ . Equip  $\mathbb{R}^2$  with the norm  $|\cdot|$  given by

$$|(x, y)| = \max\{|x|, |y|^\lambda\}$$

where  $\lambda = \log 3 / \log 4$ . Then for all  $v \in \mathbb{R}^2$ ,

$$|\Phi(v)| = 3|v|.$$

It follows that for all  $a, b \in X$  sufficiently close,

$$|f(a) - f(b)| = 3|a - b|$$

and it follows easily that  $(X, f)$  is cxc.

We now (greatly) generalize this example.

**Theorem 4.5.1 (From expanding to homothety)** *Let  $f : M \rightarrow M$  be a  $C^1$  expanding map of a compact connected  $C^1$  Riemannian manifold to itself. Then there exists a distance function  $d$  on  $M$  and constants  $\delta > 0$  and  $\rho > 1$  such that for all  $x, y \in M$ ,*

$$d(x, y) < \delta \Rightarrow d(f(x), f(y)) = \rho \cdot d(x, y)$$

*and such that balls of radius  $\leq \delta$  are connected and contractible.*

**Corollary 4.5.2 (Expanding implies cxc)** *The dynamical system  $((M, d), f)$  is cxc. Hence the metric  $d$  is unique, up to quasisymmetry.*

**Proof:** (of Corollary) We remark that  $f : M \rightarrow M$  is necessarily a covering map of degree  $D = \deg f$ .

Since  $f$  is expanding on a compact manifold, Axiom [Irreducibility] holds.

Let  $\mathcal{U}_0$  be a finite open cover of  $M$  by open balls of radius  $\delta$ . If  $U \in \mathcal{U}$  then since  $U$  is contractible we have

$$f^{-1}(U) = \bigcup_1^D \tilde{U}_i$$

where the union is disjoint and where each  $f|_{\tilde{U}_i} : \tilde{U}_i \rightarrow U$  is a homeomorphism which multiplies distances by exactly the factor  $\rho$ . Thus for each  $i$  there is an inverse branch  $g_i : U \rightarrow \tilde{U}_i$  which is a homeomorphism and which contracts distances exactly by the factor  $\rho^{-1}$ . By induction, for each  $n$  and each  $U \in \mathcal{U}_0$  there are  $D^n$  inverse branches of  $f^n$  over  $U$  which are homeomorphisms and which contract distances by  $\rho^{-n}$ . Verification of the axioms is now straightforward. The last claim follows from Theorem 2.8.2. ■

The proof of Theorem 4.5.1 occupies the remainder of this section.

**Sketch of proof.** One way to prove the theorem is to apply the geometric constructions of the previous chapter. We prefer however to give a self-contained proof using the algebra hidden behind expanding covers of Riemannian manifolds.

- I. By a celebrated result of Gromov [Gro1],  $f$  is topologically conjugate to the action of an expanding endomorphism on an infra-nilmanifold. Thus we may assume  $M$  is an infra-nilmanifold modelled on a simply connected nilpotent Lie group  $G$  and  $f$  is such an endomorphism.
- II. Let  $\tilde{f}$  denote the lift of  $f$  to the universal cover  $G$ . We shall show that there exists an associated  $\tilde{f}$ -homogeneous norm  $|\cdot| : G \rightarrow [0, \infty)$  satisfying the following properties for all  $x \in G$ :

- 1  $|x| = 0 \iff x = 1_G$ ,
- 2  $|x^{-1}| = |x|$ ,
- 3  $\exists \rho > 1$  such that  $|\tilde{f}(x)| = \rho|x|$
- 4  $|\cdot|$  is proper and continuous.

- III. For some  $0 < \epsilon \leq 1$ , the function

$$x, y \mapsto |x^{-1}y|^\epsilon$$

is bilipschitz equivalent to a left-invariant metric  $d = d_\epsilon$  on  $G$ . In the metric  $d$ , the map  $\tilde{f}$  expands distances by the constant factor  $\rho^\epsilon$ , and thus  $d$  descends to a distance on  $M$  with the desired properties.

We now begin the proof of Theorem 4.5.1.

**Infra-nilmanifolds.** For background, see [Dek]. Let  $G$  be a real, simply connected, finite dimensional, nilpotent Lie group. Then  $G \rtimes \text{Aut}(G)$  acts on  $G$  on the left via

$${}^{(g,\Phi)}x = g \cdot \Phi(x).$$

An *almost-Bieberbach group* is a torsion-free subgroup  $E < G \rtimes \text{Aut}(G)$  of the form  $L \rtimes F$  where  $L < G$  is discrete and cocompact and  $F < \text{Aut}(G)$  is finite. Recalling that  $E$  then acts freely on  $G$ , the quotient  $E \backslash G$  (which is not a coset space) is called an *infra-nilmanifold modelled on  $G$* .

**Expanding endomorphisms.** Suppose  $E$  is an almost-Bieberbach group,  $M = E \backslash G$ , and  $(g, \Phi) \in G \rtimes \text{Aut}(G)$  satisfies  $(g, \Phi)E(g, \Phi)^{-1} \subset E$ . Define

$$\tilde{f} : G \rightarrow G$$

by

$$\tilde{f}(x) = {}^{(g,\Phi)}x.$$

Then  $\tilde{f}$  descends to a map

$$f : M \rightarrow M$$

which is called an *endomorphism* of the infra-nilmanifold  $M$ . It is called *expanding* if all eigenvalues of the differential  $d\tilde{f} : \mathfrak{g} \rightarrow \mathfrak{g}$  lie outside the closed unit disk, where  $\mathfrak{g}$  is the Lie algebra of  $G$ .

We remark that

$$\tilde{f}(x)^{-1} \cdot \tilde{f}(y) = ({}^{(g,\Phi)}x)^{-1} \cdot ({}^{(g,\Phi)}y) = \Phi(x^{-1}y). \quad (4.8)$$

**Homogeneous norms.** If  $\Psi \in G \rtimes \text{Aut}(G)$ , a function  $|\cdot| : G \rightarrow [0, \infty)$  will be called a  $\Psi$ -*homogeneous norm* if it satisfies properties (1)-(4) in (II) with  $\tilde{f}$  replaced by  $\Psi$  in (3). Equation (4.8) implies that if  $f$  is given by the action of  $(g, \Phi)$ , then  $|\cdot|$  is a  $\tilde{f}$ -homogeneous norm if and only if it is a  $\Phi$ -homogeneous norm.

Since  $G$  is simply connected, the exponential map  $\exp : \mathfrak{g} \rightarrow G$  is a diffeomorphism. Hence we may identify  $\mathfrak{g}$  and  $G$ . In this identification,

$\Phi$  becomes  $d\Phi$ , which we again denote by  $\Phi$ . Thus, we may assume that  $\Phi : \mathfrak{g} \rightarrow \mathfrak{g}$  is a linear map and search for  $\Phi$ -homogeneous norms on  $\mathfrak{g}$ .

The case when  $\Phi$  is semisimple is treated in detail in [FS]. In general, we need the following development.

**Linear algebra.**

**Lemma 4.5.3** *Let  $\mathcal{V}$  be a finite-dimensional real vector space and  $\Phi \in \text{Aut}(\mathcal{V})$  have all eigenvalues strictly outside the closed unit disk. Then there exists a function*

$$|\cdot| : \mathcal{V} \rightarrow [0, \infty),$$

and a real 1-parameter family  $\Phi_t \subset \text{Aut}(\mathcal{V})$  with  $\Phi = \Phi_1$  such that for all  $v \in \mathcal{V}$  and all  $t \in \mathbb{R}$

1.  $|v| = 0 \iff v = 0$
2.  $|-v| = |v|$
3.  $|\Phi_t(v)| = e^t|v|$
4.  $|\cdot|$  is proper and continuous.

Assuming the lemma, we proceed as in [FS]. Let  $|\cdot|$  be the homogeneous norm on  $\mathfrak{g}$  given by Lemma 4.5.3 and transfer this via the exponential map to a homogeneous norm on  $G$  satisfying conditions (1)-(4) in the sketch of the proof. Now we are done with Lie algebras and work only on  $G$ . Condition (4) implies that

$$\{(x, y) \in G \times G : |x| + |y| = 1\}$$

is compact. Therefore

$$Q = \sup\{|xy| : |x| + |y| = 1\}$$

exists. For any  $x, y \in G$ , let  $t$  be so that  $e^t = |x| + |y|$ . Then

$$\begin{aligned}
 |xy| &= e^t e^{-t} |xy| \\
 &= e^t |\Phi_{-t}(xy)| \\
 &= e^t |\Phi_{-t}(x)\Phi_{-t}(y)| \\
 &\leq e^t Q \\
 &= Q(|x| + |y|)
 \end{aligned}$$

since  $|\Phi_{-t}(x)| + |\Phi_{-t}(y)| = 1$  by construction. In summary, the norm  $|\cdot|$  satisfies the additional property

$$5. |xy| \leq Q(|x| + |y|)$$

for some constant  $Q > 0$ .

**Quasimetrics.** The function

$$\varrho(x, y) = |x^{-1}y|$$

satisfies the symmetry and nondegeneracy conditions of a distance function by properties (1) and (2) of the norm  $|\cdot|$ . However, the triangle inequality fails if  $Q > 1$ ; the function  $\varrho$  is then called a *quasimetric*.

It is well-known that, given any quasimetric  $\varrho$ , there are constants  $C, \alpha > 0$  such that  $C\varrho^\alpha$  defines a metric. We outline the construction below and refer to e.g. [GdlH], §7.3 for details. Define

$$\varrho_\epsilon(x, y) = |x^{-1}y|^\epsilon$$

which is now a quasimetric with constant  $Q^\epsilon$ . Moreover, it satisfies the homogeneity property

$$\varrho_\epsilon(\Phi(x), \Phi(y)) = \rho^\epsilon \varrho_\epsilon(x, y). \quad (4.9)$$

Given  $x, y \in G$  a *chain*  $C$  from  $x$  to  $y$  is a sequence

$$x = x_0, x_1, \dots, x_n = y$$

of elements of  $G$ ; its *length* is given by

$$l_\epsilon(C) = \sum_{i=1}^n \varrho_\epsilon(x_{i-1}, x_i).$$

The set of chains from  $x$  to  $y$  is denoted  $\mathcal{C}_{xy}$ . Define a new function on pairs of points by

$$d_\epsilon(x, y) = \inf\{l_\epsilon(C) : C \in \mathcal{C}_{xy}\}.$$

The function  $d_\epsilon$  is symmetric and trivially satisfies the triangle inequality. Since  $\varrho_\epsilon$  satisfies Equation (4.9), so does  $d_\epsilon$ . Moreover, if  $Q^\epsilon < \sqrt{2}$  then for all  $x, y \in G$ , one has (*ibid*, Prop. 10)

$$(3 - 2Q^\epsilon)\varrho_\epsilon(x, y) \leq d_\epsilon(x, y) \leq \varrho_\epsilon(x, y)$$

so that the nondegeneracy condition holds and the functions  $d_\epsilon, \varrho_\epsilon$  are bilipschitz equivalent.

This completes the proof, modulo the proof of Lemma 4.5.3.

**Proof:** (Lemma 4.5.3) Assume first that  $\Phi$  lies on a 1-parameter subgroup

$$\Phi_t = \exp(\phi t)$$

for some  $\phi \in \text{End}(\mathcal{V})$ . Then the real parts of the eigenvalues of  $\phi$  have strictly positive real parts.

**Claim.** *There exists a basis for  $\mathcal{V}$  such that if  $\|\cdot\|$  is the corresponding Euclidean norm, then for all  $0 \neq v \in \mathcal{V}$  the function*

$$t \mapsto \|\Phi_t x\|$$

*is strictly increasing.*

The claim implies that for nonzero  $v$ , there is exactly one  $t(v)$  such that

$$\|\Phi_{t(v)}(v)\| = 1.$$

Define  $|0| = 0$  and for  $v \neq 0$  define

$$|v| = e^{-t(v)}.$$





where  $\lambda_i, \rho_i > 0$ ,  $I$  is the 2-by-2 identity matrix, and  $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ .

If  $i$  corresponds to a block of the second kind we set  $\lambda_i = \rho_i \cos \theta_i$ ; this is positive since this is the real part of the corresponding complex eigenvalue. By making a coordinate change of the form

$$\begin{pmatrix} 1 & & & & \\ & \delta^{-1} & & & \\ & & \dots & & \\ & & & \delta^{-(m-1)} & \end{pmatrix}$$

for an  $m$ -by- $m$  block we may assume that the off-diagonal elements are  $\delta$  in the first case and  $\delta I$  in the second, where

$$0 < \delta < \lambda_i.$$

Thus if  $\phi_i = \phi|_{\mathcal{V}_i}$  then

$$\phi_i = \lambda_i I + \delta N_i + K_i$$

where  $N_i$  is the nilpotent matrix with ones just above the diagonal,  $K_i$  is skew-symmetric, and the three terms commute pairwise.

So setting

$$\Phi_t^i = \exp(\phi_i t)$$

we have

$$\Phi_t^i = \exp((\lambda_i I + \delta N_i)t) \cdot \exp(K_i t)$$

where the second factor is orthogonal.

Let  $\langle \cdot, \cdot \rangle_i$  denote the inner product on  $\mathcal{V}_i$  corresponding to the above basis on  $\mathcal{V}_i$  and extend to  $\mathcal{V}$  in the obvious way so that the  $\mathcal{V}_i$  are orthogonal. The claim is proved once we show that for each  $i$

$$t \mapsto \|\Phi_t^i(v)\|_i$$

is strictly increasing.

We have for all  $t_0 \in \mathbb{R}$  and all  $v \neq 0$

$$\begin{aligned} \frac{d}{dt} \Big|_{t=t_0} \langle \Phi_t(v), \Phi_t(v) \rangle &= \frac{d}{dt} \Big|_{t=t_0} \langle e^{(\lambda_i I + \delta N_i)t} v, e^{(\lambda_i I + \delta N_i)t} v \rangle \\ &= 2 \langle e^{(\lambda_i I + \delta N_i)t_0}, \frac{d}{dt} \Big|_{t=t_0} e^{(\lambda_i I + \delta N_i)t} v \rangle \\ &= 2 \langle e^{(\lambda_i I + \delta N_i)t_0} v, (\lambda_i I + \delta N_i) e^{(\lambda_i I + \delta N_i)t_0} v \rangle \\ &= 2\lambda_i \langle y, y \rangle + 2\delta \langle y, N_i y \rangle \end{aligned}$$

where  $y = e^{(\lambda_i I + \delta N_i)t_0} v$ . The Cauchy-Schwarz inequality shows that  $|\langle y, N_i y \rangle| < \langle y, y \rangle$  and so since  $\delta < \lambda_i$  we have that the derivative at  $t_0$  is strictly positive and the claim is proved.

If  $\Phi$  does not lie on a 1-parameter subgroup we proceed as follows. It is well-known that  $\Phi$  lies on a 1-parameter subgroup if and only if the Jordan blocks with negative real eigenvalues occur in identical pairs. If this is not the case, we first change notation so  $\Phi = \Phi'$ . Next, let  $M_i : \mathcal{V}_i \rightarrow \mathcal{V}_i$  be given by  $-\text{id}$  if the  $i$ th Jordan block of  $\Phi$  is real with negative eigenvalue and by  $\text{id}$  otherwise, and set  $M = \oplus_i M_i : \mathcal{V} \rightarrow \mathcal{V}$ . Then  $\Phi'$  commutes with  $M$  and we set  $\Phi = M\Phi'$ . Then  $\Phi$  lies on a 1-parameter subgroup  $\Phi_t = \exp(\phi t)$  and we set  $\Phi'_t = M\Phi_t$ . Since

$$\|\Phi'_t(v)\| = \|M\Phi_t(v)\| = 1 \iff \|\Phi_t(v)\| = 1$$

we have

$$|\Phi'_t(v)| = e^t |v|$$

for all nonzero  $v$  and the proof is complete. ■

**Remarks:** In many cases, raising to a power in step (III) of the construction of  $d$  is unnecessary and a representative metric  $d$  can either be written down explicitly or is a well-studied object.

For example, suppose  $\mathfrak{g}$  is abelian (i.e. all brackets are trivial) and  $\Phi$  is diagonalizable over  $\mathbb{R}$ . This is a generalization of the baby example and one can write the metric  $d$  explicitly. The resulting gauges on the universal cover  $\mathbb{R}^n$  are studied by Tyson [Tys], §15. If not equivalent to the Euclidean gauge, these gauges are highly anisotropic: there exist a flag  $V_0 \subset V_1 \subset \dots \subset V_m = \mathbb{R}^n$  such that any qs automorphism  $h$  satisfies  $h(V_k) = V_k$ ,  $k = 1, \dots, m$ .

Another well-studied situation arises in the *Carnot-Carathéodory case*, i.e. when  $\Phi|_{\mathcal{H}} = \lambda \text{id}_{\mathcal{H}}$  on a subalgebra  $\mathcal{H}$  which generates  $\mathfrak{g}$  as a Lie algebra. In this case any two points are joined by a smooth curve with tangent in the distribution defined by  $\mathcal{H}$ . The resulting length space is a so-called *smooth Carnot-Carathéodory* metric space; cf. Pansu [Pan2]. The prototypical example is the map  $(x, y, z) \mapsto (2x, 2y, 4z)$  on the *Heisenberg manifold*  $M = H/\Gamma$  where  $H$  is the three-dimensional Heisenberg group of upper triangular matrices with ones on the diagonal and  $\Gamma$  is the lattice consisting of such matrices with integer entries.

In both cases, the conformal dimension (i.e. the infimum of the Hausdorff dimension over all quasisymmetrically equivalent spaces) is given by

$$\frac{1}{\lambda_1}(\lambda_1 + \lambda_2 + \dots + \lambda_n)$$

where  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  are the eigenvalues of  $\Phi$ .

The classification of Lie algebras admitting expanding endomorphisms is still in progress; see [DL].

## 4.6 Non-cxc maps with periodic branch points

### 4.6.1 Barycentric subdivision

Given a Euclidean triangle  $T$ , its *barycentric subdivision* is the collection of six smaller triangles formed by the three medians. Barycentric subdivision is natural with respect to Euclidean affine maps: if  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is affine, then the small triangles comprising the barycentric subdivision of  $T$  are sent by  $A$  to those comprising  $A(T)$ . If  $T$  is equilateral, the six smaller triangles

are congruent. Suppose  $T$  has side length one, and let  $B$  be an orientation-preserving affine map sending  $T$  to one of the six smaller triangles in its barycentric subdivision. Then

$$B = L \circ S \circ K$$

where  $K$  is a linear isometry,  $L$  is a translation, and

$$S = \begin{pmatrix} 1/2 & \sqrt{3}/6 \\ 0 & 1/3 \end{pmatrix}.$$

Using the naturality of barycentric subdivision and the fact that the Euclidean operator norm of  $S$  is  $(\sqrt{7} + 1)/6 \approx .608 < 1$ , it follows easily that for any triangle, under iterated barycentric subdivision, the diameters of the smaller triangles after  $n$  subdivisions tends to zero exponentially in  $n$ .

Let  $T$  as above be the Euclidean equilateral unit triangle. Equip  $T$  with an orientation and label the vertices of  $T$  as  $a, b, c$  as shown. Let  $T_1$  be one of the two smaller triangles in the first subdivision meeting at the vertex  $c$ , and let  $\phi : T_1 \rightarrow T$  be the restriction of the unique orientation-preserving Euclidean affine map fixing  $c$  and sending  $T_1$  onto  $T$ . Regard now the two-sphere  $S^2$  as the double  $T \cup \bar{T}$  of the triangle  $T$  across its boundary. Equip  $S^2$  with the complete length structure inherited from the Euclidean metric on  $T$  and its mirror image, so that the sphere becomes a CW complex  $X$  equipped with a path metric. By composing with reflections, there is a unique affinely natural extension of  $\phi$  to an orientation-preserving degree six branched covering map  $f = \phi_{\mathcal{R}} : S^2 \rightarrow S^2$  sending each of the twelve smaller triangles at level one onto  $T$  or  $\bar{T}$ ; see Figure 4.6.1.

The twelve smaller triangles give a CW structure  $\mathcal{R}(X)$  on  $X$  subdividing the original one, and we obtain a finite subdivision rule (in the sense of §4.3) with mesh going to zero. Notice, however, that this f.s.r. does not have bounded valence, since the branch point  $c$  of  $\phi_{\mathcal{R}}$  is a fixed 0-cell.

Let  $\mathcal{U}_0$  be the finite open cover of the sphere whose elements are given by the construction in Section 4.3. The discussion there implies that together,  $f : S^2 \rightarrow S^2$  and  $\mathcal{U}_0$  satisfy axioms [Irreducibility] and [Expansion], but not [Degree] in the definition of topologically cxc, and that the diameters of the elements of  $\mathcal{U}_n$  tend to zero exponentially in  $n$ .

Let  $\Gamma_f = \Gamma(f, \mathcal{U}_0)$  be the associated graph constructed in §3.2. By theorem 3.2.1, for some  $\varepsilon > 0$ , there is a homeomorphism  $\phi_f : S^2 \rightarrow \partial_\varepsilon \Gamma_f$

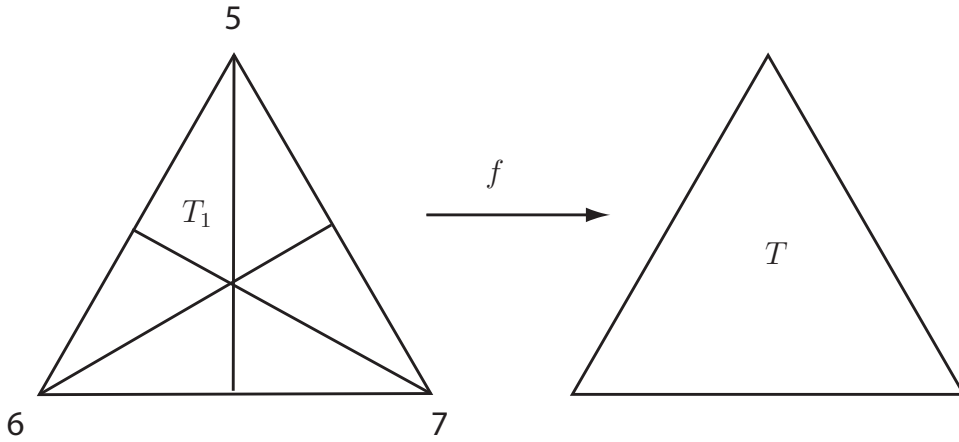


Figure 4.6.1

conjugating  $f$  to the induced map  $F$  on the boundary. Since  $P_f$  consists of a finite set of points, Proposition 3.3.8 applies and hence  $\partial_\varepsilon \Gamma_f$  fails to be doubling.

The map  $f$  is not the only dynamical system naturally associated to the barycentric subdivision rule. Let  $\mathbb{H} \subset \mathbb{C}$  denote the upper half-plane and let  $\rho : T \rightarrow \mathbb{H}$  be the unique Riemann map sending  $a \mapsto 0$ ,  $b \mapsto 1$ ,  $c \mapsto \infty$ . By Schwarz reflection, this defines a conformal isomorphism  $\rho : S^2 \rightarrow \widehat{\mathbb{C}}$ , where now  $S^2$  is the sphere endowed with the conformal structure of the path metric defined above. Let  $\psi : T_1 \rightarrow T$  be given by the unique Riemann map fixing  $c$  and sending vertices to vertices. As before, this determines an f.s.r.  $\mathcal{S}$  with an associated map  $\psi_{\mathcal{S}} : S^2 \rightarrow S^2$ . By the symmetry of the construction, the map  $g : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  given by  $\rho \circ \psi_{\mathcal{S}} \circ \rho^{-1}$  is a rational map; it is given by

$$g(z) = \frac{4}{27} \frac{(z^2 - z + 1)^3}{z^2(z-1)^2}.$$

See [CFKP].

The composition  $h' = (\phi_{\mathcal{R}}|_{T_1})^{-1} \circ (\phi_{\mathcal{S}}|_{T_1}) : T_1 \rightarrow T_1$  extends by reflection to a homeomorphism  $h'_1 : (S^2, a, b, c) \rightarrow (S^2, a, b, c)$ . Letting  $h'_0 = \text{id}$ , then

$$h'_0 \circ \phi_{\mathcal{S}} = \phi_{\mathcal{R}} \circ h'_1$$

and  $h'_0$  is isotopic to  $h'_1$  relative to the set  $\{a, b, c\}$ . That is, as postcritically finite branched coverings of  $S^2$ ,  $\phi_{\mathcal{R}}$  and  $\phi_{\mathcal{S}}$  are combinatorially equivalent.

Letting  $h_1 = h'_1 \circ \rho^{-1}$  gives  $h_0 \circ g = f \circ h_1$  with  $h_0, h_1$  isotopic relative to the set  $\{0, 1, \infty\}$ .

By lifting under the dynamics, we obtain for each  $n \in \mathbb{N}$  a homeomorphism  $h_n : \widehat{\mathbb{C}} \rightarrow S^2$  such that  $h_n \circ g = f \circ h_{n+1}$  with  $h_n \sim h_{n+1}$  relative to  $\{0, 1, \infty\}$ . Since  $f$  is uniformly expanding with respect to the length metric on  $S^2$ , the sequence of maps  $\{h_n\}$  converges uniformly to a map  $h : \widehat{\mathbb{C}} \rightarrow S^2$  for which  $hg = gh$ . Since  $g$  is locally contracting near infinity, the diameters of the preimages of the two half planes  $\mathbb{H}^\pm$  under  $g^{-k}$  which meet the point at infinity remain bounded from below as  $k \rightarrow \infty$ . Therefore  $h$  is not injective. Indeed, it is easy to see that  $h$  collapses the closure of each Fatou component to a point.

Let  $\mathcal{V}_0 = \{h^{-1}(U) : U \in \mathcal{U}_0\}$  be the open covering of  $\widehat{\mathbb{C}}$  given by pulling back the elements of  $\mathcal{U}_0$  under  $h^{-1}$  and let  $\Gamma_g = \Gamma(g, \mathcal{V}_0)$ . Then  $h$  induces an isometry  $h_\Gamma : \Gamma_g \rightarrow \Gamma_f$ . The natural map  $\phi_g : J_g \rightarrow \partial\Gamma_g$  satisfies  $\phi_f \circ h|_{J_g} = \partial h_\Gamma \circ \phi_g|_{J_g}$  and collapses the closure of every Fatou component to a point.

### 4.6.2 Expanding polynomials

For  $z = re^{i\theta}$  let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be given by  $f(z) = 1 - are^{i2\theta}$  where  $a = (1 + \sqrt{5})/2$  is the golden ratio. This map is an *expanding polynomial* in the sense of [BCM] and is studied in their Example 5.2.

The origin is a critical point which is periodic of period three, hence for each  $n \in \mathbb{N}$ ,  $f^{3n}$  is locally  $2^n$ -to-one on neighborhoods of the origin. The point  $-\beta \approx -1.7$  is a repelling fixed point with preimage  $\beta$ . Let  $I = [-\beta, \beta]$ ,  $I^- = [-\beta, 0]$ ,  $I^+ = [0, \beta]$ . Then  $f|_{I^\pm} : I^\pm \rightarrow I$  is a homeomorphism which expands Euclidean lengths by the factor  $a$ .

Let  $T_0$  be the metric tree with underlying space  $I$  and length metric given by the Euclidean length metric  $\sigma_0$ . It is easy to see that for all  $n \in \mathbb{N}$ , the set  $f^{-1}(T_n)$  is a tree which is the union of  $T_n$  together with a finite collection of smooth closed arcs  $J_i$ . Each such  $J_i$  is attached to  $T_n$  at a single endpoint which lies in  $f^{-n+1}(\{0, f(0), f(f(0))\})$ , and  $f|_{J_i}$  is a homeomorphism onto its image. Inductively, define a length metric  $\sigma_n$  on  $T_n$  by setting

$$\sigma_{n+1}|_{J_i} = a^{-1}(f|_{J_i})^*(\sigma_n).$$

Then  $f : T_{n+1} \rightarrow T_n$  multiplies the lengths of curves by the factor  $a$ .

Let  $\pi_{n+1} : T_{n+1} \rightarrow T_n$  be the map which collapses each such “new” interval  $J_i$  to the point on  $T_n$  to which it is attached. Clearly,  $\pi_n$  is distance-decreasing

for all  $n$ . Let

$$X = T_0 \xleftarrow{\pi_1} T_1 \xleftarrow{\pi_2} T_2 \dots$$

denote the inverse limit. Metrize  $X$  as follows. The diameters of the  $T_n$  are bounded by the partial sums of a convergent geometric series and thus are uniformly bounded. Hence for all  $x = (x_n), y = (y_n) \in T$ ,

$$\sup_n \sigma_n(x_n, y_n)$$

is bounded and increasing, hence convergent. It follows easily that  $T$  inherits a length metric  $\sigma$  such that the map  $f$  of  $T$  induced by  $f|_{T_{n+1}} : T_{n+1} \rightarrow T_n$  multiplies the lengths of curves by the factor  $a$ .

It is easy to see that near the origin, for each  $k \geq 1$   $X$  contains an isometrically embedded copy of the one-point union of  $2^k$  copies of a Euclidean interval of length  $a^{-k}$  where the common vertex is the origin  $(0, 0, 0, \dots)$ . This implies that  $X$  is not doubling, since (i) doubling is hereditary under passing to subspaces, and (ii) at least  $2^k$  balls of radius  $a^{-k}/2$  will be needed to cover the ball of radius  $a^{-k}$  centered at the origin. On the other hand, it is also easy to see that  $(X, f)$  satisfies the other axioms for a cxc system.

## 4.7 The $p$ -adic case

The construction of the graph  $\Gamma$  is reminiscent of certain constructions in  $p$ -adic dynamics. Below, we give a quick and partial account of  $p$ -adic dynamics in order to point out some formal similarities and major differences between our setting and the  $p$ -adic setting. References include [BH, Ben, RL1].

The main object of  $p$ -adic dynamics is to understand the iterates of rational maps with  $p$ -adic coefficients acting on  $\mathbb{P}^1(\mathbb{C}_p)$ , where  $\mathbb{C}_p$  is the metric completion of the algebraic closure of  $\mathbb{Q}_p$  endowed with the  $p$ -adic norm.  $\mathbb{C}_p$  is an algebraically closed, non-Archimedean valued field, and a complete *non-locally compact* totally disconnected ultra-metric space.

Let us note that the first difference with our setting is that  $\mathbb{C}_p$  is neither locally compact nor connected !

Since the metric on  $\mathbb{C}_p$  is an ultrametric, two balls are either disjoint, or one is contained in the other. In turn this induces a tree structure on the family of balls: the vertices are the balls of rational radii, and the edges originating from such a vertex are parametrised by the residual field  $\overline{\mathbb{F}}_p$ . If

$B \subset B'$  are two balls, then the edge joining them is made of the intermediate balls, and if  $B \cap B' = \emptyset$ , then the edge joining these balls is made of the two edges joining these balls to the smallest ball which contains both of them. This is the  $p$ -adic hyperbolic space  $\mathcal{H}$  [RL2].  $\mathcal{H}$  can be metrized to become a complete  $\mathbb{R}$ -tree i.e., a 0-hyperbolic metric space. The projective space  $\mathbb{P}^1(\mathbb{C}_p)$  is a part of the boundary of  $\mathcal{H}$ . The tree  $\mathcal{H}$  is isometric to the Bruhat-Tits building for  $\mathrm{SL}(2, \mathbb{C}_p)$  and is closely related to the Berkovich line [Ber].

We emphasize that the boundary at infinity of  $\mathcal{H}$  is larger than  $\mathbb{P}^1(\mathbb{C}_p)$ , since some intersections of balls with radii not converging to 0 may be empty, yielding points of  $\partial\mathcal{H}$  not in  $\mathbb{P}^1(\mathbb{C}_p)$ .

It turns out that rational maps send balls to balls, and that rational maps always act on the tree  $\mathcal{H}$ . So in the  $p$ -adic setting, the natural hyperbolic space  $\mathcal{H}$  on which any rational map  $f$  acts does not depend on the dynamics: it is a universal object independent of  $f$ . Another difference is that dynamics can be tame on the boundary, but never on the tree. That is, in  $\mathbb{P}^1(\mathbb{C}_p)$  the chaotic locus may be empty, but in  $\mathcal{H}$  it is always nonempty. In contrast, in our setting, the dynamics is always chaotic on the boundary  $\partial\Gamma$ , while the induced dynamics on the tree  $\Gamma$  itself is transient.

Finally, one can define, for rational maps  $R$  of degree  $d$  on  $\mathbb{C}_p$ , an invariant measure  $\mu$  such that  $R^*\mu = d\mu$ . While its metric entropy is at most  $\log d$ , there are examples for which the inequality is strict. This happens when the Julia set is contained in the hyperbolic space  $\mathcal{H}$  and the topological degree of the map on the Julia set is also strictly smaller than  $d$ ; see for instance [FRL]).

Of course, there are examples of rational maps over the  $p$ -adics for which the dynamics on the Julia set is conjugate to a full shift. In such cases, one obtains cxc maps. But generally, the  $p$ -adic case is rather different from ours, and the similarities are merely formal.





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