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# An efficient algorithm for positive realizations

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## Abstract

We combine recent ideas from the theory of positive systems to give an *efficient general algorithm* for positive realizations of transfer functions. With the help of our algorithm we determine the *minimal* order of positive realizations for a family of transfer functions, which complements an earlier result of [6]. Finally, we improve a lower-bound of [18] to indicate that our algorithm is indeed efficient in general.

*Key words:* Positive linear systems, discrete time filtering, positive realizations

## 1. Introduction

Assume we are given the transfer function

$$H(z) = \frac{p_1 z^{n-1} + \dots + p_n}{z^n + q_1 z^{n-1} + \dots + q_n},$$

where  $p_j, q_j \in \mathbb{R}$  for all  $1 \leq j \leq n$ , of a *discrete time-invariant linear SISO system* of McMillan degree  $n$ .

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We say that a triple  $\mathbf{A} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$  is an  *$n$ th order realization of  $H(z)$*  if it satisfies the condition:

$$H(z) = \mathbf{c}^T (z\mathbf{I} - \mathbf{A})^{-1} \mathbf{b}.$$

A standard result in the theory of linear systems, see, e.g., [10, Chapter 9], states that an  *$n$ th order realization of  $H(z)$*  always exists.

In this note, we are interested in the *positive realization problem*, i.e., finding  $\mathbf{A}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  with non-negative entries (and possibly of higher dimension  $M \geq n$ ). The nonnegativity restriction on the entries of  $\mathbf{A}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  reflects physical constraints in applications. Such positive systems appear, for example, in modeling of bio-systems, chemical reaction systems, and socio-economic systems, as described in detail in [10,14,15]. A thorough overview of the positive realization problem and related results has recently been given in [4], while for a direct application in filter-design we refer the reader to [5].

The *existence problem* is to decide for a given

transfer function whether any positive realization  $\mathbf{A}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  of any dimension  $M$  exists. It is well known, although maybe surprising, that the constraint of positivity may force the dimension  $M$  of realizations to be strictly larger than  $n$ , see [1], [6], [18] for different reasons why this phenomenon may occur.

The *minimality problem* is to find the lowest possible value of  $M$ . These problems have been given considerable attention over the past decade. The existence problem was completely solved in [2] and [9], cf., also [16,17,11], while a few particular cases of the minimality problem were settled in [8,13,19,3,23,22].

As these results indicate the state of the art of the theory is rather two-sided. On one hand, there exists a *general and constructive solution* [2,9] to the existence problem which, however, is *inefficient* in the sense that it yields very large dimensions, even in trivial cases. On the other hand, the minimality problem is solved only for *particular classes* of transfer functions, and a general solution seems out of reach of with the use of current methods. In this note we propose to bridge this gap by providing a *constructive, efficient general algorithm* to solve the existence problem in close-to-minimal dimensions. The idea behind our method is that we find an efficient way to reduce the case of a general transfer function to the few particular cases in which the minimality problem is already settled or, at least, strong results about it are already available.

In Section 2 we provide the necessary preliminaries and we describe the new algorithm. In Section 3 we give an illustrative example by revisiting a family of transfer functions  $H^N(z)$  from [6]. On one hand, we apply our algorithm to obtain  $(N + 1)$ -dimensional positive realizations of  $H^N(z)$ . On the other hand, with a slight modification of our algorithm, we complement the results of [6] by showing that for each  $N$  the minimal order of positive realizations of  $H^N(z)$  is equal to  $N$ . This supports our claim that the algorithm produces positive realizations in close-to-minimal dimensions. Of course, to support such a claim *in general*, one would need strong general lower-bounds on the minimal possible value  $M$  of the order of positive realizations. In Section 4 we provide a new lower-bound, improving a result of [18].

## 2. The algorithm

We begin by describing some standard preliminary results.

It is well-known that a necessary condition for the existence of positive realizations is that one of the dominant poles (i.e., the poles with maximal modulus) of  $H(z)$  be nonnegative real, and there is no loss of generality in assuming that it is located at  $\lambda_0 = 1$ , see, e.g., [2]. The transfer function  $H(z)$  is called *primitive* if  $\lambda_0$  is a unique dominant pole. It is also known, see [9]) that by the method of down-sampling the case of non-primitive transfer functions can be traced back to primitive ones. Therefore it is customary to assume that  $H(z)$  is a *primitive transfer function with dominant pole at  $\lambda_0 = 1$* . We will also assume, for technical simplicity, that  $\lambda_0 = 1$  is a *simple pole* (this makes the calculations less involved; we note that the case of a multiple dominant pole can be reduced to the simple pole case as in [17, Step 4]). The residue at  $\lambda_0 = 1$  can also be assumed to be 1 without loss of generality (see e.g., [2]).

After making these normalizing assumptions, the transfer function  $H(z)$  takes the form

$$H(z) = \frac{1}{z-1} + G(z) = \frac{1}{z-1} + \sum_{j=1}^r \sum_{i=1}^{n_j} \frac{c_j^{(i)}}{(z-\lambda_j)^i}, \quad (2.1)$$

where all the poles  $\lambda_j$  of  $G(z)$  are of modulus strictly less than 1, i.e.  $G(z)$  is asymptotically stable (note also that in (2.1), the poles  $\lambda_j$  and coefficients  $c_j^{(i)}$  are possibly complex).

### Definition 2.1

In the series expansion  $H(z) = \sum_{k=1}^{\infty} t_k z^{-k}$  the coefficients  $t_k$  are called the impulse response of  $H(z)$ .

Note that, if  $H(z) = \mathbf{c}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$  then  $t_k = \mathbf{c}^T \mathbf{A}^{k-1} \mathbf{b}$  for all  $k \geq 0$ . In particular, the  $t_k$ 's have to be non-negative for  $H(z)$  to have a positive realization. We now give the main ingredients upon which our algorithm is based. The first is the following simple but powerful result of Hadjicostis (see [12, Theorem 5]).

### Lemma 2.1 (Hadjicostis)

Let  $H(z) = \sum_{j=1}^{\infty} t_j z^{-j}$  be a rational transfer function

with nonnegative impulse response  $t_1, t_2, \dots$ . For an index  $m \geq 1$  let  $H_m(z)$  denote the transfer function corresponding to the shifted sequence  $t_m, t_{m+1}, \dots$ ,

i.e.  $H_m(z) = \sum_{j=1}^{\infty} t_{m+j-1} z^{-j}$ . Assume that  $H_m(z)$  admits a positive realization of some dimension  $k$ . Then  $H(z)$  admits a positive realization of dimension  $k + m - 1$ .

Let us apply this lemma to  $H(z)$  as given in (2.1). Note that  $H_1(z) = H(z)$  by definition, and for each  $m \geq 2$  we have  $H_m(z) = zH_{m-1}(z) - t_{m-1}$ . Hence, for each  $m \geq 1$  we have  $H_m(z) = \frac{1}{z-1} + \sum_{j=1}^r \sum_{i=1}^{n_j} \frac{c_{j,m}^{(i)}}{(z-\lambda_j)^i}$ , with the leading coefficient remaining 1, while all other coefficients  $c_{j,m}^{(i)} \rightarrow 0$  exponentially as  $m \rightarrow \infty$  (due to the asymptotic stability of  $G(z)$ ). That is, the leading coefficient becomes *very large compared to other coefficients*, and this is exactly the familiar situation of the *positive decomposition problem*, which we now turn to.

The task in the positive decomposition problem is to decompose an arbitrary transfer function  $G(z)$  as the difference  $G(z) = T_1(z) - T_2(z)$ , with  $T_1(z)$  and  $T_2(z)$  both admitting positive realizations (see [7,13,19,20]). By rescaling, one can assume that  $G(z)$  is asymptotically stable, and then the usual approach is to take a one-dimensional positive system  $T_2(z) = \frac{R}{z-w}$ , where  $0 < w < 1$  is larger than the modulus of any pole of  $G(z)$ , and  $R$  is a sufficiently large positive number. Then  $T_1(z) = G(z) + T_2(z)$  can be shown to admit a positive realization which, in some cases, turns out to be also minimal [7,13,19,20]. For our purposes, the essence of these results can be summarized as follows: *for any primitive transfer function, as long as the partial fraction coefficient of the dominant pole is much larger than all other coefficients (as in  $H_m(z)$  and  $T_1(z)$  above), there exist efficient methods to construct positive realizations*. We do not wish to list all the relevant theorems and constructions of [7,13,19,20] concerning the positive decomposition problem but let us state here, as a sample result for the reader's convenience, the powerful result, Theorem 8 of [7], which handles all transfer functions with simple poles (we changed the wording of the theorem slightly, but not its content).

**Theorem 2.2** (Benvenuti, Farina & Anderson)

$$\text{Let } H(z) = \frac{R}{z-1} + G(z) = \frac{1}{z-1} + \sum_{j=1}^{n-1} \frac{c_j}{z-\lambda_j},$$

where  $G(z)$  is a strictly proper asymptotically stable rational transfer function of order  $n$ , with simple poles. Let  $\mathcal{P}_j$  ( $j \geq 3$ ) denote the set of points in the

complex plane that lie in the interior of the regular polygon with  $j$  edges having one vertex in point 1 and center at 0.  $\mathcal{P}_j$  can formally be defined in polar coordinates as in [7]:

$$\mathcal{P}_j := \left\{ (\rho, \theta) : \rho \cos\left(\frac{(2k+1)\pi}{j} - \theta\right) < \cos\frac{\pi}{j}, \right. \\ \left. \text{for } k = 0, 1, \dots, j-1 \right\}.$$

Let  $N_2$  denote the sum of the number of negative real poles and the number of nonnegative real poles with negative residues in  $G(z)$ . Let  $N_3$  denote the number of pairs of complex conjugate poles of  $G(z)$  belonging to the region  $\mathcal{P}_3$ , and let  $N_j$  ( $j \geq 4$ ) denote the number of pairs of complex conjugate poles of  $G(z)$  belonging to the region

$$\mathcal{P}_j \setminus \bigcup_{m=3}^{j-1} \mathcal{P}_m.$$

If  $R$  is sufficiently large compared to the other coefficients  $c_j$  then  $H(z)$  admits a positive realization of dimension  $N = n + N_2 + \sum_{j \geq 3} (j-2)N_j$ .

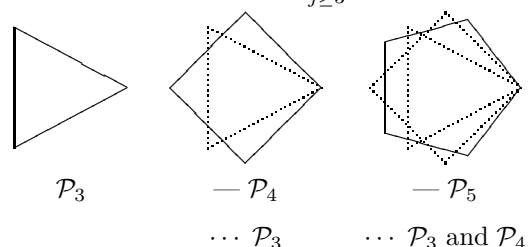


FIGURE 1. The sets  $\mathcal{P}_j$ .

The proof of this theorem is constructive (see [7]). The dimension  $N$  appearing in the theorem is not necessarily minimal but it is a very good *a priori* upper bound on the order of the realization. This theorem was later improved and generalized as follows: in [13, Corollary 2], the dimension  $N$  is slightly lowered if  $H(z)$  has at least two nonnegative poles with negative residues; in [19, Corollary 2], the *nonnegative* poles of  $G(z)$  are allowed to be *multiple*. [20, Theorem 1], handles conjugate pairs of *complex multiple* poles while [20, Theorem 2], handles *negative multiple* poles. These papers also provide a number of examples where *minimality* of the arising dimension  $N$  can be claimed (see the discussion on minimality in [13,19,20]). Finally, in [20, Theorem 4], a synthesis of all these results was given which covers the case of  $H(z) = \frac{R}{z-1} + G(z)$  for *any* asymptotically stable rational transfer function  $G(z)$  (the

price being paid for such generality is that the upper bound on  $N$  is slightly worse than in Theorem 2.2, above). All these results are constructive.

Before turning to our algorithm let us describe another important case, unrelated to positive decompositions, in which the minimality problem is already settled. This is the case of transfer functions with 3 nonnegative simple poles, as presented in [8, Theorem 3].

**Theorem 2.3** (*Benvenuti, Farina, Anderson & De Bruyne*)

Let  $H(z) = \frac{r_1}{z - \lambda_1} + \frac{r_2}{z - \lambda_2} + \frac{r_3}{z - \lambda_3}$  be a third order transfer function (i.e.  $r_1, r_2, r_3 \neq 0$ ) with distinct positive real poles  $1 = \lambda_1 > \lambda_2 > \lambda_3 > 0$ . Then  $H(z)$  has a third order positive realization if and only if the following conditions hold:

- (i)  $r_1 > 0$
- (ii)  $r_1 + r_2 + r_3 \geq 0$
- (iii)  $(1 - \bar{\eta})r_1 + (\lambda_2 - \bar{\eta})r_2 + (\lambda_3 - \bar{\eta})r_3 \geq 0$
- (iv)  $(1 - \eta)^2 r_1 + (\lambda_2 - \eta)^2 r_2 + (\lambda_3 - \eta)^2 r_3 \geq 0$  for all  $\eta$  such that  $\bar{\eta} \leq \eta \leq \lambda_3$ ,

where  $\bar{\eta}$  is the maximum of 0 and

$$\frac{1 + \lambda_2 + \lambda_3 - 2\sqrt{(\lambda_2 - \lambda_3)^2 + (1 - \lambda_2)(1 - \lambda_3)}}{3}.$$

We are now ready to combine all these preliminaries in the following

#### ALGORITHM:

Assume  $H(z)$  is given as in 2.1, and the aim is to construct a positive realization of  $H(z)$ , or to conclude that there is no such realization. For the sake of transparency let us assume for the moment that  $H(z)$  has simple poles only (and see Remark 1 below for the general case).

*Step 0.* Set  $m = 1$ ,  $H_m(z) = H(z)$ .

*Step 1.* Check whether Theorem 2.2 or 2.3 are applicable to  $H_m(z)$ . If so, then construct a positive realization of  $H_m(z)$  and apply Lemma 1 (if it is necessary, i.e. if  $m > 1$ ) to obtain a positive realization of  $H(z)$ . If not, then

*Step 2.* Check whether the first element  $t_m$  of the impulse response sequence of  $H_m(z)$  is nonnegative. If not, then conclude that there is no positive realization of  $H(z)$ . If  $t_m$  is nonnegative, then calculate  $H_{m+1}(z) = zH_m(z) - t_m$ , which corresponds to the shifted impulse response sequence  $t_{m+1}, t_{m+2}, \dots$ . Then increase the value of  $m$  to  $m + 1$  and go back to Step 1.

The existing theory of positive realizations guarantees that this algorithm terminates in a finite number of steps. Indeed, we either conclude at some stage that an element  $t_m$  of the impulse response sequence is negative, or Theorems 2.2 or 2.3 become applicable due to the fact that all partial fraction coefficients become very small compared to the leading coefficient.

#### Remark 2.1

In order not to lose the essence of the method we described the algorithm in its simplest form. However, it can easily be generalized to cover primitive transfer functions with *multiple poles*. Indeed, if multiple poles are present, then it is enough in Step 1 to replace Theorem 2.2 with the more general Theorem 4 of [20]. Also, in Step 1 it is always advantageous to check all mentioned improvements of Theorem 2.2, such as Corollary 2 in [13], and Corollary 2 in [19].  $\square$

#### Remark 2.2

There are good heuristic arguments to believe that this algorithm is efficient in terms of producing small dimensions, and definitely better than the existing algorithm of [2]. First, the partial fraction coefficients decay exponentially, so that only a few iterations are needed before Theorem 2.2 or 2.3 become applicable, and these theorems already provide minimal or close-to-minimal dimensions (see Section 3 for a numerical example). Second, the method of [2] involves the time development of an  $n$ -dimensional “cube” around the vector  $(1, 1, \dots, 1)$  and, as such, can only produce dimensions larger than  $2^n$  (usually significantly larger than  $2^n$ ).  $\square$

### 3. Example: a family of transfer functions revisited

In this section we give an example which complements a result of [6] and shows that our algorithm can lead to minimal or close-to-minimal dimensions.

Consider the family of transfer functions

$$H^N(z) = \frac{1}{z - 1} - \frac{4 \cdot (5/2)^{N-2}}{z - 0.4} + \frac{3 \cdot 5^{N-2}}{z - 0.2} \quad (3.1)$$

as in [6, Example 4].

For  $N = 2$  one can apply Theorem 2.3 and construct a 3-dimensional minimal realization of  $H^2(z)$ .

For  $N = 4$  the following 4-dimensional positive realization of  $H^4(z)$  is given in [6, Example 3] and it is shown to be minimal.

$$\mathbf{b} = (0 \ 0 \ 0 \ 1)^T, \quad \mathbf{c} = (6 \ 0 \ 0 \ 51)^T$$

and

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & \frac{63 + 4\sqrt{26}}{85} & 0 & 0 \\ 0 & \frac{22 - 4\sqrt{26}}{85} & \frac{63 - 4\sqrt{26}}{85} & 0 \\ 0 & 0 & \frac{22 + 4\sqrt{26}}{85} & 0 \end{pmatrix}, \quad (3.2)$$

Furthermore, it is proved in [6] that for any  $N \geq 4$  the minimal dimension of positive realizations of  $H^N(z)$  is *at least*  $N$ . Here we prove that an  $N$ -dimensional minimal positive realization of  $H^N(z)$  does indeed exist for every  $N$ .

Let us first apply our algorithm, word by word, to  $H^N(z)$  for  $N \geq 4$ . It is not difficult to see that each application of Step 2 decreases the value of  $N$  by 1, that is, with the notation of the algorithm  $H_1(z) = H^N(z)$ ,  $H_2(z) = H^{N-1}(z)$ ,  $H_3(z) = H^{N-2}(z)$ ,  $\dots$ ,  $H_m(z) = H^{N+1-m}(z)$ . The algorithm terminates when Theorem 2.3 becomes applicable, i.e. when  $H_m(z) = H^2(z)$ , that is  $m = N - 1$ . Then a 3-dimensional positive realization of  $H^2(z)$  is constructed, and the application of Lemma 2.1 produces a positive realization of  $H^N(z)$  of order  $3 + (m - 1) = N + 1$ .

We can further lower the dimension of realizations by invoking the realization (3.2) of  $H^4(z)$  above. Indeed, let us stop the iterations at  $m = N - 3$ , i.e. at  $H_m(z) = H^4(z)$  and make use of the realization (3.2) of  $H^4(z)$ . Then, the application of Lemma 2.1 produces a positive realization of  $H^N(z)$  of order  $4 + (m - 1) = N$ . (Note that the algorithm would only stop at  $m = N - 1$ , and the extra knowledge of formula (3.2) was necessary for this improvement.)

From this example we conclude the following facts. First, the minimal order of positive realizations of  $H^N(z)$  is  $N$  (it was already shown to be not less than  $N$  in [6]). Second, a direct application of our algorithm produces positive realizations of order  $N + 1$ , very close to the minimal order.

#### 4. Improved lower-bounds

We saw in the previous section that the minimal order of positive realizations of  $H^N(z)$  is  $N$ . Also, it is easy to calculate (see [6]) that the impulse response sequence of  $H^N(z)$  contains zeros, namely  $t_{N-1} = t_N = 0$ . A general lower-bound presented in

[18] gives that in such a case the order  $M$  of any positive realization must satisfy  $\frac{M(M+1)}{2} - 1 + M^2 \geq N$ , i.e.  $M$  is at least  $\approx \sqrt{\frac{2N}{3}}$ . In view of the actual minimal value  $M = N$  a lower-bound of the order of magnitude  $N$  would be more welcome (instead of the order of magnitude  $\sqrt{N}$ ). In this section we present such an improvement (but we note here that while the lower-bound of [18] is valid in general, the improvement here is restricted to transfer functions with positive real poles, as is the case of the example of the previous section).

Throughout this section we assume that  $H(z)$  is a given primitive transfer function of McMillan degree  $n$  with *positive real poles*, and there exists a positive integer  $k_0$ , such that for the impulse response sequence of  $H(z)$  we have  $t_{k_0} = 0$  and  $t_k > 0$  for all  $k > k_0$ . This means that  $H(z)$  is of the form

$$\frac{1}{z-1} + \sum_{j=1}^r \sum_{i=1}^{n_j} \frac{c_j^{(i)}}{(z-\lambda_j)^i} \quad (4.1)$$

where  $c_j^{(i)} \in \mathbb{R}$ ,  $0 < \lambda_j < 1$ , and  $\sum_{j=1}^r n_j = n - 1$ .

In this section we will need some further basic results from the theory of nonnegative realizations. Let the triple  $(\mathbf{h}, \mathbf{F}, \mathbf{g})$  denote an arbitrary minimal ( $n$ -dimensional) realization of  $H(z)$  (for canonical minimal realizations see e.g. [10]). Assume that there exists a matrix  $\mathbf{P}$  of size  $n \times M$  such that for some triple  $(\mathbf{c}, \mathbf{A}, \mathbf{b})$  with nonnegative entries the equalities

$$\mathbf{F}\mathbf{P} = \mathbf{P}\mathbf{A}, \quad \mathbf{P}\mathbf{b} = \mathbf{g}, \quad \mathbf{c}^T = \mathbf{h}^T\mathbf{P} \quad (4.2)$$

hold. (There is a well-known geometrical interpretation of these equalities. Namely, the columns of matrix  $\mathbf{P}$  represent the edges of a finitely generated cone  $\mathcal{P}$  in  $\mathbb{R}^n$ , such that  $\mathcal{P}$  is  $\mathbf{F}$ -invariant, and  $\mathcal{P}$  lies between the reachability cone and the observability cone corresponding to the triple  $(\mathbf{h}, \mathbf{F}, \mathbf{g})$ .) It is well-known that the triple  $(\mathbf{c}, \mathbf{A}, \mathbf{b})$  then provides a positive realization of  $H(z)$ .

#### Definition 4.1

A triple  $(\mathbf{c}, \mathbf{A}, \mathbf{b})$  which arises in such a manner is called a *cone-generated realization* of  $H(z)$ .

It is a basic result in the theory of positive realizations that a transfer function  $H(z)$  admits positive realizations if and only if it admits cone-generated realizations (see [21]). It is also conjectured that the minimal order of positive realizations can always be achieved by cone-generated realizations. It would be

interesting to see a proof of this fact, as already mentioned in the Open Problems section of [4]. Here we will present a lower estimate on the order of cone-generated realizations of  $H(z)$ . We will need the following auxiliary result.

**Lemma 4.1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) := \sum_{j=1}^r p_{(j)}(x) \lambda_j^x$$

where  $\lambda_1 > \lambda_2 > \dots > \lambda_r > 0$  and  $p_{(j)}$  denotes a polynomial (with real coefficients) of degree  $n_j$ .

Then  $f$  has at most  $R \equiv R_f := \sum_{j=1}^r (n_j + 1) - 1$  pairwise distinct real roots.

*Proof.* This lemma is well known, but let us include a proof for sake of completeness. We prove the statement by induction with respect to  $R$ . If  $R_f = 0$ , then  $f$  is a scalar multiple of an exponential function, therefore it has no roots.

Assume that the statement is valid for  $R = l$ , and let  $f(x) := \sum_{j=1}^r p_{(j)}(x) \lambda_j^x$ , with  $R_f = l + 1$ . We may assume, without loss of generality, that  $\lambda_1 = 1$ . Assume also, contrary to our statement, that  $f$  has at least  $R_f + 1 = l + 2$  pairwise distinct real roots. This means, by Rolle's theorem, that the derivative  $f'$  must have at least  $l + 1$  pairwise distinct real roots. However, it is easy to see that  $f'$  is of the same form as  $f$ , and  $R_{f'} = l$ . Indeed, the derivative  $f'(x) = q_{(1)}(x) + \sum_{j=2}^r q_{(j)}(x) \lambda_j^x$ , where  $\deg q_{(1)} = n_1 - 1$  (if  $n_1 = 0$  then  $q_{(1)}(x) \equiv 0$ ) and  $\deg q_{(j)} = n_j$  for  $2 \leq j \leq r$ . Hence, we conclude that on one hand  $R_{f'} = l$ , and, on the other hand,  $f'(x)$  has at least  $l + 1$  pairwise distinct real roots. This contradiction completes the proof.  $\square$

We are now ready to give an improvement of the lower-bound of [18].

**Theorem 4.2**

Assume that  $H(z)$  is a transfer function of McMillan degree  $n$ , with positive real poles, given as in (4.1) above. Assume also that there exists a positive integer  $k_0$ , such that for the impulse response sequence of  $H(z)$  we have  $t_{k_0} = 0$  and  $t_k > 0$  for all  $k > k_0$ . Then the dimension  $M$  of any cone-generated positive realization of  $H(z)$  satisfies  $M \geq \frac{k_0}{n-1}$ .

*Proof.* Let the triple  $(\mathbf{h}, \mathbf{F}, \mathbf{g})$  denote a minimal ( $n$ -dimensional) realization of  $H(z)$  and consider

any cone-generated positive realization  $(\mathbf{c}, \mathbf{A}, \mathbf{b})$  of  $H(z)$  arising from a matrix  $\mathbf{P}$  of size  $n \times M$ , as in 4.2. Let  $\mathbf{e}_i$  denote an arbitrary column of the matrix  $\mathbf{P}$ , and consider the sequence  $g_k^{(i)} := \mathbf{h}^T \mathbf{F}^{k-1} \mathbf{e}_i \geq 0$ . Let  $\mathbf{P}_1 := [p_{i,j}]$  be the nonnegative matrix of size  $M \times \infty$  defined by  $p_{i,j} := g_j^{(i)}$  for  $1 \leq i \leq M$  and  $1 \leq j$ .

Let  $\mathbf{K} := [k_{i,j}]$  denote the infinite Hankel matrix composed of the impulse response sequence of  $H(z)$ , i.e.  $k_{i,j} := t_{i+j-1}$ . By the assumptions imposed on  $\mathbf{P}$ , there exists a matrix  $\mathbf{Q} = [q_{i,j}]$  of size  $\infty \times M$ , with nonnegative entries, such that  $\mathbf{Q}\mathbf{P}_1 = \mathbf{K}$  holds. (This is true because the  $k$ th row of  $\mathbf{K}$  is given by  $k_{k,j} = t_{k+j-1} = \mathbf{h}^T \mathbf{F}^{j-1} (\mathbf{F}^{k-1} \mathbf{g})$ , and the vector  $\mathbf{F}^{k-1} \mathbf{g}$  lies inside the cone  $\mathcal{P}$  by assumption, therefore it can be decomposed as a linear combination of the edges  $\mathbf{e}_i$  of  $\mathcal{P}$  with nonnegative coefficients, and [one choice of] these coefficients will form the  $k$ th row of the matrix  $\mathbf{Q}$ .) Since  $(\mathbf{h}, \mathbf{F}, \mathbf{g})$  is a minimal realization, for an arbitrary column  $\mathbf{e}_i$  of the matrix  $\mathbf{P}$  the transfer function corresponding to the impulse response sequence  $g_k^{(i)} = \mathbf{h}^T \mathbf{F}^{k-1} \mathbf{e}_i$  is of the form

$$H^{(\mathbf{e}_i)}(z) = \frac{C_i}{z-1} + \sum_{j=1}^r \sum_{s=1}^{n_j} \frac{d_{j,i}^{(s)}}{(z-\lambda_j)^s}.$$

(Note that some coefficients  $C_i$  and  $d_{j,i}^{(s)}$  may be 0.) The column  $\mathbf{e}_i$  of  $\mathbf{P}$  will be called *dominant* if  $C_i \neq 0$  in  $H^{(\mathbf{e}_i)}(z)$ . Delete the non-dominant rows from the matrix  $\mathbf{P}_1$  and the corresponding columns from the matrix  $\mathbf{Q}$ . The remaining matrices (of sizes  $M_1 \times \infty$  and  $\infty \times M_1$  for some  $M_1 \leq M$ ) will be denoted by  $\mathbf{P}_1^{(dom)}$  and  $\mathbf{Q}^{(dom)}$ . We see that  $\mathbf{Q}^{(dom)} \mathbf{P}_1^{(dom)} \leq \mathbf{K}$  entrywise. Recall that we have  $t_{k_0} = 0$  by the assumption of the impulse response of  $H(z)$ . This implies that for some dominant index  $i$  ( $1 \leq i \leq M_1$ ) we must have  $g_{k_0}^{(i)} = 0$ , otherwise  $k_{1,k_0} = t_{k_0}$  would be strictly positive in the first row of  $\mathbf{g}$ . Considering now the second row of  $\mathbf{K}$  we see that  $k_{2,k_0-1} = t_{k_0} = 0$ , hence we must have  $g_{k_0-1}^{(i)} = 0$  for some dominant index  $i$ . By the same argument we see that for every  $1 \leq j \leq k_0$  we must have  $g_j^{(i)} = 0$  for some dominant index  $i$ . In other words, each of the first  $k_0$  columns of the matrix  $\mathbf{P}_1^{(dom)}$  must contain a zero, and hence there are at least  $k_0$  zero entries in  $\mathbf{P}_1^{(dom)}$ .

On the other hand,  $g_k^{(i)}$  is the impulse response of  $H^{(\mathbf{e}_i)}(z) = \frac{C_i}{z-1} + \sum_{j=1}^r \sum_{s=1}^{n_j} \frac{d_{j,i}^{(s)}}{(z-\lambda_j)^s}$ . Thus,  $g_k^{(i)} = C_i + \sum_{j=1}^{r-1} p_{(j)}(k) \lambda_j^k$ , where  $p_{(j)}$  are polynomials of degree not exceeding  $n_j - 1$ , for  $1 \leq j \leq r$ . Hence, Lemma 4.1 implies that there are at most  $R = (1 +$

$\sum_{j=1}^r n_j) - 1 = n - 1$  zeros in each row of  $\mathbf{P}_1^{(dom)}$ . This means that the number of zeros in the matrix  $\mathbf{P}_1^{(dom)}$  is at most  $M_1(n - 1)$ . Therefore we have the estimates

$$k_0 \leq \# \text{ of zeros in } \mathbf{P}_1^{(dom)} \leq M_1(n - 1) \leq M(n - 1),$$

and hence  $M \geq \frac{k_0}{n-1}$ .  $\square$

#### Remark 4.1

If we apply Theorem 4.2 to the functions  $H^N(z)$  of Section 3 we obtain  $M \geq N/2$  which is still quite far from the actual minimal value  $N$ . However, if there are more than 3 poles present in  $H(z)$  then the geometric arguments of [6] seem very hard to generalize, while Theorem 4.2 still applies. Of course, it would be interesting to improve our lower-bound to something like  $M \geq k_0$  instead of  $M \geq \frac{k_0}{n-1}$ , if possible.

#### Remark 4.2

As mentioned in the ‘‘Open Problems and New Directions’’ section of [4] it would be desirable to have tight upper and lower bounds on the minimal order of a positive realization *in general*. Note, however, that the results of [6,18] and Theorem 4.2 above are all based on the assumption that the impulse response sequence of  $H(z)$  contains at least one element of 0. The only other lower-bound known to us is that of [12] which, however, does not give any non-trivial estimates for transfer functions with nonnegative poles.

What can be said if the impulse response does not contain zeros? Unfortunately, we do not have a general approach to this case. As a first step in this direction we examined the modified family  $H^{N,\varepsilon}(z) = \frac{1}{z-1} - \frac{4 \cdot (5/2)^{N-2}}{z-0.4} + \frac{3 \cdot 5^{N-2+\varepsilon}}{z-0.2}$  for small values of  $\varepsilon$ . Note that the impulse response sequence no longer contains zeros. Due to the system being 3-dimensional we could use elementary (but tedious) geometric arguments to conclude that for small enough  $\varepsilon$  the minimal order  $M$  of positive realizations of  $H^{N,\varepsilon}(z)$  still satisfies  $M \geq N/2$ . It is not clear, however, how to generalize these arguments to transfer functions of higher degree (as in Theorem 4.2) where the geometric intuition is missing. Therefore, finding tight lower-bounds in the general case remains an open problem.

## 5. Conclusion

We have combined some recent ideas of positive system theory to describe an *efficient* algorithm to

the positive realization problem of transfer functions. As an example we examined a family of transfer functions first given in [6], and determined the minimal order of positive realizations for that family. With respect to the minimality problem we have proved new lower-bounds on the order of positive realizations of transfer functions with positive real poles, improving an earlier general result of [18].

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