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Inverse Problem for the Schrödinger Operator in an Unbounded Strip

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Abstract

We consider the operator $H := i\partial_t + \nabla \cdot (c\nabla)$ in an unbounded strip Ω in \mathbb{R}^2 , where $c(x, y) \in \mathcal{C}^3(\overline{\Omega})$. We prove adapted a global Carleman estimate and an energy estimate for this operator. Using these estimates, we give a stability result for the diffusion coefficient $c(x, y)$.

1 Introduction

Let $\Omega = \mathbb{R} \times (-\frac{d}{2}, \frac{d}{2})$ be an unbounded strip of \mathbb{R}^2 with a fixed width d . We will consider the Schrödinger equation

$$(1.1) \quad \begin{cases} Hq := i\partial_t q + \nabla \cdot (c(x, y)\nabla q) = 0 & \text{in } Q = \Omega \times (0, T), \\ q(x, y, t) = b(x, y, t) & \text{on } \Sigma = \partial\Omega \times (0, T), \\ q(x, y, 0) = q_0(x, y) & \text{on } \Omega, \end{cases}$$

where $c(x, y) \in \mathcal{C}^3(\overline{\Omega})$ and $c(x, y) \geq c_{min} > 0$. Moreover, we assume that c and all its derivatives up to order three are bounded. If we assume that q_0 belongs to $H^4(\Omega)$ and b is sufficiently regular (e.g. $b \in H^1(0, T, H^{\frac{9}{2}+\varepsilon}(\partial\Omega)) \cap H^2(0, T, H^{\frac{5}{2}+\varepsilon}(\partial\Omega))$ and some additional conditions), then (1.1) admits a solution in $H^1(0, T, H^{\frac{3}{2}+\varepsilon}(\Omega))$. We will use this regularity result later. The aim of this paper is to give a stability and uniqueness result for the coefficient $c(x, y)$ using global Carleman estimates and energy estimates. We denote by ν the outward unit normal to Ω on $\Gamma = \partial\Omega$. We denote $\Gamma = \Gamma^+ \cup \Gamma^-$, where $\Gamma^+ = \{(x, y) \in \Gamma; y = \frac{d}{2}\}$ and $\Gamma^- = \{(x, y) \in \Gamma; y = -\frac{d}{2}\}$. We use the following notations $\nabla \cdot (c\nabla u) = \partial_x(c\partial_x u) + \partial_y(c\partial_y u)$, $\nabla u \cdot \nabla v = \partial_x u \partial_x v + \partial_y u \partial_y v$, $\partial_\nu u = \nabla u \cdot \nu$.

We shall use the following notations $Q = \Omega \times (0, T)$, $\tilde{Q} = \Omega \times (-T, T)$, $\Sigma = \Gamma \times (0, T)$ and $\tilde{\Sigma} = \Gamma \times (-T, T)$.

Our problem can be stated as follows:

Is it possible to determine the coefficient $c(x, y)$ from the measurement of $\partial_\nu(\partial_t q)$ on Γ^+ ?

Let q (resp. \tilde{q}) be a solution of (1.1) associated with (c, b, q_0) (resp. (\tilde{c}, b, q_0)) satisfying some regularity properties:

- $\partial_t \tilde{q}$, $\nabla(\partial_t \tilde{q})$ and $\Delta(\partial_t \tilde{q})$ are bounded,
- q_0 is a real valued function in $\mathcal{C}^3(\Omega)$,

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- q_0 and all its derivatives up to order three are bounded.

Our main result is

$$|c - \tilde{c}|_{H^1(\Omega)}^2 \leq C |\partial_\nu(\partial_t q) - \partial_\nu(\partial_t \tilde{q})|_{L^2((-T, T) \times \Gamma^+)}^2,$$

where C is a positive constant which depends on (Ω, Γ, T) and where the above norms are weighted Sobolev norms.

We prove an adapted global Carleman estimate and an energy estimate for the operator H with a boundary term on Γ^+ . Such energy estimate has been proved in [16] for the Schrödinger operator in a bounded domain in order to obtain a controllability result. Then using these estimates and following the method developed by Imanuvilov, Isakov and Yamamoto for the Lamé system in [12], [14], we give a stability and uniqueness result for the diffusion coefficient $c(x, y)$. Note that Carleman estimates with an internal observation has been proved for the heat equation in unbounded domains in [5]. For the Schrödinger equation, [4] gives a stability result for an inverse problem in a bounded domain.

The Schrödinger equation has got properties halfway between the heat equation and the wave equation. This comes mainly from the dual-purpose character of this problem: neither parabolic nor hyperbolic.

On one hand, a physical background could be the characterization of the diffusion coefficient for a strip in geophysics. Indeed if we look for time harmonic solutions of (1.1), the problem can be written, after some changes of variables as the reconstruction of a non local potential P in a strip for the operator $-\Delta + P$. Few results for inverse problems exist in a two-dimensional strip (see [7]). For the layer $\mathbb{R}^n \times [0, h]$ with $n \geq 2$, several results exist for the stationary inverse problems (see [2], [8], [9], [13], [17], ...).

On the other hand, we can link our problem to the determination of the curvature function for a curved quantum guide (see [11], [6], [10], ...).

For the first time, the method of Carleman estimates was introduced in the field of inverse problems in the work of Bukhgeim and Klivanov [3]. A recent book by Klivanov and Timonov [15] is devoted to the Carleman estimates applied to inverse coefficient problems.

This paper is organized as follows. In section 2, we give an adapted global Carleman estimate for the operator H . In section 3, we prove an energy estimate and we give a stability result for the diffusion coefficient c .

2 Global Carleman Estimate

Let $c = c(x, y)$ be a bounded positive function in $C^3(\overline{\Omega})$ such that

Assumption 2.1. $c(x, y) \geq c_{min} > 0$, c and all its derivatives up to order three are bounded by a positive constant \tilde{C}_0 .

Let $q = q(x, y, t)$ be a function equals to zero on $\partial\Omega \times (-T, T)$ and solution of the Schrödinger equation

$$i\partial_t q + \nabla \cdot (c(x, y)\nabla q) = f.$$

We prove here a global Carleman-type estimate for q with a single observation acting on a part Γ^+ of the boundary Γ in the right-hand side of the estimate. Let $\tilde{\beta}$ be a $C^4(\Omega)$ positive function such that there exists positive constants C_0, C_1, C_{pc} which satisfy

Assumption 2.2. • $|\nabla \tilde{\beta}| \geq C_0 > 0$ in Ω , $\partial_\nu \tilde{\beta} \leq 0$ on Γ^- ,

- $\tilde{\beta}$ and all its derivatives up to order four are bounded in $\overline{\Omega}$ by C_1 ,
- $2\Re(D^2 \tilde{\beta}(\zeta, \bar{\zeta})) - c\nabla c \cdot \nabla \tilde{\beta} |\zeta|^2 + 2c^2 |\nabla \tilde{\beta} \cdot \zeta|^2 \geq C_{pc} |\zeta|^2$, for all $\zeta \in \mathbb{C}$

where

$$D^2 \tilde{\beta} = \begin{pmatrix} c\partial_x(c\partial_x \tilde{\beta}) & c\partial_x(c\partial_y \tilde{\beta}) \\ c\partial_y(c\partial_x \tilde{\beta}) & c\partial_y(c\partial_y \tilde{\beta}) \end{pmatrix}.$$

Note that the last assertion of Assumption 2.2 expresses the pseudo-convexity condition for the function β . Then, we define $\beta = \tilde{\beta} + K$ with $K = m\|\tilde{\beta}\|_\infty$ and $m > 1$. For $\lambda > 0$ and $t \in (-T, T)$, we define the following weight functions

$$(2.1) \quad \varphi(x, y, t) = \frac{e^{\lambda\beta(x, y)}}{(T+t)(T-t)}, \quad \eta(x, y, t) = \frac{e^{2\lambda K} - e^{\lambda\beta(x, y)}}{(T+t)(T-t)}.$$

Let H be the operator defined by

$$(2.2) \quad Hq := i\partial_t q + \nabla \cdot (c(x, y)\nabla q) \text{ in } \tilde{Q} = \Omega \times (-T, T).$$

We set $\psi = e^{-s\eta}q$, $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ for $s > 0$ and we introduce the following operators

$$(2.3) \quad M_1\psi := i\partial_t\psi + \nabla \cdot (c\nabla\psi) + s^2c|\nabla\eta|^2\psi,$$

$$(2.4) \quad M_2\psi := is\partial_t\eta\psi + 2cs\nabla\eta \cdot \nabla\psi + s\nabla \cdot (c\nabla\eta)\psi.$$

Then the following result holds.

Theorem 2.3. *Let H , M_1 , M_2 be the operators defined respectively by (2.2), (2.3), (2.4). We assume that Assumptions 2.1 and 2.2 are satisfied. Then there exist $\lambda_0 > 0$, $s_0 > 0$ and a positive constant $C = C(\Omega, \Gamma, T)$ such that, for any $\lambda \geq \lambda_0$ and any $s \geq s_0$, the next inequality holds:*

$$\begin{aligned} & s^3\lambda^4 \int_{-T}^T \int_{\Omega} e^{-2s\eta} \varphi^3 |q|^2 dx dy dt + s\lambda \int_{-T}^T \int_{\Omega} e^{-2s\eta} \varphi |\nabla q|^2 dx dy dt + \|M_1(e^{-s\eta}q)\|_{L^2(\tilde{Q})}^2 \\ & + \|M_2(e^{-s\eta}q)\|_{L^2(\tilde{Q})}^2 \leq C \left[s\lambda \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi |\partial_\nu q|^2 \partial_\nu \beta d\sigma dt + \int_{-T}^T \int_{\Omega} e^{-2s\eta} |Hq|^2 dx dy dt \right], \end{aligned}$$

for all q satisfying $Hq \in L^2(\Omega \times (-T, T))$, $q \in L^2(-T, T; H_0^1(\Omega))$, $\partial_\nu q \in L^2(-T, T; L^2(\Gamma))$.

Proof. If we set $\psi = e^{-s\eta}q$, we calculate $M\psi = e^{-s\eta}H(e^{s\eta}\psi)$ and we obtain:

$$M\psi = M_1\psi + M_2\psi$$

with M_1 and M_2 defined respectively by (2.3) and (2.4). Then

$$(2.5) \quad \begin{aligned} \int_{-T}^T \int_{\Omega} |M\psi|^2 dx dy dt &= \int_{-T}^T \int_{\Omega} |M_1\psi|^2 dx dy dt + \int_{-T}^T \int_{\Omega} |M_2\psi|^2 dx dy dt \\ &+ 2\Re\left(\int_{-T}^T \int_{\Omega} M_1\psi \overline{M_2\psi} dx dy dt\right), \end{aligned}$$

where \bar{z} is the conjugate of z , $\Re(z)$ its real part and $\Im(z)$ its imaginary part. We have to compute the scalar product in (2.5)

$$\Re\left(\int_{-T}^T \int_{\Omega} M_1\psi \overline{M_2\psi} dx dy dt\right) = I_{11'} + I_{12'} + I_{13'} + I_{21'} + I_{22'} + I_{23'} + I_{31'} + I_{32'} + I_{33'}.$$

Then, we have

$$(2.6) \quad I_{11'} = \Re\left(\int_{-T}^T \int_{\Omega} (i\partial_t\psi)(-is\partial_t\eta\bar{\psi}) dx dy dt\right) = -\frac{s}{2} \int_{-T}^T \int_{\Omega} \partial_t\eta |\psi|^2 dx dy dt.$$

$$\begin{aligned} I_{12'} &= \Re\left(2is \int_{-T}^T \int_{\Omega} c \partial_t\psi \nabla\eta \cdot \nabla\bar{\psi} dx dy dt\right) \\ &= s\Im\left(\int_{-T}^T \int_{\Omega} c \nabla\eta \cdot \nabla\psi \partial_t\bar{\psi} dx dy dt\right) - s\Im\left(\int_{-T}^T \int_{\Omega} c \nabla\eta \cdot \nabla\bar{\psi} \partial_t\psi dx dy dt\right). \end{aligned}$$

After an integration by parts with respect to the space variable in the first integral and to the time variable in the second integral, we obtain

$$(2.7) I_{12'} = -s\Im \left(\int_{-T}^T \int_{\Omega} \nabla \cdot (c\nabla\eta) \psi \partial_t \bar{\psi} dx dy dt \right) + s\Im \left(\int_{-T}^T \int_{\Omega} c\psi \nabla(\partial_t\eta) \cdot \nabla \bar{\psi} dx dy dt \right).$$

$$(2.8) \quad I_{13'} = s\Im \left(\int_{-T}^T \int_{\Omega} \nabla \cdot (c\nabla\eta) \psi \partial_t \bar{\psi} dx dy dt \right).$$

Note that $I_{13'}$ vanishes with the first term of $I_{12'}$.

$$(2.9) \quad \begin{aligned} I_{21'} &= \Re \left(-is \int_{-T}^T \int_{\Omega} \partial_t \eta \bar{\psi} \nabla \cdot (c\nabla\psi) dx dy dt \right) \\ &= s\Im \left(\int_{-T}^T \int_{\Omega} c\psi \nabla(\partial_t\eta) \cdot \nabla \bar{\psi} dx dy dt \right). \end{aligned}$$

Using integrations by parts, we obtain

$$(2.10) \quad \begin{aligned} I_{22'} &= 2s\Re \left(\int_{-T}^T \int_{\Omega} c\nabla \cdot (c\nabla\psi) \nabla\eta \cdot \nabla \bar{\psi} dx dy dt \right) \\ &= -s\lambda \int_{-T}^T \int_{\Omega} \varphi |\nabla\psi|^2 (\nabla \cdot (c^2\nabla\beta) + \lambda c^2 |\nabla\beta|^2) dx dy dt \\ &\quad + s \int_{-T}^T \int_{\partial\Omega} c^2 \partial_\nu \eta |\partial_\nu \psi|^2 d\sigma dt + 2s\lambda^2 \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla\beta \cdot \nabla\psi|^2 dx dy dt \\ &\quad + 2s\lambda \Re \left(\int_{-T}^T \int_{\Omega} \varphi c \sum_{i,j=1}^2 \partial_{x_i} (c\partial_{x_j}\beta) \partial_{x_i}\psi \partial_{x_j}\bar{\psi} dx dy dt \right) \\ &= -s\lambda \int_{-T}^T \int_{\Omega} \varphi |\nabla\psi|^2 (\nabla \cdot (c^2\nabla\beta) + \lambda c^2 |\nabla\beta|^2) dx dy dt \\ &\quad - s\lambda \int_{-T}^T \int_{\partial\Omega} c^2 \varphi \partial_\nu \beta |\partial_\nu \psi|^2 d\sigma dt + 2s\lambda^2 \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla\beta \cdot \nabla\psi|^2 dx dy dt \\ &\quad + 2s\lambda \Re \left(\int_{-T}^T \int_{\Omega} \varphi D^2\beta (\nabla\psi, \nabla\bar{\psi}) dx dy dt \right). \end{aligned}$$

Using integration by parts, we obtain

$$(2.11) \quad \begin{aligned} I_{23'} &= s\Re \left(\int_{-T}^T \int_{\Omega} \nabla \cdot (c\nabla\psi) \nabla \cdot (c\nabla\eta) \bar{\psi} dx dy dt \right) \\ &= s\lambda \int_{-T}^T \int_{\Omega} c\varphi |\nabla\psi|^2 (\nabla \cdot (c\nabla\beta) + \lambda c |\nabla\beta|^2) dx dy dt \\ &\quad - \frac{s\lambda^2}{2} \int_{-T}^T \int_{\Omega} \varphi \nabla \cdot (c\nabla \cdot (c\nabla\beta) \nabla\beta) |\psi|^2 dx dy dt \\ &\quad - \frac{s\lambda^3}{2} \int_{-T}^T \int_{\Omega} \varphi c |\nabla\beta|^2 \nabla \cdot (c\nabla\beta) |\psi|^2 dx dy dt \\ &\quad - \frac{s\lambda^4}{2} \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla\beta|^4 |\psi|^2 dx dy dt - \frac{s\lambda^3}{2} \int_{-T}^T \int_{\Omega} \varphi \nabla \cdot (c^2 |\nabla\beta|^2 \nabla\beta) |\psi|^2 dx dy dt \\ &\quad - \frac{s\lambda^2}{2} \int_{-T}^T \int_{\Omega} \varphi c \nabla\beta \cdot \nabla (\nabla \cdot (c\nabla\beta) + \lambda c |\nabla\beta|^2) |\psi|^2 dx dy dt \\ &\quad - \frac{s\lambda}{2} \int_{-T}^T \int_{\Omega} \varphi \nabla \cdot (c\nabla (\nabla \cdot (c\nabla\beta) + \lambda c |\nabla\beta|^2)) |\psi|^2 dx dy dt. \end{aligned}$$

And we obviously have

$$(2.12) \quad I_{31'} = s^3 \Re \left(\int_{-T}^T \int_{\Omega} c (-i \partial_t \eta \bar{\psi}) |\nabla \eta|^2 \psi \, dx \, dy \, dt \right) = 0.$$

$$(2.13) \quad \begin{aligned} I_{32'} &= 2s^3 \Re \left(\int_{-T}^T \int_{\Omega} c^2 |\nabla \eta|^2 \psi \, \nabla \eta \cdot \nabla \bar{\psi} \, dx \, dy \, dt \right) \\ &= s^3 \int_{-T}^T \int_{\Omega} c^2 |\nabla \eta|^2 \nabla \eta \cdot \nabla |\psi|^2 \, dx \, dy \, dt \\ &= s^3 \lambda^3 \int_{-T}^T \int_{\Omega} c \varphi^3 (|\nabla \beta|^2 \nabla \cdot (c \nabla \beta) + \nabla \beta \cdot \nabla (c |\nabla \beta|^2)) |\psi|^2 \, dx \, dy \, dt \\ &+ 3s^3 \lambda^4 \int_{-T}^T \int_{\Omega} c^2 \varphi^3 |\nabla \beta|^4 |\psi|^2 \, dx \, dy \, dt. \end{aligned}$$

$$(2.14) \quad \begin{aligned} I_{33'} &= s^3 \Re \left(\int_{-T}^T \int_{\Omega} c |\nabla \eta|^2 \psi \nabla \cdot (c \nabla \eta) \bar{\psi} \, dx \, dy \, dt \right) \\ &= -s^3 \lambda^3 \int_{-T}^T \int_{\Omega} c \varphi^3 |\nabla \beta|^2 \nabla \cdot (c \nabla \beta) |\psi|^2 \, dx \, dy \, dt \\ &- s^3 \lambda^4 \int_{-T}^T \int_{\Omega} c^2 \varphi^3 |\nabla \beta|^4 |\psi|^2 \, dx \, dy \, dt. \end{aligned}$$

So, by (2.6)-(2.14), we get :

$$\begin{aligned} \Re \left(\int_{-T}^T \int_{\Omega} M_1 \psi \overline{M_2 \psi} \, dx \, dy \, dt \right) &= 2s\lambda^2 \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla \beta \cdot \nabla \psi|^2 \, dx \, dy \, dt \\ &- s\lambda \int_{-T}^T \int_{\Omega} \varphi c \nabla c \cdot \nabla \beta |\nabla \psi|^2 \, dx \, dy \, dt \\ &+ 2s\lambda \Re \left(\int_{-T}^T \int_{\Omega} \varphi D^2 \beta (\nabla \psi, \nabla \bar{\psi}) \, dx \, dy \, dt \right) \\ &+ 2s^3 \lambda^4 \int_{-T}^T \int_{\Omega} c^2 \varphi^3 |\nabla \beta|^4 |\psi|^2 \, dx \, dy \, dt \\ &- s\lambda \int_{-T}^T \int_{\partial \Omega} c^2 \varphi \partial_\nu \beta |\partial_\nu \psi|^2 \, d\sigma \, dt + X, \end{aligned}$$

where

$$\begin{aligned} X &= \frac{-s}{2} \int_{-T}^T \int_{\Omega} \partial_{tt} \eta |\psi|^2 \, dx \, dy \, dt + 2s\Im \int_{-T}^T \int_{\Omega} c \psi \nabla (\partial_t \eta) \cdot \nabla \bar{\psi} \, dx \, dy \, dt \\ &- \frac{s\lambda^2}{2} \int_{-T}^T \int_{\Omega} \varphi \nabla \cdot (c \nabla \cdot (c \nabla \beta) \nabla \beta) |\psi|^2 \, dx \, dy \, dt \\ &- \frac{s\lambda^3}{2} \int_{-T}^T \int_{\Omega} \varphi c |\nabla \beta|^2 \nabla \cdot (c \nabla \beta) |\psi|^2 \, dx \, dy \, dt \\ &- \frac{s\lambda^4}{2} \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla \beta|^4 |\psi|^2 \, dx \, dy \, dt - \frac{s\lambda^3}{2} \int_{-T}^T \int_{\Omega} \varphi \nabla \cdot (c^2 |\nabla \beta|^2 \nabla \beta) |\psi|^2 \, dx \, dy \, dt \\ &- \frac{s\lambda^2}{2} \int_{-T}^T \int_{\Omega} \varphi c \nabla \beta \cdot \nabla (\nabla \cdot (c \nabla \beta) + \lambda c |\nabla \beta|^2) |\psi|^2 \, dx \, dy \, dt \\ &- \frac{s\lambda}{2} \int_{-T}^T \int_{\Omega} \varphi \nabla \cdot (c \nabla (\nabla \cdot (c \nabla \beta) + \lambda c |\nabla \beta|^2)) |\psi|^2 \, dx \, dy \, dt \\ &+ s^3 \lambda^3 \int_{-T}^T \int_{\Omega} \varphi^3 c \nabla \beta \cdot \nabla (c |\nabla \beta|^2) |\psi|^2 \, dx \, dy \, dt. \end{aligned}$$

Recall that:

$$\|M\psi\|_{L^2(\tilde{Q})}^2 = \|M_1\psi\|_{L^2(\tilde{Q})}^2 + \|M_2\psi\|_{L^2(\tilde{Q})}^2 + 2\Re(M_1\psi, \overline{M_2\psi}).$$

Then :

$$(2.15) \quad \begin{aligned} & \|M_1\psi\|_{L^2(\tilde{Q})}^2 + \|M_2\psi\|_{L^2(\tilde{Q})}^2 + 4s\lambda^2 \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla\beta \cdot \nabla\psi|^2 dx dy dt \\ & + 4s\lambda \Re \left(\int_{-T}^T \int_{\Omega} \varphi D^2\beta(\nabla\psi, \nabla\bar{\psi}) dx dy dt \right) - 2s\lambda \int_{-T}^T \int_{\Omega} \varphi c \nabla c \cdot \nabla\beta |\nabla\psi|^2 dx dy dt \\ & + 4s^3\lambda^4 \int_{-T}^T \int_{\Omega} c^2 \varphi^3 |\nabla\beta|^4 |\psi|^2 dx dy dt - 2s\lambda \int_{-T}^T \int_{\partial\Omega} c^2 \varphi \partial_\nu \beta |\partial_\nu \psi|^2 d\sigma dt \leq \|M\psi\|_{L^2(\tilde{Q})}^2 + 2|X|. \end{aligned}$$

Taking into account

- $|\tilde{\beta}| + |\nabla\tilde{\beta}| + |\nabla(\nabla \cdot (c\nabla\tilde{\beta}))| + |\nabla \cdot (\nabla(\nabla \cdot (c\nabla\tilde{\beta})))| \leq C(\tilde{C}_0, C_1)$ in Ω ,
- $|\partial_{tt}\eta| \leq C(T)\varphi^3$, $|\partial_t\varphi| \leq C(T)\varphi^2$, $\varphi \leq C(T)\varphi^3$, $\varphi^2 \leq C(T)\varphi^3$,
- $|s\Im(\int_{-T}^T \int_{\Omega} c\psi\nabla(\partial_t\eta) \cdot \nabla\bar{\psi} dx dy dt)| \leq C(T)s\lambda \int_{-T}^T \int_{\Omega} c\varphi |\nabla\beta \cdot \nabla\psi|^2 dx dy dt$
 $+ C(T)s\lambda \int_{-T}^T \int_{\Omega} c\varphi^3 |\psi|^2 dx dy dt$,

where $C(\tilde{C}_0, C_1)$ is a positive constant depending upon \tilde{C}_0, C_1 and $C(T)$ is a positive constant depending upon T . Therefore we obtain the following estimation for X (with $C(\tilde{C}_0, C_1, T)$ a positive constant depending upon \tilde{C}_0, C_1, T):

$$|X| \leq C(\tilde{C}_0, C_1, T) \left[(s\lambda^4 + s^3\lambda^3) \int_{-T}^T \int_{\Omega} \varphi^3 |\psi|^2 dx dy dt + s\lambda \int_{-T}^T \int_{\Omega} \varphi |\nabla\beta \cdot \nabla\psi|^2 dx dy dt \right].$$

The two terms of the previous estimate of $|X|$ are neglectable with respect to

$$s^3\lambda^4 \int_{-T}^T \int_{\Omega} c^2 \varphi^3 |\nabla\beta|^4 |\psi|^2 dx dy dt \quad \text{or} \quad s\lambda^2 \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla\beta \cdot \nabla\psi|^2 dx dy dt,$$

for s and λ sufficiently large. Using Assumption 2.2, we have

$$\begin{aligned} & 4s\lambda \Re \left(\int_{-T}^T \int_{\Omega} \varphi D^2\beta(\nabla\psi, \nabla\bar{\psi}) dx dy dt \right) - 2s\lambda \int_{-T}^T \int_{\Omega} \varphi c \nabla c \cdot \nabla\beta |\nabla\psi|^2 dx dy dt \\ & + 4s\lambda^2 \int_{-T}^T \int_{\Omega} \varphi c^2 |\nabla\beta \cdot \nabla\psi|^2 dx dy dt \geq C_{pc}s\lambda \int_{-T}^T \int_{\Omega} \varphi |\nabla\psi|^2 dx dy dt, \end{aligned}$$

so (2.15) becomes

$$\begin{aligned} & \|M_1\psi\|_{L^2(\tilde{Q})}^2 + \|M_2\psi\|_{L^2(\tilde{Q})}^2 + 4s^3\lambda^4 \int_{-T}^T \int_{\Omega} c^2 \varphi^3 |\nabla\beta|^4 |\psi|^2 dx dy dt \\ & + s\lambda \int_{-T}^T \int_{\Omega} \varphi |\nabla\psi|^2 dx dy dt \leq 2s\lambda \int_{-T}^T \int_{\partial\Omega} c^2 \varphi \partial_\nu \beta |\partial_\nu \psi|^2 d\sigma dt + \|M\psi\|_{L^2(\tilde{Q})}^2. \end{aligned}$$

Recall that $\partial_\nu \beta \leq 0$ on Γ^- , $c(x, y) \geq c_{min} > 0$ in $\bar{\Omega}$, $|\nabla\beta| \geq C_0 > 0$ and $\psi = e^{-s\eta}q$, then the proof is complete. \square

3 Inverse Problem

In this section, we establish a stability inequality and deduce a uniqueness result for the coefficient c . The Carleman estimate (2.5) proved in section 2 will be the key ingredient in the proof of such a stability estimate.

Let q be solution of

$$(3.1) \quad \begin{cases} i\partial_t q + \nabla \cdot (c\nabla q) = 0 & \text{in } \Omega \times (0, T), \\ q(x, y, t) = b(x, y, t) & \text{on } \partial\Omega \times (0, T), \\ q(x, y, 0) = q_0(x, y) & \text{in } \Omega, \end{cases}$$

and \tilde{q} be solution of

$$(3.2) \quad \begin{cases} i\partial_t \tilde{q} + \nabla \cdot (\tilde{c}\nabla \tilde{q}) = 0 & \text{in } \Omega \times (0, T), \\ \tilde{q}(x, y, t) = b(x, y, t) & \text{on } \partial\Omega \times (0, T), \\ \tilde{q}(x, y, 0) = q_0(x, y) & \text{in } \Omega, \end{cases}$$

where c and \tilde{c} both satisfy Assumption 2.1. If we set $u = q - \tilde{q}$, $v = \partial_t u$ and $\gamma = \tilde{c} - c$, then u and v satisfy respectively

$$(3.3) \quad \begin{cases} i\partial_t u + \nabla \cdot (c\nabla u) = \nabla \cdot (\gamma\nabla \tilde{q}) & \text{in } \Omega \times (0, T), \\ u(x, y, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, y, 0) = 0 & \text{in } \Omega, \end{cases}$$

$$(3.4) \quad \begin{cases} i\partial_t v + \nabla \cdot (c\nabla v) = \nabla \cdot (\gamma\nabla \partial_t \tilde{q}) = f & \text{in } \Omega \times (0, T), \\ v(x, y, t) = 0 & \text{on } \partial\Omega \times (0, T), \\ v(x, y, 0) = \frac{1}{i}\nabla \cdot (\gamma\nabla q_0) & \text{in } \Omega. \end{cases}$$

We extend the function v on $\Omega \times (-T, T)$ by the formula $v(x, y, t) = v(x, y, -t)$ for every $(x, y, t) \in \Omega \times (-T, 0)$.

3.1 Energy Estimate

We assume throughout this section that $\gamma \in H_0^1(\Omega)$. We introduce

$$(3.5) \quad \mathbb{E}(t) = \int_{\Omega} e^{-2s\eta(t)} |\partial_t u(t)|^2 dx dy + \int_{\Omega} \varphi^{-1}(t) e^{-2s\eta(t)} |\partial_t \nabla u(t)|^2 dx dy.$$

In this section, we will give an estimation of $\mathbb{E}(0)$.

First Step: We first give an estimation of $\int_{\Omega} e^{-2s\eta(0)} |\partial_t u(0)|^2 dx dy$.

We set $\psi = e^{-s\eta} v$. With the operator

$$(3.6) \quad M_1 \psi = i\partial_t \psi + \nabla \cdot (c\nabla \psi) + s^2 |\nabla \eta|^2 \psi,$$

we introduce, following [4],

$$\mathcal{I} = 2\Im \left(\int_{-T}^0 \int_{\Omega} M_1 \psi \bar{\psi} dx dy dt \right).$$

Assumption 3.1. $\partial_t \tilde{q}$, $\nabla(\partial_t \tilde{q})$, $\Delta(\partial_t \tilde{q})$ are bounded by a positive constant.

We have the following estimate

Lemma 3.2. *We assume that Assumption 3.1 is satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T)$ such that for any $\lambda \geq \lambda_0$ and $s \geq s_0$, we have*

$$\mathcal{I} = \int_{\Omega} e^{-2s\eta(x, y, 0)} |\partial_t u(x, y, 0)|^2 dx dy$$

and

$$\begin{aligned} |\mathcal{I}| &\leq Cs^{-3/2} \lambda^{-2} \int_{\Omega} e^{-2s\eta(x, y, 0)} (|\gamma|^2 + |\nabla \gamma|^2) dx dy \\ &\quad + Cs^{-1/2} \lambda^{-1} \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_{\nu} v|^2 d\sigma dt. \end{aligned}$$

Proof. In a first step, we calculate \mathcal{I}

$$\begin{aligned}
\mathcal{I} &= 2\Im \left(\int_{-T}^0 \int_{\Omega} (i\partial_t \psi \bar{\psi} + \nabla \cdot (c\nabla \psi) \bar{\psi} + s^2 |\nabla \eta|^2 \psi \bar{\psi}) dx dy dt \right) \\
&= 2\Re \left(\int_{-T}^0 \int_{\Omega} \partial_t \psi \bar{\psi} dx dy dt \right) \\
&= \int_{-T}^0 \int_{\Omega} \partial_t |\psi|^2 dx dy dt \\
&= \int_{\Omega} |\psi(x, y, 0)|^2 dx dy \\
&= \int_{\Omega} e^{-2s\eta(x, y, 0)} |v(x, y, 0)|^2 dx dy.
\end{aligned}$$

So, we have

$$\mathcal{I} = \int_{\Omega} e^{-2s\eta(x, y, 0)} |\partial_t u(x, y, 0)|^2 dx dy.$$

In a second step, we estimate \mathcal{I} . Using Young inequality we can write

$$\begin{aligned}
|\mathcal{I}| &\leq 2 \left(\int_{-T}^T \int_{\Omega} |M_1 \psi|^2 dx dy dt \right)^{\frac{1}{2}} \left(\int_{-T}^T \int_{\Omega} e^{-2s\eta} |v|^2 dx dy dt \right)^{\frac{1}{2}} \\
&\leq C(T) s^{-\frac{3}{2}} \lambda^{-2} \left(\|M_1 \psi\|_{L^2(\bar{Q})}^2 + s^3 \lambda^4 \int_{-T}^T \int_{\Omega} e^{-2s\eta} \varphi^3 |v|^2 dx dy dt \right)
\end{aligned}$$

with $C(T)$ a positive constant which depends on T . Then with the Carleman estimate (2.5) proved in section 2 we have

$$|\mathcal{I}| \leq C s^{-\frac{3}{2}} \lambda^{-2} \left(s \lambda \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_n v|^2 d\sigma dt + \int_{-T}^T \int_{\Omega} e^{-2s\eta} |\nabla \cdot (\gamma \nabla \partial_t \tilde{q})| dx dy dt \right),$$

where $C = C(\Omega, \Gamma^+, T)$ is a positive constant. Using Assumption 3.1, since

$$e^{-2s\eta(x, y, t)} \leq e^{-2s\eta(x, y, 0)} \quad \text{for all } t \in (-T, T),$$

we obtain for s and λ sufficiently large the estimate

$$\begin{aligned}
(3.7) \quad |\mathcal{I}| &\leq C s^{-3/2} \lambda^{-2} \int_{\Omega} e^{-2s\eta(x, y, 0)} (|\gamma|^2 + |\nabla \gamma|^2) dx dy \\
&\quad + C s^{-1/2} \lambda^{-1} \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_{\nu} v|^2 d\sigma dt.
\end{aligned}$$

where $C = C(\Omega, \Gamma^+, T)$ is a positive constant. □

Second Step: We then give an estimate of $\int_{\Omega} \varphi^{-1}(0) e^{-2s\eta(0)} |\partial_t \nabla u(0)|^2 dx dy$.

We denote

$$(3.8) \quad E(t) := \int_{\Omega} c \varphi^{-1}(t) e^{-2s\eta(t)} |\nabla v(t)|^2 dx dy,$$

where $\varphi^{-1} = \frac{1}{\varphi}$. We give an estimate for $E(0)$ in Theorem 3.4. In a first step we prove the following lemma :

Lemma 3.3. *Let v be solution of (3.4) in the following class*

$$v \in C([0, T], H^1(\Omega)), \quad \partial_{\nu} v \in L^2(0, T, L^2(\Gamma)).$$

Then the following identity holds true

$$E(\tau) - E(\kappa) = -2\Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} dx dy dt \right)$$

$$\begin{aligned}
& +\Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} (-4s\lambda + 2\lambda\varphi^{-1}) \partial_t \bar{v} \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right) \\
& -2s \int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \varphi^{-1} \partial_t \eta |\nabla v|^2 \, dx \, dy \, dt + \int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \partial_t (\varphi^{-1}) |\nabla v|^2,
\end{aligned}$$

for $f \in H_0^1(\Omega)$.

Proof. Since v is solution of (3.4) note that $\partial_t \bar{v} = i\bar{f} - i\nabla \cdot (c\nabla \bar{v})$. Therefore, we obtain the two following equalities.

$$(3.9) \quad \Re \int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} \, dx \, dy \, dt = \Re \left(-i \int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \nabla \cdot (c \nabla \bar{v}) \, dx \, dy \, dt \right),$$

$$(3.10) \quad
\begin{aligned}
& \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \partial_t \bar{v} (-4s\lambda + 2\lambda\varphi^{-1}) \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right) \\
& = \Re \left(i \int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \bar{f} (-4s\lambda + 2\lambda\varphi^{-1}) \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right) \\
& - \Re \left(i \int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} (-4s\lambda + 2\lambda\varphi^{-1}) \nabla \cdot (c \nabla \bar{v}) \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right).
\end{aligned}$$

We multiply the first equation of (3.4) by $e^{-2s\eta} \varphi^{-1} \partial_t \bar{v}$ and we integrate on $(\kappa, \tau) \times \Omega$, where $[\kappa, \tau] \subset [0, T]$. So, if we consider the real part of the obtained equality, we have

$$\begin{aligned}
0 & = \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} \varphi^{-1} |\partial_t v|^2 \, dx \, dy \, dt \right) = \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} \, dx \, dy \, dt \right) \\
& - \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} \varphi^{-1} \nabla \cdot (c \nabla v) \partial_t \bar{v} \, dx \, dy \, dt \right).
\end{aligned}$$

Then by integration by parts, we obtain

$$\begin{aligned}
0 & = \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} \, dx \, dy \, dt \right) + \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c \nabla v \cdot \nabla (e^{-2s\eta} \varphi^{-1} \partial_t \bar{v}) \, dx \, dy \, dt \right) \\
0 & = \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} \, dx \, dy \, dt \right) - 2s \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c \varphi^{-1} e^{-2s\eta} \partial_t \bar{v} \nabla v \cdot \nabla \eta \, dx \, dy \, dt \right) \\
& - \lambda \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \varphi^{-1} \partial_t \bar{v} \nabla v \cdot \nabla \beta \, dx \, dy \, dt \right) + \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \varphi^{-1} \nabla v \cdot \nabla (\partial_t \bar{v}) \, dx \, dy \, dt \right).
\end{aligned}$$

Note that

$$E(\tau) - E(\kappa) = \int_{\Omega} \int_{\kappa}^{\tau} c \partial_t (e^{-2s\eta} \varphi^{-1} |\nabla v|^2) \, dx \, dy \, dt.$$

Therefore we have

$$\begin{aligned}
0 & = \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} \, dx \, dy \, dt \right) - 2s \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c \varphi^{-1} e^{-2s\eta} \partial_t \bar{v} \nabla v \cdot \nabla \eta \, dx \, dy \, dt \right) \\
& - \lambda \Re \left(\int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \varphi^{-1} \partial_t \bar{v} \nabla v \cdot \nabla \beta \, dx \, dy \, dt \right) + \frac{1}{2} E(\tau) - \frac{1}{2} E(\kappa) \\
& + s \int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \varphi^{-1} \partial_t \eta |\nabla v|^2 \, dx \, dy \, dt - \frac{1}{2} \int_{\kappa}^{\tau} \int_{\Omega} c e^{-2s\eta} \partial_t (\varphi^{-1}) |\nabla v|^2 \, dx \, dy \, dt,
\end{aligned}$$

and the proof of Lemma 3.3 is complete. \square

Theorem 3.4. *Let v be solution of (3.4) in the following class*

$$v \in C([0, T], H^1(\Omega)), \partial_\nu v \in L^2(0, T, L^2(\Gamma)).$$

We assume that Assumptions 2.1 and 2.2 are checked. Then there exists a positive constant $C = C(\Omega, \Gamma, T) > 0$ such that

$$(3.11) \quad E(0) \leq C \left[s^2 \lambda^2 \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_\nu \beta |\partial_\nu v|^2 d\sigma dt + s\lambda \int \int_Q e^{-2s\eta} |f|^2 \right]$$

for s and λ sufficiently large.

Proof. We apply Lemma 3.3 with $\kappa = 0$ and $\tau = T$. Since $E(T) = 0$, we obtain

$$E(0) = 2\Re \left(\int_0^T \int_\Omega e^{-2s\eta} f \varphi^{-1} \partial_t \bar{v} dx dy dt \right) - \Re \left(\int_0^T \int_\Omega c e^{-2s\eta} (-4s\lambda + 2\lambda\varphi^{-1}) \partial_t \bar{v} \nabla \beta \cdot \nabla v dx dy dt \right) \\ + 2s \int_0^T \int_\Omega c e^{-2s\eta} \varphi^{-1} \partial_t \eta |\nabla v|^2 dx dy dt - \int_0^T \int_\Omega c e^{-2s\eta} \partial_t (\varphi^{-1}) |\nabla v|^2.$$

We give now estimates of the four integrals in the previous equality.

First integral: $B := 2\Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f \partial_t \bar{v} dx dy dt \right).$

Using (3.9), we have :

$$(3.12) \quad B = 2\Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f \partial_t \bar{v} dx dy dt \right) = \Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f \partial_t \bar{v} dx dy dt \right) \\ + \Re \left(-i \int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f \nabla \cdot (c \nabla \bar{v}) dx dy dt \right).$$

Recall that if we set $\psi = e^{-s\eta} v$, then $M\psi = e^{-s\eta} H(e^{s\eta} \psi) = M_1 \psi + M_2 \psi$ for $s > 0$ with

$$M_1 \psi := i \partial_t \psi + \nabla \cdot (c \nabla \psi) + s^2 c |\nabla \eta|^2 \psi,$$

$$M_2 \psi := i s \partial_t \eta \psi + 2cs \nabla \eta \cdot \nabla \psi + s \nabla \cdot (c \nabla \eta) \psi.$$

So (3.12) becomes

$$B = 2\Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f \partial_t \bar{v} dx dy dt \right) = \Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} (s \partial_t \eta \bar{\psi} + \partial_t \bar{\psi}) dx dy dt \right) \\ + \Re \left(-i \int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} (s \nabla \cdot (c \nabla \eta) \bar{\psi} + c s^2 |\nabla \eta|^2 \bar{\psi} + 2c s \nabla \eta \cdot \nabla \bar{\psi} + \nabla \cdot (c \nabla \eta) \bar{\psi}) dx dy dt \right) \\ = \Re \left(-i \int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} (i \partial_t \bar{\psi} + \nabla \cdot (c \nabla \bar{\psi})) dx dy dt \right) \\ + \Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} (s \partial_t \eta - i s \nabla \cdot (c \nabla \eta) - i c s^2 |\nabla \eta|^2) \bar{\psi} dx dy dt \right) \\ + \Re \left(-i \int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} 2c s \nabla \eta \cdot \nabla \bar{\psi} dx dy dt \right)$$

Note that

$$i \partial_t \bar{\psi} + \nabla \cdot (c \nabla \bar{\psi}) = M_1 \bar{\psi} - s^2 c |\nabla \eta|^2 \bar{\psi}.$$

Then, we obtain

$$B = \Re \left(-i \int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} (M_1 \bar{\psi} - s^2 c |\nabla \eta|^2 \bar{\psi}) dx dy dt \right) \\ + \Re \left(\int_0^T \int_\Omega e^{-2s\eta} \varphi^{-1} f e^{s\eta} (s \partial_t \eta - i s \nabla \cdot (c \nabla \eta) - i c s^2 |\nabla \eta|^2) \bar{\psi} dx dy dt \right)$$

$$+ \Re \left(-i \int_0^T \int_{\Omega} e^{-2s\eta} \varphi^{-1} f e^{s\eta} 2c s \nabla \eta \cdot \nabla \bar{\psi} \, dx \, dy \, dt \right).$$

If we come back to the function v , the previous equality becomes :

$$\begin{aligned} B &= \Re \left(-i \int_0^T \int_{\Omega} (f e^{-s\eta} \varphi^{-1} M_1(e^{-s\eta} \bar{v}) - s^2 c |\nabla \eta|^2 e^{-2s\eta} \varphi^{-1} f \bar{v}) \, dx \, dy \, dt \right) \\ &+ \Re \left(\int_0^T \int_{\Omega} e^{-2s\eta} \varphi^{-1} f (s \partial_t \eta - i s \nabla \cdot (c \nabla \eta) + i c s^2 |\nabla \eta|^2) \bar{v} \, dx \, dy \, dt \right) \\ &+ \Re \left(-i \int_0^T \int_{\Omega} e^{-2s\eta} \varphi^{-1} f 2c s \nabla \eta \cdot \nabla \bar{v} \, dx \, dy \, dt \right). \end{aligned}$$

Then there exists a positive constant $C = C(\Omega, \Gamma, T)$ such that:

$$(3.13) \quad \left| \int_0^T \int_{\Omega} e^{-2s\eta} \varphi^{-1} f \partial_t \bar{v} \, dx \, dy \, dt \right| \leq C \left[s\lambda \int \int_Q e^{-2s\eta} |f|^2 \, dx \, dy \, dt + \|M_1(e^{-s\eta} \bar{v})\|_{L^2(Q)}^2 \right. \\ \left. + s^3 \lambda^4 \int \int_Q e^{-2s\eta} \varphi^2 |v|^2 \, dx \, dy \, dt + s\lambda \int \int_Q e^{-2s\eta} |\nabla v|^2 \, dx \, dy \, dt \right].$$

Second integral: $D := \Re \left(\int_0^T \int_{\Omega} c (-4s\lambda + 2\lambda\varphi^{-1}) e^{-2s\eta} (1 + \varphi^{-1}) \partial_t \bar{v} \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right)$.

We denote by $\rho := -4s\lambda + 2\lambda\varphi^{-1}$. Using (3.10), we have

$$\begin{aligned} 2D &= \Re \left(\int_0^T \int_{\Omega} c e^{-2s\eta} \rho \partial_t \bar{v} \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right) + \Re \left(i \int_0^T \int_{\Omega} c e^{-2s\eta} \bar{f} \rho \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right) \\ &- \Re \left(i \int_0^T \int_{\Omega} c e^{-2s\eta} \rho \nabla \cdot (c \nabla \bar{v}) \nabla \beta \cdot \nabla v \, dx \, dy \, dt \right). \end{aligned}$$

If we introduce $\psi = e^{-s\eta} v$, we get:

$$\begin{aligned} 2D &= \Re \left(\int_0^T \int_{\Omega} c e^{-2s\eta} \rho \nabla v \cdot \nabla \beta (s \partial_t \eta e^{s\eta} \bar{\psi} + e^{s\eta} \partial_t \bar{\psi}) \, dx \, dy \, dt \right) + \Re \left(i \int_0^T \int_{\Omega} c e^{-2s\eta} \nabla v \cdot \nabla \beta \bar{f} \, dx \, dy \, dt \right) \\ &- \Re \left(i \int_0^T \int_{\Omega} c e^{-2s\eta} \rho \nabla v \cdot \nabla \beta [s^2 c \bar{\psi} |\nabla \eta|^2 e^{s\eta} + 2s c e^{s\eta} \nabla \bar{\psi} \cdot \nabla \eta + s e^{s\eta} \bar{\psi} \nabla \cdot (c \nabla \eta) + e^{s\eta} \nabla \cdot (c \nabla \bar{\psi})] \right). \end{aligned}$$

Therefore

$$2D = \Re \left(-i \int_0^T \int_{\Omega} c e^{-s\eta} \rho \nabla v \cdot \nabla \beta [M_1 \bar{\psi} + M_2 \bar{\psi}] \, dx \, dy \, dt \right) + \Re \left(i \int_0^T \int_{\Omega} c e^{-2s\eta} \rho \nabla v \cdot \nabla \beta \bar{f} \, dx \, dy \, dt \right).$$

Thus there exists a positive constant $C = C(\Omega, \Gamma, T)$ such that

$$(3.14) \quad |2D| \leq C \left[s\lambda \int \int_Q [|M_1(e^{-s\eta} \bar{v})|^2 + |M_2(e^{-s\eta} \bar{v})|^2] \, dx \, dy \, dt \right. \\ \left. + s\lambda \int \int_Q e^{-2s\eta} |\nabla v|^2 + s\lambda \int \int_Q e^{-2s\eta} |f|^2 \, dx \, dy \, dt \right].$$

Two last integrals:

There exists a positive constant $C = C(\Omega, \Gamma, T)$ such that

$$(3.15) \quad \left| \int_0^T \int_{\Omega} c e^{-2s\eta} \partial_t (\varphi^{-1}) |\nabla v|^2 \, dx \, dy \, dt \right| \leq C \int \int_Q e^{-2s\eta} |\nabla v|^2 \, dx \, dy \, dt,$$

and

$$(3.16) \quad \left| s \int_0^T \int_{\Omega} c \partial_t \eta e^{-2s\eta} \varphi^{-1} |\nabla v|^2 \, dx \, dy \, dt \right| \leq C s \int \int_Q \varphi e^{-2s\eta} |\nabla v|^2 \, dx \, dy \, dt.$$

Using now the Carleman estimate of Theorem 2.3 and Lemma 3.3, from (3.9)-(3.12), we deduce the existence of a positive constant $C = C(\Omega, \Gamma, T)$ such that:

$$E(0) \leq C \left[s^2 \lambda^2 \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu\beta} |\partial_{\nu} v|^2 d\sigma dt + s\lambda \int \int_Q e^{-2s\eta} |f|^2 dx dy dt \right],$$

and the proof is complete. \square

3.2 Stability Estimate

Now following an idea developed in [12] for Lamé system in bounded domains, we give an underestimate for $\mathbb{E}(0)$. We adapt the proof of lemma 3.2 of [12] to an unbounded domain.

Assumption 3.5. • q_0 is a real valued function in $C^3(\Omega)$

- q_0 and all its derivatives up to order three are bounded
- $|\nabla\beta \cdot \nabla q_0| \geq C > 0$ on Ω .

Lemma 3.6. We consider the first order partial differential operator

$$(P_0g)(x, y) = \partial_x q_0(x, y) \partial_x g(x, y) + \partial_y(x, y) \partial_y g(x, y), P_0g := \nabla q_0 \cdot \nabla g$$

where q_0 satisfies Assumption 3.5. Then there exist positive constants $\lambda_1 > 0$, $s_1 > 0$ and $C = C(\Omega, \Gamma, T)$ such that for all $\lambda \geq \lambda_1$ and $s \geq s_1$

$$s^2 \lambda^2 \int_{\Omega} \varphi_0 e^{-2s\eta_0} |g|^2 dx dy \leq C \int_{\Omega} \varphi_0^{-1} e^{-2s\eta_0} |P_0g|^2 dx dy$$

with $\eta_0(x, y) := \eta(x, y, 0)$, $\varphi_0(x, y) := \varphi(x, y, 0)$ and for $g \in H_0^1(\Omega)$.

Proof. Let $g \in H_0^1(\Omega)$. We denote by $w = e^{-s\eta_0} g$ with $\eta_0 := \eta(x, y, 0)$ and $Q_0w = e^{-s\eta_0} P_0(e^{s\eta_0} w)$, so we get $Q_0w = swP_0\eta_0 + P_0w$. Therefore, we have:

$$\begin{aligned} \int_{\Omega} \varphi_0^{-1} Q_0w \overline{Q_0w} dx dy &= s^2 \int_{\Omega} \varphi_0^{-1} |w|^2 |P_0\eta_0|^2 dx dy + \int_{\Omega} \varphi_0^{-1} |P_0w|^2 dx dy \\ &\quad + 2s\Re \left(\int_{\Omega} \varphi_0^{-1} w P_0\eta_0 \overline{P_0w} dx dy \right) \\ &= s^2 \lambda^2 \int_{\Omega} \varphi_0 |w|^2 (\nabla q_0 \cdot \nabla \beta)^2 dx dy + \int_{\Omega} \varphi_0^{-1} |P_0w|^2 dx dy - s\lambda \int_{\Omega} \nabla q_0 \cdot \nabla \beta \nabla q_0 \cdot \nabla (|w|^2) dx dy. \end{aligned}$$

So, integrating by parts, we obtain

$$\begin{aligned} \int_{\Omega} \varphi_0^{-1} Q_0w \overline{Q_0w} dx dy &= s^2 \lambda^2 \int_{\Omega} \varphi_0 |w|^2 (\nabla q_0 \cdot \nabla \beta)^2 dx dy + \int_{\Omega} \varphi_0^{-1} |P_0w|^2 dx dy \\ &\quad + s\lambda \int_{\Omega} \nabla \cdot (\nabla q_0 \cdot \nabla \beta \nabla q_0) |w|^2 dx dy. \end{aligned}$$

Thus

$$\int_{\Omega} \varphi_0^{-1} e^{-2s\eta_0} |P_0g|^2 dx dy \geq s^2 \lambda^2 \int_{\Omega} \varphi_0 |\nabla\beta \cdot \nabla q_0|^2 e^{-2s\eta_0} |g|^2 dx dy + s\lambda \int_{\Omega} \nabla \cdot (P_0\beta \nabla q_0) e^{-2s\eta_0} |g|^2 dx dy.$$

Using Assumptions 3.1 and 4.3, we can conclude for s and λ sufficiently large. \square

Then, we deduce the following result.

Lemma 3.7. *Let u be solution of (3.4). We assume that Assumptions 2.2, 3.1 and 3.5 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T)$ such that for s and λ sufficiently large, the two following estimates hold true*

$$(3.17) \quad s^2 \lambda^2 \int_{\Omega} \varphi_0 e^{-2s\eta_0} |\gamma|^2 dx dy \leq C \int_{\Omega} \varphi_0^{-1} e^{-2s\eta_0} |\partial_t u(x, y, 0)|^2 dx dy,$$

$$(3.18) \quad s^2 \lambda^2 \int_{\Omega} e^{-2s\eta_0} |\nabla \gamma|^2 dx dy \leq C \int_{\Omega} \varphi_0^{-1} e^{-2s\eta_0} (|\nabla(\partial_t u(x, y, 0))|^2 + |\gamma|^2) dx dy,$$

for $\gamma \in H_0^2(\Omega)$.

Proof. We apply Lemma 3.6 to the first order partial differential equations satisfied by

- γ given by the initial condition in (3.4)

$$P_0 \gamma := \partial_x q_0 \partial_x \gamma + \partial_y q_0 \partial_y \gamma = i \partial_t u(x, y, 0) - \gamma \Delta q_0,$$

- $\partial_x \gamma$ given by the x -derivative of the initial condition in (3.4)

$$\begin{aligned} P_0 \partial_x \gamma &:= \partial_x q_0 \partial_x (\partial_x \gamma) + \partial_y q_0 \partial_y (\partial_x \gamma) \\ &= i \partial_t (\partial_x u(x, y, 0)) - \partial_x \gamma (\Delta q_0 + \partial_{xx} q_0) - \partial_y \gamma \partial_{xy} q_0 - \gamma \partial_x (\Delta q_0), \end{aligned}$$

- $\partial_y \gamma$ given by the y -derivative of the initial condition in (3.4)

$$\begin{aligned} P_0 \partial_y \gamma &:= \partial_x q_0 \partial_x (\partial_y \gamma) + \partial_y q_0 \partial_y (\partial_y \gamma) \\ &= i \partial_t (\partial_y u(x, y, 0)) - \partial_y \gamma (\Delta q_0 + \partial_{yy} q_0) - \partial_x \gamma \partial_{xy} q_0 - \gamma \partial_y (\Delta q_0). \end{aligned}$$

Then using Lemma 3.6 and Assumption 3.5, the proof of Lemma 3.7 is complete. \square

Theorem 3.8. *Let q and \tilde{q} be solutions of (3.1) and (3.2) such that $c - \tilde{c} \in H_0^2(\Omega)$. We assume that Assumptions 2.1, 2.2, 3.1 and 3.5 are satisfied. Then there exists a positive constant $C = C(\Omega, \Gamma, T)$ such that for s and λ large enough,*

$$\int_{\Omega} \varphi_0 e^{-2s\eta_0} (|c - \tilde{c}|^2 + |\nabla(c - \tilde{c})|^2) dx dy \leq C \int_{-T}^T \int_{\Gamma^+} \varphi e^{-2s\eta} \partial_{\nu} \beta |\partial_{\nu} (\partial_t q - \partial_t \tilde{q})|^2 d\sigma dt.$$

Proof. Adding (3.17) and (3.18) we obtain (with C a positive constant) using the estimate (3.7) for $|\mathcal{I}|$ and the energy estimate (3.11) for $E(0)$

$$\begin{aligned} s^2 \lambda^2 \int_{\Omega} \varphi_0 e^{-2s\eta_0} (|\nabla \gamma|^2 + |\gamma|^2) dx dy &\leq C \int_{\Omega} \varphi_0^{-1} e^{-2s\eta_0} (|\nabla(\partial_t u(x, y, 0))|^2 + |\partial_t u(x, y, 0)|^2) dx dy \\ &\leq C(|\mathcal{I}| + E(0)) \\ &\leq C s^{-3/2} \lambda^{-2} \int_{\Omega} e^{-2s\eta_0} (|\gamma|^2 + |\nabla \gamma|^2) dx dy + C s^{-1/2} \lambda^{-1} \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_{\nu} v|^2 d\sigma dt \\ &\quad + C s^2 \lambda^2 \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_{\nu} v|^2 d\sigma dt + C s \lambda \int_Q e^{-2s\eta} |f|^2. \end{aligned}$$

So we get

$$\begin{aligned} s^2 \lambda^2 \int_{\Omega} \varphi_0 e^{-2s\eta_0} (|\nabla \gamma|^2 + |\gamma|^2) dx dy &\leq C s^2 \lambda^2 \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_{\nu} v|^2 d\sigma dt \\ &\quad + C s \lambda \int_Q e^{-2s\eta} |\nabla \cdot (\gamma \nabla \partial_t \tilde{q})|^2 dx dy dt \\ &\leq C s^2 \lambda^2 \int_{-T}^T \int_{\Gamma^+} e^{-2s\eta} \varphi \partial_{\nu} \beta |\partial_{\nu} v|^2 d\sigma dt + C s \lambda \int_Q e^{-2s\eta} (|\nabla \gamma|^2 + |\gamma|^2) dx dy dt. \end{aligned}$$

Then, for s and λ sufficiently large, the theorem is proved. \square

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