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Dynamical properties and characterization of gradient drift diffusions

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Abstract

We study the dynamical properties of the Brownian diffusions having σId as diffusion coefficient matrix and $b = \nabla U$ as drift vector. We characterize this class through the equality $D_+^2 = D_-^2$, where D_+ (resp. D_-) denotes the forward (resp. backward) stochastic derivative of Nelson's type. Our proof is based on a remarkable identity for $D_+^2 - D_-^2$ and on the use of the martingale problem. We also give a new formulation of a famous theorem of Kolmogorov concerning reversible diffusions. We finally relate our characterization to some questions about the complex stochastic embedding of the Newton equation which initially motivated of this work.

Key words: Gradient drift diffusion, Time reversal, Nelson stochastic derivatives, Kolmogorov theorem, Reversible diffusion, Stationary diffusion, Martingale problem

1 Introduction

For a general process $Z = (Z_t)_{t \in [0, T]}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have introduced in [3] the notion of *stochastic derivative* for Z at t with respect to a *differentiating* sub- σ -field \mathcal{A}^t of \mathcal{F} (resp. *forward differentiating*, *backward differentiating*). More precisely, it means that \mathcal{A}^t is such that the quantity

$$\mathbb{E} \left[\frac{Z_{t+h} - Z_t}{h} \middle| \mathcal{A}^t \right]$$

converges in probability (or for another topology) when $h \rightarrow 0$ (resp. $h \downarrow 0$, $h \uparrow 0$); the limit being called the stochastic derivatives of Z at t w.r.t. \mathcal{A}^t . When we consider Brownian diffusions of the form

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad t \in [0, T], \quad (1)$$

then, under suitable conditions, the σ -field \mathcal{F}_t^X generated by X_t is both a forward and backward differentiating σ -field for X at t . The associated derivatives are called *Nelson derivatives*, due to the Markov property of the diffusion and of its time reversal which allow to take the conditional expectation both with respect to the past \mathcal{P}_t^X and the future \mathcal{F}_t^X of the diffusion. For simplicity, we note them respectively D_+ and D_- in the sequel. Notice that these derivatives are relevant and natural quantities for Brownian diffusions: they are indeed respectively equals to the forward and the backward (up to sign) drift of X . Moreover, they exist under rather mild conditions, see *e.g.* [9,12].

We shall see that Nelson derivatives turn out to have remarkable properties when we work with diffusions of the type

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \sigma W_t, \quad t \in [0, T]. \quad (2)$$

Here, $\sigma \in \mathbb{R}$ is assumed to be constant. For instance, we shall show that the equalities $D_+ X_t = -D_- X_t$, $t \in (0, T)$, characterizes the class of stationary diffusions of the type (2) having moreover an homogeneous gradient drift (see Proposition 5). This statement is in fact quite easy to obtain. A more difficult one, which is the main result of this paper, states that a Brownian diffusion of the type (2) is a gradient diffusion - that is, its drift coefficient writes $b = \nabla_x U$ for a certain U - if and only if $D_+^2 X_t = D_-^2 X_t$ for any $t \in (0, T)$, see Theorem 6 for a precise statement. Let us notice that this result was conjectured at the end of the note [1]. Our proof is based on the discovery of a remarkable identity (Lemma 8): we can write the quantity $p_t(X_t)(D_+^2 X_t - D_-^2 X_t)$ as the divergence of a certain vector field, where p_t denotes the density of the law of X_t . Combined with the expression of the adjoint of the infinitesimal generator, we can then conclude using probabilistic arguments, especially the martingale problem. Let us moreover stress on the fact that we were able to solve our

problem with probabilistic tools, whereas its analytic transcription with the help of partial differential equations seemed more difficult to treat.

The paper is organized as follows. In section 2, we introduce some notations and we give the useful expressions of the Nelson derivatives under the conditions given by Millet, Nualart and Sanz in [9]. In section 3, we study the above mentioned characterizations and we prove our main result. In section 4, we make some remarks on the questions related to the complex stochastic embedding of the Newton equation, which have motivated this work.

2 Preliminaries on stochastic derivatives

2.1 Notations

Let $T > 0$ and $d \in \mathbb{N}^*$. The space \mathbb{R}^d is endowed with its canonical scalar product $\langle \cdot, \cdot \rangle$. Let $|\cdot|$ be the induced norm.

If $f : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a smooth function, we set $\partial_j f = \frac{\partial f}{\partial x_j}$. We denote by $\nabla f = (\partial_i f)_i$ the gradient of f and by $\Delta f = \sum_j \partial_j^2 f$ its Laplacian. For a smooth map $\Phi : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, we denote by Φ^j its j^{th} -component, by $(\partial_x \Phi)$ its differential which we represent into the canonical basis of \mathbb{R}^d : $(\partial_x \Phi) = (\partial_j \Phi^i)_{i,j}$, and by $\text{div} \Phi = \sum_j \partial_j \Phi^j$ its divergence. By convention, we denote by $\Delta \Phi$ the vector $(\Delta \Phi^j)_j$. The image of a vector $u \in \mathbb{R}^d$ under a linear map M is simply denoted by Mu , for instance $(\partial_x \phi)u$. The map $a : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is viewed as $d \times d$ matrices whose columns are denoted by a_k . Finally, we denote by $\text{div} a$ the vector $(\text{div} a_k)_k$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space on which is defined a d -dimensional Brownian motion W . For a process Z defined on $(\Omega, \mathcal{A}, \mathbb{P})$, we set \mathcal{P}_t^Z the σ -field generated by Z_s for $0 \leq s \leq t$ and \mathcal{F}_t^Z the σ -field generated by Z_s for $t \leq s \leq T$. Consider the d -dimensional diffusion process $X = (X_t)_{t \in [0, T]}$ solution of the stochastic differential equation (1) where $X_0 \in L^2(\Omega)$ is a random vector independent of W , and the functions $\sigma : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ are Lipschitz with linear growth. More precisely, we assume that σ and b satisfy the two following conditions: there exists a constant $K > 0$ such that, for all $x, y \in \mathbb{R}^d$, we have

$$\sup_{t \in [0, T]} \left[|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \right] \leq K |x - y|$$

and

$$\sup_{t \in [0, T]} \left[|b(t, x)| + |\sigma(t, x)| \right] \leq K(1 + |x|).$$

We moreover assume that b is differentiable w.r.t. x and we set $G = (\partial_x b) - (\partial_x b)^*$, i.e. $G_i^j = \partial_i b^j - \partial_j b^i$. Finally, we set $a = \sigma \sigma^*$, i.e. $a_i^j = \sum_k \sigma_i^k \sigma_j^k$.

In the sequel, we will work under the following assumption:

- (H) For any $t \in (0, T)$, the law of X_t admits a positive density $p_t : \mathbb{R}^d \rightarrow (0, +\infty)$ and we have, for any $t_0 \in (0, T)$:

$$\max_{j=1, \dots, n} \int_{t_0}^T \int_{\mathbb{R}^d} |\operatorname{div}(a_j(t, x)p_t(x))| dx dt < +\infty. \quad (3)$$

The functions

$$\frac{\operatorname{div}(a_j(t, \cdot)p_t(\cdot))}{p_t(\cdot)} \quad (4)$$

are Lipschitz.

The condition (3) will ensure us that the time reversed process $\overline{X}_t = X_{T-t}$ is again a diffusion process (see [9], Theorem 2.3). Let us moreover notice that our condition (4) is weaker than that which is imposed in Proposition 4.1 of [15]. Finally, let us remark that the positivity assumption made on p_t is quite weak when X is of the type (2): it is for instance automatically verified when we can apply Girsanov theorem in (2), that is when the Novikov condition is verified.

2.2 Stochastic derivatives of Nelson's type

In [3], we have introduced the notion of *differentiating* σ -field:

Definition 1 Set $t \in (0, T)$ and let Z be a process. We say that \mathcal{A}^t (resp. \mathcal{B}^t) is a forward differentiating σ -field (resp. backward differentiating σ -field) for Z at t if $E[\frac{Z_{t+h} - Z_t}{h} | \mathcal{A}^t]$ (resp. $E[\frac{Z_t - Z_{t-h}}{h} | \mathcal{B}^t]$) converges in probability when $h \downarrow 0$. In these cases, we define the so-called forward and backward derivatives

$$D_+^{\mathcal{A}^t} Z_t = \lim_{h \downarrow 0} E \left[\frac{Z_{t+h} - Z_t}{h} | \mathcal{A}^t \right], \quad (5)$$

$$D_-^{\mathcal{B}^t} Z_t = \lim_{h \downarrow 0} E \left[\frac{Z_t - Z_{t-h}}{h} | \mathcal{B}^t \right]. \quad (6)$$

For Brownian diffusions X of the form (1), the present turns out to be a forward and backward differentiating σ -field. Precisely, the σ -field \mathcal{F}_t^X generated by X_t is both forward and backward differentiating for X at t . Equivalently, due to the Markov property of X (resp. of its time reversal \overline{X}), \mathcal{P}_t^X (resp. $\mathcal{F}_t^{\overline{X}}$) is forward (resp. backward) differentiating for X at t . For this reason, we call

the derivatives defined by (5) and (6) stochastic derivatives of Nelson's type. Indeed, in [11] Nelson introduced the processes which have stochastic derivatives in $L^2(\Omega)$ with respect to a fixed filtration (\mathcal{P}_t) and a fixed decreasing filtration (\mathcal{F}_t) .

Henceforth, we work with the stochastic derivatives of Nelson's type for Brownian diffusions and so we simply write $D_{\pm}X$ instead of $D_{\pm}^{\mathcal{F}_t^X}X_t$. Now, we can relate the stochastic derivatives of Nelson's type to the time reversal theory:

Proposition 1 *Let X be given by (1) and satisfying assumption (H). Then X is a Markov diffusion w.r.t. the increasing filtration (\mathcal{P}_t^X) and the decreasing filtration (\mathcal{F}_t^X) . Moreover for almost all $t \in (0, T)$, \mathcal{F}_t^X is a forward and backward differentiating σ -field for X at t and*

$$D_+X_t = b(t, X_t) \tag{7}$$

$$D_-X_t = b(t, X_t) - \frac{\operatorname{div}(a(t, X_t)p_t(X_t))}{p_t(X_t)}. \tag{8}$$

Proof. The proof essentially uses Theorem 2.3 of Millet-Nualart-Sanz [9], and is divided in two steps:

1) X is a Markov diffusion w.r.t. the increasing filtration (\mathcal{P}_t^X) , so:

$$E \left[\frac{X_{t+h} - X_t}{h} \middle| \mathcal{P}_t^X \right] = E \left[\frac{1}{h} \int_t^{t+h} b(s, X_s) ds \middle| \mathcal{P}_t^X \right],$$

and

$$\begin{aligned} E \left| E \left[\frac{X_{t+h} - X_t}{h} \middle| \mathcal{P}_t^X \right] - b(t, X_t) \right| &\leq \frac{1}{h} E \int_t^{t+h} |b(s, X_s) - b(t, X_t)| ds \\ &= \frac{1}{h} \int_t^{t+h} E |b(s, X_s) - b(t, X_t)| ds. \end{aligned}$$

Using the fact that b is Lipschitz and that $t \mapsto E|X_t|$ is locally integrable (see, e.g., Theorem 2.9 in [7]), we can conclude by the differentiation Lebesgue theorem that for almost all $t \in (0, T)$:

$$\frac{1}{h} \int_t^{t+h} E |b(s, X_s) - b(t, X_t)| ds \rightarrow 0 \text{ a.s., as } h \rightarrow 0.$$

Therefore D_+X_t exists and is equal to $b(t, X_t)$.

2) Thanks to assumption (H), we can apply Theorem 2.3 in [9]. Hence $\bar{X}_t = X_{T-t}$ is a diffusion process w.r.t. the increasing filtration (\mathcal{F}_{T-t}) and whose

generator reads

$$\bar{L}_t f = \bar{b}^i \partial_i f + \frac{1}{2} \bar{a}^{ij} \partial_{ij} f$$

with $\bar{a}^{ij}(T-t, x) = a^{ij}(t, x)$ and

$$\bar{b}^i(T-t, x) = -b^i(t, x) + \frac{\operatorname{div}(a_i(t, x)p_t(x))}{p_t(x)}.$$

We have :

$$\begin{aligned} E \left[\frac{X_t - X_{t-h}}{h} \middle| \mathcal{F}_t^X \right] &= E \left[\frac{\bar{X}_{T-t} - \bar{X}_{T-t+h}}{h} \middle| \mathcal{F}_{T-t}^X \right] \\ &= -E \left[\frac{1}{h} \int_{T-t}^{T-t+h} \bar{b}(s, \bar{X}_s) ds \middle| \mathcal{F}_{T-t}^X \right]. \end{aligned} \quad (9)$$

Assumption (H) implies that

$$t \mapsto E \left| \frac{\operatorname{div}(a_i(t, X_t)p_t(X_t))}{p_t(X_t)} \right|$$

is locally integrable. Then, using the same calculations and arguments as above, we obtain that D_-X_t exists and is equal to $-\bar{b}(T-t, \bar{X}_{T-t})$. \square

Corollary 2 *If X given by (2) verifies assumption (H), we have for almost all $t \in (0, T)$:*

$$D_+X_t = b(t, X_t) \quad \text{and} \quad D_-X_t = b(t, X_t) - \sigma^2 \frac{\nabla p_t}{p_t}(X_t).$$

Remark 3 The appearance of the density p_t in the formula giving D_-X_t may seem surprising at first sight. As clear from the proof of Theorem 1, the reason for such a formula stems from the Brownian theory of time reversal. The same term was obtained by Föllmer [5] for Brownian semimartingales of the form $\int_0^t b_s ds + W_t$ with $E \int_0^T b_s^2 ds < \infty$, by relating the backward Nelson derivative and the time reversed drift. Based on the same strategy, Millet, Nualart and Sanz [9] extended the result to diffusions satisfying (H) using Malliavin calculus. Finally, this additional term can also be viewed as the result of a "grossissement de filtration" (see Pardoux [12]). Roughly speaking, when we consider a diffusion $X_t = \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s$ and \mathcal{G}_t the σ -field generated by $W_u - W_r$ for $T-t \leq u < r \leq T$, then $\bar{W}_t - \bar{W}_0$ is a \mathcal{G}_t -Brownian motion and the question sums up to writing the Doob-Meyer decomposition of $\bar{W}_t - \bar{W}_0$ in the enlarged filtration $\mathcal{H}_t = \mathcal{G}_t \vee \bar{X}_t$. In particular, knowing this answer gives the decomposition of \bar{X} with respect to its natural filtration.

Finally, we will also need the following composition formula, stated by Nelson [11] and that we prove for the diffusions we consider.

Proposition 4 Let $f \in C^{1,2}([0, T] \times \mathbb{R}^d)$ with bounded second order derivatives and let X be a diffusion of the form (2) satisfying (H). Then, for almost all $t \in (0, T)$:

$$D_{\pm}f(t, X_t) = \left(\partial_t f + (\partial_x f)D_{\pm}X_t \pm \frac{\sigma^2}{2}\Delta f \right) (t, X_t). \quad (10)$$

Proof. Let $h > 0$.

1) The forward case. The Taylor formula yields:

$$\begin{aligned} f(t+h, X_{t+h}) - f(t, X_t) &= \partial_t f(t, X_t)h + \partial_x f(t, X_t)(X_{t+h} - X_t) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n (X_{t+h}^i - X_t^i)(X_{t+h}^j - X_t^j) \partial_{ij}^2 f(t, X_t) + R(t, h) \end{aligned} \quad (11)$$

where the remainder $R(t, h)$ is given by

$$\begin{aligned} R(t, h) &= \frac{1}{2} \sum_{i,j=1}^n (X_{t+h}^i - X_t^i)(X_{t+h}^j - X_t^j) (\partial_{ij}^2 f(u_{t,h}) - \partial_{ij}^2 f(t, X_t)) \\ &\quad + h \sum_{j=1}^n (X_{t+h}^j - X_t^j) \partial_t \partial_j f(u_{t,h}) \end{aligned}$$

with $u_{t,h} = (t + \theta h, (1 - \theta)X_t + \theta X_{t+h})$ and $\theta \in (0, 1)$ depending on t and h .

We first treat the third term of the r.h.s of (11). For instance for the term $\frac{1}{h}E[(X_{t+h}^i - X_t^i)^2 | X_t]$:

$$(X_{t+h}^i - X_t^i)^2 = \left(\int_t^{t+h} b(s, X_s) ds \right)^2 + \sigma^2 (W_{t+h}^i - W_t^i)^2 + 2\sigma (W_{t+h}^i - W_t^i) \int_t^{t+h} b(s, X_s) ds. \quad (12)$$

We have by Schwarz inequality:

$$\left(\int_t^{t+h} b(s, X_s) ds \right)^2 \leq h \int_t^{t+h} b^2(s, X_s) ds.$$

Thus

$$\frac{1}{h}E \left(\int_t^{t+h} b(s, X_s) ds \right)^2 \leq \int_t^{t+h} E[b^2(s, X_s)] ds \longrightarrow 0,$$

since $t \rightarrow E|X_t|^2$ is locally integrable (see, *e.g.*, Theorem 2.9 in [7]). Again by Schwarz inequality, we deduce that $h^{-1}(W_{t+h}^i - W_t^i) \int_t^{t+h} b(s, X_s) ds$ tends to 0 in $L^1(\Omega)$. Moreover:

$$\frac{1}{h}E[(W_{t+h}^i - W_t^i)^2 | X_t] = \frac{1}{h}E[(W_{t+h}^i - W_t^i)^2] = 1.$$

We now treat the remainder of (11). The fact that $\partial^2 f$ is bounded allows to show as above that $h^{-1} \left(\int_t^{t+h} b(s, X_s) ds \right)^2 (\partial_{ij}^2 f(u_{t,h}) - \partial_{ij}^2 f(t, X_t))$ and

$$\frac{W_{t+h}^i - W_t^i}{h} \int_t^{t+h} b(s, X_s) ds (\partial_{ij}^2 f(u_{t,h}) - \partial_{ij}^2 f(t, X_t))$$

converges to 0 in $L^1(\Omega)$. Moreover

$$\begin{aligned} E \left[\frac{(W_{t+h}^i - W_t^i)^2}{h} (\partial_{ij}^2 f(u_{t,h}) - \partial_{ij}^2 f(t, X_t)) \right] \\ \leq \frac{\sqrt{E|W_{t+h}^i - W_t^i|^4}}{h} \sqrt{E|\partial_{ij}^2 f(u_{t,h}) - \partial_{ij}^2 f(t, X_t)|^2} \\ \leq C \sqrt{E|\partial_{ij}^2 f(u_{t,h}) - \partial_{ij}^2 f(t, X_t)|^2}. \end{aligned}$$

Since $\partial^2 f$ is bounded and $u_{t,h}$ tends to (t, X_t) as $h \rightarrow 0$, we can apply the bounded convergence theorem and conclude.

2) The backward case. We calculate the Taylor expansion of $-(f(t-h, X_{t-h}) - f(t, X_t))$ and we write $(X_{t-h}^i - X_t^i)^2 = (\bar{X}_{T-t+h}^i - \bar{X}_{T-t}^i)^2$. We then write the decomposition (12) for \bar{X} with its time reversed drift \bar{b} and its time reversed driving Brownian motion \bar{W} . So the computations are identical to those of the first point. \square

3 Dynamical study of gradient diffusions

3.1 First order derivatives

In this section, we only consider Brownian diffusions of type (2) with a homogeneous drift. More precisely, we work with X verifying

$$X_t = X_0 + \int_0^t b(X_s) ds + \sigma W_t, \quad t \in [0, T]. \quad (13)$$

We can then characterize the sub-class of stationary diffusions having a gradient drift vector, by means of first order Nelson derivatives. Such diffusions were already considered by many authors. A result of Kolmogorov [8] states that b is a gradient if and only if the law of X given by (13) is reversible, *i.e.* $(X_t)_{t \in [0, T]}$ and $(X_{T-t})_{t \in [0, T]}$ have the same law. In what follows, we show that another characterization of this last fact can be made with the help of Nelson derivatives. For instance, knowing that b is a gradient allows to easily construct an invariant law for X . More precisely, when $b = \nabla U$ with $U : \mathbb{R}^d \rightarrow \mathbb{R}$ regular enough and with sufficiently fast decrease at infinity, the probability

law μ defined by

$$d\mu = c^{-1} e^{\frac{2U(x)}{\sigma^2}} dx \quad \text{with} \quad c = \int_{\mathbb{R}^d} e^{\frac{2U(x)}{\sigma^2}} dx < \infty$$

is invariant for X .

We can easily prove the following:

Proposition 5 *Let X be the Brownian diffusion defined by (13). We moreover assume that X verifies assumption (H).*

- (1) *If $D_+X_t = -D_-X_t$ for any $t \in (0, T)$ then $b = \nabla U$ with $U : \mathbb{R}^d \rightarrow \mathbb{R}$ given by $U = \frac{\sigma^2}{2} \log p_t$. In particular, X is a stationary diffusion with initial law μ given by $d\mu = e^{\frac{2U(x)}{\sigma^2}} dx$.*
- (2) *Conversely, if $b = \nabla U$ with $U : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $c := \int_{\mathbb{R}^d} e^{\frac{2U(x)}{\sigma^2}} dx < \infty$ and if the law of X_0 is $d\mu = c^{-1} e^{\frac{2U(x)}{\sigma^2}} dx$, then the probability law μ is invariant for X and, for any $t \in (0, T)$, we have $D_+X_t = -D_-X_t$.*

Proof. The first point is a direct consequence of the formulae contained in Corollary 2. For the second point, the existence of the invariant law is given by a general theorem (see *e.g.* [2], Theorem 8.6.3 p.163) while the equality $D_+X_t = -D_-X_t$ comes once again from the formulae contained in Corollary 2. \square

3.2 Second order derivatives and characterization of gradient diffusions

In [14] Theorem 5.4, the authors give a very nice generalization of Kolmogorov's result [8] based on an integration by part formula from Malliavin calculus. Precisely, the drift is this time not assumed to be time homogeneous and nor the diffusion stationary. Their characterization requires that there exists one reversible law in the reciprocal class of the diffusion. In our case, we are also able to characterize a larger class of Brownian diffusions. However this further needs to use second order stochastic derivatives. The main result of our paper is the following theorem:

Theorem 6 *Let X be given by (13), verifying assumption (H), such that $b \in C^2(\mathbb{R}^d)$ with bounded derivatives, and such that for all $t \in (0, T)$ the second order derivatives of $\nabla \log p_t$ are bounded. We then have the following equivalence:*

$$D_+^2 X_t = D_-^2 X_t \quad \text{for almost all } t \in (0, T) \quad \iff \quad b \text{ is a gradient.} \quad (14)$$

- Remark 7** (1) Saying that b is a gradient means that we can write $b = \nabla U$ for a certain potential $U : \mathbb{R}^d \rightarrow \mathbb{R}$. It is equivalent, by Poincaré lemma, to verify that $G = \partial_x b - (\partial_x b)^*$ is identically zero.
- (2) When $d = 1$, that is when X is a *one-dimensional* Brownian diffusion, the equality $D_-^2 X - D_+^2 X = 0$ is always verified, see Lemma 8.
- (3) The proof we propose here is entirely based on probabilistic arguments. A more "classical" strategy for proving that $G \equiv 0$ when $D_-^2 X = D_+^2 X$ would use the fact that we then have $\text{div}(p_t G_i) = 0$ for any index i and any time $t \in (0, T)$ (see Lemma 8). For instance, when $d = 2$, this system of equalities reduces to $(\partial_1 b_2 - \partial_2 b_1) p_t = c$ on \mathbb{R}^2 , c denoting a constant. It is then not difficult to deduce that $\partial_1 b_2 = \partial_2 b_1$. In particular, b is a gradient. On the other hand this method seems hard to adapt in higher dimensions. In particular, it seems already difficult to integrate $\text{div}(p_t G) = 0$ when $d = 3$.

First of all, we need the following technical lemma which gives a remarkable identity for $D_+^2 X - D_-^2 X$:

Lemma 8 *Let X be given by (2), verifying assumption (H), such that $b \in C^{1,2}([0, T] \times \mathbb{R}^d)$ with bounded derivatives, and such that for all $t \in (0, T)$ the second order derivatives of $\nabla \log p_t$ are bounded. Therefore for any $i = 1, \dots, n$:*

$$(D_-^2 X_t - D_+^2 X_t)^i = \frac{\text{div}(p_t G_i)}{p_t}. \quad (15)$$

Recall that $G = (\partial_x b) - (\partial_x b)^*$, i.e. $G_i^j = \partial_i b^j - \partial_j b^i$.

Let us stress that the expression we obtain in (15) is the key point of our proof of Theorem 6, and that it is valid for diffusions of the type (2) and not only of the type (13).

Proof. We have, by Proposition 4:

$$D_+^2 X_t = D_+ b(t, X_t) = \left(\partial_t b + (\partial_x b) b + \frac{\sigma^2}{2} \Delta b \right) (t, X_t), \quad (16)$$

and

$$\begin{aligned} D_-^2 X_t &= D_- \left(b - \sigma^2 \frac{\nabla p_t}{p_t} \right) (t, X_t) \\ &= \left[\partial_t b + (\partial_x b) b - \frac{\sigma^2}{2} \Delta b - \sigma^2 \partial_t \frac{\nabla p_t}{p_t} - \sigma^2 (\partial_x b) \frac{\nabla p_t}{p_t} \right. \\ &\quad \left. - \sigma^2 \left(\partial_x \frac{\nabla p_t}{p_t} \right) b + \sigma^4 \left(\partial_x \frac{\nabla p_t}{p_t} \right) \frac{\nabla p_t}{p_t} + \frac{\sigma^4}{2} \Delta \frac{\nabla p_t}{p_t} \right] (t, X_t). \end{aligned}$$

With the Fokker-Planck equation $\partial_t p_t = -\operatorname{div}(p_t b) + \frac{\sigma^2}{2} \Delta p_t$ in mind, we can write:

$$\partial_t \frac{\nabla p_t}{p_t} = \nabla \frac{\partial_t p_t}{p_t} = \nabla \left(-\operatorname{div} b + \frac{\langle -b, \nabla p_t \rangle + \frac{\sigma^2}{2} \Delta p_t}{p_t} \right). \quad (17)$$

Therefore:

$$D_-^2 X_t - D_+^2 X_t = (\sigma^2 A + \sigma^4 B)(t, X_t)$$

with

$$A = -\Delta b + \nabla \operatorname{div} b - (\partial_x b) \frac{\nabla p_t}{p_t} + \nabla \frac{\langle b, \nabla p_t \rangle}{p_t} - \left(\partial_x \frac{\nabla p_t}{p_t} \right) b,$$

$$B = \left(\partial_x \frac{\nabla p_t}{p_t} \right) \frac{\nabla p_t}{p_t} + \frac{1}{2} \Delta \frac{\nabla p_t}{p_t} - \frac{1}{2} \nabla \frac{\Delta p_t}{p_t}.$$

Let us simplify A . By the Leibniz rule we have:

$$\nabla \frac{\langle b, \nabla p_t \rangle}{p_t} = (\partial_x b)^* \frac{\nabla p_t}{p_t} + \left(\partial_x \frac{\nabla p_t}{p_t} \right)^* b.$$

Since $p_t \in C^2$, the Schwarz lemma yields $\left(\partial_x \frac{\nabla p_t}{p_t} \right)^* = \left(\partial_x \frac{\nabla p_t}{p_t} \right)$. Thus

$$A = -\Delta b + \nabla \operatorname{div} b + G \frac{\nabla p_t}{p_t},$$

from which we deduce

$$A^i = \frac{\operatorname{div}(p_t G_i)}{p_t}.$$

Let us simplify B . We have:

$$2 \left[\left(\partial_x \frac{\nabla p_t}{p_t} \right) \frac{\nabla p_t}{p_t} \right]^i = 2 \sum_j \partial_i \left(\frac{\partial_j p_t}{p_t} \right) \frac{\partial_j p_t}{p_t} = \partial_i \sum_j \left(\frac{\partial_j p_t}{p_t} \right)^2.$$

But, again par the Schwarz lemma:

$$\left[\Delta \frac{\nabla p_t}{p_t} \right]^i = \sum_j \partial_j^2 \frac{\partial_i p_t}{p_t} = \partial_i \sum_j \partial_j \left(\frac{\partial_j p_t}{p_t} \right).$$

We then deduce that $B = 0$, which concludes the proof. \square

Now, we go back to the proof of Theorem 6. In order to simplify the exposition, in the sequel we assume without loss of generality that $\sigma = 1$. Let $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a bounded Lipschitz function and X^ε , for $\varepsilon > 0$, be the unique solution of

$$dX_t^\varepsilon = (b + \varepsilon \gamma)(X_t^\varepsilon) dt + dW_t, \quad t \in [0, T], \quad X_0^\varepsilon = X_0 \in L^2(\Omega). \quad (18)$$

Before proving Theorem 6, we need the following lemma, stated and proved in [6], Proposition 3.1:

Lemma 9 *Let $\phi : C[0, T] \rightarrow \mathbb{R}$ be a measurable function such that $E[\phi(X)^2]$ is finite. Then the following equality holds:*

$$\frac{\partial}{\partial \varepsilon} E[\phi(X^\varepsilon)]|_{\varepsilon=0} = E \left[\phi(X) \int_0^T \langle \gamma(X_s), dW_s \rangle \right]. \quad (19)$$

Proof. For the sake of completeness, let us briefly recall how the authors obtain (19). We can write $E[\phi(X^\varepsilon)] = E^{\mathbb{Q}^\varepsilon}[(Z^\varepsilon)^{-1}, \phi(X^\varepsilon)]$ with $d\mathbb{Q}^\varepsilon/d\mathbb{P} = Z^\varepsilon$, where

$$\begin{aligned} Z^\varepsilon &= \exp \left(-\varepsilon \int_0^T \langle \gamma(X_s^\varepsilon), dW_s \rangle - \frac{\varepsilon^2}{2} \int_0^T |\gamma(X_s^\varepsilon)|^2 ds \right) \\ &= \exp \left(-\varepsilon \int_0^T \langle \gamma(X_s^\varepsilon), dW_s^\varepsilon \rangle + \frac{\varepsilon^2}{2} \int_0^T |\gamma(X_s^\varepsilon)|^2 ds \right), \end{aligned}$$

and $W_t^\varepsilon = W_t + \varepsilon \int_0^t \gamma(X_s^\varepsilon) ds$. Note that, under \mathbb{Q}^ε , W^ε is a Brownian motion by Girsanov theorem. In particular the law of $(X^\varepsilon, W^\varepsilon)$ under \mathbb{Q}^ε is the same as the law of (X, W) under \mathbb{P} . Consequently, $E[\phi(X^\varepsilon)] = E[(Z^\varepsilon)^{-1} \phi(X)]$. Equality (19) follows now easily by Lebesgue bounded convergence. \square

Now, we go back to the proof of Theorem 6:

Proof. If b is a gradient, then for any $i \in \{1, \dots, d\}$, $G_i = 0$. So Lemma 8 yields $D_-^2 X_t - D_+^2 X_t = 0$.

Conversely, assume that $D_-^2 X_t - D_+^2 X_t = 0$ for any $t \in (0, T)$. Let $i \in \{1, \dots, d\}$, $\varepsilon \geq 0$, and X^ε be the diffusion process defined by (18) with $\gamma = G_i$. We denote by \mathcal{L}_ε the infinitesimal generator of X^ε , considered as a $(L^2(\mathbb{R}^d), \langle \cdot, \cdot \rangle)$ operator. For simplicity, $\mathcal{L} = \mathcal{L}_0$ will denote the generator of $X = X^0$. It is well-known that the adjoint $\mathcal{L}_\varepsilon^*$ of \mathcal{L}_ε writes

$$\mathcal{L}_\varepsilon^* = -\operatorname{div}[(b + \varepsilon G_i) \cdot] + \frac{1}{2} \Delta. \quad (20)$$

Let $f \in C_0^\infty(\mathbb{R}^d)$. The Dynkin formula for X reads:

$$E[f(X_t)] - f(x) = E \left[\int_0^t \mathcal{L}f(X_s) ds \right]. \quad (21)$$

But

$$\begin{aligned}
E \left[\int_0^t \mathcal{L}f(X_s) ds \right] &= \int_0^t \int_{\mathbb{R}^d} \mathcal{L}f(y) p_s(y) dy ds \\
&= \int_0^t \int_{\mathbb{R}^d} f(y) \mathcal{L}^* p_s(y) dy ds \\
&= \int_0^t E \left[f(X_s) \frac{\mathcal{L}^* p_s(X_s)}{p_s(X_s)} \right] ds.
\end{aligned} \tag{22}$$

Since for all $s \in (0, T)$, $\frac{\text{div}(p_s G_i)}{p_s}(X_s) = 0$ a.s., we deduce from (22) and (20) that:

$$E \left[\int_0^t \mathcal{L}f(X_s) ds \right] = \int_0^t E \left[f(X_s) \frac{\mathcal{L}_\varepsilon^* p_s(X_s)}{p_s(X_s)} \right] ds = E \left[\int_0^t \mathcal{L}_\varepsilon f(X_s) ds \right].$$

Therefore:

$$E[f(X_t)] - f(x) = E \left[\int_0^t \mathcal{L}_\varepsilon f(X_s) ds \right]. \tag{23}$$

So the process M defined by

$$M_t = f(X_t) - f(x) - \int_0^t \mathcal{L}_\varepsilon f(X_s) ds$$

is a $(\mathcal{P}^W, \mathbb{P})$ -martingale (recall that we decided to note \mathcal{P}_t^W the σ -field generated by W_s for $s \in [0, t]$, see section 2.1). Indeed, by the Markov property applied to X , we can write

$$E(M_t - M_s | \mathcal{P}_s^W) = E_{X_s} \left(f(X_{t-s}) - f(x) - \int_0^{t-s} \mathcal{L}_\varepsilon f(X_s) ds \right) = 0.$$

Thus the law of X solves the martingale problem associated with the Markov diffusion X^ε . But b has linear growth and since the second order derivatives of b are bounded it is also the case for G_i and so for $b + \varepsilon G_i$. This allows to apply the Stroock-Varadhan theorem (see *e.g.* [14, Th 24.1 p.170]) which establishes the existence and uniqueness of solutions for the martingale problem. Therefore X and X^ε have the same law. As a consequence, for any measurable function $\phi : C[0, T] \rightarrow \mathbb{R}$ such that $E[\phi(X)^2] < \infty$, the function $\varepsilon \mapsto E[\phi(X^\varepsilon)]$ is constant. We now apply Lemma 9 with $\gamma = G_i$. So, we have:

$$E \left[\phi(X) \int_0^T \langle G_i(X_s), dW_s \rangle \right] = 0. \tag{24}$$

Let \mathbb{Q} be the equivalent probability to \mathbb{P} given by Girsanov theorem applied to X . In particular, X is a Brownian motion under \mathbb{Q} and $\eta = d\mathbb{Q}/d\mathbb{P} \in \mathcal{F}_T^X$. Thanks to (24), we have $E^\mathbb{Q} \left[\phi(X) \eta^{-1} \int_0^T \langle G_i(X_s), dW_s \rangle \right] = 0$. Thus, since ϕ is arbitrary, Lemma 1.1.3. in [11] shows that $\int_0^T \langle G^i(X_s), dW_s \rangle = 0$ \mathbb{Q} -a.s. and then also \mathbb{P} -a.s. Then $G_i(X) \equiv 0$ by Itô isometry (under \mathbb{P}) and, since $\mathcal{L}(X_t)$ has a positive density for any $t \in (0, T)$, we finally have $G \equiv 0$. This concludes the proof. \square

4 A remark on the complex stochastic embedding of the Newton equation

From D_{\pm} one can construct a *complex* stochastic derivative

$$\mathcal{D} = \frac{D_+ + D_-}{2} + i \frac{D_+ - D_-}{2} \quad (25)$$

which extends on stochastic processes the classical derivative operator $\frac{d}{dt}$. Moreover, and contrary to D_+ and D_- , the operator \mathcal{D} has the following natural but however remarkable property:

Proposition 10 *For X given by (1) and verifying assumption (H), we have:*

$$\mathcal{D}X_t = 0 \text{ for any } t \in (0, T) \iff X \text{ is a constant process on } [0, T]. \quad (26)$$

Proof. The condition $\mathcal{D}X_t = 0$ is equivalent to $D_+X_t = D_-X_t = 0$. Thus the forward drift and the backward drift are zero. So X is a (\mathcal{P}_t^X) and (\mathcal{F}_t^X) -martingale. We can then use the arguments of Nelson in the proof of Theorem 11.11 in [10] which allow to conclude. \square

Using (25) and extending \mathcal{D} by \mathbb{C} -linearity to complex Brownian diffusions, we can easily compute

$$\mathcal{D}^2 = \frac{D_+D_- + D_-D_+}{2} + i \frac{D_+^2 - D_-^2}{2}. \quad (27)$$

Let us remark that the real part of \mathcal{D}^2 coincides with the notion of *mean acceleration* introduced by Nelson [10], equality (11.15), for which he had conjectured that it is the more relevant quantity describing an acceleration on Brownian diffusions.

Now, as an example, let us consider the analog of the Newton equation

$$\frac{d^2x}{dt^2} = -\nabla U(x) \quad (28)$$

using this new derivative \mathcal{D} acting on Brownian diffusions. More precisely, assume that the Brownian diffusion X given by (13) verifies, for any $t \in [0, T]$:

$$\mathcal{D}^2 X_t = -\nabla U(X_t). \quad (29)$$

Equation (29) is called the *stochastic embedded equation* of the Newton equation with respect to the extension \mathcal{D} (see [1]). This embedded equation contains the deterministic ordinary differential equation, as an equation written in the

sense of distributions of the Schwartz theory is an extension of the initial ordinary or partial differential equation.

As we said, (29) admits at least $t \mapsto x_t$ verifying (28) as solution. But, what can we say about uniqueness? If not, what can we say about the other solutions?

First, if X satisfies (29), we must have $D_+^2 X_t = D_-^2 X_t$ for any $t \in [0, T]$, see (27). Moreover, it is proved in [1] that, under some regularity conditions, if one searches solutions of (29) *in the class of gradient diffusions* of the type (13), the density p_t of the solution X_t is characterized via the Schrödinger equation. Our Theorem 6, which was conjectured at the end of the note [1], shows that the solutions of (29) are forced to have a gradient function as drift coefficient.

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