



**HAL**  
open science

# Distributions that are convolvable with generalized Poisson kernel of solvable extensions of homogeneous Lie groups

Ewa Damek, Jacek Dziubanski, Philippe Jaming, Salvador Pérez-Esteve

► **To cite this version:**

Ewa Damek, Jacek Dziubanski, Philippe Jaming, Salvador Pérez-Esteve. Distributions that are convolvable with generalized Poisson kernel of solvable extensions of homogeneous Lie groups. *Mathematica Scandinavica*, 2009, 105, pp.31-65. hal-00120271

**HAL Id: hal-00120271**

**<https://hal.science/hal-00120271>**

Submitted on 13 Dec 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# DISTRIBUTIONS THAT ARE CONVOLVABLE WITH GENERALIZED POISSON KERNEL OF SOLVABLE EXTENSIONS OF HOMOGENEOUS LIE GROUPS

EWA DAMEK, JACEK DZIUBANSKI, PHILIPPE JAMING & SALVADOR PÉREZ-ESTEVA

ABSTRACT. In this paper, we characterize the class of distributions on an homogeneous Lie group  $\mathfrak{N}$  that can be extended via Poisson integration to a solvable one-dimensional extension  $\mathfrak{S}$  of  $\mathfrak{N}$ . To do so, we introduce the  $\mathcal{S}'$ -convolution on  $\mathfrak{N}$  and show that the set of distributions that are  $\mathcal{S}'$ -convolvable with Poisson kernels is precisely the set of suitably weighted derivatives of  $L^1$ -functions. Moreover, we show that the  $\mathcal{S}'$ -convolution of such a distribution with the Poisson kernel is harmonic and has the expected boundary behaviour. Finally, we show that such distributions satisfy some global weak- $L^1$  estimates.

## CONTENTS

1. Introduction	2
2. Background and preliminary results	3
2.1. Homogeneous Lie algebras, norms and Lie groups	3
2.2. Haar measure and convolution of functions	4
2.3. Invariant differential operators on $\mathfrak{N}$	6
2.4. A decomposition of the Dirac distribution	8
2.5. Laplace operators and Poisson kernels	9
3. Distributions on $\mathfrak{N}$	10
3.1. Basic facts and the space $\mathcal{D}'_{L^1}$	10
3.2. The $\mathcal{S}'$ -convolution	12
3.3. Weighted spaces of distributions	14
4. Distributions that are $\mathcal{S}'$ -convolvable with the Poisson kernel	15
4.1. Extensions of distributions with the Poisson kernel	15
4.2. Regularity of the $\mathcal{S}'$ -convolution of a distribution and the Poisson kernel	16
4.3. The Dirichlet problem in $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$	18
5. Global estimates for Poisson integrals of distributions in $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$	21
References	27

---

*Date:* September 16, 2004.

*Key words and phrases.* homogeneous Lie groups, distribution,  $\mathcal{S}'$ -convolution, Poisson integrals.

Research partially financed by:

E.D., J.D., Ph.J. : *European Commission* Harmonic Analysis and Related Problems 2002-2006 IHP Network (Contract Number: HPRN-CT-2001-00273 - HARP).

E.D., J.D. : *European Commission Marie Curie Host Fellowship for the Transfer of Knowledge* “Harmonic Analysis, Nonlinear Analysis and Probability” MTKD-CT-2004-013389.

J.D. : Polish funds for science 2005–2008 (research project 1P03A03029).

S.P.-E. : Conacyt-DAIC U48633-F..

## 1. INTRODUCTION

The aim of this paper is to contribute to the understanding of the boundary behaviour of harmonic functions on one dimensional extensions of homogeneous Lie groups. More precisely, we here address the question of which distributions on the homogeneous Lie group can be extended via Poisson-like integration to the whole domain and in which sense this distribution may be recovered as a limit on the boundary of its extension. This question has been recently settled in the case of Euclidean harmonic functions on  $\mathbb{R}_+^{n+1}$  in [AGPS, AGPPE]. For sake of simplicity, let us detail the kind of results we are looking for in this context.

Let us endow  $\mathbb{R}_+^{n+1} := \{(x, t) : x \in \mathbb{R}^n, t > 0\}$  with the Euclidean laplacian. The associated Poisson kernel is then given by  $\mathbb{P}_t(x) = \frac{t}{(t^2 + x^2)^{(n+1)/2}}$  and a compactly supported distribution  $T$  can be extended into an harmonic function via convolution  $u(x, t) = \mathbb{P}_t * T$ . As  $\mathbb{P}_t$  is not in the Schwartz class, this operation is not valid for arbitrary distributions in  $\mathcal{S}'$ . The question thus arises of which distributions in  $\mathcal{S}'$  can be extended via convolution with the Poisson kernel. The first task is to properly define *convolution* and it turns out that the best results are obtained by using the  $\mathcal{S}'$ -convolution which agrees with the usual convolution of distributions when this makes sense. The space of distributions that can be  $\mathcal{S}'$ -convolved with the Poisson kernel is then the space of derivatives of properly-weighted  $L^1$ -functions. Moreover, the distribution obtained this way is a harmonic function which has the expected boundary behaviour.

In this paper, we generalize these results to one dimensional extensions of homogeneous Lie groups, that is homogeneous Lie groups with a one-dimensional family of dilations acting on it. This is a natural habitat for generalizing results on  $\mathbb{R}_+^{n+1}$  and these spaces occur in various situations. The most important to our sense is that homogeneous Lie groups occur in the Iwasawa decomposition of semi-simple Lie groups and hence as boundaries of the associated rank one symmetric space or more generally, as boundaries of homogeneous spaces of negative curvature [He]. Both symmetric spaces and homogeneous spaces of negative curvature are semi-direct products  $\mathfrak{S} = \mathfrak{N}\mathbf{R}_+^*$  of a homogeneous group  $\mathfrak{N}$  and  $\mathbf{R}_+^*$  acting by dilations in the first case, or “dilation like” automorphisms in the second. For a large class of left-invariant operators on  $\mathfrak{S}$  bounded harmonic functions can be reproduced from their boundary values on  $\mathfrak{N}$  via so called Poisson integrals. They involve Poisson kernels whose behavior at infinity is very similar to the one of  $\mathbb{P}_t$ . While for rank one symmetric spaces and the Laplace-Beltrami operator this is immediate from an explicit formula, for the most general case it has been obtained only recently after many years of considerable interest in the subject (see [BDH] and references there). Therefore, we consider a large family of kernels on which we only impose growth conditions that are similar to those of usual Poisson kernels. This allows us to obtain the desired generalizations.

In doing so, the main difficulty comes from the right choice of definition of the  $\mathcal{S}'$ -convolution, since the various choices are *a priori* non equivalent do to the non-commutative nature of the homogeneous Lie group. Once the right choice is made, we obtain the full characterization of the space of distributions that can be extended via Poisson integration. We then show that this extension has the desired properties, namely that it is harmonic if the Poisson kernel is harmonic and that the original distribution is obtained as a boundary

value of its extension. Finally, we show that the harmonic functions obtained in this way satisfy some global estimates.

The article is organized as follows. In the next section, we recall the main results on Lie groups that we will use. We then devote a section to results on distributions on homogeneous Lie groups and the  $\mathcal{S}'$ -convolution on these groups. Section 4 is the main section of this paper. There we prove the characterization of the space of distributions that are  $\mathcal{S}'$ -convolvable with Poisson kernels and show that their  $\mathcal{S}'$ -convolution with the Poisson kernel has the expected properties. We conclude the paper by proving that functions that are  $\mathcal{S}'$ -convolution of distributions with the Poisson kernels satisfy global estimates.

## 2. BACKGROUND AND PRELIMINARY RESULTS

In this section we recall the main notations and results we need on homogeneous Lie algebras and groups. Up to minor changes of notation, all results from this section that are given without proof can be found in the first chapter of [FS], although in a different order.

**2.1. Homogeneous Lie algebras, norms and Lie groups.** Let  $\mathfrak{n}$  be a real and finite dimensional nilpotent Lie algebra with Lie bracket denoted  $[\cdot, \cdot]$ . We assume that  $\mathfrak{n}$  is endowed with a family of dilations  $\{\delta_a : a > 0\}$ , consisting of automorphisms of  $\mathfrak{n}$  of the form  $\delta_a = \exp(A \log a)$  where  $A$  is a diagonalizable linear operator on  $\mathfrak{n}$  with positive eigenvalues. As usual, we will often write  $a\eta$  for  $\delta_a\eta$  and even  $\eta/a$  for  $\delta_{1/a}\eta$ . Without loss of generality, we assume that the smallest eigenvalue of  $A$  is 1. We denote

$$1 = d_1 \leq d_2 \leq \dots \leq d_n := \bar{d}$$

the eigenvalues of  $A$  listed with multiplicity. We will write

$$\Delta = \left\{ \sum_{\alpha \in F} \alpha_j d_j : F \subset \mathbb{N}^n \text{ finite} \right\}.$$

If  $\alpha$  is a multi-index, we will write  $|\alpha| = \alpha_1 + \dots + \alpha_n$  for its length and  $d(\alpha) = d_1 \alpha_1 + \dots + d_n \alpha_n$  for its weight.

Next, we fix a basis  $X_1, \dots, X_n$  of  $\mathfrak{n}$  such that  $AX_j = d_j X_j$  for each  $j$  and write  $\vartheta_1, \dots, \vartheta_n$  for the dual basis of  $\mathfrak{n}^*$ . Finally we define an Euclidean structure on  $\mathfrak{n}$  by declaring the  $X_i$ 's to be orthonormal. The associated scalar product will be denoted  $\langle \cdot, \cdot \rangle$  and the norm  $\|\cdot\|$ .

We denote by  $\mathfrak{N}$  the connected and simply connected Lie group that corresponds to  $\mathfrak{n}$ . If we denote by  $V$  the underlying vector space of  $\mathfrak{n}$  and by  $\theta_k = \vartheta_k \circ \exp^{-1}$ , then  $\theta_1, \dots, \theta_n$  form a system of global coordinates on  $\mathfrak{N}$  that allow to see  $\mathfrak{N}$  as  $V$ . Note that  $\theta_k$  is homogeneous of degree  $d_k$  in the sense that  $\theta_k(\delta_a \eta) = a^{d_k} \theta_k(\eta)$ . The group law is then given by

$$\theta_k(\eta\xi) = \theta_k(\eta) + \theta_k(\xi) + \sum_{\alpha \neq 0, \beta \neq 0, d(\alpha) + d(\beta) = d_k} c_k^{\alpha, \beta} \theta^\alpha(\eta) \theta^\beta(\xi)$$

for some constants  $c_k^{\alpha, \beta}$  and  $\theta^\alpha = \theta_1^{\alpha_1} \dots \theta_n^{\alpha_n}$ . Note that the sum above only involves terms with degree of homogeneity  $< d_k$ , that is coordinates  $\theta_1, \dots, \theta_{k-1}$ . Although the group law is written multiplicatively, we will write 0 for the identity of  $\mathfrak{N}$ .

Now we consider the semidirect products  $\mathfrak{S} = \mathfrak{N} \times \mathbb{R}_+^*$  of such a nilpotent group  $\mathfrak{N}$  with  $\mathbb{R}_+^*$ , that is, we consider  $\mathfrak{S} = \mathfrak{N} \times \mathbb{R}_+^*$  with the multiplication

$$(\eta, a)(\xi, b) = (\eta\delta_a(\xi), ab).$$

Finally, we fix an homogeneous norm on  $\mathfrak{N}$ , that is a continuous function  $x \mapsto |x|$  from  $\mathfrak{N}$  to  $[0, +\infty)$  which is  $\mathcal{C}^\infty$  on  $\mathfrak{N} \setminus \{0\}$  such that

- (i)  $|\delta_a \eta| = a|\eta|$ ,
- (ii)  $|\eta| = 0$  if and only if  $\eta = 0$ ,
- (iii)  $|\eta^{-1}| = |\eta|$ ,
- (iv)  $|\eta \cdot \xi| \leq (|\eta| + |\xi|)$ ,  $\gamma \geq 1$  and, according to [HS], we may chose  $|\cdot|$  in such a way that  $\gamma = 1$ ,
- (v) this norm satisfies Petree's inequality: for  $r \in \mathbb{R}$ ,

$$(1 + |\eta\xi|)^r \leq (1 + |\eta|)^{|r|}(1 + |\xi|)^r.$$

This inequality is obtained as follows: when  $r \geq 0$ , write

$$1 + |\xi\eta| \leq 1 + (|\eta| + |\xi|) \leq (1 + |\eta|)(1 + |\xi|)$$

and raise it to the power  $r$ . For  $r < 0$ , write

$$1 + |\xi| \leq 1 + (|\xi\eta| + |\eta^{-1}|) \leq (1 + |\xi\eta| + |\eta|) \leq (1 + |\xi\eta|)(1 + |\eta|)$$

and raise it to the power  $-r$ .

In particular,  $d(\eta, \xi) = |\eta^{-1}\xi|$  is a left-invariant metric on  $\mathfrak{N}$ .

For smoothness issues in the next sections, we will need the following notation. Let  $\Phi$  be a fixed  $\mathcal{C}^\infty$  function on  $[0, +\infty]$  such that  $\Phi = 1$  in  $[0, 1]$ ,  $\Phi(x) = x$  on  $[2, +\infty)$  and  $\Phi \geq 1$  on  $[1, 2]$ . Then for  $\mu \in \mathbb{R}$ , we will denote by  $\omega_\mu(\eta) = (1 + \Phi(|\eta|))^\mu$  which is  $\mathcal{C}^\infty$  in  $\mathfrak{N}$ . In all estimates written bellow,  $\omega_\mu$  can always be replaced by  $(1 + |\eta|)^\mu$ .

**2.2. Haar measure and convolution of functions.** If  $\eta \in \mathfrak{N}$  and  $r > 0$ , we define

$$B(\eta, r) = \{\xi \in \mathfrak{N} : |\xi^{-1}\eta| < r\}$$

the ball of center  $\eta$  and radius  $r$ . Note that  $\overline{B(\eta, r)}$  is compact.

If  $d\lambda$  denotes Lebesgue measure on  $\mathfrak{n}$ , then  $\lambda \circ \exp^{-1}$  is a bi-invariant Haar measure on  $\mathfrak{N}$ . We choose to normalize it so as to have  $|B(\eta, 1)| = 1$  and still denote it by  $d\lambda$ . Moreover, we have

$$|B(\eta, r)| = |B(0, r)| = |r \cdot B(0, 1)| = r^Q,$$

where  $Q = d_1 + \dots + d_n = \text{tr } A$  is the homogeneous dimension of  $\mathfrak{N}$ . This measure admits a polar decomposition. More precisely, if we denote by  $\mathbb{S} = \{\eta \in \mathfrak{N} : |\eta| = 1\}$ , there exists a measure  $d\sigma$  on  $\mathbb{S}$  such that for all  $\varphi \in L^1(\mathfrak{N})$ ,

$$\int_{\mathfrak{N}} \varphi(\eta) d\lambda(\eta) = \int_0^{+\infty} \int_{\mathbb{S}} \varphi(r\xi) r^{Q-1} d\sigma(\xi) dr.$$

On  $\mathfrak{S}$  the right-invariant Haar measure is given by  $\frac{d\lambda da}{a}$ .

Recall that the convolution on a group  $\mathfrak{N}$  with left-invariant Haar measure  $d\lambda$  is given by

$$f * g(\eta) = \int_{\mathfrak{N}} f(\xi)g(\xi^{-1}\eta) d\lambda(\xi) = \int_G f(\eta\xi^{-1})g(\xi) d\lambda(\xi).$$

This operation is not commutative but, writing  $\check{f}(\eta) = f(\eta^{-1})$ , we have  $f * g = (\check{g} * \check{f})^\sim$ .

We will need the following:

**Lemma 2.1.** *Let  $h$  be a  $C^\infty$  function on  $\mathfrak{N}$  supported in a compact neighborhood of 0 such that*

$$\int_{\mathfrak{N}} h(\eta) d\lambda(\eta) = 1.$$

*Set  $h_a(\eta) = a^{-Q}h(\delta_{a^{-1}}\eta)$ , then the family  $h_a$  forms a smooth compactly supported approximate identity. In particular, if  $f$  is continuous and bounded on  $\mathfrak{N}$ , then  $f * h_a \rightarrow f$  uniformly on compact sets as  $a \rightarrow 0$ .*

We will need the following elementary lemma that can be proved along the lines of [AGPPE, Lemma 9]:

**Lemma 2.2.** *For  $r, s \in \mathbb{R}$ , let*

$$I_{r,s}(\eta) = \int_{\mathfrak{N}} (1 + |\xi|)^r (1 + |\xi^{-1}\eta|)^s d\lambda(\xi).$$

*Then, if  $r + s + Q < 0$ ,  $I_{r,s}(\eta)$  is finite. Moreover, if this is the case, there is a constant  $C_{r,s}$  such that, for every  $\eta \in \mathfrak{N}$ ,*

$$I_{r,s}(\eta) \leq \begin{cases} C_{r,s}(1 + |\eta|)^{r+s+Q} & \text{if } r + Q > 0 \text{ and } s + Q > 0 \\ C_{r,s}(1 + |\eta|)^{\max(r,s)} \log(2 + |\eta|) & \text{if } r + Q = 0 \text{ or } s + Q = 0 \\ C_{r,s}(1 + |\eta|)^{\max(r,s)} & \text{else} \end{cases}.$$

*Proof.* From Peetre's inequality we immediately get the first part of the lemma.

From now on, we can assume that  $r + s + Q < 0$ . Write  $\mathfrak{N} = \Omega_1 \cup \Omega_2 \cup \Omega_3$  for a partition of  $\mathfrak{N}$  given by

$$\Omega_1 = \left\{ \xi \in \mathfrak{N} : |\xi| \leq \frac{1}{2}|\eta| \right\} \quad \text{and} \quad \Omega_2 = \left\{ \xi \in \mathfrak{N} : |\xi| > \frac{1}{2}|\eta|, |\xi^{-1}\eta| \leq \frac{1}{2}|\eta| \right\}$$

and let

$$I_i(\eta) = \int_{\Omega_i} (1 + |\xi|)^r (1 + |\xi^{-1}\eta|)^s d\lambda(\xi).$$

First, for  $\xi \in \Omega_1$ , we have  $\frac{1}{2}|\eta| \leq |\xi^{-1}\eta| \leq \frac{3}{2}|\eta|$  so that

$$\begin{aligned} I_1(\eta) &\leq C_s(1 + |\eta|)^s \int_{\Omega_1} (1 + |\xi|)^r d\lambda(\xi) \leq C_s(1 + |\eta|)^s \int_0^{\frac{|\eta|}{2}} t^{Q-1}(1 + t)^r dt \\ &\leq \begin{cases} C_{r,s}(1 + |\eta|)^{r+s+Q} & \text{if } r + Q > 0 \\ C_{r,s}(1 + |\eta|)^s \ln(2 + |\eta|) & \text{if } r + Q = 0 \\ C_{r,s}(1 + |\eta|)^s & \text{if } r + Q < 0 \end{cases}. \end{aligned}$$

Next, for  $\xi \in \Omega_2$ , we have  $\frac{1}{2}|\eta| \leq |\xi| \leq \frac{3}{2}|\eta|$ , thus

$$\begin{aligned} I_2(\eta) &\leq C_r(1 + |\eta|)^r \int_{\Omega_2} (1 + |\xi^{-1}\eta|)^s d\lambda(\xi) \leq C_r(1 + |\eta|)^r \int_0^{|\eta|/2} t^{Q-1}(1 + t)^s dt \\ &\leq \begin{cases} C_{r,s}(1 + |\eta|)^{r+s+Q} & \text{if } s + Q > 0 \\ C_{r,s}(1 + |\eta|)^r \ln(2 + |\eta|) & \text{if } s + Q = 0 \\ C_{r,s}(1 + |\eta|)^r & \text{if } s + Q < 0 \end{cases}. \end{aligned}$$

Finally, for  $\xi \in \Omega_3$ , we have  $\frac{1}{3}|\xi| \leq |\xi^{-1}\eta| \leq 3|\xi|$  so that

$$\begin{aligned} I_3(\eta) &\leq C_{r,s} \int_{\Omega_3} (1 + |\xi|)^r (1 + |\xi|)^s d\lambda(\xi) \leq C_{r,s} \int_{\mathfrak{N} \setminus \Omega_1} (1 + |\xi|)^{r+s} d\lambda(\xi) \\ &= C_{r,s} \int_{\frac{|\eta|}{2}}^{+\infty} t^{Q-1} (1+t)^{r+s} dt \leq C_{r,s} (1 + |\eta|)^{r+s+Q}. \end{aligned}$$

The proof is then complete when grouping all estimates.  $\square$

**2.3. Invariant differential operators on  $\mathfrak{N}$ .** Recall that an element  $X \in \mathfrak{n}$  can be identified with a left-invariant differential operator on  $\mathfrak{N}$  via

$$Xf(\xi) = \left. \frac{\partial}{\partial s} f(\xi \cdot \exp(sX)) \right|_{s=0}.$$

There is also a right-invariant differential operator  $Y$  corresponding to  $X$ , given by

$$Yf(\xi) = \left. \frac{\partial}{\partial s} f(\exp(sX) \cdot \xi) \right|_{s=0}.$$

Note that  $X$  and  $Y$  agree at  $\xi = 0$ . For  $X_1, \dots, X_n$  the basis of  $\mathfrak{n}$  defined in section 2.1 we write  $Y_1, \dots, Y_n$  for the corresponding right-invariant differential operators.

If  $\alpha$  is a multi-index, we will write

$$\begin{aligned} X^\alpha &= X_1^{\alpha_1} \dots X_n^{\alpha_n}, & \tilde{X}^\alpha &= X_n^{\alpha_n} \dots X_1^{\alpha_1}, \\ Y^\alpha &= Y_1^{\alpha_1} \dots Y_n^{\alpha_n}, & \tilde{Y}^\alpha &= Y_n^{\alpha_n} \dots Y_1^{\alpha_1}. \end{aligned}$$

We will write  $Z^\alpha$  if something is true for any of the above. For instance, we will use without further notice that

$$|Z^\alpha \omega_\mu| \leq C \omega_{\mu-d(\alpha)}.$$

For “nice” functions, one has<sup>1</sup>

$$\int_{\mathfrak{N}} X^\alpha f(\eta) g(\eta) d\lambda(\eta) = (-1)^{|\alpha|} \int_{\mathfrak{N}} f(\eta) \tilde{X}^\alpha g(\eta) d\lambda(\eta)$$

and

$$\int_{\mathfrak{N}} Y^\alpha f(\eta) g(\eta) d\lambda(\eta) = (-1)^{|\alpha|} \int_{\mathfrak{N}} f(\eta) \tilde{Y}^\alpha g(\eta) d\lambda(\eta).$$

As a consequence, one also has

$$\begin{aligned} X^\alpha(f * g) &= f * (X^\alpha g), & \tilde{X}^\alpha(f * g) &= f * (\tilde{X}^\alpha g), \\ Y^\alpha(f * g) &= (Y^\alpha f) * g & \text{and} & \quad \tilde{Y}^\alpha(f * g) = (\tilde{Y}^\alpha g) * f. \end{aligned}$$

Moreover, using  $X^\alpha \check{f} = (-1)^{|\alpha|} (Y^\alpha f)^\vee$  or  $\tilde{X}^\alpha \check{f} = (-1)^{|\alpha|} (\tilde{Y}^\alpha f)^\vee$  and correcting the proof in [FS], one gets

$$(X^\alpha f) * g = f * (\tilde{Y}^\alpha g) \text{ and } (\tilde{X}^\alpha f) * g = f * (Y^\alpha g).$$

Recall that a polynomial on  $\mathfrak{N}$  is a function of the form

$$P = \sum_{\text{finite}} a_\alpha \theta^\alpha$$

<sup>1</sup>in [FS] the  $\tilde{\cdot}$  is missing, this is usually harmless but not in this article.

and that its isotropic and homogeneous degrees are respectively defined by  $\max\{|\alpha|, a_\alpha \neq 0\}$  and  $\max\{d(\alpha), a_\alpha \neq 0\}$ .

For sake of simplicity, we will write the Leibniz' Formula as

$$X^\alpha(\varphi\psi) = \sum_{\beta \leq \alpha} \Lambda_{\alpha,\beta} X^\beta \varphi X^{\alpha-\beta} \psi, \quad \tilde{X}^\alpha(\varphi\psi) = \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha,\beta} \tilde{X}^\beta \varphi \tilde{X}^{\alpha-\beta} \psi.$$

Further, we may write

$$(2.1) \quad \tilde{Y}^\alpha = \sum_{\beta \in \mathcal{I}_\alpha} \tilde{\mathbb{Q}}_{\alpha,\beta} X^\beta$$

where  $\mathcal{I}_\alpha = \{\beta : |\beta| \leq |\alpha|, d(\beta) \geq d(\alpha)\}$  and  $\tilde{\mathbb{Q}}_{\alpha,\beta}$  are homogeneous polynomials of homogeneous degree  $d(\beta) - d(\alpha)$ . In the same way, any euclidean derivative can be written in terms of left or right invariant derivatives. We will only need the following in the next section: for every  $M$ , there exist polynomials  $\omega_\alpha$ ,  $|\alpha| \leq 2M$  and left-invariant operators  $X^\alpha$  such that

$$(2.2) \quad (I - \Delta)^M = \sum_{|\alpha| \leq 2M} \omega_\alpha X^\alpha.$$

Finally, we will exhibit another link among several of this objects. Let  $h_a$  be as in Lemma 2.1 and let  $f, \varphi$  be smooth compactly supported functions. Then  $\langle (X^\alpha f) * h_a, \varphi \rangle$  is

$$\begin{aligned} &= \langle X^\alpha f, \varphi * \check{h}_a \rangle = (-1)^{|\alpha|} \langle f, \tilde{X}^\alpha(\varphi * \check{h}_a) \rangle = \langle f, \varphi * (\tilde{Y}^\alpha h_a)^\vee \rangle \\ &= \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \varphi(\xi\eta) (\tilde{Y}^\alpha h_a)(\eta) z, d\lambda(\eta) d\lambda(\xi) \\ &= \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \varphi(\xi\eta) \sum_{\beta \in \mathcal{I}_\alpha} \tilde{\mathbb{Q}}_{\alpha,\beta}(\eta) (X^\beta h_a)(\eta) d\lambda(\eta) d\lambda(\xi) \\ &= \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \in \mathcal{I}_\alpha} (-1)^{|\beta|} \left( \tilde{X}^\beta \left( \tilde{\mathbb{Q}}_{\alpha,\beta}(\eta) \varphi(\xi\eta) \right) \right) h_a(\eta) d\lambda(\eta) d\lambda(\xi) \\ &= \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \in \mathcal{I}_\alpha} (-1)^{|\beta|} \sum_{\iota \leq \beta} \tilde{\Lambda}_{\beta,\iota} (\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta})(\eta) (\tilde{X}^\iota \varphi)(\xi\eta) h_a(\eta) d\lambda(\eta) d\lambda(\xi). \end{aligned}$$

As  $\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta}$  is an homogeneous polynomial, if it is not a constant, then  $\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta}(0) = 0$ . With Lemma 2.1, it follows that

$$\int_{\mathfrak{N}} (\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta})(\eta) (\tilde{X}^\iota \varphi)(\xi\eta) h_a(\eta) d\lambda(\eta) \rightarrow 0$$

uniformly with respect to  $\xi$  in compact sets, as  $a \rightarrow 0$ . On the other hand, if  $\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta}$  is a constant,

$$\begin{aligned} \int_{\mathfrak{N}} (\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta})(\eta) (\tilde{X}^\iota \varphi)(\xi\eta) h_a(\eta) d\lambda(\eta) &= (\tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta})(0) \int_{\mathfrak{N}} (\tilde{X}^\iota \varphi)(\xi\eta) h_a(\eta) d\lambda(\eta) \\ &\rightarrow \tilde{X}^{\beta-\iota} \tilde{\mathbb{Q}}_{\alpha,\beta}(0) \tilde{X}^\iota \varphi(\xi) \end{aligned}$$



as  $a \rightarrow 0$ , uniformly with respect to  $\xi$  in compact sets, again with Lemma 2.1. We thus get that  $\langle (X^\alpha f) * h_a, \varphi \rangle$  converges to

$$\int_{\mathfrak{N}} f(\xi) \sum_{\beta \in \mathcal{I}_\alpha} (-1)^{|\beta|} \sum_{\iota \leq \beta} \tilde{\Lambda}_{\beta, \iota}(\tilde{X}^{\beta-\iota} \tilde{\mathcal{Q}}_{\alpha, \beta})(0) \tilde{X}^\iota \varphi(\xi) d\lambda(\xi)$$

On the other hand  $(X^\alpha f) * h_a$  converges uniformly to  $X^\alpha f$  on compact sets, thus

$$\langle (X^\alpha f) * h_a, \varphi \rangle \rightarrow \langle X^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, \tilde{X}^\alpha \varphi \rangle.$$

As the two forms of the limit are the same for all  $f, \varphi$  with compact support, we thus get that

$$(2.3) \quad \tilde{X}^\alpha = (-1)^{|\alpha|} \sum_{\beta \in \mathcal{I}_\alpha} (-1)^{|\beta|} \sum_{\iota \leq \beta} \tilde{\Lambda}_{\beta, \iota}(\tilde{X}^{\beta-\iota} \tilde{\mathcal{Q}}_{\alpha, \beta})(0) \tilde{X}^\iota.$$

**2.4. A decomposition of the Dirac distribution.** In Section 3.1, we will need the following result about the existence of a parametrix:

**Lemma 2.3.** *For every integer  $m$  and every compact set  $K \subset \mathfrak{N}$  with  $0$  in the interior, there exists an integer  $M$ , a family of left-invariant differential operators  $X^\alpha$  of order  $|\alpha| \leq M$  and a family of functions  $\{F_\alpha\}_{|\alpha| \leq M}$  of class  $\mathcal{C}^m$  with support in  $K$  such that*

$$(2.4) \quad \sum_{\alpha} X^\alpha F_\alpha = \delta_0$$

where  $\delta_0$  is the Dirac mass at origin.

*Proof.* Let us start with the proof in the euclidean case, even though this is classical (see [Sc])

First, for  $M$  big enough, the function  $F_0$  defined on  $\mathbb{R}^d$  by  $\widehat{F}_0(\xi) = 1/(1+4\pi^2|\xi|^2)^M$  (where  $\widehat{F}$  is the Fourier transform of  $F$ ) is of class  $\mathcal{C}^m$  and satisfies  $(I - \Delta)^M F = \delta_0$  where  $\Delta$  is the Euclidean Laplace operator.

Now let  $\varphi$  be a smooth function supported in  $K$  with  $\varphi = 1$  in a neighbourhood of  $0$ . Then by Leibniz's rule, we get that  $(I - \Delta)^M (F_0 \varphi)$  is of the form

$$\varphi (I - \Delta)^M F_0 + \sum c_{\alpha\beta} \partial^\beta F_0 \partial^\alpha \varphi.$$

Note that  $\partial^\alpha \varphi = 0$  in a neighbourhood of  $0$  and that  $F_0$  is analytic away from  $0$  so that, if we set  $H = \sum_{0 < |\alpha| \leq 2M, |\beta| \leq 2M} c_{\alpha\beta} \partial^\beta F_0 \partial^\alpha \varphi$  then  $H$  is smooth and supported in  $K$ . Further, as

$(I - \Delta)^M F_0 \varphi = \varphi(0) \delta_0 = \delta_0$ , we have thus proved that there exists two functions  $G$  of class  $\mathcal{C}^m$  with support in  $K$  such that

$$(I - \Delta)^M G = \delta_0 + H.$$

To obtain (2.4), let us recall (2.2):

$$(I - \Delta)^M = \sum_{|\alpha| \leq 2M} \omega_\alpha X^\alpha.$$

It follows that, for  $\psi \in \mathcal{D}$ ,

$$\begin{aligned} \langle (I - \Delta)^M G, \psi \rangle &= \sum_{|\alpha| \leq 2M} \langle \omega_\alpha X^\alpha G, \psi \rangle = \sum_{|\alpha| \leq 2M} (-1)^{|\alpha|} \langle G, X^\alpha (\omega_\alpha \psi) \rangle \\ &= \sum_{|\alpha| \leq 2M} (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \langle G, X^{\alpha-\beta} \omega_\alpha X^\beta \psi \rangle \\ &= \sum_{|\alpha| \leq 2M} \sum_{\beta \leq \alpha} \langle X^\beta ((-1)^{|\alpha|+|\beta|} G X^{\alpha-\beta} \omega_\alpha), \psi \rangle. \end{aligned}$$

We have thus written

$$(I - \Delta)^M G = \sum_{\alpha} \sum_{\beta \leq \alpha} X^\beta ((-1)^{|\alpha|+|\beta|} G X^{\alpha-\beta} \omega_\alpha)$$

which is of the desired form.  $\square$

## 2.5. Laplace operators and Poisson kernels.

**Definition.** Let  $\mathbb{P}$  be a smooth function on  $\mathfrak{N}$  and let  $\mathbb{P}_a(\eta) = a^Q \mathbb{P}(\delta_{a^{-1}} \eta)$ . We will say that  $\mathbb{P}$  has property  $(\mathcal{R}_\Gamma)$  if it satisfies the following estimates:

- (i) there exists a constant  $C$  such that

$$\frac{1}{C} \omega_{-Q-\Gamma} \leq \mathbb{P} \leq C \omega_{-Q-\Gamma};$$

- (ii) for every left-invariant operator  $X^\alpha$ , there is a constant  $C_\alpha$  such that for every  $\eta \in \mathfrak{N}$ ,  $|X^\alpha \mathbb{P}(\eta)| \leq C_\alpha \omega_{-Q-\Gamma-d(\alpha)}(\eta)$ ,  
 (iii) for every  $k$ , there is a constant  $C_k$  such that for every  $\eta \in \mathfrak{N}$ , and every  $a > 0$ ,

$$|(a \partial_a)^k \mathbb{P}_a(\eta)| \leq C_k a^{-Q} \omega_{-Q-\Gamma}(\delta_{a^{-1}} \eta).$$

Note that several other important estimates will automatically result from this estimates.

- (1) First, by homogeneity of the left-invariant operator  $X^\alpha$ , there is a constant  $C_\alpha$  such that for every  $\eta \in \mathfrak{N}$ , and every  $a > 0$ ,

$$|X^\alpha \mathbb{P}_a(\eta)| \leq C_\alpha a^{-Q-d(\alpha)} \omega_{-Q-\Gamma-d(\alpha)}(\delta_{a^{-1}} \eta).$$

- (2) Let  $\mathbb{X} = X_{i_1}^{\alpha_1} \cdots X_{i_k}^{\alpha_k}$  be a left-invariant differential operator. Set  $d(\mathbb{X}) = d_{i_1} \alpha_1 + \cdots + d_{i_k} \alpha_k$  its weight, then the commutation rules in  $\mathfrak{n}$  imply that  $\mathbb{X} = \sum_{\beta: d(\beta)=d(\mathbb{X})} c_\beta X^\beta$ . It follows that

$$|\mathbb{X} \mathbb{P}_a(\eta)| \leq C a^{-Q-d(\mathbb{X})} \omega_{-Q-\Gamma-d(\mathbb{X})}(\delta_{a^{-1}} \eta).$$

- (3) Writing  $Y^\alpha = \sum_{\beta \in \mathcal{I}_\alpha} Q_{\alpha,\beta} X^\beta$  where  $Q_{\alpha,\beta}$  is a homogeneous polynomial of degree  $d(\beta) - d(\alpha)$ , we get that

$$|Y^\alpha \mathbb{P}_a(\eta)| \leq C a^{-Q-d(\alpha)} \omega_{-Q-\Gamma-d(\alpha)}(\delta_{a^{-1}} \eta).$$

In particular, in all estimates,  $\mathbb{P}_a$  can be replaced by  $\check{\mathbb{P}}_a$ . Also, as for the previous point,  $Y^\alpha$  may be replaced by  $\mathbb{Y} = Y_{i_1}^{\alpha_1} \cdots Y_{i_k}^{\alpha_k}$ .

- (4) The previous remark also shows that in point (ii) we may as well impose the condition for *right invariant* differential operators. This would not change the class of kernels.

*Remark* : A large class of kernels satisfying property  $\mathcal{R}_\Gamma$  is associated to left-invariant operators on  $\mathfrak{S}$ . Consider a second order left-invariant operator on  $\mathfrak{S}$  of the form

$$\mathcal{L} = \sum_{j=1}^m Y_j^2 + Y.$$

We assume the Hörmander condition *i.e.* that

$$(2.5) \quad Y_1, \dots, Y_m \text{ generate the Lie algebra of } \mathfrak{S}.$$

The image of such an operator on  $\mathbb{R}^+$  under the natural homomorphism  $(\xi, a) \rightarrow a$  is, up to a multiplicative constant,

$$(a\partial_a)^2 - \alpha a\partial_a.$$

If  $a > 0$  then there is a smooth integrable function  $\mathbb{P}_a$  on  $\mathfrak{N}$  such that the Poisson integrals

$$(2.6) \quad f * \mathbb{P}_a(\eta) = \int_{\mathfrak{N}} f(\xi) \mathbb{P}_a(\xi^{-1}\eta) \, d\lambda(\xi)$$

of  $L^\infty$  function  $f$  are  $\mathcal{L}$ -harmonic and moreover, all bounded  $\mathcal{L}$  harmonic functions are of this form. In particular,  $\mathbb{P}_a(\eta)$  is  $\mathcal{L}$ -harmonic.

The properties i) and ii) for  $\mathbb{P}$  have been proved in [BDH] - see the main theorem there for diagonal action and  $\mathcal{L}$  satisfying 2.5. iii) follows immediately from i) and the (left-invariant) Harnack inequality applied to the harmonic function  $\mathbb{P}_a(\eta)$  *i.e.*

$$|(a\partial_a)^k \mathbb{P}_a(\eta)| \leq C_k \mathbb{P}_a(\eta).$$

Our first aim will be to give a meaning to such Poisson integrals for as general as possible distributions  $f$  so as to still obtain an  $\mathcal{L}$ -harmonic functions when the kernel is  $\mathcal{L}$ -harmonic.

### 3. DISTRIBUTIONS ON $\mathfrak{N}$

**3.1. Basic facts and the space  $\mathcal{D}'_{L^1}$ .** Distributions on  $\mathfrak{N}$  are defined as on  $\mathbb{R}^n$  as the dual of the space  $\mathcal{D} := \mathcal{D}(\mathfrak{N})$  of  $C^\infty$  functions with compact support, endowed with the semi-norms given, for  $K \subset \mathfrak{N}$  compact and  $\alpha$  a multi-index, by

$$p_{K,\alpha}(\varphi) = \sup_{\eta \in K} |\partial^\alpha(\varphi)|.$$

We will write the space of distributions  $\mathcal{D}' := \mathcal{D}'(\mathfrak{N})$ , endowed with the natural family of semi-norms. Notions such as support, Schwartz class  $\mathcal{S} := \mathcal{S}(\mathfrak{N})$ , tempered distributions  $\mathcal{S}' := \mathcal{S}'(\mathfrak{N}), \dots$  are defined as for distributions on  $\mathbb{R}^n$  and the space of compactly supported distributions will be denoted  $\mathcal{E}' := \mathcal{E}'(\mathfrak{N})$ . Because of the link between left invariant derivatives and Euclidean derivatives (similar to the links between left and right invariant derivatives, *see* [FS]), these spaces are just the usual spaces of derivatives on  $\mathfrak{N}$  seen as  $V \simeq \mathbb{R}^n$ . In particular, we will use the fact that every set of distributions that is weakly bounded is also strongly bounded.

For  $T \in \mathcal{D}'$ , we define  $\check{T} \in \mathcal{D}'$  by  $\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$ , while  $X^\alpha T$  is defined by  $\langle X^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \check{X}^\alpha \varphi \rangle$ .

The definition of the convolution of two functions is easily extended to convolution of a distribution with a smooth function via the following pairings: for  $T \in \mathcal{D}'$  a distribution and  $\psi, \varphi \in \mathcal{D}$  smooth functions

- the right convolution is given by  $\langle T * \psi, \varphi \rangle = \langle T, \varphi * \check{\psi} \rangle$
- the left convolution is given by  $\langle \psi * T, \varphi \rangle = \langle T, \check{\psi} * \varphi \rangle$ .

As in the Euclidean case, one may check that  $T * \psi$  and  $\psi * T$  are both smooth.

We will now introduce the space of integrable distributions  $\mathcal{D}'_{L^1}$  and show that this is the space of derivatives of  $L^1$  functions.

**Definition.** Let  $\mathcal{B} := \mathcal{B}(\mathfrak{N})$  be the space of smooth functions  $\varphi : \mathfrak{N} \rightarrow \mathbb{C}$  such that, for every left-invariant differential operator  $X^\alpha$ ,  $X^\alpha \varphi$  is bounded.

Let  $\dot{\mathcal{B}} := \dot{\mathcal{B}}(\mathfrak{N})$  be the subspace of all  $\varphi \in \mathcal{B}(\mathfrak{N})$  such that, for every left-invariant differential operator  $X^\alpha$ ,  $|X^\alpha \varphi(u)| \rightarrow 0$  when  $|u| \rightarrow \infty$ .

We equip these spaces with the topology of uniform convergence of all derivatives.

The space  $\mathcal{D}'_{L^1} = \mathcal{D}'_{L^1}(\mathfrak{N})$  is the topological dual of  $\dot{\mathcal{B}}(\mathfrak{N})$  endowed with the strong dual topology.

Note that  $\mathcal{S}$  and  $\mathcal{C}_0^\infty$  are dense in  $\dot{\mathcal{B}}$  (but not in  $\mathcal{B}$ ) so that  $\mathcal{D}'_{L^1}$  is a subspace of  $\mathcal{S}'$ . Note also that every compactly supported distribution is in  $\mathcal{D}'_{L^1}$ .

It is also obvious that if  $T \in \mathcal{D}'_{L^1}$ ,  $\varphi \in \mathcal{B}$  and  $X^\alpha$  is left-invariant, then  $X^\alpha T \in \mathcal{D}'_{L^1}$  and  $\varphi T \in \mathcal{D}'_{L^1}$ . We will need the following characterization of this space:

**Theorem 3.1.** *Let  $T \in \mathcal{D}'(\mathfrak{N})$ . The following are equivalent*

- (i)  $T \in \mathcal{D}'_{L^1}(\mathfrak{N})$ ;
- (ii)  $T$  has a representation of the form  $T = \sum_{\text{finite}} X^\alpha f_\alpha$  where  $f_\alpha \in L^1(\mathfrak{N})$  and  $X^\alpha$  are left-invariant differential operators;
- (iii) for every  $\varphi \in \mathcal{D}(\mathfrak{N})$ , the regularization  $T * \varphi \in L^1(\mathfrak{N})$ .

*Proof.* The proof follows the main steps of the Euclidean case, see [Sc, page 131]. Denote by  $\mathcal{D}_1$  the set of all functions  $\psi \in \mathcal{D}$  such that  $\|\psi\|_\infty \leq 1$ .

*i)  $\Rightarrow$  iii)* Assume that  $T \in \mathcal{D}'_{L^1}$  and let  $\varphi \in \mathcal{D}$ . Now, note that

$$(3.7) \quad \langle T * \varphi, \psi \rangle = \langle T, \psi * \check{\varphi} \rangle$$

so, if  $\varphi$  is fixed and  $\psi$  runs over  $\mathcal{D}_1$ , the set of numbers on the right of (3.7) is bounded, thus so is the set of numbers  $\{\langle T * \varphi, \psi \rangle, \psi \in \mathcal{D}_1\}$ . But  $T * \varphi$  is a (smooth) function so this implies that  $T * \varphi \in L^1$ .

*iii)  $\Rightarrow$  ii)* Assume that, for every  $\psi \in \mathcal{D}$ ,  $T * \psi \in L^1$ , thus  $T * \check{\psi} \in L^1$ . Now, for  $\psi \in \mathcal{D}$  fixed, the set of numbers

$$\langle \check{T} * \varphi, \check{\psi} \rangle = \langle \check{T}, \check{\psi} * \check{\varphi} \rangle = \langle T, \varphi * \psi \rangle = \langle T * \check{\psi}, \varphi \rangle$$

stays bounded when  $\varphi$  runs over  $\mathcal{D}_1$ . It follows that the set of distributions  $\{\check{T} * \varphi, \varphi \in \mathcal{D}_1\}$  is bounded in  $\mathcal{D}'$  since it is a weakly bounded set.

This implies that there exists an integer  $m$  and a compact neighborhood  $K$  of 0 such that, for every function  $\psi$  of class  $\mathcal{C}^m$  with support in  $K$ ,  $\check{T} * \varphi * \psi(0)$  stays bounded when  $\varphi$  varies over  $\mathcal{D}_1$ . Using

$$\check{T} * \varphi * \psi(0) = \langle \check{T} * \varphi, \check{\psi} \rangle = \langle \check{T}, \check{\psi} * \check{\varphi} \rangle = \langle T * \psi, \varphi \rangle$$

we get that  $T * \psi \in L^1$  for every  $\psi \in \mathcal{C}^m$  with support in  $K$ .

Now, according to Lemma 2.3, we may write

$$\sum_{finite} X^\alpha F_\alpha = \delta_0$$

where the  $F_\alpha$ 's are of class  $C^m$  and are supported in  $K$ . It follows that

$$T = \sum_{finite} T * \partial^\alpha F_\alpha = \sum_{finite} \partial^\alpha (T * F_\alpha).$$

The first part of the proof shows that the  $T * F_\alpha$ 's are in  $L^1$  so that we obtain the desired representation formula.

$ii) \Rightarrow i)$  is obvious so that the proof is complete.  $\square$

**Definition.** Let  $\mathcal{B}_c := \mathcal{B}_c(\mathfrak{N})$  be the space  $\mathcal{B}(\mathfrak{N})$  endowed with the topology for which  $\varphi_n \rightarrow 0$  if,

- (i) for every left-invariant differential operator  $X^\alpha$ ,  $X^\alpha \varphi_n \rightarrow 0$  uniformly over compact sets,
- (ii) for every left-invariant differential operator  $X^\alpha$ , the  $X^\alpha \varphi_n$ 's are uniformly bounded.

The representation formula of  $T \in \mathcal{D}'_{L^1}$  given by the previous theorem shows that  $T$  can be extended to a continuous linear functional on  $\mathcal{B}_c$  so that  $\mathcal{D}'_{L^1}$  is also the dual of  $\mathcal{B}_c$ . For example, if we write  $T = f_0 + \sum_{|\alpha| \geq 1} X^\alpha f_\alpha$ , then

$$\langle T, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} = \langle f_0, 1 \rangle = \int_{\mathfrak{N}} f_0(\xi) d\lambda(\xi).$$

**3.2. The  $\mathcal{S}'$ -convolution.** Recall that if  $G \in \mathcal{S}'$  and  $\varphi \in \mathcal{S}$  then  $\check{G} * \varphi \in \mathcal{C}^\infty$  so that the following definition makes sense:

**Definition.** Let  $F, G \in \mathcal{S}'(\mathfrak{N})$ , we will say that they are  $\mathcal{S}'$ -convolvable if, for every  $\varphi \in \mathcal{S}(\mathfrak{N})$ ,  $(\varphi * \check{G})F \in \mathcal{D}'_{L^1}$ . If this is the case, we define

$$\langle F * G, \varphi \rangle = \langle (\varphi * \check{G})F, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}.$$

If  $F, G \in \mathcal{S}(\mathfrak{N})$ , then  $F$  and  $G$  are  $\mathcal{S}'$ -convolvable and the above definition coincides with the usual one. Indeed, for every  $\varphi \in \mathcal{S}(\mathfrak{N})$ ,

$$\begin{aligned} \langle F * G, \varphi \rangle &= \int_{\mathfrak{N}} F * G(\eta) \varphi(\eta) d\lambda(\eta) \\ &= \int_{\mathfrak{N}} \int_{\mathfrak{N}} F(\xi) G(\xi^{-1}\eta) \varphi(\eta) d\lambda(\xi) d\lambda(\eta) \\ &= \int_{\mathfrak{N}} \left( \int_{\mathfrak{N}} \varphi(\eta) \check{G}(\eta^{-1}\xi) d\lambda(\eta) \right) F(\xi) \cdot 1 d\lambda(\xi) \\ &= \langle (\varphi * \check{G})F, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}. \end{aligned}$$

*Remark :* There are various ways to define the  $\mathcal{S}'$ -convolution that extend the definition for functions. For  $S, T \in \mathcal{D}'(\mathfrak{N})$ , let us cite the following:

- (1)  $S$  and  $T$  are  $\mathcal{S}'_1$ -convolvable if, for every  $\varphi \in \mathcal{D}(\mathfrak{N})$ ,  $S_x \otimes T_y \varphi(xy) \in \mathcal{D}'_{L^1}(\mathfrak{N} \otimes \mathfrak{N})$ . The  $\mathcal{S}'_1$ -convolution of  $S$  and  $T$  is then defined by

$$\langle S *_1 T, \varphi \rangle = \langle S_x \otimes T_y \varphi(xy), 1 \rangle_{\mathcal{D}'_{L^1}(\mathfrak{N} \otimes \mathfrak{N}), \mathcal{B}_c(\mathfrak{N} \otimes \mathfrak{N})}.$$

- (2)  $S$  and  $T$  are  $\mathcal{S}'_2$ -convolvable if, for every  $\varphi \in \mathcal{D}$ ,  $S(\check{T} * \varphi) \in \mathcal{D}'_{L^1}(\mathfrak{N})$

$$\langle S *_2 T, \varphi \rangle = \langle S(\check{T} * \varphi), 1 \rangle_{\mathcal{D}'_{L^1}(\mathfrak{N}), \mathcal{B}_c(\mathfrak{N})}.$$

- (3)  $S$  and  $T$  are  $\mathcal{S}'_3$ -convolvable if, for every  $\varphi, \psi \in \mathcal{S}(\mathfrak{N})$ ,  $(\check{S} * \varphi)(T * \check{\psi}) \in L^1(\mathfrak{N})$ . The  $\mathcal{S}'_3$ -convolution of  $S$  and  $T$  is then defined by

$$\langle S *_3 T, \varphi * \psi \rangle = \int_{\mathfrak{N}} (\check{S} * \varphi)(\eta)(T * \check{\psi})(\eta) d\lambda(\eta).$$

It turns out that in the Euclidean case, all four definitions are equivalent and lead to the same convolution [Sh]. There are various obstructions to prove this in our situation, mostly stemming from the fact that left and right-invariant derivatives differ.

Also, one may replace the  $\mathcal{D}'_{L^1}$  space by the similar one defined with the help of right-invariant derivatives. We will here stick to the choice given in the definition above as it seems to us that this is the definition that gives the most satisfactory results.

One difficulty that arises is that the derivative of a convolution is not easily linked to the convolution of a derivative. Here is an illustration of what may be done and of the difficulties that arise. We hope that this will convince the reader that several facts that seem obvious (and are for usual convolutions of functions) need to be proved, *e.g.* that  $T * \mathbb{P}_a$  is harmonic if  $\mathbb{P}_a$  is.

**Lemma 3.2.** *Let  $S, T \in \mathcal{D}'(\mathfrak{N})$  and let  $Y$  be a right-invariant differential operator of first order. If  $S$  and  $T$  are  $\mathcal{S}'$ -convolvable, if  $YS$  and  $T$  are  $\mathcal{S}'$ -convolvable and if, for all  $\varphi \in \mathcal{S}(\mathfrak{N})$ ,  $Y((\varphi * \check{T})S) \in \mathcal{D}'_{L^1}(\mathfrak{N})$ , then*

$$Y(S * T) = (YS) * T.$$

*Proof.* As  $(Yf)g = Y(fg) - fYg$ , we get that

$$\begin{aligned} \langle Y(S * T), \varphi \rangle &= -\langle S * T, Y\varphi \rangle = -\langle ((Y\varphi) * \check{T})S, 1 \rangle \\ &= -\langle Y(\varphi * \check{T})S, 1 \rangle = -\langle Y((\varphi * \check{T})S), 1 \rangle + \langle (\varphi * \check{T})YS, 1 \rangle \\ &= 0 + \langle (YS) * T, \varphi \rangle \end{aligned}$$

the next to last equality being justified by the assumptions on  $F, G$ . □

Using this lemma inductively gives

$$Y^\alpha(S * T) = (Y^\alpha S) * T$$

provided all intermediate steps satisfy the assumption of the lemma. This is the case if  $S$  is compactly supported.

**3.3. Weighed spaces of distributions.** We will need the following weighted space of integrable distributions, introduced in the Euclidean setting in [Ho1, Ho2, Or].

**Definition.** Given  $\mu \in \mathbb{R}$  we consider

$$\omega_\mu \mathcal{D}'_{L^1}(\mathfrak{N}) := \omega_\mu \mathcal{D}'_{L^1}(\mathfrak{N}) = \{T \in \mathcal{S}'(\mathfrak{N}) : \omega_{-\mu} T \in \mathcal{D}'_{L^1}(\mathfrak{N})\}$$

with the topology induced by the map

$$\begin{aligned} \omega_\mu \mathcal{D}'_{L^1}(\mathfrak{N}) &\rightarrow \mathcal{D}'_{L^1}(\mathfrak{N}) \\ T &\mapsto \omega_{-\mu} T. \end{aligned}$$

This space admits an other representation given in the following lemma:

**Lemma 3.3.** Given  $\mu \in \mathbb{R}$ , we have

$$(3.8) \quad \omega_\mu \mathcal{D}'_{L^1}(\mathfrak{N}) = \left\{ T \in \mathcal{S}'(\mathfrak{N}) : T = \sum_{finite} X^\alpha g_\alpha, \text{ where } g_\alpha \in L^1(\mathfrak{N}, \omega_{-\mu} d\lambda) \right\}.$$

*Proof.* Let us temporarily indicate with  $\mathcal{V}$  the right hand side of (3.8). Given  $T \in \mathcal{V}$ , we can write  $T = \sum_{finite} X^\alpha (\omega_\mu f_\alpha)$ , where  $f_\alpha \in L^1$ . But then,

$$\begin{aligned} T &= \sum_{finite} \sum_{0 \leq \beta \leq \alpha} \Lambda_{\alpha, \beta} X^{\alpha - \beta} \omega_\mu X^\beta f_\alpha \\ &= \omega_\mu \sum_{finite} \sum_{0 \leq \beta \leq \alpha} \Lambda_{\alpha, \beta} \omega_{-\mu} X^{\alpha - \beta} (\omega_\mu) X^\beta f_\alpha. \end{aligned}$$

By definition, the distribution  $X^\beta f_\alpha$  belongs to  $\mathcal{D}'_{L^1}$ . Moreover, an easy computation shows that the function  $\omega_{-\mu} X^{\alpha - \beta} (\omega_\mu)$  belongs to the space  $\mathcal{B}$ . Since  $\mathcal{D}'_{L^1}$  is closed under multiplication by functions in  $\mathcal{B}$ , we conclude that  $T$  belongs to  $\omega_\mu \mathcal{D}'_{L^1}$ .

Conversely, given  $T \in \omega_\mu \mathcal{D}'_{L^1}$  we can write, by definition,  $T = \omega_\mu \sum_{finite} X^\alpha f_\alpha$ , where  $f_\alpha \in L^1$  or,  $T = \omega_\mu \sum_{finite} X^\alpha (\omega_{-\mu} g_\alpha)$ , where  $g_\alpha \in L^1(\omega_{-\mu} d\lambda)$ . Now, given  $\varphi \in \mathcal{S}$ , the pairing

$\langle T, \varphi \rangle_{\mathcal{S}', \mathcal{S}}$  can be written as

$$\sum_{finite} (-1)^{|\alpha|} \left\langle g_\alpha, \omega_{-\mu} \tilde{X}^\alpha (\omega_\mu \varphi) \right\rangle_{\mathcal{S}', \mathcal{S}} = \sum_{finite} \sum_{0 \leq \beta \leq \alpha} (-1)^{|\alpha|} \tilde{\Lambda}_{\alpha, \beta} \left\langle g_\alpha, \omega_{-\mu} \left( \tilde{X}^{\alpha - \beta} \omega_\mu \right) \tilde{X}^\beta \varphi \right\rangle_{\mathcal{S}', \mathcal{S}}.$$

We observe that for each multi-indexes  $\alpha$  and  $\beta$ , the function

$$b_{\alpha, \beta} = (-1)^{|\alpha|} \tilde{\Lambda}_{\alpha, \beta} \omega_{-\mu} \left( \tilde{X}^{\alpha - \beta} \omega_\mu \right)$$

belongs to  $\mathcal{B}$ . Thus,

$$\langle T, \varphi \rangle_{\mathcal{S}', \mathcal{S}} = \sum_{\alpha, \beta} (-1)^{|\beta|} \left\langle X^\beta \left( (-1)^{|\beta|} b_{\alpha, \beta} g_\alpha \right), \varphi \right\rangle_{\mathcal{S}', \mathcal{S}}$$

or,

$$T = \sum_{\alpha, \beta} (-1)^{|\beta|} X^\beta \left( (-1)^{|\beta|} b_{\alpha, \beta} g_\alpha \right).$$

To conclude that the distribution  $T$  belongs to  $\mathcal{V}$  we only need to observe that  $L^1(\omega_{-\mu}d\lambda)$  is closed under multiplication by functions in  $\mathcal{B}$ . This completes the proof of Lemma 3.3.  $\square$

As an immediate corollary, we get that

**Corollary 3.4.** *The space  $\omega_{\mu}\mathcal{D}'_{L^1}(\mathfrak{N})$  is closed under the action of left-invariant differential operators  $X^{\alpha}$  and under multiplication by functions in  $\mathcal{B}$ .*

#### 4. DISTRIBUTIONS THAT ARE $\mathcal{S}'$ -CONVOLVABLE WITH THE POISSON KERNEL

**4.1. Extensions of distributions with the Poisson kernel.** We are now in position to prove the following:

**Theorem 4.1.** *Let  $T \in \mathcal{S}'$  and  $\mathbb{P}$  be kernel with property  $(\mathcal{R}_{\Gamma})$ . Then the following are equivalent:*

- (i)  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}$ ,
- (ii)  $T$  is  $\mathcal{S}'$ -convolvable with  $\mathbb{P}_a$  for some  $a > 0$ ,
- (iii)  $T$  is  $\mathcal{S}'$ -convolvable with  $\mathbb{P}_a$  for each  $a > 0$ .

*Proof.* It is of course enough to prove equivalence between (i) and (ii), the equivalence with (iii) will then automatically follow.

Let us assume that  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}$ . We want to show that, if  $\varphi \in \mathcal{S}$ ,  $(\varphi * \check{\mathbb{P}}_a)T \in \mathcal{D}'_{L^1}$ . It is enough to show that  $(\varphi * \check{\mathbb{P}}_a)\omega_{Q+\Gamma} \in \mathcal{B}$ . But, for a left-invariant derivative  $X^{\alpha}$  and  $\beta \leq \alpha$ ,

$$X^{\alpha-\beta}(\varphi * \check{\mathbb{P}}_a)(\eta) = \varphi * (X^{\alpha-\beta}\check{\mathbb{P}}_a)(\eta) = \int_{\mathfrak{N}} X^{\alpha-\beta}\check{\mathbb{P}}_a(\eta\xi^{-1})\varphi(\xi) d\lambda(\xi)$$

so that

$$\begin{aligned} |X^{\alpha-\beta}(\varphi * \check{\mathbb{P}}_a)(\eta)| &\leq C \int_{\mathfrak{N}} \frac{1}{(1 + |\eta\xi^{-1}|)^{Q+\Gamma+d(\alpha)-d(\beta)}} \varphi(\xi) d\lambda(\xi) \\ &\leq C\omega_{-Q-\Gamma-d(\alpha)+d(\beta)}(\eta) \int_{\mathfrak{N}} (1 + |\xi|)^{Q+\Gamma+d(\alpha)-d(\beta)} \varphi(\xi) d\lambda(\xi) \end{aligned}$$

by Petree's inequality. As  $|X^{\beta}\omega_{Q+\Gamma}| \leq C_{\beta}\omega_{Q+\Gamma-d(\beta)}$ , it follows from Leibnitz' Rule that  $(\varphi * \check{\mathbb{P}}_a)\omega_{Q+\Gamma} \in \mathcal{B}$ . The first part of the proof is thus complete.

Conversely, let us assume that  $T$  is  $\mathcal{S}'$ -convolvable with  $P_a$  and fix  $\varphi \in \mathcal{S}$ , a non-negative function supported in  $B(0, 2)$  and such that  $\varphi = 1$  on  $B(0, 1)$ . Then

$$\begin{aligned} \varphi * \check{\mathbb{P}}_a(\eta) &= \int_{\mathfrak{N}} \varphi(\xi)\mathbb{P}_a(\eta^{-1}\xi) d\lambda(\xi) \\ &\geq C(a) \int_{B(0,1)} \frac{1}{(1 + |\eta^{-1}\xi|)^{Q+\Gamma}} d\lambda(\xi). \end{aligned}$$

But, for  $\xi \in B(0, 1)$ ,  $|\eta^{-1}\xi| \leq (|\eta| + |\xi|) \leq (1 + |\eta|)$ . It follows that,

$$\varphi * \check{\mathbb{P}}_a(\eta) \geq \frac{C(a)}{(1 + |\eta|)^{Q+\Gamma}} \geq C(a)\omega_{-Q-\Gamma}(\eta).$$



As we have already shown that  $\omega_{Q+\Gamma}\varphi * \check{\mathbb{P}}_a(\eta) \in \mathcal{B}$ , we get that  $\frac{1}{\omega_{Q+\Gamma}\varphi * \check{\mathbb{P}}_a(\eta)} \in \mathcal{B}$ . Finally, writing

$$T = \omega_{Q+\Gamma} \frac{1}{\omega_{Q+\Gamma}\varphi * \check{\mathbb{P}}_a(\eta)} (\varphi * \check{\mathbb{P}}_a)(\eta) T$$

gives the desired representation since, by hypothesis,  $(\varphi * \check{\mathbb{P}}_a)T \in \mathcal{D}'_{L^1}$ .  $\square$

**4.2. Regularity of the  $\mathcal{S}'$ -convolution of a distribution and the Poisson kernel.** We may now prove the following lemma, which allows us to represent  $T * P_a$  as a function:

**Lemma 4.2.** *Let  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}(\mathfrak{N})$  and  $\mathbb{P}$  be a kernel with property  $(\mathcal{R}_\Gamma)$ . Then, the  $\mathcal{S}'$ -convolution of  $T$  with the kernel  $\mathbb{P}_a$  is the function given by*

$$(4.9) \quad \eta \mapsto \langle \omega_{-Q-\Gamma}(\cdot)T, \omega_{Q+\Gamma}(\cdot)\check{\mathbb{P}}_a(\eta^{-1}\cdot) \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}$$

*Proof.* First note that  $\xi \mapsto \omega_{Q+\Gamma}(\xi)\check{\mathbb{P}}_a(\eta^{-1}\xi)$  is in  $\mathcal{B}$  and  $\omega_{-Q-\Gamma}(\cdot)T \in \mathcal{D}'_{L^1}$  so that (4.9) makes sense.

We want to prove that, if  $T = \sum_{finite} \omega_{Q+\Gamma}X^\alpha f_\alpha$  with  $f_\alpha \in L^1$ , and if  $\varphi \in \mathcal{S}$ , then

$$\langle T * \mathbb{P}_a, \varphi \rangle := \langle (\varphi * \check{\mathbb{P}}_a)T, 1 \rangle \text{ is equal to } \left\langle \langle \omega_{-Q-\Gamma}(\cdot)T, \omega_{Q+\Gamma}(\cdot)\check{\mathbb{P}}_a(\eta^{-1}\cdot) \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c}, \varphi(\eta) \right\rangle.$$

By linearity, it is enough to consider only one term in the sum,  $T = \omega_{Q+\Gamma}X^\alpha f$  with  $f \in L^1(\mathfrak{N})$ . But then

$$\begin{aligned} \langle \omega_{Q+\Gamma}(\varphi * \check{\mathbb{P}}_a)X^\alpha f, 1 \rangle &= (-1)^{|\alpha|} \left\langle f, \tilde{X}^\alpha(\omega_{Q+\Gamma}(\varphi * \check{\mathbb{P}}_a)) \right\rangle \\ &= (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta} \int_{\mathfrak{N}} f(\xi) \tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma}(\xi) \tilde{X}^\beta(\varphi * \check{\mathbb{P}}_a)(\xi) d\lambda(\xi). \end{aligned}$$

Further, we have

$$\tilde{X}^\beta(\varphi * \check{\mathbb{P}}_a) = \varphi * (\tilde{X}^\beta \check{\mathbb{P}}_a)(\xi) = \int_{\mathfrak{N}} \varphi(\eta) (\tilde{X}^\beta \check{\mathbb{P}}_a)(\eta^{-1}\xi) d\lambda(\eta).$$

It follows that  $\langle \omega_{Q+\Gamma}(\varphi * \check{\mathbb{P}}_a)X^\alpha f, 1 \rangle$  is

$$\begin{aligned} (4.10) \quad &= (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma}(\xi) (\tilde{X}^\beta \check{\mathbb{P}}_a)(\eta^{-1}\xi) d\lambda(\xi) \varphi(\eta) d\lambda(\eta) \\ &= (-1)^{|\alpha|} \int_{\mathfrak{N}} \left( \int_{\mathfrak{N}} f(\xi) \tilde{X}^\alpha(\omega_{Q+\Gamma}(\cdot)\check{\mathbb{P}}_a(\eta^{-1}\cdot))(\xi) d\lambda(\xi) \right) \varphi(\eta) d\lambda(\eta) \end{aligned}$$

using  $(\tilde{X}^\beta \check{\mathbb{P}}_a)(\eta^{-1}\xi) = \tilde{X}_\xi^\beta \check{\mathbb{P}}_a(\eta^{-1}\xi)$  and Leibnitz' Rule. Thus

$$\langle \omega_{Q+\Gamma}(\varphi * \check{\mathbb{P}}_a)X^\alpha f, 1 \rangle = \int_{\mathfrak{N}} \langle X^\alpha f(\xi), \omega_{Q+\Gamma}(\xi)\check{\mathbb{P}}_a(\eta^{-1}\xi) \rangle \varphi(\eta) d\lambda(\eta)$$

as claimed.

All interversions of integrals are easily justified by the fact that  $\omega_{Q+\Gamma}\check{\mathbb{P}}_a \in \mathcal{B}$ .  $\square$

**Corollary 4.3.** *Let  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}$  and  $\mathbb{P}$  be a kernel with property  $(\mathcal{R}_\Gamma)$ . Then the function  $T * \mathbb{P}_a$  is smooth. Moreover, for any left-invariant derivative  $X^\alpha$ ,  $T$  is  $\mathcal{S}'$ -convolvable with  $X^\alpha\mathbb{P}_a$  and  $X^\alpha(T * \mathbb{P}_a) = T * (X^\alpha\mathbb{P}_a)$  and for any  $k \in \mathbb{N}$ ,  $T$  is  $\mathcal{S}'$ -convolvable with  $(a\partial_a)^k\mathbb{P}_a$  and  $(a\partial_a)^k(T * \mathbb{P}_a) = T * ((a\partial_a)^k\mathbb{P}_a)$ . In particular,  $T * \mathbb{P}_a$  is harmonic if  $\mathbb{P}$  is.*

*Proof.* As the proof of the implication  $i) \Rightarrow ii)$  of Theorem 4.1 only depends on the estimates of the Poisson kernel from Section 2.5, we get with the same proof that if  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}$  then  $T$  is  $\mathcal{S}'$ -convolvable with  $X^\alpha\mathbb{P}_a$  and  $(a\partial_a)^k\mathbb{P}_a$ .

For the other assertions, we may again assume that  $T = \omega_{Q+\Gamma}X^\alpha f$ . As  $T * \mathbb{P}_a$  is a function, from (4.10) in the proof of the previous lemma, we get that

$$T * \mathbb{P}_a(\eta) = (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha,\beta} \int_{\mathfrak{N}} f(\xi) \tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma}(\xi) \tilde{X}_\xi^\beta (\check{\mathbb{P}}_a(\eta^{-1}\xi)) d\lambda(\xi).$$

It then remains to differentiate with respect to  $\eta$  under the integral.  $\square$

We will need the space  $\mathcal{D}_{L^1(\omega_\mu)}$  of all functions  $\varphi \in C^\infty$  such that, for every left-invariant partial differential operator  $X^\alpha$ ,  $X^\alpha\varphi \in L^1(\omega_\mu d\lambda)$  endowed with the topology given by the family of semi-norms

$$\|\varphi\|_{\alpha,\mu} = \sum_{\beta \leq \alpha} \left\| X^\beta \varphi \right\|_{L^1(\omega_\mu d\lambda)}.$$

We may get a more precise estimate of the Poisson integrals at fixed level.

**Proposition 4.4.** *Let  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}(\mathfrak{N})$ . For each  $a > 0$ , the  $\mathcal{S}'$ -convolution  $T * \mathbb{P}_a$  belongs to  $\mathcal{D}_{L^1(\omega_{-Q-\Gamma}d\lambda)}$ .*

*Proof.* By linearity, it is enough to prove that, if  $T = \omega_{Q+\Gamma}X^\alpha f$  for some  $f \in L^1(\mathfrak{N})$ , then  $X^\iota(T * \mathbb{P}_a) = T * X^\iota\mathbb{P}_a \in \omega_{Q+\Gamma}\mathcal{D}_{L^1}$ . But, from (4.9), we get that

$$\begin{aligned} T * X^\iota\mathbb{P}_a(\eta) &= (-1)^{|\alpha|} \left\langle f, \tilde{X}^\alpha(\omega_{Q+\Gamma}(\cdot)X^\iota\check{\mathbb{P}}_a(\eta^{-1}\cdot)) \right\rangle \\ (4.11) \quad &= (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha,\beta} \int_{\mathfrak{N}} f(\xi) \tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma}(\xi) \tilde{X}_\xi^\beta X^\iota\check{\mathbb{P}}_a(\eta^{-1}\xi) d\lambda(\xi) \end{aligned}$$

using Leibnitz' Formula and the facts that  $f \in L^1$  and  $\omega_{Q+\Gamma}(\cdot)X^\iota\check{\mathbb{P}}_a(\eta^{-1}\cdot) \in \mathcal{B}$ . Using the estimates

$$|\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma}(\xi)| \leq C(\alpha, \beta) \omega_{Q+\Gamma-d(\alpha)+d(\beta)}(\xi)$$

and

$$|\tilde{X}_\xi^\beta X^\iota\check{\mathbb{P}}_a(\eta^{-1}\xi)| \leq C(\alpha, \beta, a) \omega_{-Q-\Gamma-d(\beta)-d(\iota)}(\eta^{-1}\xi)$$

we see that the  $L^1(\omega_{-Q-\Gamma}d\lambda)$ -norm of each term of the sum in (4.11) is bounded by

$$\begin{aligned} &C \int_{\mathfrak{N}} \omega_{-Q-\Gamma}(\eta) \int_{\mathfrak{N}} |f(\xi)| \omega_{Q+\Gamma-d(\alpha)+d(\beta)}(\xi) \omega_{-Q-\Gamma-d(\beta)-d(\iota)}(\eta^{-1}\xi) d\lambda(\xi) d\lambda(\eta) \\ &= C \int_{\mathfrak{N}} |f(\xi)| \omega_{Q+\Gamma-d(\alpha)+d(\beta)}(\xi) \int_{\mathfrak{N}} \omega_{-Q-\Gamma}(\eta) \omega_{-Q-\Gamma-d(\beta)-d(\iota)}(\eta^{-1}\xi) d\lambda(\eta) d\lambda(\xi) \\ &\leq C \int_{\mathfrak{N}} |f(\xi)| \omega_{-d(\alpha)+d(\beta)}(\xi) d\lambda(\xi) \end{aligned}$$

with Lemma 2.2. As  $d(\beta) \leq d(\alpha)$  we get the desired result.  $\square$

**4.3. The Dirichlet problem in  $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$ .** We will now prove that  $T$  is the boundary value of  $T * \mathbb{P}_a$  in the  $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$  sense.

**Theorem 4.5.** *Let  $T \in \omega_{Q+\Gamma}\mathcal{D}'_{L^1}$ , then the convolution  $T * \mathbb{P}_a$  converges to  $T$  in  $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$  when  $a \rightarrow 0^+$ .*

*Proof.* We want to prove that, for  $\varphi \in \dot{\mathcal{B}}$ ,

$$(4.12) \quad \langle \omega_{-Q-\Gamma}(T * \mathbb{P}_a), \varphi \rangle_{\mathcal{D}'_{L^1}, \dot{\mathcal{B}}} \rightarrow \langle \omega_{-Q-\Gamma}T, \varphi \rangle_{\mathcal{D}'_{L^1}, \dot{\mathcal{B}}}$$

when  $a \rightarrow 0$ . It is of course enough to consider  $T = \omega_{Q+\Gamma}X^\alpha f$  with  $f \in L^1$ .

Write  $\varphi_{-Q-\Gamma} = \omega_{-Q-\Gamma}\varphi$  and  $\xi\varphi_{-Q-\Gamma}(\eta) = \varphi_{-Q-\Gamma}(\xi\eta)$ . Then  $\langle \omega_{-Q-\Gamma}(T * \mathbb{P}_a), \varphi \rangle_{\mathcal{D}'_{L^1}, \dot{\mathcal{B}}}$  is

$$\begin{aligned} &= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \tilde{X}_\xi^\alpha(\omega_{Q+\Gamma}(\xi)\tilde{\mathbb{P}}_a(\eta^{-1}\xi)) \varphi_{-Q-\Gamma}(\eta) \, d\lambda(\eta) \, d\lambda(\xi) \\ &= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta}\omega_{Q+\Gamma})(\xi) \tilde{X}_\xi^\beta(\tilde{\mathbb{P}}_a(\eta^{-1}\xi)) \varphi_{-Q-\Gamma}(\eta) \, d\lambda(\eta) \, d\lambda(\xi) \\ &= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta}\omega_{Q+\Gamma})(\xi) (\tilde{X}^\beta \tilde{\mathbb{P}}_a)(\eta^{-1}) \varphi_{-Q-\Gamma}(\xi\eta) \, d\lambda(\eta) \, d\lambda(\xi) \\ &= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta}\omega_{Q+\Gamma})(\xi) (-1)^{|\beta|} \tilde{Y}^\beta \mathbb{P}_a(\eta)_\xi \varphi_{-Q-\Gamma}(\eta) \, d\lambda(\eta) \, d\lambda(\xi). \end{aligned}$$

Now let  $\psi$  be a smooth cut-off function such that  $\psi(\eta) = 1$  if  $|\eta| \leq 1$  and  $\psi(\eta) = 0$  if  $|\eta| \geq 2$  and write  $\tilde{\psi} = 1 - \psi$ . Then  $\langle \omega_{-Q-\Gamma}(T * \mathbb{P}_a), \varphi \rangle_{\mathcal{D}'_{L^1}, \dot{\mathcal{B}}} = S_1 + S_2$  where  $S_1$  is

$$(-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta}\omega_{Q+\Gamma})(\xi) (-1)^{|\beta|} \tilde{Y}^\beta \mathbb{P}_a(\eta)_\xi \varphi_{-Q-\Gamma}(\eta) \psi(\eta) \, d\lambda(\eta) \, d\lambda(\xi)$$

while  $S_2 = (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta} S_2^\beta$  with

$$S_2^\beta = \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) (\tilde{X}^{\alpha-\beta}\omega_{Q+\Gamma})(\xi) (-1)^{|\beta|} \tilde{Y}^\beta \mathbb{P}_a(\eta)_\xi \varphi_{-Q-\Gamma}(\eta) \tilde{\psi}(\eta) \, d\lambda(\eta) \, d\lambda(\xi).$$

Let us first show that  $S_2 \rightarrow 0$ , that is each  $S_2^\beta \rightarrow 0$ . As, for  $|\eta| \geq 1$ ,

$$(1 + |\eta|/a)^{-Q-\Gamma-d(\beta)} \leq a^{Q+\Gamma+d(\beta)} |\eta|^{-Q-\Gamma-d(\beta)} \leq Ca^{Q+\Gamma+d(\beta)} (1 + |\eta|)^{-Q-\Gamma-d(\beta)},$$

so that using the estimates of derivatives of  $\mathbb{P}_a$  and  $\omega_{Q+\Gamma}$ , we get that

$$\begin{aligned}
|S_2^\beta| &\leq C \int_{\mathfrak{N}} |f(\xi)| (1 + |\xi|)^{Q+\Gamma-d(\alpha)+d(\beta)} \times \\
&\quad \times \int_{|\eta| \geq 1} a^{Q+d(\beta)} (1 + |\eta|/a)^{-Q-\Gamma-d(\beta)} (1 + |\xi\eta|)^{-Q-\Gamma} d\lambda(\eta) d\lambda(\xi) \\
&\leq C a^\Gamma \int_{\mathfrak{N}} |f(\xi)| (1 + |\xi|)^{Q+\Gamma-d(\alpha)+d(\beta)} \times \\
&\quad \times \int_{|\eta| \geq 1} (1 + |\eta|)^{-Q-\Gamma-d(\beta)} (1 + |\xi\eta|)^{-Q-\Gamma} d\lambda(\eta) d\lambda(\xi) \\
&\leq C a^\Gamma \|f\|_{L^1}
\end{aligned}$$

with Lemma 2.2. It follows that  $S_2 \rightarrow 0$ .

Let us now turn to  $S_1$ . First, from (2.1),

$$\begin{aligned}
S_1 &= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha,\beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \times \\
&\quad \times (-1)^{|\beta|} \sum_{\iota \in \mathcal{I}_\beta} \tilde{\mathbb{Q}}_{\beta,\iota}(\eta) X^\iota \mathbb{P}_a(\eta) \xi \varphi_{-Q-\Gamma}(\eta) \psi(\eta) d\lambda(\eta) d\lambda(\xi) \\
&= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha,\beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \times \\
&\quad \times (-1)^{|\beta|} \mathbb{P}_a(\eta) \sum_{\iota \in \mathcal{I}_\beta} (-1)^{|\iota|} \tilde{X}^\iota (\tilde{\mathbb{Q}}_{\beta,\iota} \xi \varphi_{-Q-\Gamma} \psi)(\eta) d\lambda(\eta) d\lambda(\xi) \\
&= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha,\beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \times \\
&\quad \times (-1)^{|\beta|} \mathbb{P}_a(\eta) \sum_{\iota \in \mathcal{I}_\beta} (-1)^{|\iota|} \sum_{\iota' \leq \iota} \tilde{\Lambda}_{\iota,\iota'} \tilde{X}^{\iota-\iota'} (\tilde{\mathbb{Q}}_{\beta,\iota} \xi \varphi_{-Q-\Gamma})(\eta) \tilde{X}^{\iota'} \psi(\eta) d\lambda(\eta) d\lambda(\xi).
\end{aligned}$$

Now, if  $\iota' \neq 0$ ,  $\tilde{X}^{\iota'} \psi$  is supported in  $1 \leq |\eta| \leq 2$ . Further, from Leibnitz' Rule,  $\varphi \in \mathcal{B}$  and Peetre's inequality we get that  $\tilde{X}^{\iota-\iota'} (\tilde{\mathbb{Q}}_{\beta,\iota} \xi \varphi_{-Q-\Gamma})(\eta)$  is bounded by  $C \omega_{-Q-\Gamma}(\xi)$  with  $C$  independent from  $\eta$ . It follows that

$$\left| \int_{\mathfrak{N}} \mathbb{P}_a(\eta) \tilde{X}^{\iota-\iota'} (\tilde{\mathbb{Q}}_{\beta,\iota} \xi \varphi_{-Q-\Gamma})(\eta) \tilde{X}^{\iota'} \psi(\eta) d\lambda(\eta) \right| \leq C \omega_{-Q-\Gamma}(\xi) \int_{1 \leq |\eta| \leq 2} \mathbb{P}_a(\eta) d\lambda(\eta).$$

Consequently, since this integral goes to 0, we have

$$\int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) (\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \mathbb{P}_a(\eta) \tilde{X}^{\iota-\iota'} (\tilde{\mathbb{Q}}_{\beta,\iota} \xi \varphi_{-Q-\Gamma})(\eta) \tilde{X}^{\iota'} \psi(\eta) d\lambda(\eta) d\lambda(\xi) \rightarrow 0.$$

This shows that  $S_1$  has same limit as

$$\begin{aligned}
S'_1 &:= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \times \\
&\quad \times (-1)^{|\beta|} \mathbb{P}_a(\eta) \sum_{\iota \in \mathcal{I}_\beta} (-1)^{|\iota|} \tilde{X}^\iota (\tilde{\mathbb{Q}}_{\beta, \iota} \xi \varphi_{-Q-\Gamma})(\eta) \psi(\eta) d\lambda(\eta) d\lambda(\xi) \\
&= (-1)^{|\alpha|} \int_{\mathfrak{N}} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \times \\
&\quad \times (-1)^{|\beta|} \mathbb{P}_a(\eta) \sum_{\iota \in \mathcal{I}_\beta} (-1)^{|\iota|} \sum_{\iota' \leq \iota} \tilde{\Lambda}_{\iota, \iota'} \tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}(\eta) \tilde{X}^\iota \xi \varphi_{-Q-\Gamma}(\eta) \psi(\eta) d\lambda(\eta) d\lambda(\xi).
\end{aligned}$$

Now, if  $\tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}$  is not a constant polynomial, then  $\tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}(0) = 0$  so that

$$(4.13) \quad \int_{\mathfrak{N}} \mathbb{P}_a(\eta) \tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}(\eta) \tilde{X}^\iota \xi \varphi_{-Q-\Gamma}(\eta) \psi(\eta) d\lambda(\eta)$$

goes to 0 when  $a \rightarrow 0$ , while if  $\tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}$  is constant, then, as  $\psi(0) = 1$ , this integral goes to

$$\tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}(0) \tilde{X}^\iota \xi \varphi_{-Q-\Gamma}(0).$$

Moreover, as (4.13) stays bounded by  $C\omega_{-Q-\Gamma}(\xi)$ , from the Dominated convergence theorem, we get that

$$\begin{aligned}
S'_1 &\rightarrow (-1)^{|\alpha|} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \times \\
&\quad \times (-1)^{|\beta|} \sum_{\iota \in \mathcal{I}_\beta} (-1)^{|\iota|} \sum_{\iota' \leq \iota} \tilde{\Lambda}_{\iota, \iota'} \tilde{X}^{\iota-\iota'} \tilde{\mathbb{Q}}_{\beta, \iota}(0) \tilde{X}^\iota \xi \varphi_{-Q-\Gamma}(0) d\lambda(\xi) \\
&= (-1)^{|\alpha|} \int_{\mathfrak{N}} f(\xi) \sum_{\beta \leq \alpha} \tilde{\Lambda}_{\alpha, \beta}(\tilde{X}^{\alpha-\beta} \omega_{Q+\Gamma})(\xi) \tilde{X}^\beta \xi \varphi_{-Q-\Gamma}(0) d\lambda(\xi)
\end{aligned}$$

where we have used Identity (2.3) in the last equality. But  $\tilde{X}^\beta \xi \varphi_{-Q-\Gamma}(0) = \tilde{X}^\beta \varphi_{-Q-\Gamma}(\xi)$  so that Leibnitz' Formula implies that this limit is

$$(-1)^{|\alpha|} \int_{\mathfrak{N}} f(\xi) \tilde{X}^\alpha (\omega_{Q+\Gamma} \varphi_{-Q-\Gamma})(\xi) d\lambda(\xi) = (-1)^{|\alpha|} \int_{\mathfrak{N}} f(\xi) \tilde{X}^\alpha \varphi(\xi) d\lambda(\xi) = \langle X^\alpha f, \varphi \rangle$$

as claimed.  $\square$

*Remark* : Assume that  $\mathbb{P}_a$  is harmonic for some left-invariant differential operator  $\mathcal{L}$ . The above result imply that given for a distribution  $T \in \omega_{Q+\Gamma} \mathcal{D}'_{L^1}$ , the function  $u = T * \mathbb{P}_a$  is a solution of the Dirichlet problem

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \mathfrak{S} \\ u|_{a=0} = T \end{cases}$$

where the boundary condition is now interpreted in the sense of convergence in  $\omega_{Q+\Gamma} \mathcal{D}'_{L^1}$  as  $a \rightarrow 0^+$ .

5. GLOBAL ESTIMATES FOR POISSON INTEGRALS OF DISTRIBUTIONS IN  $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$ 

In this section, we will prove that the Poisson integrals of distributions in  $\omega_{Q+\Gamma}\mathcal{D}'_{L^1}$  satisfy some global integrability conditions. In order to do so, we will use the following notations.

*Notation:* For a Borel set  $F \subset \mathfrak{S}$ , we denote by  $|F|$  its measure with respect to  $d\lambda da$ . A function on  $\mathfrak{S}$  is said to be in  $L^{1,\infty}(d\lambda da)$  if there exists a constant  $C$  such that, for all  $\alpha > 0$ ,

$$|\{(\eta, a) \in \mathfrak{S} : |f(\eta, a)| > \alpha\}| \leq \frac{C}{\alpha}.$$

For  $\Gamma \geq 1$ , let  $\Phi_\Gamma(\eta, a) = \frac{a^\Gamma}{(a + |\eta|)^{Q+\Gamma}}$  and note that  $\frac{1}{a}\Phi_\Gamma \in L^{1,\infty}(d\lambda da)$ . Indeed

$$\begin{aligned} |\{(\eta, a) : \frac{1}{a}\Phi_\Gamma(\eta, a) > \alpha\}| &= \int_0^{\alpha^{-1/(Q+1)}} |B(0, a^{(\Gamma-1)/(Q+\Gamma)}\alpha^{-1/(Q+\Gamma)} - a)| da \\ &= \int_0^{\alpha^{-1/(Q+1)}} \left( (a^{(\Gamma-1)/(Q+\Gamma)}\alpha^{-1/(Q+\Gamma)} - a) \right)^Q da \\ &= \frac{1}{\alpha} \int_0^1 \left( t^{(\Gamma-1)/(Q+\Gamma)} - t \right)^Q dt \end{aligned}$$

by changing variable  $t = a\alpha^{1/(Q+1)}$ . It should also be noted that  $\Phi_\Gamma \notin L^{1,\infty}\left(\frac{d\lambda da}{a}\right)$ .

We will denote by  $\mathcal{M}_\Gamma$  the set of measures  $\mu$  on  $\mathfrak{N}$  such that

$$\int_{\mathfrak{N}} (1 + |\xi|)^{-(Q+\Gamma)} d|\mu|(\xi) < +\infty.$$

For  $\mu \in \mathcal{M}_\Gamma$  and  $\eta \in \mathfrak{N}$ , let us denote by  $\mu_\eta$  the left translate of  $\mu$  by  $\eta$ , that is the measure defined by

$$\int_{\mathfrak{N}} \varphi(\xi) d\mu_\eta(\xi) = \int_{\mathfrak{N}} \varphi(\eta\xi) d\mu(\xi)$$

for all continuous functions  $\varphi$  with compact support on  $\mathfrak{N}$ . From Petree's inequality, we get that  $\mu_\eta \in \mathcal{M}_\Gamma$ . Further note that, if  $\mu \in \mathcal{M}_\Gamma$ , then

$$\mu(B(0, r)) \leq (1+r)^{Q+\Gamma} \int_{|\xi| < r} (1 + |\xi|)^{-(Q+\Gamma)} d\mu(\xi) \leq C(1+r)^{Q+\Gamma}.$$

We are now in position to prove the following:

**Theorem 5.1.** *Let  $\Gamma \geq 1$  and let  $\mathbb{P}$  be a kernel with property  $(\mathcal{R}_\Gamma)$ . If  $\mu \in \mathcal{M}_\Gamma$  then*

$$(5.14) \quad \frac{1}{a}(1 + a + |\eta|)^{-Q-\Gamma} \mu * \mathbb{P}_a(\eta) \in L^{1,\infty}(d\lambda da).$$

Moreover, for every  $a_0 > 0$ ,

$$(5.15) \quad (1 + a + |\eta|)^{-Q-\Gamma} a^{-\Gamma} \mu * \mathbb{P}_a(\eta) \chi_{\{(\eta, a) \in \mathfrak{S} : a > a_0\}}(\eta, a) \in L^\infty(d\lambda da).$$

*Proof.* Let  $E_l = \{(\eta, a) \in \mathfrak{S} \mid |\eta| \leq 1 \text{ and } a \leq 1\}$  and  $E_g = \mathfrak{S} \setminus E_l$  and let  $\eta_0 \in \mathfrak{N}$  be such that  $|\eta_0| \geq 2$ . Assume that we have proved that for every  $E \subset E_g$

$$\frac{1}{a}(1+a+|\eta|)^{-Q-\Gamma} \mu * \mathbb{P}_a(\eta) \chi_E \in L^{1,\infty}(\mathrm{d}\lambda \mathrm{d}a)$$

for every measure  $\mu \in \mathcal{M}_\Gamma$ . Without loss of generality we may assume that  $\mu$  is a positive measure. Applying this to  $E = \eta_0^{-1}E_l$  and to the left-translate  $\mu_0 \in \mathcal{M}_\Gamma$  of  $\mu$  by  $\eta_0^{-1}$  we get that

$$\begin{aligned} (1+a+|\eta|)^{-Q-\Gamma} a^{-1} \mu * \mathbb{P}_a(\eta) \chi_{E_l}(\eta, a) &= (1+a+|\eta|)^{-Q-\Gamma} a^{-1} \mu_0 * \mathbb{P}_a(\eta_0^{-1}\eta) \chi_{\eta_0^{-1}E_l}(\eta_0^{-1}\eta, a) \\ &\leq C(1+a+|\eta_0^{-1}\eta|)^{-Q-\Gamma} a^{-1} \mu_0 * \mathbb{P}_a(\eta_0^{-1}\eta) \chi_{\eta_0^{-1}E_l}(\eta_0^{-1}\eta, a) \\ &\in L^{1,\infty}(\mathrm{d}\lambda \mathrm{d}a). \end{aligned}$$

It is thus enough to prove that

$$(5.16) \quad \frac{1}{a}(1+a+|\eta|)^{-Q-\Gamma} \mu * \mathbb{P}_a(\eta) \chi_{E_g} \in L^{1,\infty}(\mathrm{d}\lambda \mathrm{d}a).$$

Note that if  $\mathbb{P}$  has property  $(\mathcal{R}_\Gamma)$  then  $(1+a+|\eta|)^{-Q-\Gamma} a^{-1} \mu * \mathbb{P}_a(\eta)$  is bounded by

$$\begin{aligned} &C \frac{a^{(\Gamma-1)}}{(1+a+|\eta|)^{Q+\Gamma}} \int_{\mathfrak{N}} \frac{\mathrm{d}\mu(\xi)}{(a+|\eta^{-1}\xi|)^{Q+\Gamma}} \\ &= C \frac{a^{(\Gamma-1)}}{(1+a+|\eta|)^{Q+\Gamma}} \left( \int_{|\xi| \leq \frac{1}{2}|\eta|} + \int_{\frac{1}{2}|\eta| \leq |\xi| \leq 2|\eta|} + \int_{2|\eta| \leq |\xi|} \right) \frac{\mathrm{d}\mu(\xi)}{(a+|\eta^{-1}\xi|)^{Q+\Gamma}} \\ &= I + II + III. \end{aligned}$$

Let us first estimate  $I$ . Note that, if  $|\xi| \leq \frac{1}{2}|\eta|$ , then

$$a + |\eta^{-1}\xi| \geq a + |\eta| - |\xi| \geq a + \frac{1}{2}|\eta| \geq C(a + |\eta|) \geq (1 + |\eta|)/2$$

since we only consider  $(\eta, a) \in E_g$ . It follows that

$$\begin{aligned} I \chi_{E_g} &\leq C \frac{a^{(\Gamma-1)}}{(1+a+|\eta|)^{Q+\Gamma}} \int_{|\xi| \leq \frac{1}{2}|\eta|} \mathrm{d}\mu(\xi) (1+|\eta|)^{-Q-\Gamma} \chi_{E_g} \\ &\leq C \frac{1}{a} \frac{a^\Gamma}{(1+a+|\eta|)^{Q+\Gamma}} \in L^{1,\infty}(\mathrm{d}\lambda \mathrm{d}a). \end{aligned}$$

Moreover, this computation also shows that  $a^{-\Gamma+1}I \in L^\infty(\mathrm{d}\lambda \mathrm{d}a)$ .

Let us now estimate  $III$ . Note that, if  $|\xi| \geq 2|\eta|$ , then

$$(5.17) \quad a + |\eta^{-1}\xi| \geq a + |\xi| - |\eta| \geq a + \frac{1}{2}|\xi| \geq C(a + |\xi|).$$

Further, as  $(\eta, a) \in E_g$  then either  $a \geq 1$  or  $|\eta| \geq 1$  in which case  $|\xi| \geq 2$ . Therefore  $a + |\eta^{-1}\xi| \geq C(1 + |\xi|)$ . It follows that

$$III \chi_{E_g} \leq C \frac{a^{(\Gamma-1)}}{(1+a+|\eta|)^{Q+\Gamma}} \int_{\mathfrak{N}} (1+|\xi|)^{-Q-\Gamma} \mathrm{d}\mu(\xi) \leq C \frac{1}{a} \frac{a^\Gamma}{(1+a+|\eta|)^{Q+\Gamma}} \in L^{1,\infty}(\mathrm{d}\lambda \mathrm{d}a).$$

Again, the same computation shows that  $a^{-\Gamma+1}III \in L^\infty(d\lambda da)$ .

We will now prove the result for  $II$ . To do so, notice first that, if  $a \geq a_0$  then

$$II \leq \frac{C}{a^{Q+1}} \int_{|\xi| \leq 2|\eta|} d\mu(\xi) \leq C \frac{1}{a} \frac{a^\Gamma}{a^{Q+\Gamma}(1+|\eta|)^{Q+\Gamma}} \leq Ca^{-Q-1}$$

and again  $a^{-\Gamma+1}II\chi_{a>a_0} \in L^\infty(d\lambda da)$ . It is thus enough to prove the result for  $II\chi_{a \leq a_0}$ .

Now note that, if  $\frac{1}{2}|\eta| \leq |\xi| \leq 2|\eta|$ , then  $(1+a+|\eta|)^{-Q-\Gamma} \leq C(1+|\xi|)^{-Q-\Gamma}$ , thus

$$II \leq C \frac{1}{a} \int_{\frac{1}{2}|\eta| \leq |\xi| \leq 2|\eta|} \Phi_\Gamma(\eta^{-1}\xi, a)(1+|\xi|)^{-Q-\Gamma} d\mu(\xi) = C \frac{1}{a} \int_{\frac{1}{2}|\eta| \leq |\xi| \leq 2|\eta|} \Phi_\Gamma(\eta^{-1}\xi, a) d\nu(\xi)$$

where  $\nu$  is a finite measure on  $\mathfrak{N}$ . Thus  $II$  is estimated with the help of the following proposition:

**Proposition 5.2.** *For every finite positive measure  $\nu$  on  $\mathfrak{N}$ , the function  $U_\nu$  defined on  $\mathfrak{S}$  by*

$$U_\nu(\eta, a) = \int_{\mathfrak{N}} \frac{a^{\Gamma-1}}{(a+|\eta^{-1}\xi|)^{Q+\Gamma}} d\nu(\xi)$$

belongs to  $L^{1,\infty}(d\lambda da)$ .

The proof will follow a simplified version of that of Theorem 1 in [Sj] which deals with the Euclidean case, for more general measures.

*Notation :* On  $\mathfrak{S}$  we denote by  $D_\infty$  the distance given by  $D_\infty((\eta, a), (\eta', a')) = \max(|\eta^{-1}\eta'|, |a-a'|)$ .

*Proof of Proposition 5.2.* We want to prove that  $U_\nu \in L^{1,\infty}(d\lambda da)$ , that is, that there exists a constant  $C \geq 0$  such that for all  $\alpha > 0$ ,

$$|\{(\eta, a) \in \mathfrak{S} : U_\nu(\eta, a) > \alpha\}| \leq \frac{C}{\alpha}.$$

For  $i_0$  a non-negative integer, let

$$K_0 = B(0, 2^{i_0}) \times ]0, 2^{i_0}].$$

It is enough to prove that there is a constant  $C \geq 0$ , independent of  $i_0$  such that, for all  $\alpha > 0$

$$|\{(\eta, a) \in \mathfrak{S} : U_\nu(\eta, a) > \alpha\} \cap K_0| \leq \frac{C}{\alpha}.$$

To do so, we will show that there is a constant  $C$  such that, for each  $\alpha > 0$ , we may construct a set  $S$  which satisfies the following properties:

- (i)  $|\{(\eta, a) : U_\nu > \alpha\} \cap K_0| \leq C|S|$ ;
- (ii)  $U_\nu(\eta, a) > \frac{\alpha}{C}$  for all  $(\eta, a) \in S$ ;
- (iii) for all  $\eta \in \mathfrak{N}$ ,

$$U_S(\eta) := \int_S \frac{a^{\Gamma-1}}{(a+|\eta^{-1}\xi|)^{Q+\Gamma}} d\lambda(\xi) da$$

satisfies  $U_S(\eta) \leq C$ .



Once this is done, we can conclude as follows

$$\begin{aligned} |\{(\eta, a) \in \mathfrak{S} : U_\nu(\eta, a) > \alpha\} \cap K_0| &\leq C|S| \leq \frac{C^2}{\alpha} \int_S U_\nu(\eta, a) \, d\lambda(\eta) \, da \\ &= \frac{C^2}{\alpha} \int_{\mathfrak{N}} U_S(\eta) \, d\nu(\eta) \\ &\leq \frac{C^3}{\alpha} \|\nu\| \end{aligned}$$

where we have respectively used Property i, ii, Fubini's Theorem and Property iii.

*Construction of the set  $S$ .*

We will use a dyadic covering of  $K_0$ :

- Set  $Q_i = B(0, 2^{i_0}) \times [2^{i_0-i-1}, 2^{i_0-i}]$ ,  $i = 0, 1, \dots$
- cover each  $Q_i$  by sets of the form

$$Q_{i,j} = B(\eta_{i,j}, 2^{i_0-i-3}) \times [2^{i_0-i-1}, 2^{i_0-i}]$$

in such a way that each element of  $Q_i$  belongs to at most  $\kappa$  sets  $Q_{i,j}$  where  $\kappa$  is a set that depends only on the group  $\mathfrak{N}$ . This is possible thanks to a covering lemma that may be found *e.g.* in [FS, Section 1.F].

We will order the  $Q_{i,j}$ 's by lexicographic order and define inductively the authorized pieces  $A_{i,j}$  and the associated set of forbidden pieces  $F_{i,j}$  as follows:

- $A_{i,j} = Q_{i,j}$  if
  - (a)  $|Q_{i,j} \cap \{(\eta, a) \in K_0 : U_\nu(\eta, a) > \alpha\}| > 0$ ,
  - (b) and  $Q_{i,j} \notin \bigcup_{(l,k) < (i,j)} F_{l,k}$ .

Else, we set  $A_{i,j} = \emptyset$ .

- if  $A_{i,j} \neq \emptyset$  we define the set of forbidden pieces as

$$F_{i,j} = \{Q_{l,k} : (l, k) > (i, j) \text{ and } D_\infty(Q_{l,k}, Q_{i,j}) < 2^{i_0-i+\frac{l-i}{Q+\Gamma}+1}\}.$$

Else we set  $F_{i,j} = \emptyset$ .

Note that the authorized pieces are disjoint and that, if  $A_{i,j} \neq \emptyset$ , then  $F_{i,j}$  has the following property:

$$(5.18) \quad \left| \bigcup_{Q \in F_{i,j}} Q \right| \leq C|A_{i,j}|.$$

*Proof of (5.18).* Assume that  $Q_{l,k} \in F_{i,j}$  and let  $(\eta, a) \in Q_{l,k}$ . Then

$$\begin{aligned} d(\eta, \eta_{i,j}) &\leq d(\eta, \mathfrak{N} \setminus B(\eta, 2^{i_0-l-3})) + D_\infty(Q_{l,k}, Q_{i,j}) + d(\eta_{i,j}, \mathfrak{N} \setminus B(\eta_{i,j}, 2^{i_0-i-3})) \\ &< 2^{i_0-l-3} + 2^{i_0-i+\frac{l-i}{Q+\Gamma}+1} + 2^{i_0-i-3} \\ &\leq 2^{i_0-i+\frac{l-i}{Q+\Gamma}+2}. \end{aligned}$$

It follows that

$$Q_{l,k} \subset B(\eta_{i,j}, C2^{i_0-i+\frac{l-i}{Q+\Gamma}}) \times [2^{i_0-l-1}, 2^{i_0-l}].$$

Now, as pieces of different order are disjoint,

$$\left| \bigcup_{Q \in F_{i,j}} Q \right| = \sum_{m=0}^{+\infty} \left| \bigcup_{Q_{i+m,k} \in F_{i,j}} Q_{i+m,k} \right|$$

and as those of a given order overlap at most  $\kappa$  times, this is

$$\begin{aligned} &\leq \kappa \sum_{m=0}^{+\infty} \left| B(\eta_{i,j}, C_2 2^{i_0-i+\frac{m}{Q+1}+1}) \times [2^{i_0-i-m-1}, 2^{i_0-i-m}] \right| \\ &\leq C \sum_{m=0}^{+\infty} 2^{(i_0-i+\frac{m}{Q+1})Q} 2^{i_0-i-m} \\ &= C 2^{(i_0-i)(Q+1)} \sum_{m=0}^{+\infty} 2^{-\frac{\Gamma}{Q+1}m} = C |A_{i,j}|. \end{aligned}$$

□

Finally, we set  $S = \bigcup_{(i,j)} A_{i,j}$ .

*Proof of Property i.*

By construction, the authorized and the forbidden pieces cover  $\{U_\nu > \lambda\} \cap K_0$  and as these overlap at most  $\kappa$  times, we obtain

$$\begin{aligned} |\{(\eta, a) \in K_0 : U_\nu(\eta, a) > \lambda\}| &\leq \sum_{(i,j)} \left( |A_{i,j}| + \sum_{Q \in F_{i,j}} |Q| \right) \\ &\leq (C+1) \sum_{(i,j)} |A_{i,j}| = (C+1) |S| \end{aligned}$$

where  $C$  is the constant in (5.18).

*Proof of Property ii.*

If  $(\eta, a) \in S$ , that is if  $(\eta, a) \in A_{i,j}$  for some  $(i, j)$  then there exists  $(\eta', a') \in A_{i,j}$  such that  $U_\nu(\eta', a') > \lambda$ . But then  $|\eta^{-1}\eta'| \leq 2^{i_0-i-2} \leq \frac{a'}{2}$  so that, for  $\xi \in \mathfrak{N}$ ,

$$\begin{aligned} a' + |\xi^{-1}\eta'| &\geq a' - |\eta^{-1}\eta'| + |\xi^{-1}\eta| \geq \frac{a'}{2} + |\xi^{-1}\eta| \\ &\geq \frac{1}{4}(a + |\xi^{-1}\eta|). \end{aligned}$$

From this, we immediately get that

$$\alpha < U_\nu(\eta', a') \leq C U_\nu(\eta, a).$$

*Proof of Property iii.*

Set  $S_i = \bigcup_j A_{i,j}$  the set of authorized pieces of order  $i$  and write

$$S_i = T_i \times [2^{i_0-i-1}, 2^{i_0-i}].$$

Set

$$U_i(\eta) = \int_{S_i} \frac{a^{\Gamma-1}}{(a + |\xi^{-1}\eta|)^{Q+\Gamma}} d\lambda(\xi) da$$

for the part of  $U_S$  issued from pieces of order  $i$ .

**Lemma 5.3.** *There exists a constant  $C_2$  such that, for all  $\eta \in \mathfrak{N}$  and all  $i \geq 0$ ,  $U_i(\eta) \leq C_2$ . Moreover, if  $p \geq -i$  and  $d(\eta, T_i) > 2^{i_0+p}$ , then  $U_i(\eta) \leq C_2 2^{-(p+i)\Gamma}$ .*

*Proof of Lemma 5.3.* By definition

$$(5.19) \quad U_i(\eta) = \int_{2^{i_0-i-1}}^{2^{i_0-i}} a^{\Gamma-1} \int_{T_i} \frac{d\lambda(\xi)}{(a + |\xi^{-1}\eta|)^{Q+\Gamma}} da$$

Thus

$$\begin{aligned} U_i(\eta) &= \int_{2^{i_0-i-1}}^{2^{i_0-i}} \frac{1}{a^{Q+1}} \int_{T_i} \frac{d\lambda(\xi)}{(1 + |\xi^{-1}\eta|/a)^{Q+\Gamma}} da \\ &= \int_{2^{i_0-i-1}}^{2^{i_0-i}} \frac{1}{a} \int_{T_i/a} \frac{d\lambda(\zeta)}{(1 + |\zeta^{-1} \cdot (\eta/a)|)^{Q+\Gamma}} da, \end{aligned}$$

changing variable  $\zeta = \xi/a$ . By translation invariance of  $d\lambda$  we thus get that

$$U_i(\eta) \leq \int_{2^{i_0-i-1}}^{2^{i_0-i}} \frac{1}{a} \int_{\mathfrak{N}} \frac{d\lambda(\zeta)}{(1 + |\zeta|)^{Q+\Gamma}} da \leq C_2.$$

Further, if  $d(\eta, T_i) > 2^{i_0+p}$  then, for  $\xi \in T_i$ ,  $|\xi^{-1}\eta| \geq 2^{i_0+p}$  so that, from (5.19), we deduce that

$$\begin{aligned} U_i(\eta) &\leq 2^{(i_0-i)\Gamma} \int_{T_i} \frac{d\lambda(\xi)}{(2^{i_0-i-1} + |\xi^{-1}\eta|)^{Q+\Gamma}} \\ &\leq 2^{(i_0-i)\Gamma} \int_{|\xi^{-1}\eta| \geq 2^{i_0+p}} \frac{d\lambda(\xi)}{|\xi^{-1}\eta|^{Q+\Gamma}} \\ &\leq 2^{(i_0-i)\Gamma} \int_{|\zeta| > 2^{i_0+p}} \frac{d\lambda(\zeta)}{|\zeta|^{Q+\Gamma}} \\ &= C 2^{-(i+p)\Gamma} \end{aligned}$$

when integrating in polar coordinates. □

Now, for every  $\eta \in \mathfrak{N}$ , there exists an  $m$  such that

$$(5.20) \quad C_2 2^{-m-1} < U_j(\eta) \leq C_2 2^{-m}$$

(where  $C_2$  is the constant of Lemma 5.3). From Lemma 5.3, we get that  $d(\eta, T_j) < 2^{i_0+(m+1)/\Gamma-j}$ .

On the other hand, by construction, if  $i \leq j$  then

$$d(T_i, T_j) > 2^{i_0-i+\frac{j-i}{Q+\Gamma}+1}$$

so that

$$d(\eta, T_i) > d(T_i, T_j) - d(\eta, T_j) > 2^{i_0-i+\frac{j-i}{Q+\Gamma}+1} - 2^{i_0+(m+1)/\Gamma-j}.$$

It follows that  $d(\eta, T_i) > 2 \cdot 2^{i_0 - i + \frac{j-i}{Q+\Gamma}} - 2^{i_0 + (m+1)/\Gamma - j}$ . Further, if  $i_0 + (m+1)/\Gamma - j < i_0 - i + \frac{j-i}{Q+\Gamma}$ , in particular, if  $j - i \geq (m+1)/\Gamma$ , then

$$d(\eta, T_i) > 2^{i_0 - i + \frac{j-i}{Q+\Gamma}}.$$

From Lemma 5.3 we then get that

$$(5.21) \quad U_i(\eta) \leq C_2 2^{-\frac{\Gamma}{Q+\Gamma}(j-i)}$$

for  $i \leq j - m$ .

We will now prove by induction on  $j$  that there exists a constant  $C_2$  for which, for every  $\eta > 0$  and every  $j \geq 0$ , there exists a permutation  $\sigma = \sigma_{j,\eta}$  of  $\{0, \dots, j\}$  such that, for each  $i \in \{0, \dots, j\}$ ,  $U_i(\eta) \leq C_2 2^{-\frac{\Gamma}{Q+\Gamma}\sigma(i)}$ .

It then immediately follows that  $\sum U_i$  is convergent and uniformly bounded as desired.

For  $j = 0$ , this is just Lemma 5.3. Assume now the hypothesis is true up to order  $j - 1$ .

Let  $\eta \in \mathfrak{N}$  and let  $m$  be such that  $C_2 2^{-m-1} \leq U_j(\eta) \leq C_2 2^{-m}$ .

— If  $m \geq j$ , then  $U_j(\eta) \leq C_2 2^{-\frac{\Gamma}{Q+\Gamma}j}$ . Further, by induction hypothesis, there exists a permutation  $\sigma_{j-1,\eta}$  of  $\{0, \dots, j-1\}$  such that, for  $i \in \{0, \dots, j-1\}$ ,  $U_i \leq 2^{-\frac{\Gamma}{Q+\Gamma}\sigma_{j-1,\eta}(i)}$ . It is then enough to extend  $\sigma_{j-1,\eta}$  by setting  $\sigma_{j,\eta}(i) = \sigma_{j-1,\eta}(i)$  if  $i < j$  and  $\sigma_{j,\eta}(j) = j$ .

— Otherwise,  $m < j$  and the (5.21) shows that, for  $i = 0, \dots, j-m-1$ ,  $U_i(\eta) \leq C_2 2^{-\frac{\Gamma}{Q+\Gamma}(j-i)}$ .

By the induction hypothesis,  $U_{j-m}(\eta), \dots, U_{j-1}(\eta)$  are bounded by  $m-1$  different elements of  $\{C_2, C_2 2^{-\frac{\Gamma}{Q+\Gamma}}, \dots, C_2 2^{-\frac{\Gamma}{Q+\Gamma}(j-1)}\}$ . But these are decreasing, so we may as well assume that they are bounded by the  $m-1$  first elements of the family. In other words, there exists a one-to-one mapping  $\sigma_1$  from  $\{j-m, \dots, j-1\}$  to  $\{0, \dots, m-1\}$  such that, for  $i = j-m, \dots, j-1$ ,  $U_i(\eta) \leq C_2 2^{-\frac{\Gamma}{Q+\Gamma}\sigma_1(i)}$ .

Finally, as  $U_j(\eta) \leq C_2 2^{-\frac{\Gamma}{Q+\Gamma}m}$ , if we set

$$\sigma_{j,\eta}(i) = \begin{cases} j-i & \text{if } i = 0, \dots, j-m-1 \\ \sigma_1(i) & \text{if } i = j-m, \dots, j-1 \\ m & \text{if } i = j \end{cases}$$

the proof of the induction is completed. □

We have thus completed the proof (5.15). □

## REFERENCES

- [AGPS] J. ALVAREZ, M. GUZMÁN-PARTIDA & U. SKÓRNIK *S'-convolvability with the Poisson kernel in the Euclidean case and the product domain case*. *Studia Mathematica* **156** (2003), 143-163.
- [AGPPE] J. ALVAREZ, M. GUZMÁN-PARTIDA & S. PÉREZ-ESTEVA *Harmonic Extensions of Distributions*. To appear in *Math. Nach.* (2006).
- [BDH] D. BURACZEWSKI, E. DAMEK, A. HULANICKI *Asymptotic behaviour of Poisson kernels on NA groups*. To appear in *Comm. Partial Differential Equations*.
- [FS] G. FOLLAND & E. M. STEIN *Hardy spaces on homogeneous groups*. Princeton University press, 1982.
- [HS] W. HEBISCH & A. SIKORA *A smooth subadditive homogeneous norm on a homogeneous group*. *Studia Math.* **96** (1990) 231-236.

- [He] E. HEINTZE *On homogeneous manifolds of negative curvature*. Math. Ann. **211** (1974), 23–34.
- [Ho1] J. HORVÁTH *Composition of hypersingular integral operators*. Applicable Analysis **7** (1978), 171–190.
- [Ho2] J. HORVÁTH *Convolution de noyaux hypersinguliers*. Séminaire Initiation à l'Analyse, (G. Choquet, M. Rogalski, J. Saint-Raymond) (1979/1980), **8**, (January 1980), 1–17.
- [Or] N. ORTNER *Sur quelques propriétés des espaces  $D'_{L^p}$  de Laurent Schwartz*. Boll. Un. Mat. Ital. B (6) **2** (1983), 353–375.
- [Sh] R. SHIRAISHI *On the definition of convolutions for distributions*. J. Sci. Hiroshima Univ., Ser. A **23** (1959), 19–32.
- [Sc] L. SCHWARTZ *Théorie des distributions*. Hermann, 1966.
- [Sj] P. SJÖGREN *Weak  $L^1$  characterizations of Poisson integrals, Green potentials, and  $H^p$  spaces*. Trans. Amer. Math. Soc. **233** (1977), 179–196.

E.D., J.D.: INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCLAW, 50-384 WROCLAW, PL. GRUNWALDZKI 2/4, POLAND

*E-mail address:* edamek@math.uni.wroc.pl, dziuban@math.uni.wroc.pl

PH.J.: MAPMO-FÉDÉRATION DENIS POISSON, UNIVERSITÉ D'ORLÉANS, BP 6759, F 45067 ORLÉANS CEDEX 2, FRANCE

*E-mail address:* Philippe.Jaming@univ-orleans.fr

S.P.E.: INSTITUTO DE MATEMÁTICAS, UNIDAD CUERNAVACA, UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO, CUERNAVACA, MORELOS 62251, MÉXICO

*E-mail address:* salvador@matcuer.unam.mx