Nazarov's uncertainty principles in higher dimension

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ABSTRACT. In this paper we prove that there exists a constant C such that, if S, Σ are subsets of $\mathbb{R}^d$ of finite measure, then for every function $f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)|^2 \, \mathrm{d}x \le C e^{C \min\left(|S||\Sigma|, |S|^{1/d} w(\Sigma), w(S)|\Sigma|^{1/d}\right)} \left(\int_{\mathbb{R}^d \setminus S} |f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(x)|^2 \, \mathrm{d}x \right)$$

where $\widehat{f}$ is the Fourier transform of f and $w(\Sigma)$ is the mean width of Σ. This extends to dimension $d \ge 1$ a result of Nazarov [Na] in dimension $d = 1$.

1. INTRODUCTION

An uncertainty principle is a mathematical result that gives limitations on the simultaneous localization of a function and its Fourier transform. There are many statements of that nature, the most famous being due to Heisenberg-Pauli-Weil when localization is measured in terms of smallness of dispersions and to Hardy when localization is measured in terms of fast decrease of the functions. We refer the reader to the surveys [FS, BD] and to the book [HJ] for further references and results.

We will need a few notations before going on. In this paper d will be a positive integer, all subsets of $\mathbb{R}^d$ considered will be measurable and we will denote by $|S|$ the Lebesgue measure of S. The Fourier transform is defined for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{2i\pi \langle x, \xi \rangle} \, \mathrm{d}x$$

and extended to all of $L^2(\mathbb{R}^d)$ in the usual way.

In this paper, we are interested in another criterium of localization, namely smallness of support. For instance, it is well known that if a function is compactly supported, then its Fourier transform is an entire function and can therefore not be compactly supported. We may then ask what happens if a function f and its Fourier transform $\widehat{f}$ are only small outside a compact set? This leads naturally to the following definition:

Definition.
Let S, Σ be two Borel subsets of $\mathbb{R}^d$. Then we will say that

— (S, Σ) is an annihilating pair (a-pair in short) if the only function f that is supported in S and such that its Fourier transform $\widehat{f}$ is supported in Σ is $f = 0$;

— (S, Σ) is a strong annihilating pair (strong a-pair in short) if there exists a constant $C = C(S, \Sigma)$ such that for every $f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)|^2 \, \mathrm{d}x \le C \left(\int_{\mathbb{R}^d \setminus S} |f(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(x)|^2 \, \mathrm{d}x \right).$$

This notion has been extensively studied in the case S is a compact set by Logvinenko and Sereda [LS], Paneah [Pa1, Pa2], Havin and Jöricke [HJ] and Kovrijkine [Ko], see also [HJ]. In this case the class of all Σ's for which (S, Σ) is a strong a-pair is characterized. Moreover, if S is convex, there are fairly good estimates of the constant $C(S, \Sigma)$ in terms of the geometry of S and Σ.

For sets S, Σ that are sublevel sets of quadratic forms, the problem has been studied by Shubin, Vakilian, Wolff [SVW] and by Demange [De1, De2].

Here we will focus on the case of S, Σ being of finite Lebesgue measure. This was first studied by Benedicks [Be] who proved that in this case (S, Σ) is an a-pair, and a little abstract nonsense allows to prove that in this case (S, Σ) is also a strong a-pair, see [BD]. This last fact was proved with a different method by Amrein and Berthier [AB]. Unfortunally both proofs do not give any estimate on the constant $C(S, \Sigma)$. By using a randomization of Benedicks proof and an extension of a lemma of Turan, Nazarov [Na] showed that in dimension 1, the constant is of the form $C(S, \Sigma) = Ce^{C|S||\Sigma|}$. It was thought for some time that Nazarov's method would extend to higher dimension to give a constant of the same form. This is far from the expected optimal which is thought to be obtained by taking S, Σ balls of radius R and f a Gaussian function, which gives $C(S, \Sigma) = Ce^{CR^2} = e^{C(|S||\Sigma|)^{1/d}}$.

The aim of this paper is to push Nazarov's technique as far as possible and thus improve the $Ce^{C|S||\Sigma|}$ constant when the geometry of Σ is suitable. Using the recent extension of Nazarov's Turan lemma to higher dimension by Fontes-Merz [FM], we will prove the following result:

Theorem.
There exists a constant C such that, for every sets $S, \Sigma \subset \mathbb{R}^d$ of finite Lebesgue measure and for every $f \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |f(x)|^2 \, dx \leq Ce^{C \min\left(|S||\Sigma|, |S|^{1/d} w(\Sigma), w(S)|\Sigma|^{1/d}\right)} \left(\int_{\mathbb{R}^d \setminus S} |f(x)|^2 \, dx + \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(x)|^2 \, dx \right)$$

where $w(\Sigma)$ is the mean width of Σ.

In particular, if S or Σ has a geometry that is close to a ball, this is in accordance with what is supposed to be the optimal result.

The remaining of this paper is devoted to the proof of this theorem. In order to do so, we first extend to higher dimension the random periodization technique. Then we recall the Turan type estimates we will need. The last section is then devoted to the proof of the theorem.

2. RANDOM PERIODIZATION

2.1. **Preliminaries.**
For any integer d, let $SO(d)$ denote the group of rotations on $\mathbb{R}^d$. Denote by $d\nu_d$ the normalized Haar measure on $SO(d)$. Then there exists a constant $C = C(d)$ such that, for every $u \in \mathbb{S}^{d-1}$, the unit sphere $\mathbb{S}^{d-1}$ of $\mathbb{R}^d$, and every function $f \in L^1(\mathbb{R}^d)$

$$\int_{SO(d)} \int_0^{+\infty} f\big(v\, \rho(u)\big) v^{d-1} \, dv \, d\nu_d(\rho) = C \int_{\mathbb{R}^d} f(x) \, dx.$$

2.2. **The higher dimensional Lattice Averaging Lemma.**
The following lemma was proved by Nazarov in dimension $d = 1$.

Lemma 2.1 (Lattice Averaging Lemma)**.**
Let $d \geq 1$ be an integer, then for every $\varphi \in L^1(\mathbb{R}^d)$, $\varphi \geq 0$, the following estimates hold

$$\int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\big(v\, \rho(k)\big) \, dv \, d\nu_d(\rho) \simeq \int_{\|x\| \geq 1} \varphi(x) \, dx$$

and

$$\text{(2.1)} \qquad \int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\left(\frac{\rho(k)}{v}\right) \mathrm{d}v \, \mathrm{d}\nu_d(\rho) \simeq \int_{\|x\| \geq 1/2} \varphi(x) \, \mathrm{d}x.$$

Here, as usual, by $A \simeq B$ we mean that there exists a constant C depending only on d such that $\frac{1}{C}B \leq A \leq CB$.

Proof. With (2.1), we get

$$
\begin{aligned}
\int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \varphi\big(v\,\rho(k)\big) \mathrm{d}v \, \mathrm{d}\nu_d(\rho) \quad &\simeq \quad \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{SO(d)} \int_1^2 \varphi\big(v\|k\|\,\rho(k/\|k\|)\big) \, v^{d-1}\mathrm{d}v \, \mathrm{d}\nu_d(\rho) \\
&= \quad C \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{1 \leq \|x\| \leq 2} \varphi(\|k\|x) \, \mathrm{d}x \\
&= \quad C \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{1}{\|k\|^d} \int_{\|k\| \leq \|x\| \leq 2\|k\|} \varphi(x) \, \mathrm{d}x \\
&= \quad C \int_{\|x\| \geq 1} \varphi(x) \sum_{\|k\| \leq \|x\| \leq 2\|k\|} \frac{1}{\|k\|^d} \, \mathrm{d}x \\
&\simeq \quad \int_{\|x\| \geq 1} \varphi(x) \, \mathrm{d}x
\end{aligned}
$$

since, for $\|x\| \geq 1$,

$$\sum_{\|k\| \leq \|x\| \leq 2\|k\|} \frac{1}{\|k\|^d} \simeq \frac{\left|\left\{u \in \mathbb{R}^d \; : \; \|x\|/2 \leq u \leq \|x\|\right\}\right|}{\|x\|^d} = |B(0,1) \setminus B(0,1/2)|.$$

For the second statement, one first changes v into $1/v$ and the remaining of the proof is similar. $\square$

Definition.
For a function $f \in L^2(\mathbb{R})$, $\rho \in SO(d)$ and $v > 0$, we define the periodization $\Gamma_{\rho,v}(t) = \Gamma_{\rho,v}(f)(t)$ of the function f by

$$\Gamma_{\rho,v}(t) = \frac{1}{\sqrt{v}} \sum_{k \in \mathbb{Z}^d} f\left(\frac{\rho(k+t)}{v}\right).$$

The series in the definition of $\Gamma_{\rho,v}$ converges in $L^2(\mathbb{T}^d)$ and represents a periodic function. An easy computation shows that the Fourier coefficients of $\Gamma_{\rho,v}$ are $\widehat{\Gamma_{\rho,v}}(m) = \sqrt{v}\widehat{f}\big(v^{\,t}\rho(m)\big)$ for $m \in \mathbb{Z}^d$.

Notation.
In the sequel, v will be considered as a random variable equidistributed on the interval $(1,2)$ and ρ as a random variable equidistributed on $SO(d)$. The expectation with respect to these random variables will be denoted by $\mathbb{E}_{\rho,v}$

2.3. Properties of random periodizations.
From the Lattice Averaging Lemma we shall derive the following simple but useful properties of the random periodization.

Proposition 2.2.
Let $d \geq 1$ be an integer and $C = C(d)$ be the constant defined in Lemma 2.1. Let $S \subset \mathbb{R}^d$ be a set of finite measure and let $f \in L^2(\mathbb{R}^d)$ be supported in S. Then

    (i) *for all $v \in (1,2)$, $\left|\left\{\, t \in (0,1) \; : \; \Gamma_{\rho,v}(t) \neq 0 \,\right\}\right| \leq 2^d|S|$;*

(ii) $\mathbb{E}_{\rho,v}\big(\|\Gamma_{\rho,v}\|_{L^2(0,1)}^2\big) \leq 2|\widehat{f}(0)|^2 + 2C\|f\|_{L^2(\mathbb{R}^d)}^2 \leq 2(|S| + C)\|f\|_{L^2(\mathbb{R}^d)}^2.$

Proof. i) The set of all points $t \in [0,1]^d$ for which the summand $f\left(\frac{\rho(k+t)}{v}\right)$ in the series defining $\Gamma_{\rho,v}$ does not vanish equals $v\,{}^t\rho(S) \cap \big([0,1]^d + k\big)$. Therefore,

$$|\{t \in [0,1]^d\ :\ \Gamma_{\rho,v}(t) \neq 0\}| \leq \sum_{k \in \mathbb{Z}^d} |v\,{}^t\rho(S) \cap \big([0,1]^d + k\big)| = |v\,{}^t\rho(S)| \leq 2^d|S|.$$

ii) Parseval's Identity gives

$$\mathbb{E}_{\rho,v}\big(\|\Gamma_{\rho,v}\|_{L^2(\mathbb{T}^d)}^2\big) = \mathbb{E}_{\rho,v}\left(\sum_{k \in \mathbb{Z}^d} |\widehat{\Gamma_{\rho,v}}(k)|^2\right) = \mathbb{E}_{\rho,v}\big(|\widehat{\Gamma_{\rho,v}}(0)|^2\big) + \mathbb{E}_{\rho,v}\left(\sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\widehat{\Gamma_{\rho,v}}(k)|^2\right).$$

But $|\widehat{\Gamma_{\rho,v}}(0)|^2 = v|\widehat{f}(0)|^2 \leq 2|\widehat{f}(0)|^2$, and, with the Lattice Averaging Lemma,

$$
\begin{aligned}
\mathbb{E}_{\rho,v}\left(\sum_{m \in \mathbb{Z}^d \setminus \{0\}} |\Gamma_{\rho,v}(m)|^2\right) &= \int_{SO(d)} \int_1^2 \left(\sum_{m \in \mathbb{Z}^d \setminus \{0\}} v|\widehat{f}\big(v\,\rho(m)\big)|^2\right) \mathrm{d}v\,\mathrm{d}\nu_d(\rho) \\
&\leq 2\int_{SO(d)} \int_1^2 \left(\sum_{m \in \mathbb{Z}^d \setminus \{0\}} |\widehat{f}\big(v\,\rho\zeta(m)\big)|^2\right) \mathrm{d}v\,\mathrm{d}\nu_d(\rho) \\
&\leq 2C\int_{\mathbb{R}^d} |\widehat{f}\big(\rho(\xi)\big)|^2\mathrm{d}\xi = 2C\|f\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned}
$$

It remains to notice that

$$|\widehat{f}(0)|^2 = \left|\int_S f(x)\mathrm{d}x\right|^2 \leq |S|\int_S |f(x)|^2\mathrm{d}x = |S|\|f\|_{L^2(\mathbb{R})}^2.$$

$\square$

Definition.
Let $\Sigma \subset \mathbb{R}$ be a measurable set with, $0 \in \Sigma$. We consider the lattice $\Lambda = \Lambda(\rho,v) := \{v\,{}^t\rho(j)\ :\ j \in \mathbb{Z}^d\}$ and denote $\mathcal{M}_{\rho,v} = \{k \in \mathbb{Z}^d\ :\ v\,{}^t\rho(k) \in \Sigma\} = \Lambda \cap \Sigma$.

Proposition 2.3.
With the previous notations

(i) $\mathbb{E}_{\rho,v}\big(\operatorname{card}\mathcal{M}_{\rho,v} - 1\big) \leq C|\Sigma|$, *in particular $\mathcal{M}_{\rho,v}$ is almost surely finite;*

(ii) $\mathbb{E}_{\rho,v}\left(\sum_{m \in \mathbb{Z}^d \setminus \mathcal{M}_{\rho,v}} |\widehat{\Gamma_{\rho,v}}(m)|^2\right) \leq 2C\int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi.$

Proof. i) Since $\operatorname{card}\mathcal{M}_{\rho,v} = 1 + \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \chi_\Sigma\big(v\,{}^t\rho(m)\big)$, we have

$$
\begin{aligned}
\mathbb{E}_{\rho,v}\big(\operatorname{card}\mathcal{M}_{\rho,v} - 1\big) &= \int_{SO(d)} \int_1^2 \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \chi_\Sigma\big(v\,{}^t\rho(k)\big)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho) \\
&\leq C\int_{\mathbb{R}^d} \chi_\Sigma(x)\,\mathrm{d}x = C|\Sigma|.
\end{aligned}
$$

ii) From the expression of $\widehat{\Gamma_{\rho,v}}$ we get that $\mathbb{E}_{\rho,v}\left(\displaystyle\sum_{m\in\mathbb{Z}^d\setminus\mathcal{M}_{\rho,v}}|\widehat{\Gamma_{\rho,v}}(k)|^2\right)$ is

$$= \int_{SO(d)}\int_1^2\left(\sum_{m\in\mathbb{Z}^d\setminus\{0\}}v\big|\widehat{f}\big(v\,{}^t\rho(m)\big)\big|^2\chi_{\mathbb{R}^d\setminus\Sigma}\big(v\,{}^t\rho(m)\big)\right)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho)$$

$$\leq 2\int_{SO(d)}\int_1^2\left(\sum_{m\in\mathbb{Z}^d\setminus\{0\}}\big|\widehat{f}\big(v\,{}^t\rho(mk)\big)\big|^2\chi_{\mathbb{R}^d\setminus\Sigma}\big(v\,{}^t\rho(m)\big)\right)\,\mathrm{d}v\,\mathrm{d}\nu_d(\rho)$$

$$\leq 2C\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\chi_{\mathbb{R}\setminus\Sigma}(\xi)\,\mathrm{d}\xi = 2C\int_{\mathbb{R}^d\setminus\Sigma}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi.$$

by Lemma 2.1. $\qquad\square$

3. A Turan Lemma

3.1. Nazarov and Fontes-Merz' Turan Lemmas.

For sake of completeness, we will recall here the Turan type estimates of trigonometric polynomials we will need.

Theorem (Nazarov's Turan Lemma [Na])

Let $P(t)=\displaystyle\sum_{k=1}^m c_k e^{2i\pi r_k t}$ *with* $c_k\in\mathbb{C}\setminus\{0\}$, $r_1<\cdots<r_m\in\mathbb{Z}$, *be a trigonometric polynomial of order* $\operatorname{ord}P=m$ *and let* E *be a measurable subset of* $\mathbb{T}$. *Then*

$$(3.2)\qquad \sup_{z\in\mathbb{T}}|P(z)|\leq\left(\frac{14}{|E|}\right)^{m-1}\sup_{z\in E}|P(z)|.$$

The original theorem of Turan deals with sets E that are arcs. The extension to higher dimension has been obtained in [FM] using a clever induction on the dimension.

Corollary (Fontes-Merz's Turan Lemma [FM])

Let $d\geq 1$ *be an integer and let*

$$p(z_1,\ldots,z_d)=\sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d}c_{k_1,\ldots,k_d}z_1^{r_{1,k_1}}\cdots z_d^{r_{r_d,k_d}}$$

with $r_{i,k_i}\in\mathbb{Z}$ *be a polynomial in* d *variables. Then, for every measurable set* $E\subset\mathbb{T}^d$,

$$\sup_{z\in\mathbb{T}^d}|p(z)|\leq\left(\frac{14d}{|E|}\right)^{m_1+\cdots+m_d}\sup_{z\in E}|p(z)|.$$

The quantity $m_1+\cdots+m_d$ is called the order of p (with the usual convention that we take the most compact possible representation of p) and is denoted by $\operatorname{ord}p$. In general

$$m_1+\cdots+m_d\leq d\max m_i\leq d\operatorname{Card}\operatorname{Spec}p$$

while

$$\operatorname{Card}\operatorname{Spec}p\leq(m_1+1)\cdots(m_d+1).$$

3.2. **An estimate of the average order.**

The notion of order of a polynomial suggests the following definition of the order of a subset of $\mathbb{Z}^d$.

Definition.

Let $M \subset \mathbb{Z}^d$ be a finite set, we will say that M is of order k and write $\operatorname{ord} M = k$ if there exists integers $m_1, \ldots, m_d$ with $m_1 + \cdots + m_d = k$ such that the projection of M on the i-th coordinate axis has m_i elements.

Finally, if $\Lambda = A\mathbb{Z}^d$ is a lattice and $M \subset \Lambda$ is finite, we will call $\operatorname{ord} M = \operatorname{ord} A^{-1} M$.

Note that

$$m_1 = \sum_{k \in \mathbb{Z}} \sup_{k' \in \mathbb{Z}^{d-1}} \chi_M(k, k')$$

with similar expressions for the other m_i's.

In order to estimate the order of the set $\mathcal{M}_{\rho,v}$ introduced before Proposition 2.3, the easiest is to bound the order by the cardinal of the set, which amounts to bounding the supremum by the sum over $k' \in \mathbb{Z}^{d-1}$ in the above expression. One then gets $\mathbb{E}_{\rho,v}(\operatorname{ord} \mathcal{M}_{\rho,v} - d) \leq C|\Sigma|$. This shows in particular that it is enough to estimate this quantity when Σ is a relatively compact open set.

The proof of the uncertainty principle in the next section will then give a constant $Ce^{C|S||\Sigma|}$ in Nazarov's result. We will slightly improve this. in order to do so, let us introduce the following quantities:

— the average width: for a relatively compact open set Σ and for $\rho \in SO(d)$, let $P_\rho(\Sigma)$ be the projection of Σ on the span of $\rho(1, 0, \ldots, 0)$. We define

$$w(\Sigma) = \int_{SO(d)} |P_\rho(\Sigma)| \, \mathrm{d}\nu_d(\rho)$$

the average width of Σ. If Σ is a ball, this is just its diameter.

— let us also introduce the measure μ on $\mathbb{R}^d$ defined by

$$\mu(\Sigma) = \inf \left\{ \sum_{i \in I} \min(r_i, r_i^d) : \{B(x_i, r_i)\}_{i \in I} \text{ is a cover of } \Sigma \right\}.$$

Note that $\mu(\Sigma) \leq C|\Sigma|$ since the d-dimensional Hausdorff measure is the Lebesgue measure.

We will now prove the following:

Proposition 3.1.

Let Σ be a relatively compact open set with $0 \in \Sigma$. We consider a random lattice $\Lambda = \Lambda(\rho, v) := \{v\,^t\rho(j) : j \in \mathbb{Z}^d\}$ and denote $\mathcal{M}_{\rho,v} = \{k \in \mathbb{Z}^d : v\,^t\rho(k) \in \Sigma\} = \Lambda \cap \Sigma$. Then $\mathbb{E}_{\rho,v}(\operatorname{ord} \mathcal{M}_{\rho,v} - d) \leq C \min(\mu(\Sigma), w(\Sigma))$.

Proof. Let

$$m_{\rho,v}(\Sigma) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \sup_{k' \in \mathbb{Z}^{d-1}} \chi_\Sigma(v^t\rho(k, k')).$$

It is enough to prove that

$$(3.3) \qquad\qquad \mathbb{E}_{\rho,v}(m_{\rho,v}(\Sigma)) \leq C \min(\mu(\Sigma), w(\Sigma)).$$

As pointed out above, $\mathbb{E}_{\rho,v}(m_{\rho,v}(\Sigma)) \leq C|\Sigma|$. In particular, if Σ is a ball of radius r, $\mathbb{E}_{\rho,v}(m_{\rho,v}(\Sigma)) \leq Cr^d$

On the other hand

$$m_{\rho,v} := m_{\rho,v}(\Sigma) \leq \sum_{k \in \mathbb{Z} \setminus \{0\}} \sup_{y \in \mathbb{R}^{d-1}} \chi_\Sigma(^t\rho(vk, y))$$

and the one-dimensional lattice averaging lemma then gives

$$
\begin{aligned}
\mathbb{E}_{\rho,v}(m_{\rho,v}) &\leq \int_{SO(d)} \int_{|x|\geq 1} \sup_{y\in\mathbb{R}^{d-1}} \chi_{B\left(\rho(a),r\right)}(x,y)\,\mathrm{d}x\,\mathrm{d}\nu_d(\rho) \\
&\leq C \int_{SO(d)} \int_{|x|\geq 1} \chi_{P_{t_\rho}}(x)\,\mathrm{d}x\,\mathrm{d}\nu_d(\rho) \\
&\leq Cw(\Sigma).
\end{aligned}
$$

In particular, if Σ is a ball of radius r, then $\mathbb{E}_{\rho,v}\big(m_{\rho,v}(\Sigma)\big) \leq Cr$. To conclude, it is enough to note that $m_{\rho,v}(\Sigma \cup \Sigma') \leq m_{\rho,v}(\Sigma) + m_{\rho,v}(\Sigma')$ and that if $\Sigma \subset \Sigma'$ then $m_{\rho,v}(\Sigma) \leq m_{\rho,v}(\Sigma')$. Covering Σ with balls then gives the desired result. $\qquad\square$

The result above is essentially sharp as the following example shows. For simplicity, we will give the example in dimension $d = 2$. Let $N \geq 1$ be an integer and let $R \gg 1$ be two real numbers. Let

$$
\Sigma_N = \bigcup_{j=0}^{N-1} B\left(R(\cos\frac{2\pi}{N}j, \sin\frac{2\pi}{N}j), \frac{1}{2} \right).
$$

That is, Σ_N is the union of N discs regularly placed on a big circle, see the figure below.
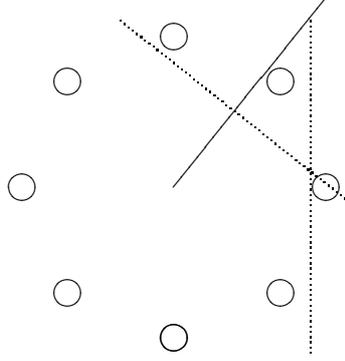
The set Σ_N :

Note that each line orthogonal to a line through the origin meets at most two circles. Moreover, these circles have radius $\leq 1/2$ thus, for k fixed, at most two segments $\{{}^t\rho(vk, vk'), v \in (1,2)\}$ can intersect Σ_N. Therefore, the $\sup_{k'}$ in the formula defining m_1 can be bounded below by $\frac{1}{2}\sum_{k'}$.

Then, for each k, $\#\{k' \in \mathbb{Z} : {}^t\rho(vk, vk') \cap \Sigma_N \neq 0\} \leq 2$ thus

$$
m_{\rho,v}(\Sigma_N) \geq \frac{1}{2} \sum_{k\in\mathbb{Z}\setminus\{0\}} \sum_{k'\in\mathbb{Z}} \chi_{\Sigma_N}\big({}^t\rho(vk, vk')\big) \simeq |\Sigma_N| \simeq N \simeq w(\Sigma_N)
$$

with the Latice Averaging Lemma.

4. Conclusion

The remaining of the proof follows the path of Nazarov's original argument. We include it here for sake of completeness.

Let us write $\nu(\Sigma) = \min\big(w(\Sigma), \mu(\Sigma)\big)$. First, it is enough to prove that there exists a constant $C = C(d)$ such that

$$
\int_{\Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi \leq Ce^{C\nu(|S|^{1/d}\Sigma)} \int_{\mathbb{R}^d\setminus\Sigma} |\widehat{f}(\xi)|^2\,\mathrm{d}\xi
$$

for every $f \in L^2(S)$. Moreover, using a scaling argument, it is enough to show that, if $|S| = 2^{-d+1}$, then for every set Σ and every $f \in L^2(S)$,

$$\int_\Sigma |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \leq C e^{C\mu(\Sigma)} \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

Set $\Gamma_{\rho,v}(t) = \Gamma_{\rho,v}(f)(t)$ the random periodization of f. Then, setting $E_{\rho,v} = \{t \in (0,1) : \Gamma_{\rho,v}(t) = 0\}$, we have by Proposition 2.2 $i)$ that $|E_{\rho,v}| \geq 1 - 2^d |S| = \frac{1}{2}$.

Next, set $\mathcal{M}_{\rho,v} := \{m \in \mathbb{Z}^d : v \, {}^t\rho(m) \in \Sigma \cup \{0\}\}$ and decompose $\Gamma_{\rho,v} = P_{\rho,v} + R_{\rho,v}$ where

$$P_{\rho,v}(t) = \sum_{m \in \mathcal{M}_{\rho,v}} \widehat{\Gamma_{\rho,v}}(m) e^{2i\pi mt}$$

while

$$R_{\rho,v}(t) = \sum_{m \in \mathbb{Z}^d \setminus \mathcal{M}_{\rho,v}} \widehat{\Gamma_{\rho,v}}(m) e^{2i\pi mt}.$$

By Proposition 2.3 $ii)$,

$$\mathbb{E}_{\rho,v}\big(\|R_{\rho,v}\|_{L^2(0,1)}^2\big) = \mathbb{E}_{\rho,v}\left(\sum_{m \in \mathbb{Z}^d \setminus \mathcal{M}_{\rho,v}} |\widehat{\Gamma_{\rho,v}}(m)|^2\right) \leq 2C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi,$$

hence

$$\mathbb{E}_{\rho,v}\left(\|R_{\rho,v}\|_{L^2(0,1)}^2 > 4C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi\right) < \frac{1}{2}.$$

On the other hand, by Proposition 3.1,

$$\mathbb{E}_{\rho,v}\big(\mathrm{ord}\, P_{\rho,v}\big) \leq C\mu(\Sigma) + d$$

and therefore

$$\mathbb{E}_{\rho,v}\big(\mathrm{ord}\, P_{\rho,v} > 2\big(C\mu(\Sigma) + d\big)\big) < \frac{1}{2}.$$

We thus get that the two events

(1) $\|R_{\rho,v}\|_{L^2(0,1)}^2 \leq 4C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi$,

(2) $\mathrm{ord}\, P_{\rho,v} \leq 2\big(C\mu(\Sigma) + d\big)$

happen simultaneously with non-zero probability, while the two events

(3) $|E_{\rho,v}| \geq \dfrac{1}{2}$,

(4) $|\widehat{f}(0)|^2 \leq |\widehat{P_{\rho,v}}(0)|^2$

are certain. We will now take $v \in (1,2)$, $\rho \in \mathbb{S}^{d-1}$ such that all four events hold simultaneously.

Further, by definition $\Gamma_{\rho,v} = 0$ on $E_{\rho,v}$, that is $P_{\rho,v}$ and $-R_{\rho,v}$ coincide on $E_{\rho,v}$. It follows that

$$\int_{E_{\rho,v}} |P_{\rho,v}(x)|^2 \, \mathrm{d}x = \int_{E_{\rho,v}} |R_{\rho,v}(x)|^2 \, \mathrm{d}x.$$

Hence

$$\left|\left\{x \in E_{\rho,v} : |P_{\rho,v}(x)|^2 \geq 16C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi\right\}\right| \leq \frac{1}{4}$$

and, as $|E_{\rho,v}| \geq \frac{1}{2}$, we get that $|\tilde{E}_{\rho,v}| \geq \frac{1}{4}$ where

$$\tilde{E}_{\rho,v} = \left\{x \in E_{\rho,v} : |P_{\rho,v}(x)| \leq 4\left(C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi\right)^{\frac{1}{2}}\right\}.$$

We can now apply Turan's Lemma and get

$$
\begin{aligned}
|\widehat{f}(0)|^2 \quad \leq \quad |\widehat{P_{\rho,v}}(0)|^2 &\leq \left(\sum_{k \in \mathbb{Z}^d} |\widehat{P_{\rho,v}}(k)| \right)^2 \leq \left(\sup_{x \in \mathbb{T}^d} |P_{\rho,v}(x)| \right)^2 \\
&\leq \quad \left[\left(\frac{14d}{|\tilde{E}_{\rho,v}|} \right)^{\mathrm{ord} P_{\rho,v} - 1} \sup_{x \in \tilde{E}_{\rho,v}} |P_{\rho,v}(x)| \right]^2 \\
&\leq \quad \left[\left(\frac{14d}{1/4} \right)^{\mathrm{ord} P_{\rho,v} - 1} 4 \left(C \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \right]^2 \\
&\leq \quad Ce^{C\nu(\Sigma)} \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.
\end{aligned}
$$

If we now apply this to $f_y(x) = f(x)e^{-2i\pi xy}$ instead of f and to the set $\Sigma_y = \Sigma - y$ instead of Σ, we obtain that

$$
|\widehat{f}(y)|^2 \leq Ce^{C\nu(\Sigma)} \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.
$$

and integrating this over Σ gives

$$
\int_{\Sigma} |\widehat{f}(y)|^2 \, \mathrm{d}y \leq C|\Sigma| e^{C\mu(\Sigma)} \int_{\mathbb{R}^d \setminus \Sigma} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi
$$

as claimed. □

The values of the constants may be tracked and linked to those of the Random Averaging Lemma, but we do not expect these constants to be any near to optimal (as they are already not optimal in dimension 1) so we will not pursue this.

Note also that, with mutadis mutandis the same proof as in [Na] we obtain the following corollary:

Corollary 4.1.
Let S, Σ be two measurable subsets of $\mathbb{R}^d$ and let C be the constant of the main theorem. Then, for every $p \in (0,2)$ and every $f \in L^p(\mathbb{R}^d)$ with spectrum in Σ,

$$
\|f\|_{L^p}^p \leq Ce^{Cp|S||\Sigma|} \int_{\mathbb{R}^d \setminus S} |f(x)|^p \, dx.
$$

References

[AB] W. O. AMREIN & A. M. BERTHIER *On support properties of L^p-functions and their Fourier transforms.* J. Functional Analysis **24** (1977) 258–267.

[Be] M. BENEDICKS *On Fourier transforms of functions supported on sets of finite Lebesgue measure.* J. Math. Anal. Appl. **106** (1985) 180–183.

[BD] A. BONAMI & B. DEMANGE *A survey on the uncertainty principle for quadratic forms.* to appear in Collecteana Math.

[De1] B. DEMANGE *Principes d'incertitude associés à des formes quadratiques non dégénérées.* Thèse de l'université d'Orléans, 2004.

[De2]   B. Demange *Inégalités d'incertitude associées à des fonctions homogènes.* C. R. Math. Acad. Sci. Paris **340** (2005) 709–714.

[FS]   G. B. Folland & A. Sitaram *The uncertainty principle — a mathematical survey.* J. Fourier Anal. Appl. **3** (1997) 207–238.

[FM]   N. Fontes-Merz *A multidimensional version of Turán's lemma.* Jour. Approx. Theory, **140** (2006) 27–30.

[HJ]   V. Havin & B. Jöricke *The uncertainty principle in harmonic analysis.* Springer-Verlag, Berlin, 1994.

[HJ]   V. Havin & B. Jöricke *The uncertainty principle in harmonic analysis.* Commutative harmonic analysis, III, 177–259, 261–266, *Encyclopaedia Math. Sci.*, **72**, Springer, Berlin, 1995.

[Ko]   O. Kovrijkine *Some results related to the Logvinenko-Sereda theorem.* Proc. Amer. Math. Soc. **129** (2001) 3037–3047.

[LS]   V. N. Logvinenko & Ju. F. Sereda, *Equivalent norms in spaces of entire functions of exponential type. (Russian)* Teor. FunkciĭFunkcional. Anal. i Priložen. Vyp. **20** (1974) 102–111, 175.

[Na]   F. L. Nazarov *Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. (Russian)* Algebra i Analiz **5** (1993) 3–66; *translation in* St. Petersburg Math. J. **5** (1994) 663–717.

[Pa1]   B. Paneah *Support-dependent weighted norm estimates for Fourier transforms.* J. Math. Anal. Appl. **189** (1995) 552–574.

[Pa2]   B. Paneah *Support-dependent weighted norm estimates for Fourier transforms. II.* Duke Math. J. **92** (1998) 35–353.

[SVW]   C. Shubin, R. Vakilian & T. Wolff *Some harmonic analysis questions suggested by Anderson-Bernoulli models.* Geom. Funct. Anal. **8** (1998) 932–964.

[Vi]   N. Ja. Vilenkin *Fonctions spéciles et théorie de la représentation de groupes* Dunod, Paris, 1969.

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