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Transport equations with rough force fields and applications to Vlasov-Poisson equation.

Delphine Salort

December 13, 2006

Abstract

In this article, we study the three dimensional Vlasov-Poisson equation. We give new conditions on the initial data using low moments to control the force field E , the density ρ in L^∞ and to have information concerning the smoothness of the solution locally in time. In particular, we introduce a new approach which consists in establishing moments effects for linear transport equations with rough force fields.

1 Introduction

In this article, we study the three dimensional Vlasov-Poisson equation given by

$$\begin{cases} \partial_t f + \xi \nabla_x f \pm E \cdot \nabla_\xi f = 0 \\ f(0, x, \xi) = f^{in}(x, \xi) \end{cases} \quad (1)$$

where f^{in} is a positive measurable function, with $E = \varphi * \rho$,

$$\varphi(x) = \frac{x}{|x|^3} \quad \text{and} \quad \rho(t, x) = \int f(t, x, \xi) d\xi.$$

This equation models the evolution of a system of particles in gravitational (for sign $-$) or coulombian (for sign $+$) interactions. The solution $f(t, x, \xi)$ models the microscopic density of particles which are, at time t , at position x with velocity ξ , and $\rho(t, x)$ models the probability of finding a particle at t at position x . Finally $E(t, x)$ models the (electrostatic or gravitational, depending on the sign) potential created by ρ .

Let $x \in \mathbb{R}$. In this article, we use $x + 0$ and $x - 0$ to denote $x + \varepsilon$ and $x - \varepsilon$ where $\varepsilon > 0$ is taken arbitrarily small.

Many articles have been devoted to the study of the Vlasov-Poisson equation in dimension three. Concerning weak solutions, the reader can refer to the articles by A. A. Arsen'ev in [1], [2], and by E. Hörst and R. Hunze in [14]. In the case of smooth initial data, the reader can refer to the article by J. Batt in [5], J. Batt and G. Rein in [6], C. Bardos and P. Degond in [4], of E. Hörst in [11], [12] and [13], of K. Pfaffelmoser in [18] and of J. Schaeffer in [20]. Concerning the study of the Vlasov-Poisson with propagation of moments, we refer the reader to the articles by P.-L. Lions and B. Perthame in [15] (see also references therein), of F. Castella

[8], of I. Gasser P.-E. Jabin and B. Perthame in [10] (see also the article by B. Perthame [17] and references therein).

In this article we focus on the following problem, to which we will refer by \mathcal{P} . Prove that :

- 1. E belongs to a certain space C^α with $\alpha \in [0, 1[$
- 2. ρ belongs to L^∞
- 3. the solution is smooth if the initial data is smooth using the boundedness condition (see the article by E. Hörst [11])

using as little information as possible on the moments of the initial data.

Let us make two preliminary remarks about problem \mathcal{P} .

- First, the three points above are related for two reasons. On the one hand, the relation $\rho = \nabla(\Delta)^{-1}E$ binds the force field E and ρ ; on the other hand, enough smoothness in E leads to precise information on the characteristics and hence on the solution by means of the initial data.
- Secondly, by proving the second point for a given solution, we also gain uniqueness of the solution to the Vlasov-Poisson equation. Indeed, G. Loeper showed in [16] that if the density ρ is bounded, then uniqueness of weak solutions to the Vlasov-Poisson equation follows. Note that this result does not provide any information pertaining to the existence of such a solution.

To devise answers to problem \mathcal{P} , we introduce a new approach to the study of the Vlasov-Poisson equation. One of the key arguments we used involved a precise study of characteristics using moments effects for transport equations with rough force fields. This approach brought new answers to the question stated above with little constraints on the moments of the initial data. Let us mention that I. Gasser P.-E. Jabin and B. Perthame in [10] studied the Vlasov-Poisson's equation by using moments effects established for the free transport equation.

Before stating our results, let us first recall the results obtained by P.-L. Lions et B. Perthame in [15] concerning the problem \mathcal{P} .

1.1 Known results concerning the problem \mathcal{P} .

In the case where the initial data is bounded with more than 6 moments in $L^1_{x,\xi}$, P.-L. Lions and B. Perthame have shown in [15] the following result (among other properties of the solution to the Vlasov-Poisson equation).

Theorem 1 [15]

Assume that $f^{in} \in L^\infty(\mathbb{R}^6)$ and that

$$\|(1 + |\xi|)^m f^{in}\|_{L^1_{x,\xi}} < +\infty \quad \text{with } m > 6.$$

Then

$$E \in \mathcal{C}(\mathbb{R}^+)(\mathcal{C}^\alpha(\mathbb{R}^3)) \quad \text{with} \quad \alpha < \frac{m-6}{3+m}.$$

E belonging to $L^\infty([0, T] \times \mathbb{R}^3)$, they deduce that the characteristics (X, V) are little perturbations of the corresponding trajectories given by the free transport equation. Indeed,

$$X(t, x, \xi) = x + t\xi + R_1(t, x, \xi) \quad \text{and} \quad V(t, x, \xi) = \xi + R_2(t, x, \xi)$$

where here

$$R_1(t, x, \xi) = \int_0^t (t-s)E(s, X(s))ds \quad \text{and} \quad R_2(t, x, \xi) = \int_0^t E(s, X(s))ds.$$

They deduce the following control

$$\|R_1\|_{L_{x,\xi}^\infty} \leq |t|^2 \|E\|_{L^\infty([0,T] \times \mathbb{R}^3)} \quad \text{and} \quad \|R_2\|_{L_{x,\xi}^\infty} \leq |t| \|E\|_{L^\infty([0,T] \times \mathbb{R}^3)}.$$

This control on E and on the characteristics allows them to prove the following result which in turn implies on the one hand sufficient conditions on the initial data to show that the density ρ belongs to $L^\infty([0, T] \times \mathbb{R}^3)$ and on the other hand a result of propagation of smoothness of the solution for smooth initial data.

Theorem 2 [15]

Assume that $f^{in} \in L^\infty(\mathbb{R}^6)$ and that

$$\|(1 + |\xi|)^{6+0} f^{in}\|_{L_{x,\xi}^1} < +\infty.$$

Then, the boundedness condition is satisfied (see E. Hörst [11] for the definition) which implies that if f^{in} is smooth, then the solution of the Vlasov-Poisson equation f is smooth.

Assume furthermore that for all $R > 0$ and for all $T > 0$

$$\text{supess} \left\{ f^{in}(y + t\xi, w), |x - y| \leq Rt^2, |\xi - w| \leq Rt \right\} \in L^\infty([0, T] \times \mathbb{R}_x^3(L_\xi^1)). \quad (2)$$

Then, there exists a solution of the Vlasov-Poisson equation (1) such that $\rho \in L^\infty([0, T] \times \mathbb{R}_x^3)$.

1.2 Results

We obtain two results.

The first one gives us the following controls on the force field E .

Theorem 3 Assume that

$$\|(1 + |\xi|)^{\frac{5}{2}+0} f^{in}\|_{L_{x,\xi}^2} < +\infty.$$

Let $p \geq 3$. Then there exist $T_0 > 0$, and $C > 0$, such that for all $A > 0$,

$$\sup_{|B| \leq A} \|E\|_{L_{T_0}^p(C^{1-\frac{3}{p}+0}(B))} \leq CA \quad \text{if} \quad \|(1 + |\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}} < +\infty \quad (3)$$

with $1 - \frac{3}{p} + 0 < 1$ and

$$\sup_{|B| \leq A} \|E\|_{L_{T_0}^p(C^{0+0}(B))} \leq CA^{\frac{3}{p}} \quad \text{if} \quad \|(1 + |\xi|)^{2-\frac{1}{3p}+0} f^{in}\|_{L_{x,\xi}^{3+0}} < +\infty.$$

Remark 1 *In Theorem 3, the moments condition on the initial data is weaker than that in Theorem 1. Conversely, the control of E given by Theorem 3 only holds over compact sets for the space variable and after averaging over small time intervals.*

Theorem 3 does not give control over E in $L^\infty([0, T] \times \mathbb{R}^3)$; so we cannot follow the same strategy as that adopted by P.-L. Lions and B. Perthame in [15] to prove that the trajectories are a small perturbation of those given by the free transport equation. To obtain a good approximation of the characteristics by free transport, we will prove the following weaker condition using moments effect

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \int_0^t |E(s, X(s, x_0, \xi_0))|^p ds < +\infty \quad \text{where } p \geq 1. \quad (4)$$

We shall see that to obtain a formal estimate such as (4), bounded initial data with strictly less than 6 moments in L^1 are allowed. In counterpart, estimate (4) gives us weaker control on R_1 and R_2 when t is small than was obtained by P.-L. Lions and B. Perthame in [15].

The second result provides conditions on the initial data to obtain smooth solutions for smooth initial data and also a bound on the density of probability ρ in $L^\infty([0, T_0] \times \mathbb{R}^3)$ for T_0 small enough.

Theorem 4 *Let $\alpha = \frac{19}{21}$ and $p \geq 3$. Assume that*

$$\|(1 + |\xi|)^{\frac{5}{2}+0} f^{in}\|_{L_{x,\xi}^2} < +\infty.$$

Then, if

$$\|(1 + |\xi|)^{2\alpha+0} f^{in}\|_{L_{x,\xi}^{3+0}} < +\infty,$$

there exists $T_0 > 0$ such that the boundedness condition is satisfied (see E. Hörst [11] for the definition) which implies that if the initial data is smooth, then the solution of (1) is smooth on $[0, T_0]$.

Let $p \geq 3$. Assume that

$$\|(1 + |\xi|)^{2(1+\frac{3}{p}(\alpha-1))+0} f^{in}\|_{L_{x,\xi}^{3+0}} < +\infty$$

and that for all $R > 0$, for all $t \in [0, T_0]$,

$$\text{supess} \left\{ f^{in}(y + t\xi, w), |x - y| \leq Rt^{p'+1}, |\xi - w| \leq Rt^{p'} \right\} \in L^\infty([0, T_0] \times \mathbb{R}_x^3(L_\xi^1)). \quad (5)$$

Then, there exists $T_0 > 0$, such that there exists a unique solution of the Vlasov-Poisson equation (1) such that the density of probability $\rho \in L^\infty([0, T_0] \times \mathbb{R}_x^3)$.

Remark 2 *Theorem 4 shows, in particular that $\rho \in L^\infty([0, T_0] \times \mathbb{R}^3)$, for bounded initial data satisfying condition (5) and which have strictly less than 6 moments in L^1 . The control of characteristics being worse than if $E \in L^\infty([0, T_0] \times \mathbb{R}^3)$, one is forced to add an additional constraint when compared to condition (2), namely condition (5). On the other hand, to obtain the smoothness propagation for the solution to (1), formal control that one obtains on the trajectories with the approach using moments effects is enough to obtain the same conclusions locally in time as if one had from the beginning $E \in L^\infty([0, T_0] \times \mathbb{R}^3)$ as in Theorem 2.*

The remainder of the article is organized as follows.

- In the second section, we reduce the proof of Theorems 3 and 4 to proving Theorems 6 and 7 over an approximate system which gives rigorous meaning to the formal computations on the characteristics.
- In the third section, we study linear transport equations with a rough force field. This study is crucial to the proof of Theorems 6 and 7.
- In the fourth section, we use the results obtained in the third section to prove Theorem 6.
- The last section is devoted to the proof of Theorem 7. This proof is based on the preceding section results and on a new moment effect along characteristics.

2 Reduction to an approximate system.

We consider for all $n \in \mathbb{N}^*$ the following equation

$$\begin{cases} \partial_t f_n + \xi \nabla_x f_n \pm E_n \cdot \nabla_\xi f_n = 0 \\ f_n(0, x, \xi) = f^{in}(x, \xi) \end{cases} \quad (6)$$

where $E_n = \varphi_n * \rho_n$ with

$$\varphi_n = \frac{x}{(|x|^2 + \frac{1}{n})^{\frac{3}{2}}} \quad \text{and} \quad \rho_n = \int f_n(t, x, \xi) d\xi.$$

The following theorem proved by E. Hörst in [11] allows for an explicit representation of the solution using characteristics.

Theorem 5 *Let I be a time interval containing 0. Then, there exists a unique solution f_n of equation (6) with*

- E_n continuous on $I \times \mathbb{R}^3$
- $E_n(t, \cdot) \in L^\infty \cap Lip$
- $E_n \in L^\infty([0, T] \times \mathbb{R}^d)$.

The solution f_n can be written in terms of the initial data as

$$f_n(t, x, \xi) = f^{in}(X_n, V_n) \quad \text{where} \quad \begin{cases} \dot{X}_n = V_n \\ \dot{V}_n = E_n \end{cases} \quad (7)$$

Let $t_0 \in I$. We denote by $(X(t_0, t, x, \xi), V(t_0, t, x, \xi))$ the solution to equation (7) with (x, ξ) as initial data at time t_0 . In particular

$$(X(t_0, t, x, \xi), V(t_0, t, x, \xi))^{-1} = (X(t, t_0, x, \xi), V(t, t_0, x, \xi)).$$

To prove Theorems 3 and 4, it suffices to prove the following estimates on the approximate system (6), which are uniform with respect to $n \in \mathbb{N}^*$.

Theorem 6 *Let $p \geq 3$. Assume that*

$$\|(1 + |\xi|)^{\frac{5}{2}+0} f^{in}\|_{L^2} < +\infty.$$

Then, there exist T_0 in $]0, 1]$, and a constant $C > 0$ such that for all $n \in \mathbb{N}^$, for all $A > 0$*

$$\sup_{|B| \leq A} \|E_n\|_{L_{T_0}^p(C^{1-\frac{3}{p}+0}(B))} \leq CA \quad \text{if} \quad \|(1 + |\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}} < +\infty \quad (8)$$

with $1 - \frac{3}{p} + 0 < 1$ and

$$\sup_{|B| \leq A} \|E_n\|_{L_{T_0}^p(C^{0+0}(B))} \leq CA^{\frac{3}{p}} \quad \text{if} \quad \|(1 + |\xi|)^{2-\frac{1}{3p}+0} f^{in}\|_{L_{x,\xi}^{3+0}} < +\infty. \quad (9)$$

Moreover, there exists a constant C such that $\forall n \in \mathbb{N}^$*

$$\|\partial_t E_n\|_{L_{[0,T_0]}^\infty(L^2(\mathbb{R}^3))} \leq C. \quad (10)$$

Theorem 6 allows us to prove Theorem 3. Indeed, estimate (9) gives us uniform estimates on E_n with some smoothness in space and estimate (10) gives us uniform estimates on E_n with some smoothness in time, allowing us to pass to the limit to the Vlasov-Poisson equation.

The following Theorem, obtained using Theorem 6, implies Theorem 4. It states that the characteristics are a little perturbation of those arising from the free transport equation; and the perturbation is uniform with respect to $n \in \mathbb{N}^*$.

Theorem 7 *Let $\alpha = \frac{19}{21}$ and $p \geq 3$. Assume that*

$$\|(1 + |\xi|)^{\frac{5}{2}+0} f^{in}\|_{L^2} < +\infty \quad \text{and} \quad \|(1 + |\xi|)^{2(1+\frac{3}{p}(\alpha-1))+0} f^{in}\|_{L_{x,\xi}^{3+0}} < +\infty.$$

Then, there exists $T_0 > 0$, there exists $C > 0$ such that for all $n \in \mathbb{N}^$, for all $(t_1, t_2) \in [0, T_0]^2$, for all $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$ the following estimate holds*

$$|X_n(t_1, t_2, x, \xi) - x + (t_1 - t_2)\xi| \leq C|t_1 - t_2|^{p'+1} \quad \text{and} \quad |V_n(t_1, t_2, x, \xi) - \xi| \leq C|t_1 - t_2|^{p'}. \quad (11)$$

To prove the first point of Theorem 4, we use a result from E. Hörst in [11] which states that propagation of smoothness is a consequence of the result that characteristics are a uniformly bounded (with respect to $n \in \mathbb{N}^*$) perturbation of the characteristics of free transport equation. Taking $p = 3$ in Theorem 7, we obtain that if the initial data has $2\alpha + 0$ moments in $L_{x,\xi}^{3+0}$ the characteristics satisfy Hörst's condition for propagation of smoothness for the solution to (1).

To prove the second point of Theorem 4, we follow the strategy adopted by P.-L. Lions and B. Perthame in [15]. We observe that

$$f(t, x, \xi) \leq \sup_{(y,w)} \{f^{in}(y + t\xi, w), |y - x| \leq R|t|^{1+\frac{1}{p'}}, |w - \xi| \leq R|t|^{\frac{1}{p'}}\}$$

and so $\rho \in L^\infty([0, T_0] \times \mathbb{R}^3)$.

In the following, we first study the linear transport equation with rough force field, and then use this study to prove Theorems 6 and 7.

3 Linear transport equation with rough force field.

In this section, we assume that we are in dimension $d \geq 1$ and we consider the linear transport equation with a rough force field $F(t, x)$ given by

$$\begin{cases} \partial_t f + \xi \nabla_x f + F \cdot \nabla_\xi f = 0 \\ f(0, x, \xi) = f^{in}(x, \xi). \end{cases} \quad (12)$$

Later on E_n is substituted with F , in equation (12) to get equation (6). In particular, in this section, we never use the fact that, for the approximate system (6), E_n can be written explicitly with respect to the density ρ_n .

If the force field F is smooth enough, then, there exists a unique solution of equation (12) which can be written explicitly in terms of the initial data by

$$f(t, x, \xi) = f^{in}(X(t, 0, x, \xi), V(t, 0, x, \xi))$$

where for all $(t_1, t_2) \in \mathbb{R}^2$, $(X(t_1, t_2, x, \xi), V(t_1, t_2, x, \xi))$ is the solution of the system

$$\begin{cases} \dot{X}(t_2) = V(t_2, X(t_2)) \\ \dot{V}(t_2) = F(t_2, X(t_2)) \end{cases} \quad (13)$$

with

$$X(t_1, t_1, x, \xi) = x \quad \text{and} \quad V(t_1, t_1, x, \xi) = \xi$$

as initial data.

In this section we endeavour to study this equation qualitatively using only weak norms on the force field F . More precisely, given $p \geq 1$, and $T > 0$, constants involving F will only depend on

$$S_{T,p} = \sup_{|B| \leq 1} \|F\|_{L^p_{[0,T]}(L^\infty(B))} \quad (14)$$

where B is a ball of \mathbb{R}^d of size 1.

We will first explain what are the motivations behind the study of this equation to show Theorems 6 and 7, then proceed to state the results obtained on equation (12) and finally prove them.

3.1 Motivations.

Reasons to study this equation to prove Theorem 6 and 7 are twofold.

- First, it provides us with information on the characteristics in terms of the smoothness of F ; which will be later useful to prove theorem 7 and gain uniform control on the characteristics.
- Also, such a study enables us to control the force field E_n which is directly related to the density ρ_n by studying ρ_n , seen as the density corresponding to the solution of the linear transport equation (12). This allows us to apply the study made on (12) and hence prove a uniform bound on E_n needed for Theorem 6.

More precisely, we focus here on the study of the characteristics and on moments effects regarding equation (12). Indeed, one of the key estimates that we have to prove to obtain Theorem 6 is that for all $p \geq 3$, for T small enough, there exists a constant C independent of $n \in \mathbb{N}^*$ such that

$$\|E_n\|_{L_T^p(L_B^\infty)} \leq C(|B|).$$

To control $\|E_n\|_{L_T^p(L_B^\infty)}$, the idea is the following. First, we split the force field E_n in two parts. Let $\gamma \in \mathcal{D}(\mathbb{R}^3)$ be a function such that $0 \leq \gamma \leq 1$ and $\gamma \equiv 1$ on $B(0, 1)$. Let $n \in \mathbb{N}^*$. We write for all $(t, x) \in \mathbb{R} \times \mathbb{R}^3$

$$E_n(t, x) = E_n^1(t, x) + E_n^2(t, x)$$

with

$$E_n^1(t, x) = \int \varphi_n(y)(1 - \gamma)(y)\rho_n(t, x - y)dy.$$

This decomposition has been used for the study of the Vlasov-Poisson equation by P.-L. Lions and B. Perthame in [15] and by F. Castella in [8].

Study of E_n^1 . E_n^1 is the convolution between a uniformly (with respect to $n \in \mathbb{N}^*$) smooth function and the density of probability ρ_n . Moreover, there exists a constant C such that for all $n \in \mathbb{N}^*$,

$$\|(1 - \gamma)\rho_n\|_{L^2} \leq C.$$

Hence, E_n^1 is easily estimated by

$$\|E_n^1\|_{L^\infty(\mathbb{R} \times \mathbb{R}^3)} \leq \sup_{(t,x) \in \mathbb{R} \times \mathbb{R}^3} \left| \int \varphi_n(y)(1 - \gamma)(y)\rho_n(x - y)dy \right| \leq C\|(1 + |\xi|)^{\frac{3}{2}+0} f^{in}\|_{L_{x,\xi}^2}. \quad (15)$$

Study of E_n^2 . E_n^2 is harder to deal with because it is the convolution between ρ_n and the function $\gamma\varphi_n$ which becomes rough when $n \rightarrow +\infty$. To tackle this difficulty, we shall make use of the fact that for each $x \in \mathbb{R}^3$ the density function $\rho_n(\cdot)\gamma(x-\cdot)$ involved is truncated in space to a ball of fixed size. More precisely, we observe that if the trajectories do not remain for too long in a compact set, then truncation in space of the density ρ_n (recall that ρ_n can be written using characteristics) provides additional moments over Hölder inequalities, averaging over time. Let us note that moments effects have been established for the free transport equation for the Euclidean metric (see B. Perthame [17] and references therein) and for non-trapped metrics in [19]. The setting is different here since we add a rough force field term of the transport equation. Indeed, we only have an a priori estimate on E_n in the space $L_t^p(L_B^\infty)$ where B is a ball of fixed size. To address the difficulty of a rough force field, we use estimates on the characteristics in term of measures, which do not require assumptions on the smoothness of these characteristics (see [19] for a similar strategy).

Remark 3 *We estimate E_n on balls and not on the whole space \mathbb{R}^3 . This is due to the fact the moments effects disappearing when we consider $\sup_{x \in \mathbb{R}^3} \rho_n(\cdot)\gamma(x-\cdot)$.*

3.2 Results obtained.

The fundamental properties that we will use on the transport equation (12) are given by the following propositions. The first one gives the following approximation on the characteristics.

Proposition 1 *Let $T > 0$ et $p \geq 1$. Then, for all (t_1, t_2) , for all $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$,*

$$|X(t_1, t_2, x, \xi) - x + (t_1 - t_2)\xi| \leq C(S_{p,T}, T)(1 + |\xi|)^{\frac{1}{p}} \quad (16)$$

and

$$|V(t_1, t_2, x, \xi) - \xi| \leq C(S_{p,T}, T)(1 + |\xi|)^{\frac{1}{p}}. \quad (17)$$

Remark 4 • *This proposition, gives us an approximation of the characteristics by those given by the free transport equation which is enough precise to allow us to obtain moments effects and propagation of moments on the solution of 12.*

- *However this approximation of the characteristics by those of the free transport equation gets worse as ξ grows, and it is still a long way from providing enough control over characteristics to prove Theorem 7. This first approximation will nevertheless be crucial to prove Theorem 7.*

The second proposition establishes that the solution effectively propagates moments, and a local in time moments effect.

Proposition 2 *Let $T > 0$, $p \geq 1$ and $\alpha \geq 0$. Then, for all $q \geq 1$, there exists a constant C such that for all $t \in [0, T]$ the following estimate holds*

$$\|(1 + |\xi|)^\alpha f(t)\|_{L_{x,\xi}^q} \leq C(\alpha, T, S_{T,p}) \|(1 + |\xi|)^\alpha f^{in}\|_{L_{x,\xi}^q}.$$

Moreover, the following moments effects occur. Let $\gamma \in \mathcal{D}(\mathbb{R}^d)$. Then, for all $q \geq 1$, for all $\alpha \geq 0$, there exists a constant C such that for all ball $B \subset \mathbb{R}^d$ of size 1 the following estimate holds

$$\left\| \sup_{x \in B} \gamma(x \cdot \cdot) (1 + |\xi|)^{\alpha + \frac{1}{qp'}} f(t) \right\|_{L_{T,x,\xi}^q} \leq C(\alpha, S_{p,T}, T, q) \|(1 + |\xi|)^\alpha f^{in}\|_{L_{x,\xi}^q}. \quad (18)$$

Remark 5 *We gain $\frac{1}{qp'}$ moments in L^q with respect to the Hölder inequalities and propagation of moments.*

3.3 Qualitative study of equation (12).

This part is devoted to the proof of Propositions 1 and 2. Given a certain $p \geq 1$, to make notations clearer, in both of the proofs we will choose $T \leq 1$ such that $S_{p,T} \leq 1$; the general case can be treated similarly.

3.3.1 Proof of Proposition 1.

We first study characteristics on small time intervals whose length is inversely proportional to $|\xi|$, to make sure that $X(t, x, \xi)$ stays in a ball of size 1. Then, using a time-velocity splitting, we prove that the characteristics (X, V) are a perturbation of those given by the free transport equation as stated in Proposition 1.

Remark 6 *Let us note that the idea of splitting in time and frequency or in time and velocity has been used in many contexts. We refer to the article by H. Bahouri and J.-Y. Chemin in [3] for the wave equation, to the article by N. Burq, P. Gérard and N. Tzvetkov in [7] for the Schrödinger equation and to [19] for the Liouville equation.*

Local study in time and velocity of characteristics. The following Lemma holds

Lemma 1 *Let $p \geq 1$ and $T \leq 1$ such that $S_{p,T} \leq 1$. Then, for all $(t_0, t_1, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d$, for all $t_2 \in [0, T]$ with $|t_2 - t_0| \leq \min(\frac{T}{4}, \frac{1}{4|\xi|})$, for all $x \in \mathbb{R}^d$*

$$X(t_1, t_2, x, \xi) \in B(X(t_1, t_0, x, \xi), 1).$$

Proof of Lemma 1. Applying Taylor's formula at order two with integral remainder, we obtain that for all $(t_1, t_2, x, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$

$$X(t_1, t_2, x, \xi) = x + (t_2 - t_1)\xi + \int_{t_1}^{t_2} (t_2 - s)F(s, X(t_1, s, x, \xi))ds.$$

We deduce that

$$X(t_1, t_2, x, \xi) - X(t_1, t_0, x, \xi) = (t_2 - t_0)\xi + (t_2 - t_0) \int_{t_1}^{t_0} F(s, X(s))ds + \int_{t_0}^{t_2} (t_2 - s)F(s, X(s))ds.$$

Using the fact that $|t_2 - t_0| \leq \min(T, \frac{1}{4|\xi|})$, we obtain

$$\left| X(t_1, t_2, x, \xi) - X(t_1, t_0, x, \xi) \right| \leq \frac{1}{4} + \frac{1}{2}S_{p,T} \leq \frac{3}{4}$$

which proves Lemma 1. □.

Splitting in time and velocity. We have

$$X(t_1, t_2, x, \xi) = x + (t_2 - t_1)\xi + \int_{t_1}^{t_2} (t_2 - s)F(s, X(t_1, s, x, \xi))ds.$$

To obtain Proposition 1, it suffices to prove that there exists a constant C such that for all $(t_1, t_2, x, \xi) \in [0, T] \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ the following estimate holds

$$\left\| \int_{t_1}^{t_2} F(s, X(t_1, s, x, \xi))ds \right\|_{L^\infty(\mathbb{R}^d)} \leq C(1 + |\xi|)^{\frac{1}{p}}. \quad (19)$$

To prove (19), we split the integral with little time interval of length less than $\min(\frac{T}{4}, \frac{1}{4|\xi|})$ for which we know that $X(t_1, t_2, x, \xi)$ stay in a ball of size 1. We obtain

$$\left| \int_0^t F(s, X(t_1, s)) ds \right| \leq \int_{t_1}^{t_2} |F(s, X(t_1, s))| ds = \sum_{k=0}^N \int_{t^k}^{t^{k+1}} |F(s, X(t_1, s))| ds$$

where $N \sim |\xi|$ and $|t^k - t^{k+1}| \sim \frac{1}{4|\xi|}$. On each time interval $[t^k, t^{k+1}]$, the trajectory X staying in a ball of size 1, we deduce that

$$\int_{t^k}^{t^{k+1}} |F(s, X(t_1, s))| ds \leq |t^k - t^{k+1}|^{\frac{1}{p'}} S_{p,T}.$$

Hence we have

$$\int_0^t |F(s, X(t_1, s))| ds \leq \sum_{k=0}^N |t^k - t^{k+1}|^{\frac{1}{p'}} \leq (1 + |\xi|)^{\frac{1}{p}}$$

and Proposition 1 follows. \square

3.3.2 Proof of Proposition 2.

The proof of Proposition 2 follows from the properties in Proposition 1 on the characteristics.

Propagation of moments. For all $p \geq 1$, the following estimate holds

$$\|(1 + |\xi|)^\alpha f\|_{L_{x,\xi}^p} = \|(1 + |\xi|)^\alpha f^{in}(X, V)\|_{L_{x,\xi}^p}.$$

Using inequality (17), we obtain that

$$\|(1 + |\xi|)^\alpha f^{in}(X, V)\|_{L_{x,\xi}^p} \leq C \|(1 + |V|)^\alpha f^{in}(X, V)\|_{L_{x,\xi}^p}.$$

Making the change of variables $(X, V) \rightarrow (x, \xi)$ we get propagation of moments as in Proposition 2.

Moments effects. Let us first prove the following lemma which states that the trajectories do not stay a long time in a compact set and then prove estimate (18) of Proposition 2.

Lemma 2 *There exists a constant C such that for all $\alpha > 0$, for all $t_1 \in [0, T]$, for all $|\xi| \geq 1$,*

$$\mu\{t_2 \in [0, T], X(t_1, t_2, x, \xi) \in B(\alpha)\} \leq \frac{C(1 + |\alpha|)}{(1 + |\xi|)^{\frac{1}{p'}}$$

where μ denotes the Lebesgue measure.

Proof of Lemma 2. Let t_0 and t_1 in $[0, T]$. According to estimate (16) of Proposition 1, there exists a constant C such that for all $t_2 \in [0, T]$ the following estimate

$$|X(t_1, t_2, x, \xi) - X(t_1, t_0, x, \xi)| \geq |t_2 - t_0| |\xi| - C(1 + |\xi|)^{\frac{1}{p}} \quad \text{holds}$$

which proves Lemma 2. □

Let $q \geq 1$, $\alpha \geq 0$ and B a ball of size 1. Let us define

$$D = \left\| \sup_{y \in B} \gamma(y - \cdot) (1 + |\xi|^\alpha) f_n \right\|_{L^q_{T,x,\xi}}^q.$$

$$D = \int_0^T \int_x \int_\xi (1 + |\xi|)^{\frac{1}{p'}} \left(\frac{(1 + |\xi|)}{1 + |V|} \right)^{\alpha q - \frac{1}{p'}} (1 + |V|)^{\alpha q - \frac{1}{p'}} f^q(X, V) \left(\sup_{y \in B} \gamma(y - \cdot) \right)^q dt dx d\xi.$$

Making the change of variables $(X, V) \rightarrow (x, \xi)$ which is a diffeomorphism of Jacobian equal to 1, we obtain that

$$D = \int_0^T \int_x \int_\xi (1 + |V|)^{\frac{1}{p'}} \left(\frac{1 + |V|}{1 + |\xi|} \right)^{\alpha q - \frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi.$$

Estimate (17) implies that

$$D \leq C \int_0^T \int_x \int_\xi (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi.$$

We split the right hand side of the above estimate in two parts. In the first part given by

$$D_1 = \int_0^T \int_x \int_{B(0,1)} (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi,$$

we integrate in ξ on the unit ball. We deduce immediatly that

$$D_1 \leq C \|f^{in}\|_{L^q}^q.$$

The second part is given by

$$D_2 = \int_0^T \int_x \int_{cB(0,1)} (1 + |\xi|)^{\frac{1}{p'}} (1 + |\xi|)^{\alpha q - \frac{1}{p'}} (f^{in})^q(x, \xi) \left(\sup_{y \in B} \gamma(y - X) \right)^q dt dx d\xi.$$

Using Lemma 2 which states that the trajectories do not stay a long time in a compact, we deduce by first integrating with respect to the time at fixed x and $\xi \in cB(0, 1)$ that there exists a constant C such that

$$\int_0^T \left(\sup_{y \in B} \gamma(y - X(s, 0, x, \xi)) \right)^q ds \leq \frac{C}{1 + |\xi|^{\frac{1}{p'}}}.$$

Hence we deduce that

$$D_2 \leq C \|(1 + |\xi|)^{\alpha - \frac{1}{p'q}} f^{in}\|_{L^q_{x,\xi}}^q$$

which ends the proof of the moments effects in Proposition 2. □

We now use the preceding study on the linear transport equation (12) to prove the uniform estimates given by Theorems 6 and 7 on the approximate system (6) of the Vlasov-Poisson equation.

4 Proof of Theorem 6.

The proof proceeds in three steps.
First, we prove uniform estimates on

$$\|E_n\|_{L_{T_0}^p(L^\infty(B))}.$$

using the moments effects in Proposition 2. Then, we show inequality (8) of Theorem 6 using Littlewood-Paley theory. Finally, in the last step, we prove inequalities (9) and (10) of Theorem 6. The first one is obtained by interpolation between estimate (8) with $p = 3$ and an another estimate which we will obtain without making use of dispersive effects and the second inequality (10) is relatively classical.

Definition 1 *Let $n \in \mathbb{N}^*$ and $p \geq 3$. We define*

$$T_0^n = \sup_{t \leq 1} \{ \sup_{|B| \leq 1} \|E_n\|_{L_t^p(L^\infty(B))} \leq 1 \}.$$

In particular, for all $n \in \mathbb{N}^*$, T_0^n is positive because $E_n \in L_{[0,T]}^\infty(L_x^\infty)$ according to Theorem 5.

4.1 Estimates of $\|E_n\|_{L_{T_0}^p(L^\infty(B))}$.

Proposition 3 *Let $p \geq 3$ and f^{in} an initial data such that*

$$\|(1 + |\xi|)^{\frac{3}{2}+0} f^{in}\|_{L_{x,\xi}^2} < +\infty \quad \text{and} \quad \|(1 + |\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x,\xi}^{p+0}} < +\infty.$$

Then, there exists $T_0 > 0$ such that for all $n \in \mathbb{N}^$ we have*

$$\sup_{|B| \leq 1} \|E_n\|_{L_{T_0}^p(L^\infty(B))} \leq 1.$$

Proof of Proposition 3. There exists a function $\tilde{\gamma} \in \mathcal{D}(\mathbb{R}^3)$ such that

$$|E_n^2(t, x)| \leq \left| \int \tilde{\gamma}(y) \varphi_n(y) \gamma(y) \rho_n(x - y) dy \right| \leq C \|\tilde{\gamma} \varphi_n\|_{L^{p'-0}} \|\gamma(x - \cdot) \rho_n\|_{L^{p+0}}.$$

As $p \geq 3$, we know that there exists a constant C such that for all $n \in \mathbb{N}^*$,

$$\|\tilde{\gamma} \varphi_n\|_{L^{p'-0}} \leq C,$$

we deduce that

$$\left| \int \varphi_n(y) \gamma(y) \rho_n(x - y) dy \right| \leq C \|\gamma(x - \cdot) \rho_n\|_{L^{p+0}}.$$

Using Hölder inequalities, we deduce that

$$\|\gamma(x - \cdot) \rho_n\|_{L^{p+0}} = \|\gamma(x - \cdot) f_n\|_{L_x^{p+0}(L_\xi^1)} \leq C \|\gamma(x - \cdot) (1 + |\xi|^{\frac{3}{p'}+0}) f_n\|_{L_{x,\xi}^{p+0}}.$$

Let B be a ball such that $|B| \leq 1$. Then we have

$$\|E_n^2\|_{L_{T_0^n}^{p+0}(L_B^\infty)} \leq C \left\| \sup_{x \in B} \gamma(x-\cdot) (1 + |\xi|^{\frac{3}{p'}+0}) f_n \right\|_{L_{T_0^n, x, \xi}^{p+0}}.$$

We now apply Proposition 2 with $q = p + 0$ and $\alpha = \frac{3}{p'} + 0$ and we obtain that

$$\|E_n^2\|_{L_{T_0^n}^{p+0}(L_B^\infty)} \leq C \|(1 + |\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x, \xi}^{p+0}}$$

hence

$$\sup_{|B| \leq 1} \|E_n^2\|_{L_{T_0^n}^p(L^\infty(B))} \leq C(0)(T_0^n)^{0+0} C \|(1 + |\xi|)^{\frac{3p-1}{pp'}+0} f^{in}\|_{L_{x, \xi}^{p+0}}. \quad (20)$$

Observe that $(T_0^n)^{0+0} \rightarrow 0$ when $T_0^n \rightarrow 0$; we deduce by a bootstrap argument that

$$\sup_{n \in \mathbb{N}^*} T_0^n > 0.$$

Combining this with estimate (15) which is independent of $n \in \mathbb{N}^*$, we obtain Proposition 3. \square

4.2 Proof of the estimate (8).

We assume to simplify the proof that $|A| = 1$, the general case can follow using the triangular inequality. Let $f \in C^\alpha$ where $0 < \alpha < 1$. Then, there exists $C_\alpha > 0$ such that

$$\|f\|_{C^\alpha} \leq C_\alpha \sup_{q \in \mathbb{N}} 2^{q\alpha} \|\Delta_q f\|_{L^\infty}$$

where Δ_q is an operator of frequency localization in a ring of size 2^q (see for example the article by J.-Y. Chemin [9] for a precise definition of Δ_q and for the proof of the above result). Concerning the smooth part E_n^1 of the force field E_n , we have

$$\|E_n^1\|_{L_{[0, T_0]}^\infty(C^1)} \leq C \|(1 + |\xi|)^{\frac{3}{2}+0} f^{in}\|_{L_{x, \xi}^2}. \quad (21)$$

Let $\psi \in \mathcal{D}(B)$ where B is a ball of size 1. Let us estimate ψE_n^2 . let $q \geq 1$, then

$$\Delta_q(\psi E_n^2)(x) = \psi \Delta_q E_n^2(x) + [\Delta_q(\psi E_n^2) - \psi \Delta_q E_n^2](x).$$

Consider the following Lemma which can be deduced from standard properties.

Lemma 3 *Let a, b be two functions. Then, for all $q \geq 1$, for all $x \in \mathbb{R}^d$, for all $\alpha \geq 0$, there exists a constant C_α such that*

$$\left| \Delta_q(ab) - a \Delta_q b \right|(x) \lesssim 2^{-q} \|a\|_{C^1} \|b\|_{L^\infty(B(x,1))} + C_\alpha 2^{q(\frac{d}{2}-1-\alpha)} \|a\|_{C^1} \|b\|_{L_x^2}.$$

Proof of Lemma 3. There exists a function $h \in \mathcal{S}(\mathbb{R}^d)$ such that for all $x \in \mathbb{R}^d$ the following equality holds

$$\Delta_q(ab) - a\Delta_q b(x) = 2^{qd} \int h(2^q(x-y)) \frac{(a(x) - a(y))}{|x-y|} |x-y| b(y) dy.$$

We have

$$|\Delta_q(ab) - a\Delta_q b(x)| \leq I_1(x) + I_2(x)$$

where

$$I_1(x) = \|a\|_{\mathcal{C}^1} 2^{qd} \int_{y \in B(x,1)} |x-y| |h(2^q(x-y)) b(y)| dy \quad \text{and}$$

$$I_2(x) = \|a\|_{\mathcal{C}^1} 2^{qd} \int |h(2^q(x-y))| |x-y| |b(y)| dy.$$

We have

$$I_1(x) \leq C 2^{-q} \|a\|_{\mathcal{C}^1} \|b\|_{L^\infty(B(x,1))}.$$

To control the term I_2 , we multiply above and below by $(2^q|x-y|)^\alpha$, and using Hölder inequalities, we deduce that

$$I_2(x) \leq C_\alpha 2^{q(\frac{d}{2}-1-\alpha)} \|\nabla a\|_{\mathcal{C}^1} \|b\|_{L_x^2},$$

which proves Lemma 3. □

We deduce from Lemma 3 that

$$\left| [\Delta_q(\psi E_n^2) - \psi \Delta_q E_n^2](x) \right| \leq C 2^{-q} \left(\|\nabla \psi\|_{L^\infty} \|E_n^2\|_{L^\infty(B(x,1))} + \|E_n^2\|_{L^2} \right).$$

The following Lemma allows us to control

$$\|E_n^2\|_{L_{[0,T_0]}^\infty(L_x^2)}.$$

Lemma 4 *There exists a constant C such that for all $n \in \mathbb{N}^*$ the following estimate holds*

$$\|E_n^2\|_{L_{[0,T_0]}^\infty(L_x^2)} \leq C \|(1 + |\xi|)^{\frac{3}{2}+0} f^{in}\|_{L_{x,\xi}^2}.$$

Proof of Lemma 4. Using Young inequalities, we obtain that

$$\|E_n^2\|_{L_x^2} = \|\gamma \varphi_n * \rho_n\|_{L_x^2} \leq C \|\gamma \varphi_n\|_{L_x^1} \|\rho_n\|_{L_x^2}.$$

Hölder inequalities ensure that

$$\|\rho_n\|_{L_x^2} \leq C \|(1 + |\xi|)^{\frac{3}{2}+0} f_n\|_{L_{x,\xi}^2}.$$

Using the fact that for all $t \in [0, T_0]$, the solution f_n of equation (6) propagates moments uniformly with respect to $n \in \mathbb{N}^*$, we deduce Lemma 4. □

Hence we only need to control

$$\psi \Delta_q E_n^2(x) = \psi \left[\widetilde{\Delta}_q(\gamma \varphi_n) * \Delta_q(\widetilde{\gamma}(x-\cdot) \rho_n) \right],$$

where $\widetilde{\Delta}_q$ is an operator of localization in frequency on a ring of size 2^q and where $\widetilde{\gamma} \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with $\widetilde{\gamma} \equiv 1$ on $B(0, 2)$. Bernstein injections give us

$$\|\Delta_q E_n^2\|_{L^\infty(B)} \leq \|\widetilde{\Delta}_q(\gamma \varphi_n)\|_{L^1} \sup_{x \in B} \|\Delta_q(\widetilde{\gamma}(x \cdot) \rho_n)\|_{L^\infty} \leq 2^{\frac{3q}{p+0}} \sup_{x \in B} \|\widetilde{\gamma}(x \cdot) \rho_n\|_{L^{p+0}}.$$

The following Lemma holds.

Lemma 5 *Let $\gamma \in \mathcal{D}(\mathbb{R}^3)$. Then, there exists a constant C which only depends of the measure of the support of γ and of $\|\gamma\|_{C^1}$ such that for all $q \geq 1$*

$$\|\Delta_q(\gamma \frac{x}{|x|^3})\|_{L^1} \leq C 2^{-q}.$$

Proof of Lemma 5. We assume for example that $\gamma \in \mathcal{D}(B(0, 1))$. We write

$$\Delta_q(\gamma \frac{x}{|x|^3}) = \Delta_q(\frac{x}{|x|^3})\gamma + \left[\Delta_q(\frac{x}{|x|^3})\gamma - \Delta_q(\frac{x}{|x|^3})\gamma \right].$$

We have

$$\left\| \gamma \Delta_q(\frac{x}{|x|^3}) \right\|_{L^1} \leq C \left\| \Delta_q(\frac{x}{|x|^3}) \right\|_{L^1} \leq C 2^{-q}.$$

Thus only remains the study of the commutator

$$\left[\Delta_q(\frac{x}{|x|^3})\gamma - \Delta_q(\frac{x}{|x|^3})\gamma \right] = S(x).$$

There exists a function $h \in \mathcal{S}(\mathbb{R}^3)$ such that

$$S(x) = 2^{qd} \int h(2^q(x-y))[\gamma(x) - \gamma(y)] \frac{y}{|y|^3} dy.$$

Let $a \in \mathcal{D}(\mathbb{R}^3)$ with $a \equiv 1$ on the ball $B(0, 2)$. We split S in two parts

$$S(x) = S_1(x) + S_2(x)$$

with

$$S_1(x) = 2^{qd} \int h(2^q(x-y))[\gamma(x) - \gamma(y)]a(y) \frac{y}{|y|^3} dy.$$

Study of S_1 . We have

$$S_1(x) = 2^{q(d-1)} \int h(2^q(x-y)) \frac{(\gamma(x) - \gamma(y))}{|x-y|} 2^q |x-y| a(y) \frac{y}{|y|^3} dy.$$

We deduce that

$$|S_1(x)| \leq 2^{q(d-1)} \|\gamma\|_{C^1} \int |h|(2^q(x-y)) 2^q |x-y| a(y) \frac{y}{|y|^3} |dy|.$$

Using Young inequalities, we get

$$\|S_1\|_{L^1} \leq C2^{-q} \left\| a \frac{y}{|y|^3} \right\|_{L^1} \leq C2^{-q}.$$

Study of S_2 . We have

$$S_2(x) = 2^{qd} \int h(2^q(x-y))(\gamma(x) - \gamma(y))(1-a(y)) \frac{y}{|y|^3} dy.$$

Here, in the integral, the truncated function $1-a$ implies that y leaves outside the ball $B(0, 2)$. So we have $\gamma(y) \equiv 0$ in the integral. If moreover $x \in {}^cB(0, 1)$ we also have $\gamma(x) = 0$ because the support of γ is contained in the ball $B(0, 1)$. Hence S_2 is compactly supported in the ball $B(0, 1)$. So we have

$$\|S_2\|_{L^1} \leq C\|S_2\|_{L^2}.$$

Multiplying above and below by $|x-y|$, we obtain that

$$|S_2(x)| \leq \|\gamma\|_{C^1} 2^{q(d-1)} \int |h|(2^q(x-y))2^q|x-y| \left| (1-a(y)) \frac{y}{|y|^3} \right| dy.$$

Applying Young inequalities, we obtain that

$$\|S_2\|_{L^1} \leq C\|S_2\|_{L^2} \leq C2^{-q} \left\| (1-a(y)) \frac{y}{|y|^3} \right\|_{L^2} \leq C2^{-q}$$

and Lemma 5 is proved. \square

We write

$$\gamma\varphi_n(x) = \frac{x^3}{|x|} \psi_n \quad \text{where} \quad \psi_n(x) = \frac{1}{(1+n|x|^2)^{\frac{3}{2}}} \gamma.$$

The function ψ_n is uniformly bounded with respect to $n \in \mathbb{N}^*$ in C^1 and has the same support as γ ; we deduce by Lemma 5 that there exists a constant C independent of $n \in \mathbb{N}^*$ such that

$$\|\Delta_q E_n^2\|_{L^\infty(B)} \leq C2^{q(\frac{3}{p}-1-(\frac{3}{p}-\frac{3}{p+0}))} \sup_{x \in B} \|(\tilde{\gamma}(x-\cdot))\rho_n\|_{L^{p+0}}.$$

In particular, we remark that for all $p \geq 1$, $-\frac{3}{p} + 1 + (\frac{3}{p} - \frac{3}{p+0}) < 1$. So we have

$$\|E_n^2\|_{C^{1-\frac{3}{p}+0}(B)} \leq C \sup_{x \in B} \|(\tilde{\gamma}(x-\cdot))\rho_n\|_{L^{p+0}} \leq C \sup_{x \in B} \|\tilde{\gamma}(x-\cdot)(1+|\xi|)^{\frac{3}{p}+0} f_n\|_{L^{p+0}}$$

with $1 - \frac{3}{p} + 0 < 1$. Combined with the moments effects in Proposition 2, we get the required estimate. \square

4.3 Proof of estimates (9) and (10).

4.3.1 Proof of the estimate (9).

The proof of this estimate follows by interpolation between estimate (8) with $p = 3$ and the estimate on E_n^2 given by the following Lemma.

Lemma 6 *There exists a constant C such that for all $n \in \mathbb{N}^*$ we have*

$$\|E_n^2\|_{L^\infty_{[0, T_0]}(\mathcal{C}_x^{0+0})} \leq C \|(1 + |\xi|)^{2+0} f^{in}\|_{L_{x, \xi}^{3+0}}. \quad (22)$$

Proof of Lemma 6. Let $t \in [0, T_0]$. Using Hölder inequalities, we deduce that

$$\|E_n^2(t)\|_{L_x^\infty} \leq C \|\gamma \rho_n\|_{L^{3+0}} \|\varphi_n\|_{L^{\frac{3}{2}-0}} \leq C \|(1 + |\xi|)^{2+0} f_n\|_{L_{x, \xi}^{3+0}}.$$

Using the fact that the solution of equation (6) propagates moments uniformly with respect to $n \in \mathbb{N}^*$, we deduce that

$$\|E_n^2\|_{L^\infty([0, T] \times \mathbb{R}^3)} \leq C \|(1 + |\xi|)^{2+0} f^{in}\|_{L_{x, \xi}^{3+0}}.$$

One can then rewrite the proof given to show estimate (8) using Littlewood-Paley theory, substituting L^∞ with the norm \mathcal{C}^{0+0} for the space variable, which concludes the proof of Lemma 6. \square

We can now show inequality (9) by interpolation between estimate (22) and estimate (8) with $p = 3$ and using the uniform bound obtained for E_n^1 in (21). \square

4.3.2 Proof of estimate (10).

When we integrate with respect to the variable ξ the equation (6), we find that

$$\partial_t \rho_n = \nabla_x \int_\xi f_n \xi d\xi.$$

Let us denote

$$g_n(t, x) = \int_\xi f_n \xi d\xi.$$

Using Littlewood-Paley theory, we obtain that

$$\|\partial_t E_n\|_{L^2} \sim \left(\sum_{q \in \mathbb{Z}} \|\Delta_q \partial_t E_n\|_{L^2}^2 \right)^{\frac{1}{2}}.$$

We need only prove that there exists a sequence $(c_q)_{q \in \mathbb{Z}}$ in ℓ^2 such that

$$\|\Delta_q \partial_t E_n\|_{L^2} \leq c_q.$$

We have

$$\|\Delta_q \partial_t E_n\|_{L^2} \leq C 2^q \|\tilde{\Delta}_q \varphi_n\|_{L^1} \|\Delta_q g_n\|_{L^2} \leq C \|\Delta_q g_n\|_{L^2}.$$

The last step consists in proving that

$$\|g_n\|_{L^2} < +\infty.$$

Applying Hölder inequalities, we obtain that

$$\|g_n\|_{L^2} \leq C \|(1 + |\xi|)^{\frac{5}{2}+0} f_n\|_{L^2_{x,\xi}}.$$

Using propagation of moments, we deduce estimate (10). \square

5 Proof of Theorem 7.

The idea behind our proof of Theorem 7 is to use the control on the trajectories given by Proposition 1 during the study of the linear transport equation with a rough force field. Even though this control is rather crude since the approximation gets worse as ξ grows, it provides enough information to ensure that two trajectories with different initial velocity will end up far from each other after a given time. This allows us to prove moments effects on the solution along each trajectory and to have the following control on E_n who gives us Theorem 7 directly.

Proposition 4 *Let $p \geq 3$. Then, there exists a constant C , there exists $T_0 > 0$, such that for all $n \in \mathbb{N}^*$, the following estimate holds*

$$\sup_{(x_0, \xi_0) \in \mathbb{R}^6} \|E_n^2(s, X_n(s, x_0, \xi_0))\|_{L^p([0, T])} \leq C.$$

Proof of Proposition 4. We prove the estimate for $p = 3$. The case $p \geq 3$ is obtained by interpolation between the estimate of Lemma 6 obtained without dispersive properties and the estimate of Proposition 4 obtained with $p = 3$.

Let $(x_0, \xi_0) \in \mathbb{R}^6$. We have

$$|E_n^2(s, X_n(s, x_0, \xi_0))| \leq C \|\gamma(X_n(s, x_0, \xi_0) - \cdot) \rho_n\|_{L^{3+0}}. \quad (23)$$

We are going to gain moments on the initial data using the fact that

$$s \rightarrow X_n(s, 0, x, \xi) - X_n(0, s, x_0, \xi_0)$$

do not stay a long time in a compact and uniformly with respect to x using the approximation on trajectories given by Proposition 1. To understand what happens, let us place in a simpler case where the characteristic X_n is that given by the free transport equation. We are reduced to understand when we can say that the function

$$\tilde{X}(s, \xi, \xi_0) : s \rightarrow x - x_0 + s(\xi + \xi_0)$$

do not stay a long time in a compact set uniformly with respect to the variable x . We observe in particular that the more ξ is near $-\xi_0$, the more the trajectory \tilde{X} stay a long time in a compact set. In our context, we must furthermore manage the fact that the trajectories are a perturbation about $|\xi|^{\frac{1}{3}}$ of those given by the free transport equation. Hence, we are going to split the integral $\int f_n(t, x, \xi) d\xi = \rho_n(t, x)$ in two areas which will be later optimized. Let $1 > \alpha \geq \frac{1}{3}$ be a parameter which will be later optimized.

- The first part is localized in the ball $B(-\xi_0, \beta|\xi_0|^\alpha)$ where β is a constant taken large enough. In this area, the dispersion is bad. In this area, to gain moments, hence we have interest to use the fact that the size of the ball $B(-\xi_0, \beta|\xi_0|^\alpha)$ where velocities leave are relatively small. Hence, here the estimate is done by using Hölder inequalities.
- The second area is localized outside the ball $B(-\xi_0, \beta|\xi_0|^\alpha)$. In this area where $-\xi_0$ is far enough of ξ , we can use some moments effects in order to control the term of the estimate (23).

Definition 2 Let $(x_0, \xi_0) \in \mathbb{R}^3 \times \mathbb{R}^3$. We denote by

$$A_{\xi_0} = B(-\xi_0, \beta|\xi_0|^\alpha)$$

where β is a large enough constant and $B_{\xi_0} = {}^c A_{\xi_0}$. Let $A \subset \mathbb{R}^3$ be a subset of velocity. We denote by

$$\Gamma_A(x_0, \xi_0) = \int_0^T \int_x \int_A (1 + |\xi|)^{6+0} (f^{in})^{3+0}(x, \xi) \gamma(X_n(s, 0, x, \xi) - X_n(0, s, x_0, \xi_0)) dt dx d\xi.$$

In the following, we assume that $|\xi_0| \geq 1$. To deal with the case where $|\xi_0| \leq 1$, the splitting is not necessary and it suffices to apply the same strategy that the one adopted below for the area with dispersion. We split the density ρ_n as follows

$$\rho_n(t, x) = \rho_n^1(t, x) + \rho_n^2(t, x) \quad \text{with} \quad \rho_n^1(t, x) = \int_{A_{\xi_0}} f_n(t, x, \xi) d\xi.$$

The area A_{ξ_0} without dispersion. The area where ξ leaves in A_{ξ_0} is estimate without using dispersive effects. We obtain using Hölder estimates that

$$\|\gamma(X_n(s, x_0, \xi_0) - \cdot) \rho_n^1\|_{L^{3+0}(\mathbb{R}^3)} \leq C \|f_n\|_{L^{3+0}_{\mathbb{R}^3 \times A_{\xi_0}}} (1 + |\xi_0|)^{2\alpha} \leq C \|(1 + |\xi|)^{2\alpha} f_n\|_{L^{3+0}_{\mathbb{R}^3 \times A_{\xi_0}}}.$$

Using the propagation of moments given by Proposition 2, we obtain that

$$\|\gamma(X_n(s) - \cdot) \rho_n^1\|_{L^{3+0}(\mathbb{R}^3)} \leq C \|(1 + |\xi|)^{2\alpha} f^{in}\|_{L^{3+0}_{x, \xi}}. \quad (24)$$

We have now to estimate

$$\|\gamma(X_n(s, x_0, \xi_0) - \cdot) \rho_n^2\|_{L^{3+0}([0, T_0] \times \mathbb{R}^3)}.$$

Applying Hölder inequalities and the change of variable $(X_n, V_n) \rightarrow (x, \xi)$, as it been done previously, we deduce that

$$\|\gamma(X_n(s, x_0, \xi_0) - \cdot) \rho_n^2\|_{L^{3+0}([0, T_0] \times \mathbb{R}^3)}^{3+0} \leq C \Gamma_{V_n^{-1}({}^c A_{\xi_0})}(x_0, \xi_0). \quad (25)$$

Insofar as V_n is a perturbation about $|\xi|^{\frac{1}{3}}$ of the identity, we deduce that we can replace in the integral (25) the velocity domain of integration $V_n^{-1}({}^c A_{\xi_0})$ by the domain ${}^c A_{\xi_0}$ at the cost of decreasing the constant β . We are reduced to estimate $\Gamma_{{}^c A_{\xi_0}}(x_0, \xi_0) = \Gamma_{B_{\xi_0}}(x_0, \xi_0)$.

The area B_{ξ_0} with dispersion. In the case where ξ leaves in B_{ξ_0} , the following Lemma gives us a dispersive effect and moments effect which allow us to control E along trajectories.

Lemma 7 *There exists a constant C such that for all $n \in \mathbb{N}^*$, $(x_0, \xi_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, $\xi \in B_{\xi_0}$*

$$\mu \left\{ s, \tilde{X}_n(s, x, \xi) = X_n(s, 0, x, \xi) - X_n(0, s, x_0, \xi_0) \in B(0, 1) \right\} \leq \frac{C}{\max(|\xi|, |\xi_0|)^{\alpha - \frac{1}{3}}} \quad (26)$$

where μ denotes the Lebesgue measure; and

$$\Gamma_{B_{\xi_0}}(x_0, \xi_0) \leq C \|(1 + |\xi|)^{2 + \frac{1}{9} - \frac{\alpha}{3}} f^{in}\|_{L_{x, \xi}^{3+0}}^{3+0}.$$

Proof of Lemma 7. To simplify calculations, we assume that $x = x_0$; the case when $x \neq x_0$ follows essentially by translation, which does not affect dispersion phenomena. Using Proposition 1, we obtain that

$$\left| \tilde{X}_n(s, x, \xi) \right| \geq s |\xi + \xi_0| - C \max(|\xi|, |\xi_0|)^{\frac{1}{3}}.$$

As $\xi \in B_{\xi_0}$, we have

$$|\xi + \xi_0| \geq \max(|\xi_0|, |\xi|)^\alpha.$$

We deduce that

$$\left| \tilde{X}_n(s, x, \xi) \right| \geq s \max(|\xi_0|, |\xi|)^\alpha - C \max(|\xi|, |\xi_0|)^{\frac{1}{3}}$$

which proves the first point of Lemma 7. By first integrating with respect to time in $\Gamma_{B_{\xi_0}}(x_0, \xi_0)$ and then applying estimate (26), we prove the second part of the Lemma. \square

Optimizing the parameter α so that the loss of moments between 2α and $2 + \frac{1}{9} - \frac{\alpha}{3}$ being optimal, we obtain that $\alpha = \frac{19}{21}$; which proves Proposition 4 for $p = 3$. By interpolating between the estimate given by Proposition 4 with $p = 3$ and the estimate given by Lemma 6, and using estimate (21), we obtain Proposition 4. \square

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