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# Multilayered Decision Problems

Dmitrii Lozovanul\*, Stefan Pickl†

## Abstract

If we generalize the Kyoto game which was presented at Lamsade-Dimacs workshop in 2004, we obtain a decision problem which can be described by a multilayered structure. This structure represents a hidden multiobjective control problem of a time-discrete systems with given starting and final states. The dynamics of the system are controlled by  $p$  actors (players). Each of the players intends to minimize his own integral-time cost of the system's passages using a certain admissible trajectory. At each stage (level) decisions are made by the players.

Nash Equilibria conditions can derived and algorithms for solving dynamic games in positional form are described. The existence theorem for Nash equilibria is related with the introduction of an auxiliary dynamic c-game.

We present the decision problem in that c-game which is defined on a special layered structure. The algorithmic principle which exploits this special structure for the decision processes will be described. New complexity results are presented and first numerical results are discussed.

**Key words :** Decision Theory, c-Game, Layered Network

## 1 Introduction - The General Model

In the following we describe the general underlying model for our decision problem. This is a short summary. For details we refer to [3, 4].

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\*Academy of Sciences, Institute of Mathematics and Computer Science, Academyst. 5, Kishinev MD-2028, Moldova. {lozovanu}@math.md

†Department of Computer Science, Universität der Bundeswehr München, 85577 Neubiberg- München, Germany. {stefan.pickl}@unibw.de

Let  $L$  be a discrete dynamical system with the set of states  $X \subseteq \mathbb{R}^m$ . At every time-step  $t = 0, 1, 2, \dots$  the state of  $L$  is  $x(t) \in X$ . Two states  $x_0$  and  $x_f$  are given in  $X$ , where  $x_0 = x(0)$  represents the starting point of  $L$  and  $x_f$  is the state into which the system  $L$  must be brought, i.e.  $x_f$  is the final state of  $L$ . We assume that the system  $L$  reaches the final state  $x_f$  at the time step  $T(x_f)$  such that

$$T_1 \leq T(x_f) \leq T_2,$$

where  $T_1$  and  $T_2$  are given. The dynamics of the system are described as follows

$$x(t+1) = g_t(x(t), u(t)), \quad t = 0, 1, 2, \dots$$

where

$$x(0) = x_0$$

and  $u(t) \in \mathbb{R}^m$  represents the vector of control parameters.

For  $u(t)$  at any time step  $t$  let there be given a nonempty set  $U_t(x(t))$  such that

$$u(t) \in U_t(x(t)), \quad t = 0, 1, 2, \dots, \quad (1)$$

i.e.  $U_t(x(t))$  is the admissible (decision) set for vector of control parameters at the time-step  $t$  when the state of system  $L$  is  $x = x(t) \in X$ . We assume that the derivatives  $g_t(x(t), u(t))$  are known and uniquely determine  $x(t+1)$  for given  $x(t)$  and  $u(t)$  at every moment of time  $t = 0, 1, 2, \dots$ . In addition we assume that at each point in time  $t$  the cost  $c_t(x(t), x(t+1))$  is known with  $c_t(x(t), x(t+1)) = c_t(x(t), g_t(x(t), u(t)))$  of system's passage from the state  $x(t)$  to the state  $x(t+1)$ .

Let

$$x_0 = x(0), x(1), x(2), \dots, x(t), \dots$$

be the trajectory generated by given vectors of control parameters

$$u(0), u(1), \dots, u(t-1), \dots$$

Either this trajectory passes through the state  $x_f$  at time  $T(x_f)$  or it does not pass through  $x_f$ . By

$$F_{x_0 x_f}(u(t)) = \sum_{t=0}^{T(x_f)-1} c_t(x(t), g_t(x(t), u(t))) \quad (2)$$

we denote the integral-time cost of system's passage from  $x_0$  to  $x_f$  if  $T_1 \leq T(x_f) \leq T_2$ ; otherwise we stipulate  $F_{x_0 x_f}(u(t)) = \infty$ .

### Multiobjective Control Problem

The multiobjective control problem is defined in the following way:  
 Minimize the function  $F_{x_0x_f}(u(t))$  which is defined by (2) according to (1).

Thus, we consider the discrete control problem which consists of cases with a fixed number of stages ( $T_1 = T_2$ ) and the number of stages is not limited ( $T_1 = 1, T_2 = \infty$ ).

Before we deal with the decision problem we introduce in the following section the special structure of Nash equilibria.

## 2 Problem Formulation for Determining a Nash Equilibrium

In order to characterize suitable Nash equilibria we consider the dynamic system  $L$  over discrete moments in time  $t = 0, 1, 2, \dots$ . At every time-step  $t$  the state of this  $L$  is  $x(t) \in X \subseteq \mathbb{R}^m$ . The dynamics of the system  $L$  are controlled by  $p$  players and it is described as follows

$$x(t + 1) = g_t(x(t), u^1(t), u^2(t), \dots, u^p(t)), \quad t = 0, 1, 2, \dots \tag{3}$$

Here  $x(0) = x_0$  is the starting point of the system  $L$  and  $u^i(t) \in \mathbb{R}^{m_i}$  represents the vectors of control parameters of player  $i, i \in \{1, 2, \dots, p\}$ . The state  $x(t + 1)$  of the system  $L$  at time-step  $t + 1$  is obtained uniquely if the state  $x(t)$  at the time-step  $t$  is known and the players  $1, 2, \dots, p$  fix their vectors of control parameters  $u^1(t), u^2(t), \dots, u^p(t)$ , respectively. For each player  $i, i \in \{1, 2, \dots, p\}$  the admissible sets  $U_t^i(x(t))$  for the vectors of control parameters  $u^i(t)$  are given, i.e.

$$u^i(t) \in U_t^i(x(t)), \quad t = 0, 1, 2, \dots; \quad i = \overline{1, p}.$$

We assume that  $U_t^i(x(t + 1)), t = 0, 1, 2, \dots; i = \overline{1, p}$ , are non-empty finite sets and that

$$U_t^i(x(t)) \cap U_t^j(x(t)) = \emptyset, \quad i \neq j, \quad t = 0, 1, 2, \dots$$

We assume that the players  $1, 2, \dots, p$  fix their vectors of control parameters  $u^1(t), u^2(t), \dots, u^p(t); t = 0, 1, 2, \dots$ , respectively, and the starting state  $x_0$  and final state  $x_f$  of the system  $L$  are known. Then for fixed vectors of control parameters  $u^1(t), u^2(t), \dots, u^p(t)$  either a unique trajectory  $x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$

from  $x_0$  to  $x_f$  exists and  $T(x_f)$  represents the time-moment when the state  $x_f$  is reached, or such trajectory from  $x_0$  to  $x_f$  does not exist. We denote by

$$F_{x_0x_f}^i(u^1(t), u^2(t), \dots, u^p(t)) = \sum_{t=0}^{T-1} c_t^i(x(t), g_t(x(t), u^1(t), u^2(t), \dots, u^p(t)))$$

the integral-time cost of system's passage from  $x_0$  to  $x_f$  for the player  $i$ ,  $i \in \{1, 2, \dots, p\}$  if the vectors  $u^1(t), u^2(t), \dots, u^p(t)$  generate a trajectory

$$x_0 = x(0), x(1), x(2), \dots, x(T(x_f)) = x_f$$

from  $x_0$  to  $x_f$  such that

$$u^i(t) \in U_t^i(x(t)), t = 0, 1, 2, \dots, T(x_f) - 1,$$

and

$$T_1 \leq T(x_f) \leq T_2.$$

Otherwise we stipulate

$$F_{x_0x_f}^i(u^1(t), u^2(t), \dots, u^p(t)) = \infty.$$

Note that  $c_t^i(x(t), g_t(x(t), u^1(t), u^2(t), \dots, u^p(t))) = c_t^i(x(t), x(t+1))$  represent the costs of the system's passage from the state  $x(t)$  to the state  $x(t+1)$  at the stage  $[t, t+1]$  for the player  $i$ . Then we obtain the following problem on networks:

### Problem: Decision Problem on Networks

If we find vectors of control parameters

$$u^{1*}(t), u^{2*}(t), \dots, u^{i-1*}(t), u^{i*}(t), u^{i+1*}(t), \dots, u^{p*}(t)$$

which satisfy the following condition

$$\begin{aligned} & F_{x_0x_f}^i(u^{1*}(t), u^{2*}(t), \dots, u^{i-1*}(t), u^{i*}(t), u^{i+1*}(t), \dots, u^{p*}(t)) \leq \\ & \leq F_{x_0x_f}^i(u^{1*}(t), u^{2*}(t), \dots, u^{i-1*}(t), u^i(t), u^{i+1*}(t), \dots, u^{p*}(t)) \\ & \forall u^i(t) \in \mathbb{R}^{m_i}, t = 0, 1, 2, \dots; i = \overline{1, p} \end{aligned}$$

(the expression  $i = \overline{1, p}$  is equivalent to  $i = \{1, \dots, p\}$ ), then we get a solution in the sense of Nash for our control problem on a network. In the following we present a sketch of the algorithmic principle.

### 3 Algorithmic Determination - Layered Structure

If we search for an suitable algorithmic principle we can orientate on similar problems of shortest path on a network. For that reason we present an application of Dijkstra's algorithm for a multiobjective version of the optimal paths problem in a weighted directed graph. The algorithm is able to determine the stationary Pareto strategy  $s^* \in S$  of the players for the multiobjective control problem on the network  $(G, X, c^1, c^2, \dots, c^p, x_0, x_f, T_1, T_2)$  with an arbitrary starting position  $x \in X$  and given final positions  $x_f \in X$ . We then obtain a tree which obtains all Pareto optimal paths from every  $x \in X$  to  $x_f$ .

### 4 Complexity Results

If we consider the following

**INSTANCE:** [*Multiobjective Decision Problem*]

Let  $L$  be a dynamic system with a finite set of states  $X$ ,  $|X| = n$ . We interpret these states as nodes of a graph  $G(X, E)$  where  $X$  is now the set of nodes (i.e. the states of our dynamic system) and  $E$  is a set of edges which have the following property:

At every discrete moment of time  $t = 0, 1, 2, \dots$  the state of the system  $L$  is  $x(t) \in X$ . Note, that here we associate  $x(t)$  with an *abstract element*. Furthermore we have a multi-objective control problem on the network  $(G, X, c^1, c^2, \dots, c^p, x_0, x_f)$  with  $p$  players: *The dynamics of the system  $L$  is described by a directed graph of passages in the graph  $G = (X, E)$ . Two states  $x_0$  and  $x_f$  are chosen in  $X$ , where  $x_0$  is a starting point of the system  $L$ ,  $x_0 = x(0)$ , and  $x_f$  is a final state of the system, i.e.  $x_f$  is a state in which the system  $L$  must be brought. An *edge*  $e = (x, y)$  signifies the possibility of passages of the system  $L$  from the state  $x = x(t)$  to the state  $y = x(t + 1)$  at any point in time  $t = 0, 1, 2, \dots, T(x_f)$ . (We assume that the system  $L$  reaches the final state  $x_f$  at the time step  $T(x_f)$ ). For simplicity we assume that the graph  $G = (X, E)$  is connected. To each edge  $e = (x, y) \in E$  of the graph of passages  $p$  functions  $c_e^1(t), c_e^2(t), \dots, c_e^p(t)$  are assigned, where  $c_e^i(t)$  expresses the cost of system's passage from the state  $x = x(t)$  to the state  $y = x(t + 1)$  at the stage  $[t, t + 1]$  for the player  $i$ . For the stationary case the functions  $c_e^i(t)$  do not depend on  $t$ .*

According to the algorithmic principle the existence of a partition  $X = X^1(t) \cup X^2(t) \cup \dots \cup X^p(t)$ ,  $(X^i(t) \cup X^j) = \emptyset, i \neq j$  can be proved. Here,  $X^i(t)$  correspond to the set of positions of player  $i$  at time-step  $t$ . The proposed algorithm determines the optimal strategy  $s^*$  which is closely related with a distinguished partition for the set  $X$ .

According to following input parameter we define the problem for the instance *Multi-objective Decision Problem*:

**Instance Multiobjective Decision Problem**

Input Parameter:  $n, p, T$

**PROBLEM:** Determine a stationary Pareto strategy.

Then we can prove that

**Theorem:** Our constructive algorithm determines for the instance *Multiobjective Decision Problem* Pareto stationary strategies of the players for every given starting position  $x$  and final position  $x_f$ . The running-time of the algorithm is  $O(n^3Tp)$ .

## 5 Conclusion

Games which are defined on networks are very interesting from a practical and theoretical point of view. We present a decision problem in the context of emissions trading. A special structure is exploited to obtain a polynomial algorithm. First numerical results will be discussed.

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