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▶ To cite this version:

Claude Lamboray. An Axiomatic characterization of the prudent order preference function. 2006. hal-00119582

HAL Id: hal-00119582
https://hal.science/hal-00119582v1

Preprint submitted on 11 Dec 2006

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An axiomatic characterization of the prudent order preference function

Claude Lamboray*

Abstract

In this paper, we will axiomatize a preference function that associates to a profile of linear orders the set of its corresponding prudent orders. We will introduce axioms that will restrict the set of linear orders to the set of prudent orders. By slightly adapting these axioms, the prudent order preference function can be fully characterized. Finally, we will characterize the extended prudent order preference function by introducing an additional Condorcet-like criterion.

Key words : Prudent Orders, Axiomatization

1 Introduction

Arrow and Raynaud[1] introduced a set of axioms that a ranking rule which combines a profile of linear orders into a compromise ranking should verify. Among these, axiom V' states that the compromise ranking should be a so-called prudent order. Intuitively, a prudent order is a linear order such that the strongest opposition against this solution is minimal, which is considered by the authors to be an interesting compromise ranking when working in an industrial or business-like context.

Apart from the works of Arrow and Raynaud [1] and Debord [3], prudent orders have also been analyzed by Lansdowne [9, 10] who compared their properties to other social ordering rules. However, the particular question of characterizing the set of prudent orders has not been addressed yet. This will be the topic of this paper.

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A characterization will be useful to highlight the particularities of prudent orders with respect to other common social ordering rules. The results presented in this paper can also be seen as a first step toward characterizing other prudent ordering rules, such as for instance the ranked pairs rule proposed by Tideman [15, 18].

Let us emphasize that, in our setting, the type of solution which we will characterize will be neither a ranking, nor a choice subset, but a *set* of rankings. This has also been the case in Young's [17] axiomatization of the set of Kemeny orders. A major difference however with the Kemeny model is that the prudent order model is "ordinal". In the literature, we can find characterizations of ordinal ranking models by Barbera [2], Pirlot [11] and Fortemps and Pirlot [5], although these authors were working in very different contexts.

The size of the set of prudent orders can be rather large in comparison to other common social ordering rules. This has been pointed out by Debord [3], who performed simulations to estimate the number of prudent orders for small profiles.

However, from a progressive decision aid perspective, the use of prudent orders as possible compromise rankings does make sense. Sometimes, we do not necessarily aim at finding directly one compromise ranking, but we can also be interested in depicting a whole range of possible compromise rankings. That is why, on the one hand, we want to keep the set of possible compromise rankings as large as possible in order to leave enough room for a progressive refinement. On the other hand, we want to restrict the whole set of linear orders to those which can be reasonably considered as potential compromise solutions.

This paper is organized as follows. First, we are going to recall the concept of a prudent order. We will introduce the relevant axioms in section 3. In section 4, we will present results related to the set of prudent orders, whereas in section 5 we will refine the set of prudent orders by taking into account an additional Condorcet criterion. Finally, we will end the paper with a conclusion.

2 Prudent orders

We denote by $\mathcal{O}$ the set of all the linear orders (transitive, complete and asymmetric binary relations) on a finite set of n alternatives $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$. Let $u = (O_1, O_2, \ldots, O_q) \in \mathcal{O}^q$ be a profile of q linear orders.

Given a profile u, we define majority margins $B_{ij} = |\{k : (a_i, a_j) \in O_k\}| - |\{k : (a_j, a_i) \in O_k\}|$ $\forall i, j$. It is easy to see that $\forall i, j$, $B_{ij} + B_{ji} = 0$. Furthermore, the majority margins of a profile of linear orders are either all even or all odd (see Debord [4]). If they are all even, then the number of linear orders belonging to the profile must also be even. If they are all odd, then the number of linear orders belonging to the profile must also be odd.

A linear extension O of a relation R is a linear order that contains R: $R \subseteq O$. We will denote by $\mathcal{E}(R)$ the set of all the linear extensions of relation R.

If $u = (O_1, O_2, \ldots, O_q)$ is a first profile and $u' = (O'_1, O'_2, \ldots, O'_{q'})$ is a second profile, then we will denote by $u + u'$ the profile $(O_1, O_2, \ldots, O_q, O'_1, O'_2, \ldots, O'_{q'})$. We denote by $(a_i a_j x)$ a linear order where a_i is followed by a_j and then by the alternatives x with x being an arbitrary permutation of the alternatives $\mathcal{A} \setminus \{a_i, a_j\}$. Furthermore, we denote by $-x$ the reverse permutation of x. Finally, the strict majority relation M is defined as follows:

$$\forall i, j \quad (a_i, a_j) \in M \iff B_{ij} > 0.$$

Let $\lambda \in \{-q, \ldots 0, \ldots, q\}$ and let us define the cut-relations $R_{>\lambda}$ and $R_{\geq\lambda}$ as follows: $\forall i \neq j$, $B_{ij} \geq \lambda \iff (a_i, a_j) \in R_{\geq\lambda}$ and $B_{ij} > \lambda \iff (a_i, a_j) \in R_{>\lambda}$.

We say that a relation R contains a cycle if there exists a subset of alternatives a_{i_1}, $a_{i_2}, \ldots, a_{i_p}$ such that $(a_{i_1}, a_{i_2}) \in R$, $(a_{i_2}, a_{i_3}) \in R$, ..., $(a_{i_p}, a_{i_1}) \in R$. When λ is large, then $R_{>\lambda}$ is empty and consequently does not contain any cycle. By gradually decreasing the cut value, some ordered pairs will be added to the corresponding strict cut-relation. Let β be the smallest value such that the corresponding strict cut relation is acyclic:

$$\beta = \min\{\lambda \in \{-q, \ldots, O, \ldots, q\} : R_{>\lambda} \text{ is acyclic }\}.$$

Let us note that, consequently, the relation $R_{\geq\beta}$ must contain at least one cycle. A prudent order $O_P \in \mathcal{O}$ is defined as a linear order that extends the relation $R_{>\beta}$:

$$R_{>\beta} \subseteq O_P. \tag{1}$$

We will characterize a function $\mathcal{PO}$, called prudent order preference function, that associates to every profile u the set of all the linear extensions of $R_{>\beta}$:

$$\begin{aligned} \mathcal{PO}(u) &= \{O_P \in \mathcal{O} : R_{>\beta} \subseteq O_P\} \\ &= \mathcal{E}(R_{>\beta}). \end{aligned}$$

Since it is always possible to extend an acyclic relation into a linear order (see Szpilrajn[13]), the set of prudent orders will never be empty. Arrow and Raynaud justified such a compromise ranking O_P to be prudent by the fact that ordered pairs that belong to the relation $R_{>\beta}$ are pairs with no contradiction and a high majority. If these ordered pairs would not belong to the final compromise ranking, there would be a large and non-divided majority against such a ranking.

It can be shown that equation 1 is equivalent to stating that O_P is a linear order that, in a way, minimizes the strongest opposition against this ranking, the value of this strongest opposition being exactly equal to β.

$$\max_{(a_i,a_j)\notin O_P, i\neq j} B_{ij} = \beta \leq \max_{(a_i,a_j)\notin O, i\neq j} B_{ij} \quad \forall O \in \mathcal{O}. \tag{2}$$

Equivalently, a prudent order O_P is a linear order that maximizes the weakest link. In fact, since $B_{ij} + B_{ji} = 0$, equation 2 can be rewritten as follows:

$$\min_{(a_i,a_j)\in O_P} B_{ij} \geq \min_{(a_i,a_j)\in O} B_{ij} \quad \forall O \in \mathcal{O}. \tag{3}$$

Another interpretation of prudent orders worth mentioning here has been highlighted by Debord [3]. Let us suppose that the profile u is such that the strict majority relation is not a linear order. Let us now consider any linear order $O \in \mathcal{O}$ and let μ_O be the minimal number of times that one has to add O to u such that the majority relation of the profile $u + \mu_O O$ corresponds exactly to the linear order O. In fact, μ_O corresponds to the necessary strength of the linear order O to impose itself as the majority relation. We define $\mu_{min} = \min_{O\in\mathcal{O}} \mu_O$.

Theorem 1 *Debord (1987) [3]*
Let u be a profile such that the strict majority relation is not a linear order. O is a prudent order if and only if the majority relation of the profile $u + \mu_{min} O$ is equal to O.

Hence, a prudent order can be interpreted as a linear order that one has to add the smallest number of times to the profile so that the majority relation of the new profile corresponds exactly to this linear order.

Several algorithms can be imagined that construct prudent orders. Let us mention three of them: Kohler's rule [8], Arrow and Raynaud's rule [1] and the Ranked Pairs rule proposed by Tideman[15, 18]. Furthermore, the first two of these three algorithms verify a form of sequential prudence (see Arrow and Raynaud [1], Lansdowne [9]). However,

unlike conjectured by Arrow and Raynaud, Lansdowne [10] highlighted that Kohler's rule and Arrow and Raynaud's rule may not be sufficient to find the whole set of prudent orders.

There is a straightforward approach to enumerate all the prudent orders (see also Debord [4]). First, use Kohler's algorithm to find one prudent order O_P. Find the strongest opposition against this ranking, which means the largest B_{ij} such that $(a_j, a_i) \in O_P$. This value corresponds to β. Compute $R_{>\beta}$. Enumerating all the prudent orders then comes down to enumerating all the linear extensions of $R_{>\beta}$. A constant amortized time algorithm for enumerating linear extensions, that is an algorithm that runs in $O(|\mathcal{PO}(u)|)$, is presented in Pruesse and Ruskey [12].

The reader may wonder if $R_{>\beta}$ can be any possible acyclic relation on the set of alternatives $\mathcal{A}$. The answer will be given by the following proposition, which states that the set of linear extensions of any acyclic relation on the set of alternatives $\mathcal{A}$ can be seen as the set of prudent orders corresponding to a certain profile.

Proposition 1 *For every acyclic relation R on the set of alternatives $\mathcal{A}$, there exists a profile u of linear orders such that $\mathcal{PO}(u) = \mathcal{E}(R)$.*

Proof: If R is a linear order, then u trivially consists of this linear order, i.e. $\mathcal{PO}(R) = R = \mathcal{E}(R)$. If R is not a linear order, then we construct the profile u as follows. For every ordered pair (a_i, a_j) such that $(a_i, a_j) \in R$, we consider the two linear orders V_{ij}^1 and V_{ij}^2:

$$V_{ij}^1 = (a_i a_j x) \quad V_{ij}^2 = (-x a_i a_j).$$

The profile u then consists of all the linear orders V_{ij}^1 and V_{ij}^2 such that $(a_i, a_j) \in R$:

$$u = \sum_{(a_i,a_j)\in R} V_{ij}^1 + V_{ij}^2.$$

This will lead to the following preference margins matrix:

$$B_{ij} = \begin{cases} 2 & \text{if } (a_i, a_j) \in R \\ -2 & \text{if } (a_j, a_i) \in R \\ 0 & \text{otherwise} \end{cases}$$

One may show that in this case, $\beta = 0$, and, consequently, $\mathcal{PO}(u) = \mathcal{E}(R_{>\beta}) = \mathcal{E}(R)$. $\square$

Let us illustrate the prudent order preference function on the following example that can be found in Taylor [14]. There are five alternatives a, b, c, d and e and the profile consists of 7 linear orders:

O_1	$abcde$
O_2	$adbec$
O_3	$adbec$
O_4	$cdbea$
O_5	$cdbae$
O_6	$bcdae$
O_7	$ecdba$

We thus have the following majority margins:

	a	b	c	d	e
a	.	-1	-1	-1	3
b	1	.	1	-1	5
c	1	-1	.	3	1
d	1	1	-3	.	5
e	-3	-5	-1	-5	.

In this case, $\beta = 1$, since $R_{>1}$ is acyclic, whereas $R_{\geq 1}$ contains a cycle since $B(c, d) \geq 1, B(d, b) \geq 1$ and $B(b, c) \geq 1$. Consequently, the relation

$$R_{>\beta} = R_{>1} = \{(a, e), (b, e), (c, d), (d, e)\}.$$

The set of prudent orders thus corresponds to all the linear extensions of this relation. These 12 prudent orders are listed below:

1	$acbde$	7	$cadbe$
2	$abcde$	8	$bcade$
3	$cabde$	9	$cdabe$
4	$acdbe$	10	$cbdae$
5	$cbade$	11	$cdbae$
6	$bacde$	12	$bcdae$

3 The axioms

In this section, we are going to introduce the axioms that we will need to characterize the prudent order preference function. More generally, a preference function f is a procedure that combines a profile u into a non-empty set of linear orders $f(u)$.

$$\begin{array}{rccl} f: & \mathcal{O}^q & \mapsto & P(\mathcal{O}) \setminus \emptyset \\ & u & \rightarrow & f(u). \end{array}$$

In general, the strict majority relation M contains cycles, which is commonly referred to as Condorcet's paradox. However, in case M is acyclic, then the first axiom says that this information must be contained in the set of solutions.

Axiom 1 *Condorcet Consistency (CC):*
If M is acyclic, then:

$$f(u) \subseteq \mathcal{E}(M).$$

In other words, this means that, if M is acyclic and if $(a_i, a_j) \in M$, then a_i must be preferred to a_j in all the linear orders of $f(u)$. This axiom implies that, if M is a linear order, then this linear order is the unique solution of the preference function.

Lemma 1 *If f verifies Condorcet Consistency and if M is a linear order, then $f(u) = \{M\}$.*

A stronger version of axiom CC says that, if M is acyclic, then $f(u)$ corresponds exactly to all the linear extensions of this relation M.

Axiom 2 *Strong Condorcet Consistency (SCC):*
If M is acyclic, then:

$$f(u) = \mathcal{E}(M).$$

It is easy to see that Strong Condorcet Consistency implies Condorcet Consistency.

Let u_E be a profile such that $B_{ij} = 0 \forall i, j$. Adding such a profile to u_E to a given profile will not alter the result.

Axiom 3 *E-invariance (EI):*

$$f(u + u_E) = f(u).$$

The next axioms says that if the size of the profile is odd and we create a new profile by taking twice the initial profile, then the set of compromise solutions may only increase.

Axiom 4 *Weak homogeneity (WH):*
If q is odd, then:

$$f(u) \subseteq f(u+u).$$

A stronger version of this axiom simply says that if we double an odd profile, then the result does not change at all.

Axiom 5 *Homogeneity (H):*
If q is odd, then:

$$f(u) = f(u+u)$$

Homogeneity implies weak homogeneity.

Before presenting our main axiom, let us introduce the concept of updating a linear order O by switching tow adjacent alternatives a_i and a_j:

Definition 1 *Let us consider a linear order O and an ordered pair (a_i, a_j). We say that the linear order O' is an update of O in favor of pair (a_i, a_j) if $O = (...a_j a_i ...)$ is such that a_j directly precedes a_i and $O' = (...a_i a_j ...)$ is obtained by reversing a_j and a_i in O.*

The majority oriented profile update procedure then consists in applying an update for each pair by taking into account the majority situation for this pair. More formally, let us consider a profile $u = (O_1, \ldots, O_q)$ with a majority margin matrix B. Let us furthermore suppose that $q \geq |M|$, i.e. q is at least as large as the number of ordered pairs belonging to the majority relation of that profile. We will construct a new profile by doing the following for every pair $\{a_i, a_j\}$ $(i \neq j)$:

- If $B_{ij} > 0$, then let O'_k be an update in favor of pair (a_i, a_j) of a linear order O_k of profile u that has not been updated yet. Replace O_k with O'_k.

- If $B_{ij} = 0$, then
  - Do nothing.
    OR
  - Let O'_k be an update in favor of pair (a_i, a_j) of a linear order O_k of profile u that has not been updated yet. Replace O_k with O'_k.
    OR

- Let O'_k be an update in favor of pair (a_j, a_i) of a linear order O_k of profile u that has not been updated yet. Replace O_k with O'_k.

Let us denote by u^{update} the profile obtained at the end of this procedure. We call this a majority oriented profile update procedure because:

- If $B_{ij} > 0$, then there is a strict majority of rankings in the initial profile u that prefer a_i over a_j. Updating a linear order in favor of (a_i, a_j) only confirms this idea.
- If $B_{ij} = 0$, then there are as many rankings in the profile u that prefer a_i over a_j than there are rankings that prefer a_j over a_i. For such a pair, three possibilities can naturally be considered:
  - We will do nothing since we do not want to discriminate between a_i and a_j.
  - We will update a linear order in favor of (a_i, a_j), which will break the indifference by improving the situation of a_i with respect to a_j.
  - We will update a linear order in favor of (a_j, a_i), which will break the indifference by improving the situation of a_j with respect to a_i.

The next axioms says that if we update the profile u into a new profile u^{update} by applying the above mentioned procedure, then the set of compromise solutions either stays the same or shrinks.

Axiom 6 *Majority Oriented Profile Convergence (MOPC):*
Let u be a profile and let u^{update} be the profile obtained using the majority oriented profile update procedure. Then:

$$f(u^{update}) \subseteq f(u).$$

The axiom means that if we update the profile in the direction of the majority relation, then the set of compromise rankings can possibly converge. Let us note that for pairs $\{a_i, a_j\}$ such that $B_{ij} = 0$, by breaking the indifference between a_i and a_j in a certain direction, or by leaving the indifference untouched, different profiles u^{update} can be constructed. This will eventually pull the set of compromise solutions $f(u^{update})$ in possibly different directions. Whatever choice will be made, the profile u^{update} will always be considered as "compatible" with the majority relation and the new set $f(u^{update})$ will always be contained in the set $f(u)$.

We will show that CC, MOPC and EI imply SCC.

Proposition 2 *If f verifies Condorcet Consistency, Majority Oriented Profile Convergence and E-invariance then f verifies Strong Condorcet Consistency.*

Proof: Let u be a profile such that the strict majority relation M is acyclic. CC tells us that, since M is acyclic, $f(u) \subseteq \mathcal{E}(M)$. We are now going to show that $\mathcal{E}(M) \subseteq f(u)$, which will complete the proof.

Let $O \in \mathcal{E}(M) \Rightarrow M \subseteq O$. We are going to show that $O \in f(u)$.

Let us denote by $D = O \setminus M$ the ordered pairs that belong to the linear order O but not to the strict majority relation M.

Let B be the majority margins of profile u. It is easy to see that $B_{ij} = 0 \iff (a_i, a_j) \in D$ or $(a_j, a_i) \in D$. In fact, if $B_{ij} = 0$, then $(a_i, a_j) \notin M$ and $(a_j, a_i) \notin M$. Since O is complete, either $(a_i, a_j) \in D$ or $(a_j, a_i) \in D$. Reciprocally, $(a_i, a_j) \in D \Rightarrow (a_i, a_j) \notin M \Rightarrow B_{ij} \leq 0$. Furthermore, it is impossible that $B_{ij} < 0$, since this would mean that $B_{ji} > 0 \Rightarrow (a_j, a_i) \in M \Rightarrow (a_j, a_i) \in O$. This is a contradiction since we supposed $(a_i, a_j) \in D \Rightarrow (a_i, a_j) \in O$. The asymetry property of linear orders does not allow that both $(a_i, a_j) \in O$ and $(a_j, a_i) \in O$.

For every $(a_i, a_j) \in O$, let us consider the following two linear orders:

$$V_{ij}^1 = (a_i a_j x).$$

$$V_{ij}^2 = (-x a_j a_i).$$

Let us add these linear orders to profile u:

$$u' = u + \sum_{(a_i, a_j) \in O} V_{ij}^1 + V_{ij}^2.$$

Since for every pair $(a_i, a_j) \in O$, V_{ij}^1 and V_{ij}^2 are two opposite orders and since f verifies E-invariance, we must have:

$$f(u') = f(u). \tag{4}$$

Let us note that the majority margins B' of profile u' are exactly the same as the majority margins B of profile u. We are going to apply the majority oriented update procedure to u' as follows:

For every pair $\{a_i, a_j\}$, do the following:

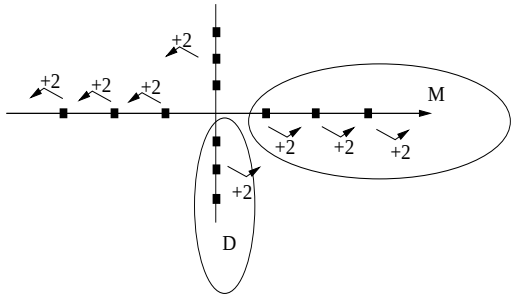


Figure 1: The transformation of the majority margins of a profile u' into the majority margins of profile u'' .

- If $B'_{ij} > 0$, then we are going to update V^2_{ij} in favor of (a_i, a_j). Note that V^2_{ij} actually belongs to profile u' since $B'_{ij} > 0 \Rightarrow B_{ij} > 0 \Rightarrow (a_i, a_j) \in O$.

- If $B'_{ij} = 0$ and $(a_i, a_j) \in D$, then we are going to update V^2_{ij} in favor of (a_i, a_j). Note that V^2_{ij} actually belongs to profile u' since $(a_i, a_j) \in D \Rightarrow (a_i, a_j) \in O$.

Let us denote by B'' the majority margins of the profile u'' obtained at the end of the procedure. In fact, B'' can be obtained from the majority margins B' by shifting two units to the right the positive pairs ($B'_{ij} > 0$), and consequently shift two units to the left the negative pairs $(B'_{ij} < 0)$. Furthermore, the pairs such that $B'_{ij} = 0$ and $(a_i, a_j) \in D$ are shifted to the right whereas the pairs such that $B'_{ij} = 0$ and $(a_i, a_j) \notin D$ are shifted to the left. This is schematically represented in figure 1, where a square represents a majority margin evaluation B'_{ij} of a pair (a_i, a_j) and the vertical axis represents the mid-value 0.

Since we suppose that f verifies Majority Oriented Profile Convergence, we must have:

$$f(u'') \quad \subseteq \quad f(u'). \tag{5}$$

Let us denote by M'' the strict majority relation of the profile u'':

- $(a_i, a_j) \in M \Rightarrow B'_{ij} = B_{ij} > 0 \Rightarrow B''_{ij} = B'_{ij} + 2 > 0 \Rightarrow (a_i, a_j) \in M''$.
- $(a_i, a_j) \in D \Rightarrow B'_{ij} = B_{ij} = 0 \Rightarrow B''_{ij} = B'_{ij} + 2 > 0 \Rightarrow (a_i, a_j) \in M''$.

Consequently, $M \cup D = O \subseteq M'' \Rightarrow O = M''$ (M'' is an asymmetric relation), and hence M'' is a linear order. By applying lemma 1, we thus have that $f(u'') = \{O\}$. Given equation 4 and 5, we can finally conclude that $O \in f(u)$. □

We will also use a slightly different version of the MOPC axiom, namely Majority Oriented Profile Invariance, which says that changing the profile in the sense of the majority does not alter the result at all.

Axiom 7 *Majority Oriented Profile Invariance (MOPI):*
Let u be a profile and let u^{update} be a profile obtained using the majority oriented profile update procedure. If the strict majority relation of u^{update} contains at least one cycle, then:

$$f(u^{update}) = f(u).$$

Axiom MOPI is the same as axiom MOPC, except that the inclusion is replaced by an equality, under the condition that the strict majority relation of profile u^{update} contains cycles. It means that if we obtain u^{update} by applying a majority oriented update procedure to profile u, and the new profile u^{update} contains cycles (either existing cycles of profile u or new cycles created through the update procedure), then the set of compromise rankings must stay the same.

Let us note that removing the cyclicity condition of profile u^{update} from this axiom will lead to a contradiction with axiom SCC. In fact, if the strict majority relation of profile u^{update}, denoted by M^{update}, is acyclic, then the strict majority relation of profile u, denoted by M, must also be acyclic, since one can show that $M \subseteq M^{update}$. According to SCC, $f(u) = \mathcal{E}(M)$ and $f(u^{update}) = \mathcal{E}(M^{update})$. If we suppose that $M \subset M^{update}$, then it can happen that $f(u^{update}) \subset f(u)$.

4 Characterization

First, we are going to show that the prudent order preference function verifies the axioms introduced so far.

Proposition 3 *The prudent order preference function verifies Condorcet Consistency, Strong Condorcet Consistency,E-Invariance, Weak Homogeneity, Homogeneity, Majority Oriented Profile Convergence and Majority Oriented Profile Invariance.*

Proof: It is easy to see that prudent orders verify EI, WH and H. Let us first prove that that they also verify SCC, i.e. if M is acyclic, then $\mathcal{PO}(u) = \mathcal{E}(M)$. Either M is complete or not.

- M is complete.
  Let $\gamma = \max\{B_{ij} : (a_i, a_j) \notin M\} : M = R_{>\gamma}$ and so $R_{>\gamma}$ is cycle-free. There exists $(a_i, a_j) \notin M$ such that $B_{ij} = \gamma$. Since M is complete and $(a_i, a_j) \notin M$, $(a_j, a_i) \in M$. Consequently, $R_{\geq\gamma}$ contains a cycle. Hence, $\beta = \gamma$ and $\mathcal{PO}(u) = \mathcal{E}(R_{>\gamma}) = \mathcal{E}(R_{>\beta}) = \mathcal{E}(M)$.

- M is not complete.
  By definition, $M = R_{>0}$ and so $R_{>0}$ is cycle-free. Since M is not complete, there must exist two alternatives a_i and a_j such that $(a_i, a_j) \notin R_{>0}$ and $(a_j, a_i) \notin R_{>0}$. Since $B_{ij} \leq 0, B_{ji} \leq 0$ and $B_{ij} + B_{ji} = 0$, we must have that $B_{ij} = B_{ji} = 0$. Consequently $R_{\geq 0}$ contains a cycle. Hence $\beta = 0$, and so $R_{>\beta} = R_{>0} = M$. Consequently,$\mathcal{PO}(u) = \mathcal{E}(M)$.

SCC implies CC.

Let us now prove that prudent orders also verify MOPC. Let B be the majority margins of profile u and let B^{update} be the majority margins of profile u^{update}. Furthermore, let β be the optimal cut-value for profile u and let β^{update} be the optimal cut-value for profile u^{update}.

If $\beta < 0$, then the strict majority relation of profile u is a linear order. The strict majority relation of the profile u^{update} must then be exactly the same linear order. In that case we have that $\mathcal{PO}(u) = \mathcal{PO}(u^{update})$ and so MOPC is verified. Let us from now on suppose that $\beta \geq 0$.

Let us cut relation B at level β $(= R_{>\beta})$ and let us cut relation B^{update} at level $\beta + 2$ $(= R^{update}_{>\beta+2})$. One may check that if $\beta \geq 0$, then:

$$R^{update}_{>\beta+2} \quad = \quad R_{>\beta}. \tag{6}$$

Since $R_{>\beta}$ is acyclic, so is $R^{update}_{>\beta+2}$. Consequently, $\beta^{update} \leq \beta$, and so:

$$R_{>\beta} = R^{update}_{>\beta+2} \subseteq R^{update}_{>\beta^{update}}.$$

This means that $\mathcal{PO}(u^{update}) = \mathcal{E}(R^{update}_{>\beta^{update}}) \subseteq \mathcal{E}(R_{>\beta}) = \mathcal{PO}(u)$, which proves MOPC.

Let us finally show that prudent orders also verify MOPI. Let us suppose that the strict majority relation of profile u^{update} contains at least one cycle.

- $\beta < 0$.
  This means that the strict majority relation of profile u is a linear order. Consequently, the strict majority relation of profile u^{update} is also a linear order. We are not interested in this case in the MOPI axiom, since we suppose that the strict majority relation of profile u^{update} contains at least one cycle.

- $\beta = 0$
  We know already that $\beta^{update} \leq \beta + 2 = 2$ (see equation 6). We furthermore know that $\beta^{update} > 0$, since we suppose that the strict majority relation of profile u^{update} contains at least one cycle. Since $\beta = 0$, the profiles u and u^{update} must be even and consequently the majority margins and the optimal cut values take only even values. Hence, $\beta^{update} = 2$. Following equation 6, we have that $R^{update}_{>\beta^{update}} = R_{>\beta}$, which means that $\mathcal{PO}(u) = \mathcal{PO}(u^{update})$.

- $\beta \geq 1$
  In that case, we can show that $R^{update}_{\geq\beta+2} = R_{\geq\beta}$. Since $R_{\geq\beta}$ contains at least one cycle, then this means that $R^{update}_{\geq\beta+2}$ also contains at least one cycle and consequently $\beta^{update} = \beta+2$. Following equation 6, we have that $R^{update}_{>\beta^{update}} = R_{>\beta}$, which means that $\mathcal{PO}(u) = \mathcal{PO}(u^{update})$.

We thus showed that, if the the strict majority relation of profile u^{update} contains at least one cycle, then $\mathcal{PO}(u) = \mathcal{PO}(u^{update})$. This proves MOPI.

□

Let us now present our first result. In fact, we will show that if i) we want to use the axioms Condorcet Consistency, Majority Profile Convergence, E-Invariance and Weak Homogeneity and ii) we want to have a set of possible compromise solutions as large as possible, then we must use the prudent order preference function.

Let us insist on the interpretation of keeping the set of compromise rankings as large as possible. In a progressive decision aid approach, it can be interesting to keep the set of compromise solutions as large as possible. Since it is useless to consider all the linear orders, the above mentioned axioms will restrict the set of possible compromise solutions to all the prudent orders.

Theorem 2 *The prudent order preference function is the largest preference function (in the sense of the inclusion) that verifies Condorcet Consistency, E-Invariance, Weak Homogeneity and Majority Oriented Profile Convergence.*

Proof: We are going to show that any preference function f that verifies the above mentioned axioms is such such that

$$f(u) \subseteq \mathcal{PO}(u)$$

Since, by proposition 3, prudent orders verify these axioms, the proof will be complete.

Let us suppose that the size of profile u is even. Let B be the majority margins of this profile and let β be the optimal cut-value of this profile. Hence, $\mathcal{PO}(u) = \mathcal{E}(R_{>\beta})$. If $\beta < 0$, then this means that the strict majority relation is a linear order. In that case, lemma 1 tells us that $f(u) = \{M\} = \mathcal{PO}(u)$. Let us from now on suppose that $\beta \geq 0$.

Let us consider the following relations:

$$\Lambda_1 = \{(a_i, a_j) : B_{ij} = 2\}.$$

$$\Lambda_2 = \{(a_i, a_j) : B_{ij} = 4\}.$$

$$\Lambda_3 = \{(a_i, a_j) : B_{ij} = 6\}.$$

$$\vdots$$

$$\Lambda_p = \{(a_i, a_j) : B_{ij} = \beta\}.$$

For every $(a_i, a_j) \in \Lambda_s$, $s = 1, \ldots, p$, we are going to consider the following linear orders:

$$\forall k = 1, \ldots, s : V_{ij}^{1k} = (a_j a_i x).$$

$$\forall k = 1, \ldots, s : V_{ij}^{2k} = (-x a_j a_i).$$

For every $(a_i, a_j) \in R_{>\beta}$, we are going to consider the following linear orders:

$$\forall k = 1, \ldots, p : V_{ij}^{1k} = (a_j a_i x)$$

$$\forall k = 1, \ldots, p : V_{ij}^{2k} = (-x a_j a_i)$$

We are going to define a new profile u^0 as follows:

$$u^0 = u + \sum_{(a_i,a_j)\in\Lambda_1} (V_{ij}^{11} + V_{ij}^{21}) + \sum_{k=1}^{2} \sum_{(a_i,a_j)\in\Lambda_2} (V_{ij}^{1k} + V_{ij}^{2k}) + \ldots$$

$$+\sum_{k=1}^{p}\sum_{(a_i,a_j)\in\Lambda_p}(V_{ij}^{1k}+V_{ij}^{2k})+\sum_{k=1}^{p}\sum_{(a_i,a_j)\in R_{>\beta}}(V_{ij}^{1k}+V_{ij}^{2k}).$$

Let us denote by B^0 the majority margins of profile u^0. In fact, B^0 is linked to B in the following way:

$$\forall i,j=1,\ldots,n \quad \begin{cases} B_{ij}^0 = B_{ij}-\beta & \text{if } B_{ij}>\beta \\ B_{ij}^0 = B_{ij}+\beta & \text{if } B_{ij}<-\beta \\ B_{ij}^0 = 0 & \text{otherwise} \end{cases}$$

Hence, the strict majority relation M^0 of profile u^0 is in fact equal to $R_{>\beta}$, and so M^0 is acyclic. By applying axiom CC, we can tell that:

$$f(u^0)\subseteq\mathcal{E}(M^0)=\mathcal{E}(R_{>\beta})=\mathcal{PO}(u).$$

Let us apply the majority oriented update procedure to u^1 in the following way: For every pair $\{a_i,a_j\}$, do the following:

- If $B_{ij}^0>0$, then $(a_i,a_j)\in R_{>\beta}$. Let us update V_{ij}^{11} in favor of (a_i,a_j) by reversing the preference between a_j and a_i.
- If $B_{ij}^0=0$ and $(a_i,a_j)\in\Lambda_p$, then let us update V_{ij}^{11} in favor of (a_i,a_j) by reversing the preference between a_j and a_i.

Let us denote by u^1 the profile obtained at the end of this procedure. Let us denote by B^1 the corresponding preference margins. B^1 can be obtained from the majority margins B^0 of the profile u^0 by shifting to the right the positive pairs $(B_{ij}^0>0)$, and, consequently, to the left the negative pairs $(B_{ij}^0<0)$. Furthermore, the pairs such that $B_{ij}^0=0$ and $(a_i,a_j)\in\Lambda_p$ are shifted to the right whereas the pairs such that $B_{ij}^0=0$ and $(a_j,a_i)\in\Lambda_p$ are shifted to the left. The remaining pairs such that $B_{ij}^0=0$ and $(a_i,a_j)\notin\Lambda_p$ and $(a_j,a_i)\notin\Lambda_p$ simply do not move. The transformation from B^0 into B^1 is schematically illustrated in figure 2.

By applying axiom MOPC, we know that:

$$f(u^1)\subseteq f(u^0)\subseteq\mathcal{PO}(u).$$

Let us apply the majority oriented update procedure to u^1. For every pair $\{a_i,a_j\}$, do the following:

- If $B_{ij}^1>0$, then $(a_i,a_j)\in R_{>\beta}\cup\Lambda_1$. Let us update V_{ij}^{12} in favor of (a_i,a_j) by reversing the preference between a_j and a_i.

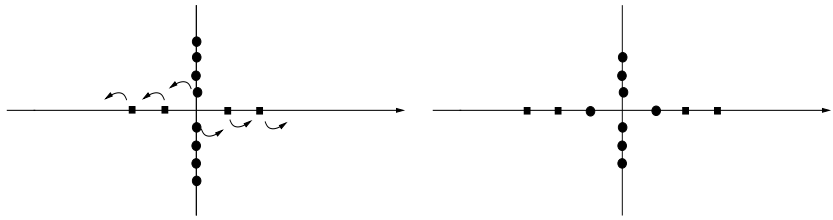


Figure 2: The transformation from B^0 into B^1.

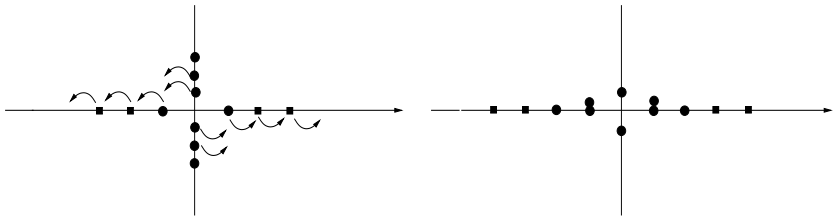


Figure 3: The transformation from B^1 into B^2.

- If $B^1_{ij} = 0$ and $(a_i, a_j) \in \Lambda_{p-1}$, then let us update V^{11}_{ij} in favor of (a_i, a_j) by reversing the preference between a_j and a_i.

Let us denote by u^2 the profile obtained at the end of this procedure. Let us denote by B^2 the corresponding preference margins. The transformation from B^1 into B^2 is schematically represented in figure 3.

By reapplying axiom MOPC, we can tell that:

$$f(u^2) \subseteq f(u^1) \subseteq f(u^0) \subseteq \mathcal{PO}(u).$$

By reapplying these arguments p times, we finally get:

$$f(u^p) \subseteq f(u^{p-1}) \subseteq \ldots \subseteq f(u^2) \subseteq f(u^1) \subseteq f(u^0) \subseteq \mathcal{PO}(u).$$

In fact, the profile u^p can be written as follows, where we denote by $\bar{V}^{2k}_{ij} = (a_i a_j x)$ the linear order that has been obtained by reversing a_j and a_i in V^{1k}_{ij} ($k = 1, \ldots, p$).

$$u^p = u + \sum_{(a_i,a_j)\in\Lambda_1} (\bar{V}^{21}_{ij} + V^{21}_{ij}) + \sum_{k=1}^{2} \sum_{(a_i,a_j)\in\Lambda_2} (\bar{V}^{2k}_{ij} + V^{2k}_{ij}) + \ldots$$

$$+\sum_{k=1}^{p}\sum_{(a_i,a_j)\in\Lambda_p}(\bar{V}_{ij}^{2k}+V_{ij}^{2k})+\sum_{k=1}^{p}\sum_{(a_i,a_j)\in R_{>\beta}}(\bar{V}_{ij}^{2k}+V_{ij}^{2k}).$$

In fact, u^p can be written as $u+u_E$, where u_E is a profile where all the majority margins are zero. Using axioms EI, we thus have that $f(u^p)=f(u)$. Consequently:

$$f(u)\subseteq\mathcal{PO}(u).$$

This completes the proof for even profiles. Let us now suppose that u has an odd size. We then create an even profile by taking the profile u twice. Applying the previous result to the even profile $u+u$ and using axiom WH, we get:

$$f(u)\subseteq f(u+u)\subseteq\mathcal{PO}(u+u)=\mathcal{PO}(u).$$

□

Using similar axioms, the following theorem fully characterizes the prudent order preference function.

Theorem 3 *The prudent order preference function is the only preference function that verifies Strong Condorcet Consistency, E-Invariance, Homogeneity and Majority Oriented Profile Invariance.*

Proof: We know from proposition 3 that the prudent order preference function verifies SCC, MOPI, EI and H.

Let us suppose that the size of profile u is even. Let B be the majority margins of this profile and let β be the optimal cut-value of this profile. Hence, $\mathcal{PO}(u)=\mathcal{E}(R_{>\beta})$. If $\beta\leq 0$, then this means that the strict majority relation is acyclic and consequently axiom SCC tells us that $f(u)=\mathcal{E}(M)=\mathcal{E}(R_{>\beta})=\mathcal{PO}(u)$. Let us from now on suppose that the majority relation contains at least one cycle and consequently $\beta>0$.

As in the proof of theorem 2, let us define a profile u^0 with majority margins B^0 and an acyclic strict majority relation M^0. Applying axiom SCC, we have:

$$f(u^0)=\mathcal{E}(M^0)=\mathcal{E}(R_{>\beta})=\mathcal{PO}(u).$$

As in the proof of theorem 2, let us define a profile u^1 by applying the majority oriented profile update procedure to u^0. Let us denote by M^1 the strict majority relation of profile u^1. In fact, $M^1=R_{>\beta}\cup\Lambda_p=R_{\geq\beta}$, which must contain at least one cycle. Hence M^1 is not acyclic and we can apply axiom MOPI:

$$f(u^1)=f(u^0)=\mathcal{PO}(u).$$

As in the proof of theorem 2, let us define a profile u^2 by applying the majority oriented profile update procedure to u^1. Let us denote by M^2 the strict majority relation of profile u^2. In fact, $M^1 \subseteq M^2$ and since M^1 is not acyclic, M^2 is not acyclic. Consequently, we can reapply axiom MOPI:

$$f(u^2) = f(u^1) = \mathcal{PO}(u).$$

By reapplying the same argument p times (as in the proof of theorem 2), and by using the axiom EI, we finally get:

$$f(u) = f(u^p) = f(u^{p-1}) = \ldots = f(u^2) = f(u^1) = f(u^0) = \mathcal{PO}(u).$$

This completes the proof for even profiles. In case, the profile u is odd, we apply the previous result to the even profile $u + u$ and using axiom H we have:

$$f(u) = f(u + u) = \mathcal{PO}(u + u) = \mathcal{PO}(u)$$

□

In comparison to theorem 2, we strengthened Condorcet Consistency by Strong Condorcet Consistency, and Weak Homogeneity by Homogeneity. Furthermore, Majority Profile Convergence was replaced by Majority Profile Invariance, although the latter does not imply the first.

Let us emphasize the independence between the five axioms used in the characterization of theorem 3.

1. Strong Condorcet Consistency
   The preference function that associates to every profile the whole set of linear orders $g_1(u) = \mathcal{O}\ \forall u$ trivially verifies MOPI, EI and H, but is clearly not SCC.

2. E-Invariance

   Let us consider the following four linear orders:

O_1	$abcde$	O_3	$deabc$
O_2	$eabcd$	O_4	$cdeab$

   Let us denote by $u^* = (O_1, O_2, O_3, O_4)$ the profile that consist of these 4 linear orders. The following majority margins B^* are associated with this profile u^*:

B^*	a	b	c	d	e
a	.	4	2	0	-2
b	-4	.	2	0	-2
c	-2	-2	.	2	0
d	0	0	-2	.	2
e	2	2	0	-2	.

We are going to define a new preference function g_2 as follows:

$$g_2(u) = \begin{cases} \mathcal{O} & \text{if } u = u^* \\ \mathcal{PO}(u) & \text{otherwise} \end{cases}$$

Hence g_2 corresponds to the prudent order preference function except for the profile u^*. It is easy to see that g_2 verifies SCC. It also verifies H. Let u be an odd profile. Hence we know that $u \neq u^*$ since u^* is even. We thus have that $g_2(u) = \mathcal{PO}(u)$. Furthermore, $u + u \neq u^*$ since, for instance, the linear order $abcde$ only appears once in the profile u^*. We thus have that $g_2(u + u) = \mathcal{PO}(u + u)$.

g_2 also verifies MOPI. Let u be a profile and let u^{update} be the profile obtained after applying the majority oriented profile update procedure to u. On the one hand, we have that $u \neq u^*$ because we would need a profile of at least 7 linear orders are needed to apply the majority oriented profile update procedure, given the majority margins B^*. A similar argument shows that $u^{update} \neq u^*$. Hence $g_2(u) = \mathcal{PO}(u) = \mathcal{PO}(u^{update}) = g_2(u^{update})$.

However, g_2 odes not verify EI: add the two linear orders $abcd$ and $dcba$ to profile u^*. We then have that $g_2(u^*) = \mathcal{O}$ and $g_2(u^* + abcd + dcba) = \mathcal{PO}(u^* + abcd + dcba) \neq \mathcal{O}$.

3. Homogeneity

   Let us consider the preference margin matrix B^* and the preference margin matrix B' compatible in the sense of the majority with a profile yielding a preference margin matrix B^*.

B^*	a	b	c
a	.	3	-1
b	-3	.	3
c	1	-3	.

B'	a	b	c
a	.	2	-2
b	-2	.	2
c	2	-2	.

   We are going to define a new preference function g_3 as follows, where B denotes the majority margin matrix of the profile u.

$$g_3(u) = \begin{cases} \mathcal{O} & \text{if } \exists \lambda \in \mathbb{N} : B = B^* + \lambda B' \\ \mathcal{PO}(u) & \text{otherwise} \end{cases}$$

   Hence g_3 corresponds to the prudent order preference function except for profiles with majority margins $B^* + \lambda B'$.

   Such a procedure g_3 verifies SCC and EI. It also verifies MOPI.

	u	update in favor of	u^{update}
O_1	$dcba$	(a,b)	$dcab$
O_2	$cabd$	(a,c)	$acbd$
O_3	$cbda$	(b,c)	$bcda$
O_4	$dbac$	(b,d)	$bdac$
O_5	$dcab$	(c,d)	$cdab$
O_6	$adbc$	(d,a)	$dabc$
O_7	$abcd$	n.a.	$abcd$
O_8	$abcd$	n.a	$abcd$
O_9	$abcd$	n.a	$abcd$
O_{10}	$bcda$	n.a	$bcda$
O_{11}	$cbda$	n.a	$cbda$
O_{12}	$dabc$	n.a	$dabc$
O_{13}	$dabc$	n.a	$dabc$
O_{14}	$cbda$	n.a	$cbda$
O_{15}	$bacd$	n.a.	$bacd$

Table 1: The profile u before and after the update procedure.

However, g_3 does not verify H. Let u be a profile with majority margins equal to B^*. Consequently, $g_3(u) = \mathcal{O}$. However, $g_3(u+u) = \{abc\}$. Although the size of profile u is odd, $g_3(u) \neq g_3(u+u)$.

4. Majority Profile Invariance

Kemeny orders [6] can be defined as follows:

$$g_4(u) = \{O_K \in \mathcal{O} : \sum_{(a_i,a_j)\in O_K} B_{ij} \geq \sum_{(a_i,a_j)\in O} B_{ij} \quad \forall O \in \mathcal{O}\}.$$

Kemeny orders verify SCC, EI, H but not MOPI, as will be shown by the example depicted in table 1. The majority margins of this profile will be:

B	a	b	c	d
a	.	**1**	**1**	-3
b	-1	.	**3**	**3**
c	-1	-3	.	**3**
d	**3**	-3	-3	.

$\rightarrow \quad g_4(u) = \{bcda\}$

Let us apply the majority oriented profile update procedure to u as shown in table 1. Let us then compute the Kemeny orders of the profile u^{update} with preference margins B^{update}:

B^{update}	a	b	c	d
a	.	3	3	-5
b	-3	.	5	5
c	-3	-5	.	5
d	5	-5	-5	.

$\rightarrow \quad g_4(u) = \{abcd\}$

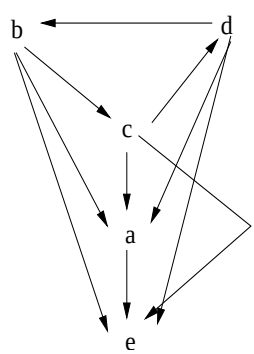
Hence $g_4(u^{update}) \neq g_4(u)$.

Let us note that the same counter examples can be used to check the independence of the axioms of theorem 2:

1. g_1 verifies EI,WH and MOPC but not CC.
2. g_2 verifies CC,WH and MOPC, but not EI.
3. g_3 verifies CC,EI and MOPC, but not WH.
4. g_4 verifies CC,EI and WH, but not MOPC.

5 Extended prudent orders

Let us come back to Taylor's example introduced in section 2. The strict majority relation of this profile can be graphically represented as follows:



This strict majority graph seems to indicate that b, c and d could be put before a in a compromise ranking. However, the prudent orders 1-9 do not follow this argument. One reason for this is that prudent order preference function does not verify what Truchon [16] calls the extended Condorcet Criterion. Let us introduce this additional condition.

Given a strict majority relation M, we say that $TC(M)$ is the top cycle of M if it is the smallest possible subset of $\mathcal{A}$ such that for all $a_i \in TC(M)$ and for all $a_j \in \mathcal{A} \setminus TC(M)$ we have $(a_i, a_j) \in M$. We partition $\mathcal{A}$ into ordered subsets $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ with $\mathcal{A}_i = TC(M|\mathcal{A} \setminus \cup_{j<i} \mathcal{A}_j)$. We call this ordered partition the top-cycle partition (see for instance Truchon[16] or Klamler[7] for further comments on this partition). In Taylors example, the top-cycle partition consists of three blocs: $\mathcal{A}_1 = \{b, c, d\}, \mathcal{A}_2 = \{a\}$ and $\mathcal{A}_3 = \{e\}$.

If the strict majority relation is acyclic, then we know already from axiom CC that

$$\forall a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l,\ k < l : \quad (a_i, a_j) \in O\ \forall O \in f(u).$$

The following axiom then says that, also for profiles with a non-acyclic strict majority relation, the top-cycle partition should not be contradicted by any solution belonging to the set of compromise rankings.

Axiom 8 *Extended Condorcet Criterion (XCC):*
Let us suppose that the strict majority relation contains at least one cycle and let $\mathcal{A}_1$, $\mathcal{A}_2, \ldots, \mathcal{A}_p$ be the top-cycle-partition. We say that a preference function f verifies the Extended Condorcet criterion if:

$$\forall a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l,\ k < l : \quad (a_i, a_j) \in O\ \forall O \in f(u).$$

Given a top-cycle partition, we can very naturally define the following partial order T:

$$(a_i, a_j) \in T \iff a_i \in \mathcal{A}_k \text{ and } a_j \in \mathcal{A}_l \text{ and } k < l$$

Axiom XCC simply says that $f(u) \subseteq \mathcal{E}(T)$. In order to incorporate this axiom into the prudent order model, we define the following new preference function $\mathcal{XPO}$, called the extended prudent order preference function :

$$\forall u \quad \mathcal{XPO}(u) = \mathcal{E}(R_{>\beta} \cup T)$$

The set of compromise rankings thus corresponds to all the linear extensions of the relation $R_{>\beta} \cup T$. We will show that $R_{>\beta} \cup T$ is acyclic. Since we can always extend an acyclic relation into a linear order (see Szpilrajn[13]), the set $\mathcal{E}(R_{>\beta} \cup T)$ is never empty and consequently $\mathcal{XPO}$ is a true preference function.

Proposition 4 *The relation $R_{>\beta} \cup T$ is acyclic.*

Proof: If $\beta < 0$, then M is a linear order and $M = R_{>\beta} = T$. Consequently, $R_{>\beta} \cup T = R_{>\beta}$ is acyclic. Let us suppose $\beta \geq 0$. Let us suppose by contradiction that $R_{>\beta} \cup T$ contains a cycle. Since this cycle cannot appear inside a bloc of the top-cycle-partition (by definition T is empty inside a bloc and $R_{>\beta}$ is acyclic), there must exist a_i and a_j such that $a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l, k < l$ and $(a_j, a_i) \in R_{>\beta}$. Hence, $B_{ji} > \beta \geq 0 \Rightarrow (a_j, a_i) \in M \Rightarrow (a_i, a_j) \notin M$. This is impossible since we supposed that a_i belongs to a higher bloc in the top-cycle partition than a_j.

□

It is clear that $\forall u, \mathcal{XPO}(u) \subseteq \mathcal{PO}(u)$. In Taylor's example, the set $\mathcal{XPO}$ will consist of the prudent orders 10-12.

Interestingly, the extended prudent order preference function verifies all the axioms used in theorem 2.

Proposition 5 *The extended prudent order preference function verifies Condorcet Consistency, Majority Oriented Profile Convergence, E-Invariance, Weak Homogeneity and Condorcet Criterion.*

Proof: EI and WH are easy to check. Furthermore, XCC is verified by construction.

Let us show that CC is verified. If M is acyclic, then $R_{>\beta} = M$ (see proposition 3). Furthermore, $T \subseteq M \Rightarrow T \subseteq R_{>\beta} \Rightarrow R_{>\beta} \cup T = R_{>\beta} = M$. Consequently, $\mathcal{XPO}(u) = \mathcal{E}(R_{>\beta} \cup T) = \mathcal{E}(M)$.

Let us show that MOPC is verified. We denote by M the strict majority relation and T the top-cycle relation of profile u. We denote by M^{update} the strict majority relation and T^{update} the top-cycle relation of profile u^{update}. Furthermore, let β be the optimal cut-value for profile u and let β^{update} be the optimal cut-value for profile u^{update}. We know from proposition 3 that $\mathcal{PO}(^{update}) = \mathcal{E}(R^{update}_{>\beta^{update}}) \subseteq \mathcal{PO}(u) = \mathcal{E}(R_{>\beta})$. Hence:

$$R_{>\beta} \subseteq R^{update}_{>\beta^{update}}.$$

We are now going to show that $T \subseteq T^{update}$, which will prove MOPC since $\mathcal{PO}(u^{update}) = \mathcal{E}(T^{update} \cup R^{update}_{>\beta^{update}}) \subseteq \mathcal{E}(T \cup R_{>\beta}) = \mathcal{XPO}(u)$.

Let us show that if $(a_i, a_j) \in T$, then $(a_i, a_j) \in T^{update}$. Since $(a_i, a_j) \in T$, there exists two blocks in the top-cycle partition $\mathcal{A}_k, \mathcal{A}_l$ such that $a_i \in \mathcal{A}_k, a_j \in \mathcal{A}_l$ and such that $k < l$. Let $\mathcal{A}'_{k'}$ be the block of the top-cycle partition of M^{update} to which a_i belongs

and let $\mathcal{A}'_{l'}$ be the block of the top-cycle partition of M^{update} to which a_j belongs. We want to show that $k' < l'$, which will prove that $(a_i, a_j) \in T^{update}$. Let us suppose by contradiction that $k' \geq l'$. If $k' > l'$, then this means that $(a_j, a_i) \in M^{update}$, since $a_i \in \mathcal{A}'_{k'}$ and $a_j \in \mathcal{A}'_{l'}$. This is impossible since we know that $(a_i, a_j) \in M, M \subseteq M^{update}$ and M^{update} is asymmetric. If $k = l$, then this means that a_i and a_j belong to the same block in the top-cycle partition of M^{update}. Let us denote this block by $\mathcal{A}' = \mathcal{A}'_{k'} = \mathcal{A}'_{l'}$. Let us consider the following subsets of alternatives: $D'_1 = \mathcal{A}' \cap \mathcal{A}_1$, $D'_2 = \mathcal{A}' \cap \mathcal{A}_2$, $\ldots$, $D'_p = \mathcal{A}' \cap \mathcal{A}_p$. Let $D^+ = \cup_{r=1}^{k} D'_r$and let $D^- = \mathcal{A} \setminus D^+$. We know that both D^+ and D^- are non-empty since $\{a_i\} \subseteq D'_k \subseteq D^+$ and $\{a_j\} \subseteq D'_l \subseteq D^-$. We know that $\forall x \in D^+$ and $\forall y \in D^-$, $(x, y) \in M \Rightarrow (x, y) \in M^{update}$. Hence $\mathcal{A}'$ cannot be a block of the top-cycle partition of M^{update} since D^+ is dominating D^-. This proves that $T \subseteq T'$. □

It will now be easy to show that if i) we want to use the axioms Condorcet Consistency, E-Invariance, Weak Homogeneity, Majority Profile Convergence and Extended Condorcet Criterion and ii) we want to have a set of compromise solutions as large as possible, then we must use the extended prudent order preference function.

Theorem 4 *The extended prudent order preference function is the largest preference function (in the sense of the inclusion) that verifies the Condorcet Consistency, E-Invariance, Weak Homogeneity, Majority Profile Convergence and Extended Condorcet Criterion.*

Proof: We know from theorem 2 that axioms CC, MOPC, EI and WH imply:

$$f(u) \subseteq \mathcal{E}(R_{>\beta})$$

Axiom XCC implies that

$$f(u) \subseteq \mathcal{E}(T)$$

Combining these two inclusions, we get:

$$f(u) \subseteq \mathcal{E}(T \cup R_{>\beta}) = \mathcal{XPO}(u)$$

Since the extended prudent order preference function verifies the 5 axioms (see proposition 5), it is consequently the largest preference function that verifies the 5 axioms. This completes the proof. □

Let us finally check the independence between the axioms CC, EI, WH, MOPC and XCC.

1. The preference function $f_1(u) = \mathcal{E}(T)$ verifies EI, WH, MOPC, XCC but not CC. Consider for instance a profile with three alternatives $\{a, b, c, d\}$ and with the following acyclic strict majority relation $M = \{(a, b), (a, c), (c, d)\}$. Then $f_1(u) = \mathcal{O}$, and consequently $f_1(u) \not\subseteq \mathcal{E}(M)$.

2. The preference function f_2 will be defined as follows (see preference function g_2):

$$f_2(u) = \begin{cases} \mathcal{O} & \text{if } u = u^* \\ \mathcal{XPO}(u) & \text{otherwise} \end{cases}$$

   f_2 verifies CC, WH, MOPC, XCC but not EI.

3. The preference function f_3 will be defined as follows (see preference function g_3):

$$f_3(u) = \begin{cases} \mathcal{O} & \text{if } \exists \lambda \in \mathbb{N} : B = B^* + \lambda B' \\ \mathcal{XPO}(u) & \text{otherwise} \end{cases}$$

   f_3 verifies CC, EI, MOPC, XCC, but not WH.

4. The preference function $f_4(u) = g_4(u)$ (Kemeny orders) verifies CC, EI, WH, XCC but not MOPC.

5. The prudent order preference function $f_5(u) = \mathcal{PO}(u)$ verifies CC, EI, WH, MOPC but not XCC. Consider for instance Taylor's example.

6 Conclusion

In this work we presented a first axiomatic characterization of a preference function that associates to a profile of linear orders the whole set of prudent orders. Among the axioms that we introduced, the axioms of Majority Oriented Profile Convergence and Invariance are the most specific of the prudent approach.

The main issue of future work will be to analyze and characterize other prudent ranking rules in the same axiomtatic framework. As a first illustration, we introduced the extended prudent order preference function, which could be characterized by simply considering an additional axiom.

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