HAL
open science

Chaos synchronization: from the genesis to polytopic observers

Gilles Millérioux, Jamal Daafouz

To cite this version:

Gilles Millérioux, Jamal Daafouz. Chaos synchronization: from the genesis to polytopic observers. 1st IFAC Conference on Analysis and Control of Chaotic Systems, CHAOS06, Jun 2006, Reims, France. pp.CDROM. hal-00119538

HAL Id: hal-00119538

https://hal.science/hal-00119538v1

Submitted on 11 Dec 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

CHAOS SYNCHRONIZATION: FROM THE GENESIS TO POLYTOPIC OBSERVERS

Gilles Millérioux [1] **Jamal Daafouz** [2]
gilles.millerioux@esstin.uhp-nancy.fr,
jamal.daafouz@ensem.inpl-nancy.fr

CRAN-ESSTIN 2, Rue Jean Lamour 54519
Vandoeuvre-Les-Nancy Cedex (France)
CRAN-ENSEM 2, Avenue de la Foret de Haye, 54516
Vandoeuvre-Les-Nancy Cedex (France)

Abstract: This paper focuses on chaos synchronization. The history of this famous problem is traced from an automatic control point of view. First, it is recalled the two synchronization schemes introduced by Pecora and Carroll in 1990 called heterogeneous and homogeneous driving. Then, it is described how chaos synchronization has turned into an observer design problem during the years 1997s. Finally, the interest of the authors of this paper to a special class of observers, named polytopic observers is motivated.

1. INTRODUCTION

In control theory, as far as chaos is concerned, people are interested in two major problems: control of chaos for which a survey is proposed in (Fradkov and Evans, 2002) and synchronization of chaos. This paper should not be considered as an exhaustif survey paper but it aims at tracing the history of the famous problem known as *chaos synchronization* frequently encountered in chaos based encryption/decryption techniques. In the years 1990, works of Hubler have been shown that driving systems with aperiodic signals could induce some interesting behaviors like nonlinear resonances or stimulation of particular modes. The idea of using a special aperiodic signal, namely a chaotic signal, to drive a nonlinear system originates from the pioneering works of (Pecora and Carroll, 1990) (Pecora and Carroll, 1991). Their works focused on a special configuration involving two systems coupled so that behavior of the second is dependent on the behavior of the first, but conversely, the first is not influenced by the behavior of the second. The first system producing the chaotic signal is called the *drive* system. The second system forced by the chaotic signal, also called *driving signal* is named the *response*. Such a driving is often referred to as *unidirectional coupling* and distinguishes from the bidirectional coupling. Under suitable conditions, the *response* exhibits a chaotic regime synchronized with the *drive*. Such a fact is known as *chaos synchronization*. There exist several definitions of synchronization, see (Blekhman *et al.*, 1997), but we can consider that synchronization between two systems is characterized by coincident temporal behaviors.

Chaos synchronization appears as an amazing phenomena since, at first glance, it would be reasonable that two chaotic systems, due to the property of sensitive dependence on initial conditions, would defy synchronization. Such a paradox has stimulated the researchers for a large decade.

[1] Gilles Millerioux works at the Henri Poincare University of Nancy in the Research Center on Automatic Control of Nancy (CRAN)
[2] Jamal Daafouz works at the Institut National Polytechnique de Lorraine in the Research Center on Automatic Control of Nancy (CRAN)

In 1997, several papers (Nijmeijer and Mareels, 1997)(Itoh *et al.*, 1997)(Millerioux, 1997)(Grassi and Mascolo, 1997) have brought out a connection between the chaos synchronization problem and the design of observers, an issue borrowed from the control theory. Indeed, it has been shown that the problem of achieving a synchronization between a *drive* chaotic system and a *response* system through a unidirectional coupling is highly connected to a state reconstruction issue. As a matter of fact, since 1999, the chaos synchronization problem has really entered the control scene and has become a popular open problem from the control theory point of view (Blondel *et al.*, 1999).

General state reconstruction approaches have been presented in the literature. Relative recent results on observer-based methods can be found in (Cruz and Nijmeijer, 2000) for the Extended Kalman Filters, in (Huijberts *et al.*, 2001) for observers with linearizable dynamics, in (Pogromsky and Nijmeijer, 1998) for observers derived from the concept of absolute stability , in (Sira-ramirez and Cruz-hernandez, 2001) for observers dedicated to systems having a Generalized Hamiltonian Forms, in (Boutat-Baddas *et al.*, 2004) for sliding mode observers dedicated to systems which are not linearizable by output injection and with observability bifurcation. However, they don't really take into account the specificity of the chaotic motion. And yet, one of the interesting specificities is that the state vector of a chaotic system lies in a compact set which has a special structure. As a result, given a chaotic attractor, the state vector is bounded and the bounds can be componentwise a priori known. Such a fact has been taken into consideration through the concept of *polytopic observer* during the year 2003 (Millerioux and Daafouz, 2003).

The content of the paper is the following. The Section 2 gives some basic backgrounds on chaotic systems. Section 3 recalls the different configurations encountered in (Pecora and Carroll, 1991) wherein the authors address the chaos synchronization problem (year 1991). The distinct issues to be solved when tackling such a problem are formulated. Then, it is shown how this formulation has turned into an observer design problem (years 1997s). Finally, in Section 4, the special class of observers called polytopic observers and introduced in (Millerioux and Daafouz, 2003) in the year 2003 is presented. The interest of using such a class in the context of the chaos synchronization problem is highlighted.

2. DEFINING CHAOS

Autonomous discrete-time nonlinear dynamics also called maps can be written in the explicit form :

$$x_{k+1} = f(x_k), \qquad x(k=0) = x_0 \tag{1}$$

n corresponds to the dimension of the system, $x_k \in \mathbb{R}^n$ is the state vector,
Those systems are called *autonomous* since the discrete time k does not appear explicitly in the equation (1). Conversely, if k appears explicitly, the system is said to be *non autonomous*.

The solution of (1) from the initial condition x_0, is a sequence of points called *iterated sequence*, or *discrete phase trajectory*, or *orbit*. The explicit solution denoted $x(k, x_0)$ of (1) can generally not be expressed in terms of known elementary and transcendental functions but actually, we are only interested in the steady-states the iterated sequence may reach. They can be more or less complex, here are the most often encountered.

Equilibrium point: An equilibrium point (also called fixed point) is characterized by $x_{k+1} = x_k = x^* \ \ \forall k > k_0$

Periodic orbit of order K: A periodic orbit of order K (also called cycle of finite period K) obeys $x_{k+K} = x_k$ and $x_{k+K'} \neq x_k \ \ \forall k > k_0$ with $K' < K$

Chaotic orbit: A chaotic orbit can be viewed in some sense as an infinite period trajectory and thus obeys $x_{k+K} \neq x_K \ \ \forall k > k_0$ and $\forall K$, x_k is bounded.

This definition is not sufficient to really characterizing chaos. As a matter of fact, it should be observed that the word "chaos" has not exactly the same meaning everywhere. A first definition, cited in (Mira, 1987), has been given in 1950. This definition states that chaos is related to the existence of infinite sequences, each of them having an infinite number of unstable cycles and their limit points when the period tends toward infinity. It follows that the state space has some singularities with fractal structure inducing a sensitivity with respect to initial conditions.

A more formal definition is due to Devaney (Devaney, 1989). Let (M, d) denote a compact metric space, and let $f : M \to M$ be a continuous map.

Definition 1. *The discrete dynamical system (1) is said to be* chaotic *if the following conditions are fulfilled:*
(C1) (Sensitive dependence on initial conditions) *There exists a quantity $\varepsilon > 0$ such that for any $x_0 \in M$ and any $\delta > 0$, there exists a point $y_0 \in M$ with $d(x_0, y_0) < \delta$ and an integer $k \geq 0$ such that $d(x_k, y_k) \geq \varepsilon$;*
(C2) (One-sided topological transitivity) *There exists some $x_0 \in M$ with $(x_k)_{k\geq 0}$ dense in M;*

(C3) (Density of periodic points) *The set* $D = \{x_0 \in M;\ \exists k > 0,\ x_k = x_0\}$ *is dense in* M.

Chaotic orbits give rise in the phase space to fractal structures called *strange attractors*. They can be characterized by some numerical quantities as the dimension, the entropy or the *Lyapunov Exponents* (LE). LE are particularily interesting because we can derive some dimension and entropy measures.

Let $M \subset \mathbb{R}^n$ denote a compact manifold and let $f : M \to M$ be a differentiable map. Let f^k denote the function resulting from k compositions of f with itself, let $D_x f = (\partial f_i / \partial x_k^{(j)})_{ij}$ be the Jacobean of f, the Jacobean of f^k is denoted $D_x f^k = D_{f^{k-1}(x_k)} f \cdot D_{f^{k-2}(x_k)} f) \ldots D_{f(x_k)} f \cdot D_x f$ and the adjoint of $D_x f^k$ is denoted $D_x f^{k*}$.

The Lyapunov Exponents are defined as the logarithm of the eigenvalues of the following quantity :

$$\Lambda_x = \lim_{k \to +\infty} (D_x f^{k*} \cdot D_x f^k)^{1/2k}$$

They capture the asymptotic behavior of the derivative along almost every orbit. When they are positive, the Lyapunov Exponents indicate expansion in some direction whereas when they are negative, they indicate contraction in some other direction. When a chaotic systems has at least two positive Lyapunov exponents, it is referred to as *hyperchaotic*.

Now, define a splitting of M induced by $\mathbb{R}^l \times \mathbb{R}^{n-l}$. The *Conditional Lyapunov Exponents* are defined as the logarithm of the eigenvalues of the two following quantities :

$$\Lambda_{x,l} = \lim_{k \to +\infty} (D_{x,l} f^{k*} \cdot D_{x,l} f^k)^{1/2k}$$

$$\Lambda_{x,n-l} = \lim_{k \to +\infty} (D_{x,n-l} f^{k*} \cdot D_{x,n-l} f^k)^{1/2k}$$

where $D_{x,l} f^k$ and $D_{x,n-l} f^k$ are the $l \times l$ and $(n-l) \times (n-l)$ diagonal blocks of the full Jacobian $D_x f^k$.
The existence of these limits, for almost all $x_k \in M$, is discussed in (Osedelec, 1968)(Mendes, 1998).

3. FROM THE GENESIS TO THE OBSERVER CONFIGURATION

This section recalls the genesis of the chaos synchronization problem highlighted in (Pecora and Carroll, 1991). Two configurations are distinguished: the heterogeneous and the homogeneous driving.

3.1 Heterogeneous driving

Let us consider the chaotic dynamical system (1), called *drive*, partitioned in two subsystems g_1 et g_2 as follows:

$$\begin{cases} w_{k+1} &= g_1(w_k, v_k) \\ v_{k+1} &= g_2(w_k, v_k) \end{cases} \tag{2}$$

with dim $w_k = n - m$ and dim $v_k = m$. The vector v_k consists of the *driving signal*. *Heterogeneous driving* corresponds to the situation when v_k forces a second system called *response* governed by:

$$z_{k+1} = r(z_k, v_k) \tag{3}$$

with dim $z_k = p$. It may correspond to a pendulum (*response*) forced by a signal emanating from a chaotic oscillator (*drive*).

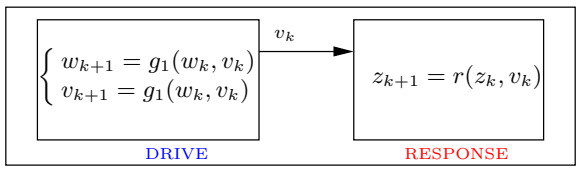

Fig. 1. Heterogeneous driving

In (Pecora and Carroll, 1991), the authors arises the question whether the *response* exhibits a stable chaotic behavior. Assuming that the *response* is actually chaotic, Pecora et Carroll suggest to investigate such an issue through the variational equation:

$$\Delta z_{k+1} = r(z'_k, v_k) - r(z_k, v_k) \tag{4}$$

Assuming that $z'_k - z_k$ is small, (4) can be rewritten :

$$\Delta z_{k+1} = D_z r \Delta z_k \tag{5}$$

where $D_z r$ stands for the Jacobean of r with respect to z_k.
The (at least) local stability of the *response* means that two distinct trajectories of (3), starting at distinct but nearby initial conditions z_0 and z'_0 will converge one to the other. It is well known that the stability of a nonlinear system behaving close to its equilibrium point can be assessed by examining the corresponding Jacobean matrix. The convergence of any trajectory to this point can be proved by considering its eigenvalues since in such a situation, they are constant. On the other hand, the Jacobean is no longer constant but time-varying when the system is chaotic. As a result, examining the eigenvalues is no longer a valid procedure. A solution to this problem is to resort to the computation of the Lyapunov exponents since they extend the notion of eigenvalues. The problem arising here lies in that the dynamics of the *response* not only depends on

z_k but on an external value v_k too. As a result, the Conditional Lyapunov Exponents (CLE) are more suited. Their computation requires the knowledge of the Jacobean derived from the full system *drive-response* (2)-(3). Then, the CLE are the Lyapounov Exponents of the diagonal blocks associated to the *response* state variables.

A necessary condition for the *response* to be stable is that all the CLE are negative. Sufficiency is not guaranteed because the Lyapunov Exponents are some quantities which are obtained by linearization along the trajectory of the *response*. It's a local concept which does not provide any information on the admissible initializations ensuring a stable behavior. Furthermore, if several chaotic regimes coexist in the phase space, the regime exhibited by the *response* may not be identical to the *drive* one.

3.2 Homogeneous driving

In (Pecora and Carroll, 1991), the authors investigate the simplified problem when $r = g_1$ and $p = n - m$, that is when the *response* is a replica of one of the subsystem of the *drive* (Fig. 2). It is then described by:

$$z_{k+1} = g_1(z_k, v_k) \tag{6}$$

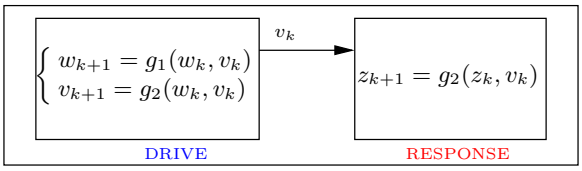

Fig. 2. Homogeneous driving

Such a configuration corresponds to a so-called *homogeneous driving*. Similar to the *heterogeneous driving*, the (at least) local stability of the *response* means that two distinct trajectories of (6), starting at distinct but nearby initial conditions z_0 and z_0', will converge one to the other. The signal v_k forcing both the subsystem g_1 of (2) and the *response* (6), stability means that, for z_k belonging to an non empty set $U \subseteq \mathbb{R}^n$, the following equality can hold :

$$\lim_{k\to\infty} \|z_k - w_k\| = 0 \tag{7}$$

Unlike the *heterogeneous driving*, (2) and (6) are ensured to exhibit the same chaotic behavior. We say that they are synchronized.

The problem of chaos synchronization with *homogeneous driving* can be summed up in the following issue :

Issue 1

a) finding out a partitioning g_1, g_2 of the *drive*
b) finding out a driving signal v_k
c) finding out a set U of suitable initial conditions for the *response*

in order to ensure the synchronization (7).

Some additional issues may be investigated:

d) what is the impact of some disturbances acting on the *drive* or on the driving signal ?
e) what is the impact of some parameters mismatch between the *drive* and the *response* ?

It is worth emphsizing that state reconstruction, an issue we usually have to face in control theory, consists in estimating the non accessible state variables through the available data, say the output of the system. As a result, a key point is that, whenever the *response* (6) guarantees (7), we can consider that it acts as an observer. The available data correspond to the v_k's of the driving signal while z_k "mirrors" the behavior of the non accessible variable w_k of the *drive*. In 1997, several papers (Nijmeijer and Mareels, 1997)(Itoh *et al.*, 1997)(Millerioux, 1997)(Grassi and Mascolo, 1997) have brought out a connection between the chaos synchronization formulated as in the Issue 1 and the design of observers, The next section aims at detailing such an observer approach.

3.3 The observer approach

Based on the previous central consideration, the formulation of the chaos synchronization issue has been converted into the following terms. Let us consider a nonlinear autonomous system described by

$$\begin{cases} x_{k+1} &= f(x_k) \\ y_k &= h(x_k) \end{cases} \tag{8}$$

with dim $x_k = n$ and dim $y_k = m$. Assume that (8) exhibits a chaotic behavior. For reconstructing the state x_k of this system, a general structure can be suggested :

$$\begin{cases} \hat{x}_{k+1} &= \tilde{f}(\hat{x}_k) + l(y_k, \hat{x}_k) \\ \hat{y}_k &= h(\hat{x}_k) \end{cases} \tag{9}$$

with l a function to be specified in order to achieve:

$$\lim_{k\to\infty} \|Tx_k - \hat{x}_k\| = 0 \quad \forall \hat{x}_0 \in U \tag{10}$$

where U is a non empty set of initial conditions. T is a constant matrix of appropriate dimension. If only a part of x_k are reconstructed, (9) is a reduced observer. If all the components of x_k are reconstructed, T is the identity matrix.

The *drive-response* setup is called *observer configuration* with driving signal y_k (Fig. 3). Most often,

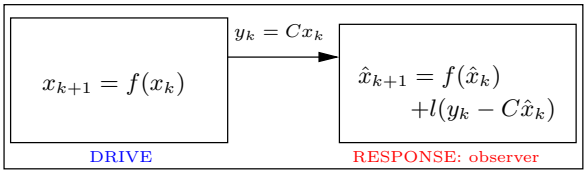

Fig. 3. Observer configuration

the output equation is linear and reads $\hat{y}_k = C\hat{x}_k$. The special setting $C = [\mathbf{0}\ \mathbf{1}_m]$ would correspond to the fact that the driving signal is a subvector of x_k. The setting $\tilde{f} = g_2$, $l = 0$, dim $\hat{x}_k = n - m$ and $T = [\ \mathbf{1}_{n-m}\ \mathbf{0}]$ corresponds to an *homogeneous driving*. Others settings correspond to an *heterogeneous driving* with $r(\hat{x}_k, y_k) = \tilde{f}(\hat{x}_k) + l(y_k, \hat{x}_k)$. As a consequence, we can claim that the observer configuration is generic and includes the two configurations first introduced by Pecora and Carroll. The Issue 1 turns into:

Issue 1'

a) finding out a function l and $\tilde{f}$
b) finding out an output y_k (or a matrix C)
c) finding out a set U of suitable initial conditions $\hat{x}_0$ for the observer

in order to ensure the synchronization (10).

The issue *a)* often reduces to checking for a special function of the form $l(y_k, \hat{x}_k) = \mathcal{L}(y_k - C\hat{x}_k)$, with $\mathcal{L}$ called the *gain matrix* while $\tilde{f} = f$. The matrix $\mathcal{L}$ may be either constant or time-varying. When $U = \mathbb{R}^n$, the synchronization is said to be global.

4. POLYTOPIC OBSERVERS

The concept of polytopic observers has been motivated in the introduction. The key idea is based on the consideration that a broad class of chaotic systems (8) can admit an equivalent convexified form, that can be rewritten:

$$\begin{cases} x_{k+1} &= \mathcal{A}(\rho_k)x_k + E(y_k) \\ y_k &= Cx_k \end{cases} \tag{11}$$

with

$$\mathcal{A}(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k)\bar{A}_i, \quad \rho_k = \Psi(x_k) \tag{12}$$

Ψ being an adequate function rendering ρ_k available from the output y_k of the system. The vector ξ_k belongs to the compact set $\mathcal{S} = \{\mu_k \in \mathbb{R}^N, \mu_k = (\mu_k^{(1)}, \ldots, \mu_k^{(N)})^T,\ \mu_k^{(i)} \geq 0\ \ \forall i$ and $\sum_{i=1}^{N} \mu_k^{(i)} = 1\}$. Owing to the convexity of $\mathcal{S}$, the whole set of matrices $\bar{A}_i$ defines a polytope, denoted $\mathcal{D}_{\mathcal{A}}$, with corresponding convex hull denoted $\mathbf{Co}\mathcal{A}\{\bar{A}_1, \ldots, \bar{A}_N\}$. The $\bar{A}_i$'s are constant matrices and are named vertices. In (Millerioux *et al.*, 2005), the conditions under which such a rewriting is possible are provided along with the computational aspects for finding out the $\bar{A}_i$'s. Therein, it is explained why (11) includes piecewise linear maps, Lur'e systems and most of chaotic systems with polynomial nonlinearities such as the Logistic map, the Henon map, the Mandelbrot map.

For the reconstruction of the state x_k, the following observer can be suggested :

$$\begin{cases} \hat{x}_{k+1} &= \mathcal{A}(\rho_k)\hat{x}_k + E(y_k) + \mathcal{L}(\rho_k)(y_k - \hat{y}_k) \\ \hat{y}_k &= C\hat{x}_k \end{cases} \tag{13}$$

where $\mathcal{L}$ is a time-varying gain, depending on the available time-varying parameter ρ_k. It it is straightforward to show that the state reconstruction error $\epsilon_k \triangleq x_k - \hat{x}_k$ obtained from (11) and (13) is governed by :

$$\epsilon_{k+1} = (\mathcal{A}(\rho_k) - \mathcal{L}(\rho_k)C)\epsilon_k \tag{14}$$

As a result, the dynamics of the state reconstruction is linear in ϵ_k but is time-varying since the matrices depend on the parameter ρ_k. Thus, (14) can be viewed as a special Linear Parameter Varying (LPV) system whose global stability has to be ensured. The gain $\mathcal{L}$ is suggested to obey :

$$\mathcal{L}(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}(\rho_k)\bar{L}_i \tag{15}$$

with the ξ_k's having to coincide with the ones involved in the polytopic decomposition (12) (this constraint can be always fulfilled since ξ_k depends on ρ_k which is on-line available). (15) means that $\mathcal{L}$ is enforced to take values in a polytope $\mathcal{D}_{\mathcal{L}}$ with convex hull $\mathbf{Co}\mathcal{L}\{\bar{L}_1, \ldots, \bar{L}_N\}$, the $\bar{L}_i$'s having to be determined. Expressing $\mathcal{L}$ like (15) and using (12) causes (14) to turn into :

$$\epsilon_{k+1} = \sum_{i=1}^{N} \xi_k^{(i)}(\bar{A}_i - \bar{L}_i C)\epsilon_k \tag{16}$$

which represents the dynamics of a system having a polytopic form.

Theorem 1. *((Millerioux and Daafouz, 2003)) System (16) converges globally toward zero if there exist some symmetric matrices P_i, matrices G_i and F_i such that, $\forall (i,j) \in \{1, \ldots, N\} \times \{1, \ldots, N\}$, the following set of Linear Matrix Inequalities is feasible*

$$\begin{bmatrix} P_i & (\bullet)^T \\ G_i\bar{A}_i - F_iC & G_i + G_i^T - P_j \end{bmatrix} > 0 \tag{17}$$

and $\mathcal{L}(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}\bar{L}_i$ with $\bar{L}_i = G_i^{-1}F_i$

It can be shown that the time-varying gain $\mathcal{L}(\rho_k) = \sum_{i=1}^{N} \xi_k^{(i)}\bar{L}_i$ with $\bar{L}_i = G_i^{-1}F_i$ ensures the existence of a Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^+$, defined by $V(\epsilon_k, \xi_k) = \epsilon_k^T \mathcal{P}_k \epsilon_k$ with

$\mathcal{P}_k = \sum_{i=1}^{N} \xi_k^{(i)} P_i$ and $\xi_k \in \mathcal{S}$, called poly-quadratic Lyapunov function which fulfills :

$$V(\epsilon_{k+1}, \xi_{k+1}) - V(\epsilon_k, \xi_k) < 0 \quad \forall \xi_k \in \mathcal{S} \qquad (18)$$

Eq. (18) is equivalent to the poly-quadratic stability of (16) and is sufficient for global asymptotic stability.

To conclude, the observer approach associated with the convexified form makes possible to handle the chaos synchronization problem formulated in the Issue 1' in a systematic and tractable way and is summed up in the following points:

a) $l(y_k, \hat{x}_k) = \mathcal{L}(y_k - C\hat{x}_k)$ with $\mathcal{L}$ computed from the solutions of (17), $\tilde{f} = f$
b) $y_k = Cx_k$ with C such that (17) is feasible
c) the set U of admissible initial conditions $\hat{x}_0$ is $U = \mathbb{R}^n$ since the convergence is global

5. CONCLUSION

Since the work of (M. *et al.*, 1993), there has been tremendous interest in using chaos synchronization for the purpose of secure communication. Various chaotic cryptosystems have been proposed and several methods for hiding an information have been suggested. Chaos synchronization using the polytopic approach has been extended in this context. An efficient encryption/decryption setup can resort to unknown input polytopic observers, the information to be encrypted acting as the unknown input.

REFERENCES

Blekhman, I.I., A.L. Fradkov, H. Nijmeijer and A.Y Pogromsky (1997). On self-synchronization and controlled synchronization. *Systems and Control letters* **31**(5), 299–305.

Blondel, V. D., E. D. Sontag, M. Vidyasagar and J. C. Willems (1999). *Open Problems in Mathematical Systems and Control Theory.* Communication and Control Engineering. Springer Verlag.

Boutat-Baddas, L., J.P. Barbot, D. Boutat and R. Tauleigne (2004). Sliding mode observers and observability singularity in chaotic synchronization. *Mathematical Problems in Engineering* (1), 11–31.

Cruz, C. and H. Nijmeijer (2000). Synchronization through filtering. *International Journal of Bifurcation and Chaos* **110**(4), 763–775.

Devaney, R. L. (1989). *An introduction to Chaotic Dynamical Systems.* Addison-Wesley. Redwood City, CA.

Fradkov, A. and R. Evans (2002). Control of chaos: Survey 1997-2000. In: *15-th IFAC World Congress.* Barcelona, Spain.

Grassi, G. and S. Mascolo (1997). Nonlinear observer design to synchronize hyperchaotic systems via a scalar signal. *IEEE Trans. Circuits. Syst. I: Fundamental Theo. Appl* **44**(10), 1011–1014.

Huijberts, H.J.C., T. Lilge and H. Nijmeijer (2001). Nonlinear discrete-time synchronization via extended observers. *International Journal of Bifurcation and Chaos* **11**(7), 1997–2006.

Itoh, M., C.W. Wu and L.O. Chua (1997). Communications systems via chaotic signals from a reconstruction viewpoint. *International Journal of Bifurcation and Chaos* **7**(2), 275–286.

M., Cuomo K., Oppenheim A. V. and Strogatz S. H. (1993). Robustness and signal recovery in a synchronized chaotic system. *International Journal of Bifurcation and Chaos* **3**(6), 1629–1638.

Mendes, R. V. (1998). Conditional exponents, entropies and a measure of dynamical self-organization. *Phys. Letters* **248**, 167–171.

Millerioux, G. (1997). Chaotic synchronization conditions based on control theory for systems described by discrete piecewise linear maps. *International Journal of Bifurcation and Chaos* **7**(7), 1635–1649.

Millerioux, G. and J. Daafouz (2003). Polytopic observer for global synchronization of systems with output measurable nonlinearities. *International Journal of Bifurcation and Chaos* **13**(3), 703–712.

Millerioux, G., F. Anstett and G. Bloch (2005). Considering the attractor structure of chaotic maps for observer-based synchronization problems. *Mathematics and Computers in Simulation* **68**(1), 67–85.

Mira, C. (1987). *Chaotic Dynamics.* World Scientific Publishing Co.. Singapore.

Nijmeijer, H. and I.M.Y. Mareels (1997). An observer looks at synchronization. *IEEE Trans. Circuits. Syst. I: Fundamental Theo. Appl* **44**, 882–890.

Osedelec, V. I. (1968). *Moscow Math. Soc.*

Pecora, L.M. and T.L. Carroll (1990). Synchronization in chaotic systems. *Phys. Rev. Lett.* **64**, 821–824.

Pecora, L.M. and T.L. Carroll (1991). Driving systems with chaotic signals. *Phys. Rev. A* **44**(8), 2374–2383.

Pogromsky, A. and H. Nijmeijer (1998). Observer-based robust synchronization of dynamical systems. *International Journal of Bifurcation and Chaos* **8**(11), 2243–2254.

Sira-ramirez, H. and C. Cruz-hernandez (2001). Syn-
chronization of chaotic systems :a generalized hamiltonian approach. *International Journal of Bifurcation and Chaos* **11**(5), 1381–1395.